Supplemental Material to "Experimental Time-Resolved Interference with Multiple Photons of Different Colors"

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MEASUREMENT OF $g^{(2)}$

In our experiment, conditioned on detection of write-out photons, we measure the second-order autocorrelation function $g^{(2)}$ of read-out photons[1], which is defined as:

$$g^{(2)} = \frac{p_{23|1}}{p_{2|1}p_{3|1}},$$

where $p_{2|1}$, $p_{3|1}$ represent the photon detecting probability at detector 2 and 3 conditioned on photon detection at 1, $p_{23|1}$ represent the coincident detecting probability between 2 and 3 (Shown in FIG.1).

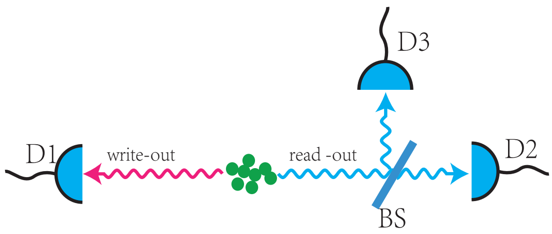

FIG. 1: **Measurement of $g^{(2)}$.**

TRANSFER MATRIX

As a general method to reconstruct a linear optical network as a black box, tomographic measurement with both single-photon inputs and two-photon inputs are necessary. In our case, since the optical network is rather simple, we make use of a different approach instead by modeling the network with a number of parameters and compare theoretically estimated results with measured results using a maximal likelihood method [2]. This method requires much less experimental overhead. Based on structure of the network, the transfer matrix $\mathcal{M}_k$ $(k = 0, 1, 2, 3)$ for different internal phases can be modeled in terms of BS reflectivity $R_1 \sim R_3$, phase φ_k ($\varphi_0 \sim \varphi_3$, phase setting from 0 to $3\pi/2$) and 9 efficiencies p_{mn}.

$$\mathcal{M}_0 = \begin{pmatrix} -\sqrt{p_{11}R_2R_3} & \sqrt{p_{12}}\left(e^{-i\varphi_0}\sqrt{T_1T_3} - \sqrt{R_1T_2R_3}\right) & \sqrt{p_{13}}\left(ie^{-i\varphi_0}\sqrt{R_1T_3} + i\sqrt{T_1T_2R_3}\right) \\ i\sqrt{p_{21}R_2T_3} & \sqrt{p_{22}}\left(i\sqrt{R_1T_2T_3} + ie^{-i\varphi_0}\sqrt{T_1R_3}\right) & \sqrt{p_{23}}\left(\sqrt{T_1T_2T_3} - e^{-i\varphi_0}\sqrt{R_1R_3}\right) \\ \sqrt{p_{31}T_2} & -\sqrt{p_{32}R_1R_2} & i\sqrt{p_{33}T_1R_2} \end{pmatrix}, \tag{1}$$

where $T_i = 1 - R_i$, p_{mn} is proportional to path-dependent transmission efficiency from input port n to output port m, and all the parameters are real numbers. Other matrices $\mathcal{M}_k$ $(k = 1, 2, 3)$ have similar forms as $\mathcal{M}_0$, just by changing the corresponding phase φ_k.

To experimentally reconstruct these transfer matrices, for each setting of φ we perform three measurements and in each measurement we inject a single photon into one of the three input ports and measure the photon detection probabilities in all output ports. In this way, we can get 9 (3×3) detection probabilities for each phase setting. Measured results for all phase settings are given below:

$$E_0 = \begin{pmatrix} 0.0842 & 0.0157 & 0.1503 \\ 0.0845 & 0.1696 & 0.0108 \\ 0.1027 & 0.0958 & 0.0872 \end{pmatrix}, \tag{2}$$

$$E_1 = \begin{pmatrix} 0.0848 & 0.0904 & 0.0868 \\ 0.0838 & 0.0953 & 0.0814 \\ 0.1008 & 0.0967 & 0.0886 \end{pmatrix}, \tag{3}$$

$$E_2 = \begin{pmatrix} 0.0849 & 0.1703 & 0.0140 \\ 0.0858 & 0.0139 & 0.1550 \\ 0.1031 & 0.0954 & 0.0880 \end{pmatrix}, \tag{4}$$

$$E_3 = \begin{pmatrix} 0.0854 & 0.0944 & 0.0779 \\ 0.0834 & 0.0897 & 0.0877 \\ 0.1026 & 0.0939 & 0.0860 \end{pmatrix}, \tag{5}$$

where $E_{k,mn}$ is the measured detection probability of output port m when a single photon is injected from input port n and the internal phase is set to φ_k. In principle, the measured probability $E_{k,mn}$ should be proportional to the transfer matrix element $|\mathcal{M}_{k,mn}|^2$. Therefore, we define a function of

$$f = \sum_{k=0}^{3} \sum_{m=1}^{3} \sum_{n=1}^{3} \left| |\mathcal{M}_{k,mn}|^2 - E_{k,mn} \right|^2, \tag{6}$$

and optimize all parameters used in our modeling of the network to minimize f numerically. Through this optimization, we get a best estimate for the matrix parameters. By direct calculation with these parameters, we reconstruct the transfer matrices as following:

$$\mathcal{M}_0 = \begin{pmatrix} -0.5617 & 0.2435 + 0.0128i & -0.0119 + 0.7498i \\ 0.5626i & -0.0124 + 0.7950i & -0.2056 - 0.0119i \\ 0.6177 & -0.5974 & 0.5723i \end{pmatrix}, \tag{7}$$

$$\mathcal{M}_1 = \begin{pmatrix} -0.5606 & -0.2526 - 0.5197i & 0.4855 + 0.2854i \\ 0.5615i & 0.5059 + 0.3111i & 0.2552 + 0.4827i \\ 0.6166 & -0.5963 & 0.5712i \end{pmatrix}, \tag{8}$$

$$\mathcal{M}_2 = \begin{pmatrix} -0.5576 & -0.7867 - 0.0809i & 0.0756 - 0.2164i \\ 0.5584i & 0.0788 - 0.2118i & 0.7512 + 0.0752i \\ 0.6132 & -0.5931 & 0.5681i \end{pmatrix}, \tag{9}$$

$$\mathcal{M}_3 = \begin{pmatrix} -0.5607 & -0.2833 + 0.5204i & -0.4861 + 0.2569i \\ 0.5616i & -0.5065 + 0.2813i & 0.2837 - 0.4833i \\ 0.6167 & -0.5964 & 0.5714i \end{pmatrix}. \tag{10}$$

Please note that the proportional coefficient between $E_{k,mn}$ and $|\mathcal{M}_{k,mn}|^2$ is 0.27, which corresponds to the product of our source efficiency of 0.45 and average detection efficiency of 0.6 for photons in the output ports.

FIDELITY LIMITATIONS

In our experiment, as our read-out photons are not ideal single photons, there may exist two photons at one input port(FIG.2). These high order events would affect the interference and decrease the fidelity obtained. Here, we mainly consider ideal situation A and two imperfect situations B & C:

A: each input and output port exist one photon.

B: two photons at input port si ,one photon at sj and no photons at the rest input port$(i, j = 1, 2, 3, i \neq j)$.

C: two photons at input port si, one photon at the other two input ports sj, $sk(i, j, k = 1, 2, 3, i \neq j \neq k,\ j > k)$.

Following reference[3], the electric field operators at input ports are described as :

$$E^{+}_{si}(t) = \zeta_{si}(t, \omega_{si}) a_{si}, \tag{11}$$

where $\zeta_{si}(t, \omega_{si})$ is the spatio-temporal mode function describing photon wavepacket shown in FIG.1.(d) in the main text. Given the transfer matrix $\mathcal{U}$, field operators at three output ports d1, d2, d3 are,

$$\{E^{+}_{d1}(t), E^{+}_{d2}(t), E^{+}_{d3}(t)\}^T = \mathcal{U}\{E^{+}_{s1}(t), E^{+}_{s2}(t), E^{+}_{s3}(t)\}^T. \tag{12}$$

In ideal situation A, the input state is:

$$|\Psi_{in1}\rangle = a^{\dagger}_{s1} a^{\dagger}_{s2} a^{\dagger}_{s3} |vac\rangle. \tag{13}$$

To obtain the probability for photon detections in output ports d1, d2, d3 at times $t_1 = t_3 + \delta t_1$, $t_2 = t_3 + \delta t_2$, t_3, we have to apply the field operators $E^{\pm}_{d1}(t_1)$, $E^{\pm}_{d2}(t_2)$, $E^{\pm}_{d3}(t_3)$, the joint probability is,

$$P_{co1}(\delta t_1, \delta t_2) = \int_{-\infty}^{+\infty} dt_3 \langle \Psi_{in1}| E^{-}_{d3}(t_3) E^{-}_{d2}(t_2) E^{-}_{d1}(t_1) E^{+}_{d1}(t_1) E^{+}_{d2}(t_2) E^{+}_{d3}(t_3) |\Psi_{in1}\rangle. \tag{14}$$

Conditioned on detection of write-out photons, average probability of existence of read-out photons at input port is P_{ro}=0.45, where the probability is P_1=0.86 for only one photon and P_2=0.14 for existence of two photons deduced from an average $g^{(2)}$=0.25. The probability to detect a photon after the network is P_d=0.6 where fiber collection efficiency and detector efficiency are included. The final probability distribution to detect three photons with different time in this situation would be:

$$P'_{co1}(\delta t_1, \delta t_2) = P^3_{ro} P^3_1 P^3_d P_{co1}(\delta t_1, \delta t_2), \tag{15}$$

which can be acquired through numerical methods.

In situation B, the input state is:

$$|\Psi_{in2}\rangle = a^{\dagger}_{si} a^{\dagger}_{si} a^{\dagger}_{sj} |vac\rangle. \tag{16}$$

Similarly, to obtain the probability for photon detections in output ports d1, d2, d3 at times $t_1 = t_3 + \delta t_1$, $t_2 = t_3 + \delta t_2$, t_3, we also apply the field operators $E^{\pm}_{d1}(t_1)$, $E^{\pm}_{d2}(t_2)$, $E^{\pm}_{d3}(t_3)$, the joint probability would be,

$$P_{co2}(\delta t_1, \delta t_2) = \sum_{i,j} \int_{-\infty}^{+\infty} dt_3 \langle \Psi_{in2}| E^{-}_{d3}(t_3) E^{-}_{d2}(t_2) E^{-}_{d1}(t_1) E^{+}_{d1}(t_1) E^{+}_{d2}(t_2) E^{+}_{d3}(t_3) |\Psi_{in2}\rangle. \tag{17}$$

and the final probability distribution to detect three photons in this situation is:

$$P'_{co2}(\delta t_1, \delta t_2) = P^2_{ro}(1 - P_{ro}) P_1 P_2 P^3_d P_{co2}(\delta t_1, \delta t_2). \tag{18}$$

In situation C, the input state is

$$|\Psi_{in3}\rangle = a^{\dagger}_{si} a^{\dagger}_{si} a^{\dagger}_{sj} a^{\dagger}_{sk} |vac\rangle, \tag{19}$$

which have 3 different combinations with different i, j, k. For the output ports, two photons at di, one photon at dj, $dk(i, j, k = 1, 2, 3, i \neq j \neq k,\ j > k)$, which also have 3 combinations. Here, for instance, two photons at $d1$ with photon detected time $t_1 = t_3 + \delta t_1$, $t_4 = t_3 + \delta t_4$, one photon at $d2$ with detected time $t_2 = t_3 + \delta t_2$, one photon at $d3$ with detected time t_3. Similar with Eq.17, the coincident probability would be

$$P_{co3}(\delta t_1, \delta t_2, \delta t_4) = \sum_{i,j} \int_{-\infty}^{+\infty} dt_3 \langle \Psi_{in3}| E^{-}_{d1}(t_4) E^{-}_{d3}(t_3) E^{-}_{d2}(t_2) E^{-}_{d1}(t_1) E^{+}_{d1}(t_1) E^{+}_{d2}(t_2) E^{+}_{d3}(t_3) E^{+}_{d1}(t_4) |\Psi_{in3}\rangle. \tag{20}$$

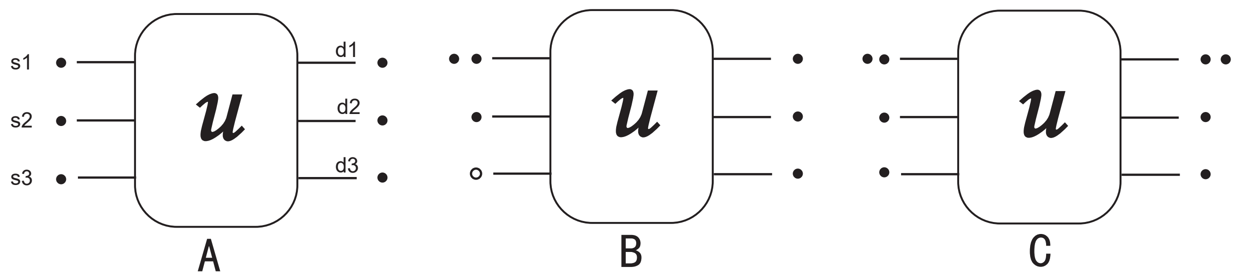

FIG. 2: **Different combinations of injecting photons.** Solid circle represents one photon and empty circle means no photon here.

As our recording system would only record photons detected first, at detected channel with two photons, there exist two cases: I. the first photon is detected and time $\min(\delta t_1, \delta t_4)$ is recorded with probability P_d. II. the first photon isn't detected while the latter one is detected and time $\max(\delta t_1, \delta t_4)$ is recorded with probability (1-P_d)P_d. The final probability distribution at two cases are:

$$\begin{aligned} P'_I(\delta t_2, \min(\delta t_1, \delta t_4)) &= P_{ro}^3 P_1^2 P_2 P_d^2 P_d P_{co3}(\delta t_1, \delta t_2, \delta t_4) \\ P'_{II}(\delta t_2, \max(\delta t_1, \delta t_4)) &= P_{ro}^3 P_1^2 P_2 P_d^2 P_d (1-P_d) P_{co3}(\delta t_1, \delta t_2, \delta t_4). \end{aligned} \tag{21}$$

the distribution is classified with same δt_2 and $\min(\delta t_1, \delta t_4)$(case I) or $\max(\delta t_1, \delta t_4)$(case II). The other 2 combinations of photon distribution at output ports are considered similarly and summed together, then we can acquire the joint probability distribution at situation C.

Based on the joint probability distribution of three situations mentioned above, we estimate the fidelity from ideal case 1 to 0.976. Considering much higher order events, the fidelity would become worse, so we briefly owe the main limitations of fidelity to the single photon quality $g^{(2)}$. Other imperfections such as inhomogeneity of pulse shape, phase instability of the linear optic network, and laser frequency instability will also contribute to the infidelity.

[1] P. Grangier, G. Roger, and A. Aspect, Europhysics Letters (EPL) **1**, 173 (1986).
[2] M. Tillmann, B. Dakić, R. Heilmann, S. Nolte, A. Szameit, and P. Walther, Nature Photonics **7**, 540 (2013).
[3] T. Legero, T. Wilk, A. Kuhn, and G. Rempe, Applied Physics B **77**, 797 (2003).