

Comparing inconsistency of Pairwise Comparison Matrices depending on entries

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Abstract Pairwise comparisons have been a long standing technique for comparing alternatives/criteria and their role has been pivotal in the development of modern decision making methods. Since several types of pairwise comparison matrices (e.g. multiplicative, additive, fuzzy) are proposed in literature, in this paper, we investigate, for which type of matrix, decision makers are more coherent when they express their subjective preferences. By performing an experiment, we found that the additive approach provides the worst level of coherence.

Keywords Multi-criteria decision making · Pairwise Comparison Matrix · Consistency evaluation · Abelian linearly ordered group · Behavioral Operations Research

1 Introduction

Pairwise comparisons matrices (PCMs) have been long used in psychophysical research to judge and compare sensory intensities (Thurstone, 1927; Gulliksen, 1959). This technique has also gained popularity in decision analysis (Koczkodaj et al, 2016) as they are deemed to be more precise than direct judgements (Por and Budescu, 2017; Millet, 1997). In fact, it allows a decision maker to compare two alternatives at a time, thus reducing the complexity of a decision-making problem, especially when the set of alternatives is large.

Formally, let $X = \{x_1, x_2, \dots, x_n\}$ be a set of decision elements such as criteria or alternatives, the entry a_{ij} of a PCM quantifies the preference intensity of x_i over x_j . In the literature, several types of preference evaluations are considered:

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- if $a_{ij} \in]0, +\infty[$ represents a preference ratio, then $A = (a_{ij})$ is a *multiplicative* PCM (Barzilai and Golany, 1990);
- if $a_{ij} \in \mathbb{R} =]-\infty, +\infty[$ represents a preference difference, then $A = (a_{ij})$ is an *additive* PCM (Barzilai, 1998);
- if $a_{ij} \in]0, 1[$ reflects a preference degree, then $A = (a_{ij})$ is a *fuzzy* PCM (Tanino, 1984; Chiclana et al, 2009a).

Thus, the entry a_{ij} of a PCM always expresses a relative preference of x_i over x_j ; it does not include a measurement unit. As an example, if we have to decide on a car purchase, we express the preference ratio, the preference difference or the preference degree (multiplicative, additive or fuzzy preference, respectively) of one criterion over one other (e.g. price over the fuel consumption) with respect to the selection of a car (we do not compare *euro* and *km/l* but we express subjective preferences).

The foremost used type of evaluation, at least with respect to the number of real-world applications, is the preference ratio, adopted among others by Saaty (1980) in the theory of the Analytic Hierarchy Process (AHP). Several scales have been proposed (Meesariganda and Ishizaka, 2017), but the most popular is the finite scale $\{\frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Several techniques have also been proposed to derive the priorities from a PCM (Barzilai, 1997; Cook and Kress, 1988; Jones and Mardle, 2004; Ishizaka and Lusti, 2006; Cavallo and D’Apuzzo, 2012a). However, a priority makes sense only if the decision maker has a minimum level of coherence. For example, if the decision-maker prefers A over B and B over C , then, by transitivity, A should be preferred over C . If the number of intransitive relations is too high, the calculated priorities are not reliable.

In the literature, it has never been studied for which preference evaluation model (i.e. additive, multiplicative or fuzzy) the decision maker is more (or less) coherent. This paper aims at filling this gap. In particular, we perform an experiment in order to measure the coherence of the participants when they express their subjective preferences by means of additive, multiplicative and fuzzy PCMs.

Experiments have been long used in psychology (Wixted, 2017). Experimental psychology employs human participants to study a variety of topics, including among others sensation and perception, memory, cognition, learning, motivation, emotion. Laboratory experiments have then grown in economics culminating in the Economics Nobel Prize in 2002; it was divided equally between Daniel Kahneman “*for having integrated insights from psychological research into economic science, especially concerning human judgement and decision-making under uncertainty*” and Vernon L. Smith “*for having established laboratory experiments as a tool in empirical economic analysis, especially in the study of alternative market mechanisms*” (https://www.nobelprize.org/nobel_prizes/economic-sciences/laureates/2002/). Experiments have also been used in Behavioral Operations Research (White et al, 2017). Keeney et al (1990) asked subjects to provide a direct ranking of alternatives and then to solve the problem with the Multi Attribute Utility (MAU); in a final ranking,

80% changed their initial ranking, mostly in agreement with the MAU ranking. Linares (2009) asked 18 students to rank cars by applying AHP; an automatic algorithm removed the inconsistencies and a new ranking was generated. In a questionnaire, the majority of the students said that when intransitivities were removed, their preferences were not better represented. Huizingh and Vrolijk (1997) asked participants to select a room to rent; they observed that participants were more satisfied with the AHP result than with a random selection. Ishizaka et al (2011) statistically compared the ranking of AHP with three additional rankings given by subject during the experiment: one at the beginning, one after providing the pairwise comparisons and one after learning the ranking provided by AHP. Rankings were found similar and moreover AHP helped the decision-makers to reformulate their rankings by taking into account the suggestions made by AHP. Later, Ishizaka and Siraj (2018) replicate the experiment with another decision problem and the multi-criteria decision methods AHP, MACBETH and SMART. The same observation was found.

In this paper, an experimental approach is also used to compare the three different preference evaluation approaches (i.e. multiplicative, additive and fuzzy) proposed in the literature for dealing with PCMs. Participants have been asked to express subjective preferences about a decision making problem with four criteria and five alternatives; it has been found that the worst level of coherence occurs when participants express preferences differences (e.g. additive approach).

The paper is organized as follows: Section 2 introduces preliminaries useful in the sequel; Section 3 describes the experiment; Section 4 provides results of the experiment and discusses them; Section 5 provides conclusions and future work.

2 An algebraic approach to Pairwise Comparison Matrices

Since there is not a unique way for representing the preference intensities under the form of PCM, Cavallo and D'Apuzzo (2009) and Cavallo (2014) propose a unified approach based on Abelian linearly ordered groups (Alo-groups), i.e. commutative groups equipped with an ordering relation. In this context, it is possible to define a consistency condition (i.e. a cardinal transitivity condition of preferences on triplets of decision elements) such that, if it holds, the decision maker is considered fully coherent and his/her judgements are not contradictory (for conditions weaker than consistency see (Cavallo and D'Apuzzo, 2015) and (Cavallo and D'Apuzzo, 2016)). Several authors are following this approach based on Alo-group (e.g. Hou (2016), Xia and Chen (2015), Ramík (2015)).

2.1 Alo-groups

As the following definition shows, an Alo-group is a commutative group equipped with an ordering relation:

Definition 1 Let G be a non-empty set, $\odot : G \times G \rightarrow G$ a binary operation on G , \leq a weak order on G . Then, $\mathcal{G} = (G, \odot, \leq)$ is an *Abelian linearly ordered group* (Alo-group for short), if (G, \odot) is an Abelian group and

$$a \leq b \Rightarrow a \odot c \leq b \odot c. \quad (1)$$

An *isomorphism* between two Alo-groups $\mathcal{G} = (G, \odot, \leq)$ and $\mathcal{H} = (H, \circ, \leq)$ is a bijection $\psi : G \rightarrow H$ that is both a lattice isomorphism and a group isomorphism, that is:

$$a < b \Leftrightarrow \psi(a) < \psi(b) \quad \text{and} \quad \psi(a \odot b) = \psi(a) \circ \psi(b). \quad (2)$$

Let us denote with: e the identity element; $a^{(-1)}$ the symmetric of a with respect to \odot ; \div the inverse operation of \odot (i.e. $a \div b = a \odot b^{(-1)}$). By definition, an Alo-group \mathcal{G} is a *lattice ordered group* (Birkhoff, 1984), that is there exists $\max\{a, b\}$, for each $a, b \in G$; thus, we can consider the following operation, called \mathcal{G} -distance, between two elements:

$$d_{\mathcal{G}} : (a, b) \in G \times G \rightarrow d_{\mathcal{G}}(a, b) = \max\{a \div b, b \div a\} \in G. \quad (3)$$

Let $n \in \mathbb{N}_0$ and $a \in G$; then, the *(n)-natural- power* (Cavallo and D'Apuzzo, 2009) $a^{(n)}$ of $a \in G$ is:

$$a^{(n)} = \begin{cases} e, & \text{if } n = 0 \\ \odot_{i=1}^n a, & \text{if } n \geq 1. \end{cases} \quad (4)$$

2.1.1 Real continuous Alo-groups

An Alo-group $\mathcal{G} = (G, \odot, \leq)$ is called *continuous* if the operation \odot is continuous (Cavallo and D'Apuzzo, 2009), and *real* if G is a subset of the real line \mathbb{R} and \leq is the weak order on G inherited from the usual order on \mathbb{R} . From now on, we will assume that $\mathcal{G} = (G, \odot, \leq)$ is a real continuous Alo-group, with G an open interval. Under these assumptions, the equation $x^{(n)} = a$ has a unique solution (Cavallo and D'Apuzzo, 2009); thus, it is reasonable to consider the following notions of *(n)-root* and *\mathcal{G} -mean*.

Definition 2 (Cavallo and D'Apuzzo, 2009) For each $n \in \mathbb{N}$ and $a \in G$, the *(n)-root* of a , denoted by $a^{(\frac{1}{n})}$, is the unique solution of the equation $x^{(n)} = a$, that is:

$$\left(a^{(\frac{1}{n})}\right)^{(n)} = a.$$

Definition 3 (Cavallo and D'Apuzzo, 2009) The \mathcal{G} -mean $m_{\mathcal{G}}(a_1, a_2, \dots, a_n)$ of the elements a_1, a_2, \dots, a_n of G is

$$m_{\mathcal{G}}(a_1, a_2, \dots, a_n) = \begin{cases} a_1 & \text{for } n=1, \\ (\odot_{i=1}^n a_i)^{(1/n)} & \text{for } n \geq 2. \end{cases}$$

Proposition 1 (Cavallo and D'Apuzzo, 2009) Let $\psi : G \rightarrow H$ be an isomorphism between real continuous Alo-groups $\mathcal{G} = (G, \odot, \leq)$ and $\mathcal{H} = (H, \circ, \leq)$, with G and H open intervals. Then, the following equalities hold:

$$\begin{aligned} m_{\mathcal{G}}(g_1, g_2, \dots, g_n) &= \psi^{-1}(m_{\mathcal{H}}(\psi(g_1), \psi(g_2), \dots, \psi(g_n))); \\ m_{\mathcal{H}}(h_1, h_2, \dots, h_n) &= \psi(m_{\mathcal{G}}(\psi^{-1}(h_1), \psi^{-1}(h_2), \dots, \psi^{-1}(h_n))). \end{aligned}$$

Examples of real continuous Alo-groups are the following ones:

Multiplicative Alo-group. $\mathcal{R}^+ = (\mathbb{R}^+, \cdot, \leq)$, where \cdot is the usual multiplication on \mathbb{R} , $e = 1$ and \mathcal{R}^+ -mean operator is the geometric mean

$$m_{\mathcal{R}^+}(a_1, \dots, a_n) = \sqrt[n]{\prod_{i=1}^n a_i}. \quad (5)$$

\mathcal{R}^+ -norm of $a \in \mathbb{R}^+$ is $\|a\|_{\mathcal{R}^+} = \max\{a, \frac{1}{a}\}$, \mathcal{R}^+ -distance between a and b is $d_{\mathcal{R}^+}(a, b) = \max\{\frac{a}{b}, \frac{b}{a}\}$.

Additive Alo-group. $\mathcal{R} = (\mathbb{R}, +, \leq)$, where $+$ is the usual addition on \mathbb{R} , $e = 0$ and \mathcal{R} -mean operator is the arithmetic mean

$$m_{\mathcal{R}}(a_1, \dots, a_n) = \frac{\sum_{i=1}^n a_i}{n}. \quad (6)$$

\mathcal{R} -norm of $a \in \mathbb{R}$ is $\|a\|_{\mathcal{R}} = \max\{a, -a\} = |a|$ (i.e. the absolute value of a), \mathcal{R} -distance between a and b is $d_{\mathcal{R}}(a, b) = \max\{a - b, b - a\} = |a - b|$.

Fuzzy Alo-group. $\mathcal{I} = (]0, 1[, \otimes, \leq)$, where $\otimes :]0, 1[^2 \rightarrow]0, 1[$ is the operation defined by

$$a \otimes b = \frac{ab}{ab + (1-a)(1-b)},$$

$e = 0.5$ and \mathcal{I} -mean operator is given by the following mean:

$$m_{\mathcal{I}}(a_1, \dots, a_n) = \frac{\sqrt[n]{\prod_{i=1}^n a_i}}{\sqrt[n]{\prod_{i=1}^n a_i} + \sqrt[n]{\prod_{i=1}^n (1-a_i)}}. \quad (7)$$

\mathcal{I} -norm of $a \in]0, 1[$ is $\|a\|_{\mathcal{I}} = \max\{a, 1-a\}$, \mathcal{I} -distance between a and b is the following one:

$$d_{\mathcal{I}}(a, b) = \max \left\{ \frac{a(1-b)}{a(1-b) + (1-a)b}, \frac{b(1-a)}{b(1-a) + (1-b)a} \right\}.$$

Two real continuous Alo-groups $\mathcal{G} = (G, \odot, \leq)$ and $\mathcal{H} = (H, \circ, \leq)$, with G and H open intervals, are isomorphic (Cavallo and D'Apuzzo, 2009).

Isomorphisms between multiplicative and additive Alo-groups are the following ones:

$$h : x \in \mathbb{R}^+ \rightarrow \log x \in \mathbb{R} \quad h^{-1} : y \in \mathbb{R} \rightarrow e^y \in \mathbb{R}^+. \quad (8)$$

Isomorphisms between multiplicative and fuzzy Alo-groups are the following ones:

$$g : t \in \mathbb{R}^+ \rightarrow \frac{x}{x+1} \in]0, 1[, \quad g^{-1} : y \in]0, 1[\rightarrow \frac{y}{1-y} \in \mathbb{R}^+. \quad (9)$$

2.2 PCMs over real continuous Alo-groups

Let $\mathcal{G} = (G, \odot, \leq)$ be a real continuous Alo-group, with G an open interval. Let us consider a set $X = \{x_1, x_2, \dots, x_n\}$ of decision elements, such as criteria or alternatives, then a PCM over $\mathcal{G} = (G, \odot, \leq)$ is defined as follows:

$$A = \begin{matrix} & \begin{matrix} x_1 & x_2 & \cdots & x_n \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{matrix} & \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \end{matrix}, \quad (10)$$

where a_{ij} quantifies the preference intensity of x_i over x_j . Thus, $a_{ij} = e$ if and only if there is indifference between x_i and x_j , $a_{ij} > e$ if and only if x_i is strictly preferred to x_j , whereas $a_{ij} < e$ expresses the reverse preference. From now on, we assume that each PCM satisfies \mathcal{G} -reciprocity property proposed by Cavallo and D'Apuzzo (2009):

Definition 4 $A = (a_{ij})$ is a \mathcal{G} -reciprocal PCM if verifies the condition:

$$a_{ji} = a_{ij}^{(-1)} \quad \forall i, j \in \{1, \dots, n\}.$$

If the decision maker is fully coherent when he/she expresses his/her preferences by means of a PCM $A = (a_{ij})$ defined over \mathcal{G} , then $A = (a_{ij})$ is \mathcal{G} -consistent.

Definition 5 $A = (a_{ij})$ is a \mathcal{G} -consistent PCM, if verifies the following condition:

$$a_{ik} = a_{ij} \odot a_{jk} \quad \forall i, j, k \in \{1, \dots, n\}. \quad (11)$$

\mathcal{G} -consistency generalizes several consistency notions proposed in literature, such as multiplicative consistency (Saaty, 1980), additive consistency (Barzilai, 1998) and fuzzy multiplicative consistency, called multiplicative transitivity by Tanino (1984, 1988) (see also (Chiclana et al, 2009b; Herrera-Viedma et al, 2004)).

In order to measure how much a PCM is far from \mathcal{G} -consistency, Cavallo and D'Apuzzo (2009) propose a consistency index, which is expressed as \mathcal{G} -mean of \mathcal{G} -distances from \mathcal{G} -consistency; i.e. \mathcal{G} -mean of \mathcal{G} -distances between a_{ik} and $a_{ij} \odot a_{jk}$.

Definition 6 Let $A = (a_{ij})$ be a \mathcal{G} -reciprocal PCM of order $n \geq 3$. Then, its \mathcal{G} -consistency index is:

$$I_{\mathcal{G}}(A) = \left(\bigodot_{(i,j,k) \in T} d_{\mathcal{G}}(a_{ik}, a_{ij} \odot a_{jk}) \right)^{\left(\frac{1}{|T|}\right)},$$

with $T = \{(i, j, k) : i < j < k\}$ and $|T| = \frac{n(n-2)(n-1)}{6}$ its cardinality.

Proposition 2 *The following statements hold:*

1. $I_{\mathcal{G}}(A) \geq e$;
2. $I_{\mathcal{G}}(A) = e \Leftrightarrow A$ is \mathcal{G} -consistent.

Moreover, \mathcal{G} -consistency index satisfies properties proposed by Brunelli (2016) and Brunelli and Fedrizzi (2015) (see Cavallo and D'Apuzzo (2012b)).

Proposition 3 (Cavallo and D'Apuzzo, 2009) *Let $\mathcal{G} = (G, \odot, \leq)$ and $\mathcal{H} = (H, \circ, \leq)$ be two real continuous Alo-groups, with G and H open intervals, and $\psi : G \rightarrow H$ an isomorphism. Then, for each PCM $A = (a_{ij})$ over \mathcal{G} , the following equality holds:*

$$\psi(I_{\mathcal{G}}(A)) = I_{\mathcal{H}}(\psi(A)),$$

where $\psi(A) = (\psi(a_{ij}))$.

Remark 1 It is important to note that, Proposition 3 allows us to compare inconsistency of PCMs defined over different Alo-groups (see Example (2) and Example (3)).

2.2.1 Multiplicative Approach

Reciprocity is expressed as follows:

$$a_{ji} = \frac{1}{a_{ij}} \quad \forall i, j \in \{1, \dots, n\}.$$

Consistency and consistency index in Definition 5 and Definition 6 are expressed as follows:

$$a_{ik} = a_{ij} \cdot a_{jk} \quad \forall i, j, k \in \{1, \dots, n\}, \quad (12)$$

$$\begin{aligned}
I_{\mathcal{R}^+}(A) &= \left(\prod_{i=1}^{n-2} \prod_{j=i+1}^{n-1} \prod_{k=j+1}^n d_{\mathbb{R}^+}(a_{ik}, a_{ij} \cdot a_{jk}) \right)^{\frac{6}{n(n-2)(n-1)}} = \\
&= \left(\prod_{i=1}^{n-2} \prod_{j=i+1}^{n-1} \prod_{k=j+1}^n \max\left\{ \frac{a_{ik}}{a_{ij}a_{jk}}, \frac{a_{ij}a_{jk}}{a_{ik}} \right\} \right)^{\frac{6}{n(n-2)(n-1)}}.
\end{aligned} \tag{13}$$

Example 1 Let us consider the reciprocal multiplicative PCM in (Cavallo and D'Apuzzo, 2009):

$$A = \begin{pmatrix} 1 & \frac{1}{7} & \frac{1}{7} & \frac{1}{5} \\ 7 & 1 & \frac{1}{2} & \frac{1}{3} \\ 7 & 2 & 1 & \frac{1}{9} \\ 5 & 3 & 9 & 1 \end{pmatrix};$$

thus, its consistency index is the following one:

$$\begin{aligned}
I_{\mathcal{R}^+}(A) &= \sqrt[4]{d_{\mathcal{R}^+}(a_{13}, a_{12} \cdot a_{23}) \cdot d_{\mathcal{R}^+}(a_{14}, a_{12} \cdot a_{24}) \cdot d_{\mathcal{R}^+}(a_{14}, a_{13} \cdot a_{34}) \cdot d_{\mathcal{R}^+}(a_{24}, a_{23} \cdot a_{34})} = \\
&= \sqrt[4]{2 \cdot 4.2 \cdot 12.6 \cdot 6} = 5.02.
\end{aligned}$$

2.2.2 Additive Approach

Reciprocity is expressed as follows:

$$a_{ji} = -a_{ij} \quad \forall i, j \in \{1, \dots, n\}.$$

Consistency and consistency index in Definition 5 and Definition 6 are expressed as follows:

$$a_{ik} = a_{ij} + a_{jk} \quad \forall i, j, k \in \{1, \dots, n\}, \tag{14}$$

$$\begin{aligned}
I_{\mathcal{R}}(A) &= \frac{6}{n(n-2)(n-1)} \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^n d_{\mathbb{R}}(a_{ik}, a_{ij} + a_{jk}) = \\
&= \frac{6}{n(n-2)(n-1)} \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^n (\max\{a_{ik} - (a_{ij} + a_{jk}), (a_{ij} + a_{jk}) - a_{ik}\}).
\end{aligned} \tag{15}$$

Example 2 Let us consider the following reciprocal additive PCM:

$$B = \begin{pmatrix} 0 & 2 & 3 & 7 \\ -2 & 0 & 3 & 5 \\ -3 & -3 & 0 & 2 \\ -7 & -5 & -2 & 0 \end{pmatrix};$$

thus, its consistency index is the following one:

$$\begin{aligned} I_{\mathcal{R}}(B) &= \frac{d_{\mathcal{R}}(a_{13}, a_{12} + a_{23}) + d_{\mathcal{R}}(a_{14}, a_{12} + a_{24}) + d_{\mathcal{R}}(a_{14}, a_{13} + a_{34}) + d_{\mathcal{R}}(a_{24}, a_{23} + a_{34})}{4} = \\ &= \frac{2 + 0 + 2 + 0}{4} = 1. \end{aligned}$$

Interestingly, although they are expressed on two different scales, values of inconsistency indices from different representations of preferences are comparable. For instance, by applying the isomorphism in (8), we have that the consistency index of the multiplicative PCM $h^{-1}(B)$ is $I_{R^+}(h^{-1}(B)) = 2.72$ that is smaller than $I_{R^+}(A)$ in Example 1; thus, B is more consistent than A .

2.2.3 Fuzzy Approach

Reciprocity is expressed as follows:

$$a_{ji} = 1 - a_{ij} \quad \forall i, j \in \{1, \dots, n\}.$$

Consistency and consistency index in Definition 5 and Definition 6 are expressed as follows:

$$a_{ik} = \frac{a_{ij} \cdot a_{jk}}{a_{ij} \cdot a_{jk} + (1 - a_{ij}) \cdot (1 - a_{jk})} \quad \forall i, j, k \in \{1, \dots, n\}, \quad (16)$$

$$I_{\mathcal{I}}(A) = \frac{a}{a + b}, \quad (17)$$

with

$$\begin{aligned} a &= \left(\prod_{i=1}^{n-2} \prod_{j=i+1}^{n-1} \prod_{k=j+1}^n d_{\mathcal{I}}(a_{ik}, a_{ij} \otimes a_{jk}) \right)^{\frac{6}{n(n-2)(n-1)}} = \\ &= \left(\prod_{i=1}^{n-2} \prod_{j=i+1}^{n-1} \prod_{k=j+1}^n \max \left\{ \frac{a_{ik}(1 - (a_{ij} \otimes a_{jk}))}{a_{ik}(1 - (a_{ij} \otimes a_{jk})) + (1 - a_{ik})(a_{ij} \otimes a_{jk})}, \frac{(a_{ij} \otimes a_{jk})(1 - a_{ik})}{(a_{ij} \otimes a_{jk})(1 - a_{ik}) + (1 - (a_{ij} \otimes a_{jk}))a_{ik}} \right\} \right)^{\frac{6}{n(n-2)(n-1)}} \end{aligned}$$

and

$$\begin{aligned} b &= \left(\prod_{i=1}^{n-2} \prod_{j=i+1}^{n-1} \prod_{k=j+1}^n (1 - d_{\mathcal{I}}(a_{ik}, a_{ij} \otimes a_{jk})) \right)^{\frac{6}{n(n-2)(n-1)}} = \\ &= \left(\prod_{i=1}^{n-2} \prod_{j=i+1}^{n-1} \prod_{k=j+1}^n 1 - \max \left\{ \frac{a_{ik}(1 - (a_{ij} \otimes a_{jk}))}{a_{ik}(1 - (a_{ij} \otimes a_{jk})) + (1 - a_{ik})(a_{ij} \otimes a_{jk})}, \frac{(a_{ij} \otimes a_{jk})(1 - a_{ik})}{(a_{ij} \otimes a_{jk})(1 - a_{ik}) + (1 - (a_{ij} \otimes a_{jk}))a_{ik}} \right\} \right)^{\frac{6}{n(n-2)(n-1)}} \end{aligned}$$

Example 3 Let us consider the following reciprocal fuzzy PCM:

$$C = \begin{pmatrix} 0.5 & 0.67 & 0.86 & 0.67 \\ 0.33 & 0.5 & 0.75 & 0.33 \\ 0.14 & 0.25 & 0.5 & 0.14 \\ 0.33 & 0.67 & 0.86 & 0.5 \end{pmatrix}.$$

Its consistency index can be computed by applying isomorphisms in (9); thus, we have:

$$I_{\mathcal{I}}(C) = g(I_{\mathcal{R}^+}(g^{-1}(C))) = 0.59.$$

We stress that $I_{\mathcal{R}^+}(g^{-1}(C)) = 1.41$ is smaller than both $I_{\mathcal{R}^+}(A)$ in Example 1 and $I_{R^+}(h^{-1}(B))$ in Example 2; thus C is more consistent than both A and B .

3 The experiment

In order to measure the inconsistency of different types of PCMs, we perform an experiment; we use an opinion survey with a sample of 120 students of Department of Architecture of University of Naples “Federico II”. The students completed the survey during a class period and received instructions to fill out it. On average, the experiment lasted one hour.

In a problem inspired by an example provided by Saaty (1987), participants were asked to choose their preferred university based on the following four criteria:

- C_1 . **Distance from Napoli.** As far as distance is concerned, the farther away from Napoli an university is perceived to be better (but not necessarily linearly);
- C_2 . **Reputation of the University.** It concerns how the university is rated;
- C_3 . **Number of students in a class.** It is assumed that small classes are preferred to large ones because more attention is ensured to the students;
- C_4 . **Attractiveness of the city.** It concerns how happy the participant felt at the city.

Thus, participants were asked to compare pairwise the criteria with respect to the selection of a University and to compare pairwise five universities on the set $X = \{x_1 = \text{University of Sannio}, x_2 = \text{University “La Sapienza”}, x_3 = \text{University of Bologna}, x_4 = \text{Politecnico Milano}, x_5 = \text{University of Barcelona}\}$ with respect to each criterion. Hierarchy of the problem is shown in Figure 1.

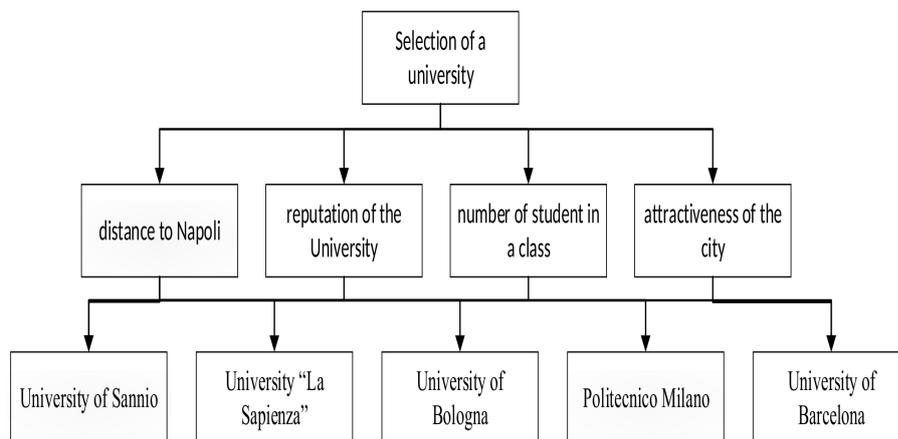


Fig. 1 Hierarchy of the problem

The goal of this experimentation is to evaluate the coherence of the participants by means of the following approaches:

- M.** The pairwise comparisons are expressed as preference ratio on the interval $]0, +\infty[$ (i.e. multiplicative approach on $]0, +\infty[$);
- S.** The pairwise comparisons are expressed as preference ratio on Saaty scale $\{\frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ (i.e. multiplicative approach on Saaty scale; we stress that, although there is not an Alo-group defined on Saaty scale, since multiplicative PCMs over Saaty scale are used in several real world applications, they are also considered in our experiment);
- A.** The pairwise comparisons are expressed as preference difference on the interval $] - \infty, +\infty[$ (i.e. additive approach);
- F.** The pairwise comparisons are expressed as a preference degree on the interval $]0, 1[$ (i.e. fuzzy approach).

Thus, we built four opinion surveys Q^M , Q^S , Q^A and Q^F ; each Q^j , with $j \in \{M, S, A, F\}$, includes five questionnaires:

- Q_0^j for pairwise comparing criteria C_1, C_2, C_3 , and C_4 with respect to the goal (i.e. the selection of a university);
- Q_1^j for pairwise comparing universities in X with respect to C_1 ;
- Q_2^j for pairwise comparing universities in X with respect to C_2 ;
- Q_3^j for pairwise comparing universities in X with respect to C_3 ;
- Q_4^j for pairwise comparing universities in X with respect to C_4 .

In the design of the experiment, we decided that each participant is allowed to complete only one survey. This “between-subject” designed experiment is more appropriate for our purpose than a “within-subject” designed experiment, where participants are exposed to more than one survey. It has been long observed that within-subject cannot be used when independence of multiple exposure is not warranty. Poulton (1973) found that within-subject experiments have a range effect. In fact, the respondents have a reference point when they are responding the second survey. Greenwald (1976) criticizes within-subject designs based on the effects of practice, sensitization and carry-over that confound causality. The reader can refer to Charness et al (2012) for further details about the two experimental methods, and to Ishizaka and Siraj (2018) for a recent application of the between-subject method. In particular, Ishizaka and Siraj (2018) evaluated three multi-criteria decision-making tools where each participant used only one tool. In addition, the statistical units included in the sample come from the same population (homogeneous for age and level degree); thus, a low within group variability is expected. Thus, we have that:

- survey Q^M , including questionnaires $Q_0^M, Q_1^M, Q_2^M, Q_3^M$ and Q_4^M , is given to 30 participants m_1, \dots, m_{30} and we asked them to express their preference ratios on $]0, +\infty[$. In this way, for each participant m_k , with $k \in \{1, \dots, 30\}$, we obtained 5 multiplicative PCMs $M_0^k, M_1^k, M_2^k, M_3^k$ and M_4^k on $]0, +\infty[$;
- survey Q^S , including questionnaires $Q_0^S, Q_1^S, Q_2^S, Q_3^S$ and Q_4^S , is given to 30 participants s_1, \dots, s_{30} and we asked them to express their preference ratios on Saaty scale. In this way, for each participant s_k , with $k \in \{1, \dots, 30\}$, we obtained 5 multiplicative PCMs $S_0^k, S_1^k, S_2^k, S_3^k$ and S_4^k on Saaty scale;

- survey Q^A , including questionnaires $Q_0^A, Q_1^A, Q_2^A, Q_3^A$ and Q_4^A , is given to 30 participants a_1, \dots, a_{30} and we asked them to express their preference differences on $] - \infty, +\infty[$. In this way, for each participant a_k , with $k \in \{1, \dots, 30\}$, we obtained 5 additive PCMs $A_0^k, A_1^k, A_2^k, A_3^k$ and A_4^k on $] - \infty, +\infty[$;
- survey Q^F , including questionnaires $Q_0^F, Q_1^F, Q_2^F, Q_3^F$ and Q_4^F , is given to 30 participants f_1, \dots, f_{30} and we asked them to express their preference degrees on $]0, 1[$. In this way, for each participant f_k , with $k \in \{1, \dots, 30\}$, we obtained 5 fuzzy PCMs $F_0^k, F_1^k, F_2^k, F_3^k$ and F_4^k on $]0, 1[$.

As an example, Figure 2 shows questionnaire Q_2^M ; we stress that the preference ratios are chosen in $[1, +\infty[$ and not in $]0, +\infty[$, because, for each pair of universities, we ask to choose the preferred one; the remaining entries of the related PCMs are computed by means of reciprocity.

We stress that each pairwise comparison expresses a subjective relative preference of a decision element over one other and it does not include a measurement unit; it is a preference ratio, a preference difference or a preference degree (e.g. preference of C_1 over C_2 with respect to selection of a university, or the preference of x_1 over x_2 with respect to the reputation of the University).

Please, for each pair of universities, select the preferred university with respect to its reputation and write your preference ratio by choosing a number in $[1, +\infty[$. If you are indifferent between two universities, you have to write a preference ratio equal to 1 and to select both the universities.

E.g. a preference ratio equal to 3.5 means that the selected university is preferred three times and half with respect to the other one.

<input type="radio"/> University of Sannio	<input type="radio"/> "Sapienza" of Roma	Preference ratio = ____
<input type="radio"/> University of Sannio	<input type="radio"/> University of Bologna	Preference ratio = ____
<input type="radio"/> University of Sannio	<input type="radio"/> Politecnico of Milano	Preference ratio = ____
<input type="radio"/> University of Sannio	<input type="radio"/> University of Barcellona	Preference ratio = ____
<input type="radio"/> "Sapienza" of Roma	<input type="radio"/> University of Bologna	Preference ratio = ____
<input type="radio"/> "Sapienza" of Roma	<input type="radio"/> Politecnico of Milano	Preference ratio = ____
<input type="radio"/> "Sapienza" of Roma	<input type="radio"/> University of Barcellona	Preference ratio = ____
<input type="radio"/> University of Bologna	<input type="radio"/> Politecnico of Milano	Preference ratio = ____
<input type="radio"/> University of Bologna	<input type="radio"/> University of Barcellona	Preference ratio = ____
<input type="radio"/> Politecnico of Milano	<input type="radio"/> University of Barcellona	Preference ratio = ____

Fig. 2 Questionnaire Q_2^M for comparing the five universities with respect to criterion C_2 by means of the multiplicative approach.

In order to establish for which type of PCM the participants are more coherent when they express subjective preferences, firstly, for each $i \in \{0, 1, 2, 3, 4\}$, we computed the following means of consistency indices:

$$m_{\mathcal{R}^+}(I_{\mathcal{R}^+}(M_i^1) \dots, I_{\mathcal{R}^+}(M_i^k) \dots, I_{\mathcal{R}^+}(M_i^{30})); \quad (18)$$

$$m_{\mathcal{R}^+}(I_{\mathcal{R}^+}(S_i^1) \dots, I_{\mathcal{R}^+}(S_i^k) \dots, I_{\mathcal{R}^+}(S_i^{30})); \quad (19)$$

$$m_{\mathcal{R}}(I_{\mathcal{R}}(A_i^1) \dots, I_{\mathcal{R}}(A_i^k) \dots, I_{\mathcal{R}}(A_i^{30})); \quad (20)$$

$$I_i^F = m_{\mathcal{I}}(I_{\mathcal{I}}(F_i^1) \dots, I_{\mathcal{I}}(F_i^k) \dots, I_{\mathcal{I}}(F_i^{30})). \quad (21)$$

By item 1. in Proposition 2, we have that (18) and (19) are geometric means (5) in $[1, +\infty[$, (20) is an arithmetic mean (6) in $[0, +\infty[$ and (21) is a fuzzy \mathcal{I} -mean (7) in $[0.5, 1[$. Thus, in order to compare the inconsistencies of PCMs defined over different Alo-groups, by Proposition 1 and isomorphisms in (8) and (9), we obtain the following fuzzy \mathcal{I} -means on the same interval $[0.5, 1[$:

$$I_i^M = g(m_{\mathcal{R}^+}(I_{\mathcal{R}^+}(M_i^1) \dots, I_{\mathcal{R}^+}(M_i^k) \dots, I_{\mathcal{R}^+}(M_i^{30}))); \quad (22)$$

$$I_i^S = g(m_{\mathcal{R}^+}(I_{\mathcal{R}^+}(S_i^1) \dots, I_{\mathcal{R}^+}(S_i^k) \dots, I_{\mathcal{R}^+}(S_i^{30}))); \quad (23)$$

$$I_i^A = g(h^{-1}(m_{\mathcal{R}}(I_{\mathcal{R}}(A_i^1) \dots, I_{\mathcal{R}}(A_i^k) \dots, I_{\mathcal{R}}(A_i^{30}))))); \quad (24)$$

4 Results and discussion

The results of the experiment described in the previous section are shown in Table 1; where:

- column I^M contains $I_0^M, I_1^M, I_2^M, I_3^M$ and I_4^M in (22);
- column I^S contains $I_0^S, I_1^S, I_2^S, I_3^S$ and I_4^S in (23);
- column I^A contains $I_0^A, I_1^A, I_2^A, I_3^A$ and I_4^A in (24);
- column I^F contains $I_0^F, I_1^F, I_2^F, I_3^F$ and I_4^F in (21);
- column p contains p value of ANOVA (ANalysis Of VAriance) test;
- column F contains F value of ANOVA test.

	I^M	I^S	I^A	I^F	p	F
$i = 0$	0.69	0.76	0.96	0.72	2.48E-8	15.15
$i = 1$	0.72	0.76	0.99	0.73	3.28E-10	19.43
$i = 2$	0.73	0.77	0.98	0.73	0.00143	5.52
$i = 3$	0.7	0.73	0.98	0.71	0.00083	5.96
$i = 4$	0.78	0.76	0.99	0.73	0.00004	8.59

Table 1 Means of consistency indices of the PCMs built by surveys Q^M, Q^S, Q^A, Q^F and ANOVA results.

It can be seen from Table 1 that the multiplicative case (I^M) is the least inconsistent in four cases ($i = 0, 1, 2, 3$) out of five; we recall that I^M, I^S, I^A and I^F assume value in $[0.5, 1[$ and the greater their value the more inconsistency is. The most inconsistent is always the additive case (I^A). These differences are significant with an ANOVA test at a confidence threshold $p = 0.05$; indeed, p value is always smaller than the confidence threshold and F value is always higher than critical value $F_{crit} = 2.69$. However, if we remove the additive case (Table 2), then the differences are not anymore significant at $p=0.05$; indeed, all p values are higher than the confidence threshold and the F values are

	I^M	I^S	I^F	p	F
$i = 0$	0.69	0.76	0.72	0.067	2.79
$i = 1$	0.72	0.76	0.73	0.132	2.07
$i = 2$	0.73	0.77	0.73	0.058	2.94
$i = 3$	0.7	0.73	0.71	0.287	1.26
$i = 4$	0.78	0.76	0.73	0.496	0.70

Table 2 Means of consistency indices of the PCMs built by surveys Q^M, Q^S, Q^F and ANOVA results.

smaller than the critical value $F_{crit} = 3.10$.

The additive method is therefore to be avoided for a better consistency.

5 Conclusions and future work

In this paper, an experiment is performed in order to compare, by a behavioral point of view, three different preference evaluation approaches proposed in literature for dealing with PCMs; i.e. multiplicative, additive and fuzzy approach.

Multiplicative, additive and fuzzy preferences share the same algebraic structure (Cavallo and D’Apuzzo, 2009), i.e. Alo-group. Alo-groups and related isomorphisms are necessary to show the formal equivalence between different representations and to naturally extend concepts and properties from one representation to another one. By quoting Fraleigh (2002), an isomorphism is defined as “the concept of two systems being structurally identical”. In this paper, we use properties of Alo-groups and related isomorphisms for comparing inconsistency of multiplicative, additive and fuzzy PCMs, in a real decision making problem.

By means of opinion surveys, participants have been asked to express subjective preferences about a decision making problem with four criteria and five alternatives; the experiment shows that by expressing “preference ratios” (i.e. multiplicative preferences) or “preference degrees” (i.e. fuzzy preferences), the participants are more coherent than when they express “preference differences” (i.e. additive preferences).

Of course, participants when they express their preferences have not in mind the isomorphisms and, moreover, it could be that the notion of “preference difference” has a less understandable meaning with respect to “preference ratio” or “preference degree”. Thus, Behavioral Operations Research is an interesting approach when human behavior needs to be examined and taken into consideration.

Our future work will be directed to apply our experimental approach to other decision problems. Moreover, we plan to carry out similar experiments in order to analyze conditions weaker than consistency, i.e. a cardinal transitivity, in both individual and collective decision procedures, such as ordinal transitivity (Cavallo and D’Apuzzo, 2015), that is the minimal logical requirement

that decision maker's preferences should satisfy, and weak consistency (Cavallo and D'Apuzzo, 2016) that ensures reliability to a priority vector proposed by Cavallo and D'Apuzzo (2012a).

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