FRACTAL ANALYSIS AND SYNTHESIS OF RAIN FIELDS FOR RADIO COMMUNICATION SYSTEMS

by

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This thesis is submitted in partial fulfilment of its requirements for the award of the degree of Doctor of Philosophy of the University of Portsmouth
The work presented in this thesis was carried out in collaboration with CCLRC-Rutherford Appleton Laboratory
for my beloved husband Rob
and my parents, Sue and Gerry.

*Tréde conaittig brethemnas: gáis, féige, físs.*

(Three things which judgment demands: wisdom, penetration, knowledge)
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ABSTRACT

This thesis has the aim of introducing fractal methods for the analysis and synthesis of rain fields into the field of radio communication systems. To this end, the fractal nature of rain rate contours as measured by meteorological radar was verified, using different techniques including the area-perimeter relationship, the box-counting dimension, and power spectral density function analysis. The fractal dimension of these contours was found to be \(-1.2\), and the different methods of calculation agreed with each other. Scaling results were also exhibited by the distribution of number of contours with respect to their enclosed area.

Multifractal analysis of the radar measured rain fields showed that rain rate fields display multifractal behaviour, as described in the literature. However, log rain rate fields have a straight line \(K(q)\) function, indicating that for these fields monofractal methods of analysis and synthesis may be used. This is of particular interest to the communications engineering community, who are not concerned with the extreme events that require multifractals to correctly categorise them, and are already accustomed to dealing with observables on a logarithmic basis.

A study of the physical and phenomenological aspects of rain was conducted, with particular emphasis on the impact of measuring device resolution and scaling limits on the calculation of the fractal dimension of rain fields. Also investigated were the differences between stratiform, convective and frontal rain events, the results of which led to the inclusion of climatologically based parameters into the rain field simulator proposed. The simulator uses a discrete additive cascade process to produce simulated monofractal log rain rate fields, which are visually and statistically realistic. The calculated value for one of the parameters, \(H = \frac{1}{3}\), related to the power spectral density function exponent, shows that log rain rate is antipersistent, and that log rain rate has long range anticorrelation.

The procedure required to convert from rain rate fields \(R(x,y,z,t)\) (mm/hr) to attenuation along a path \(A_{dB}(p,t)\) (dB) was detailed. It was found that the Met Office’s Nimrod rain radar database does not have a spatial resolution high enough to be able to accurately use radar derived attenuation data as a substitute for measurements made on site diversity links \(~10\)km apart. The fractal rain field simulator can scale the data in space to any size resolution required, without adversely affecting the statistics and spatial behaviour of the simulated field.

Attenuation time series derived from the simulated rain fields were created. In order to compare them statistically with measured attenuation time series, cumulative distribution functions were calculated from a database of measured and simulated events. The results give reasonable agreement, but emphasise the need for more measured data in order to more accurately characterise the wide range of variability present in attenuation events. Similar conclusions were drawn from the results of the diversity gain comparison performed between the measured and simulated data. The time series were also applied to the case study of a switching algorithm for an Earth-space radio system using site diversity as a fade mitigation technique. The inputs into
such a switching algorithm were defined and discussed, including a simple short-term attenuation predictor. The behaviour of the switching algorithm with the simulated data was contrasted with the behaviour with measured site diversity data, with similar results. Finally, potential areas of improvement and further work were identified.
Acknowledgements

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Over the course of this project I have discussed many ideas and concepts with fellow research colleagues, in the context of meetings, seminars and informal gatherings. I would like in particular to thank: Dr Kevin Paulson for sharing with me his interest and understanding in fractals and rain; Cristina Enjamio, Portsmouth, for interaction and fruitful discussions concerning her PhD research in rain fields and their dynamic behaviour; Dr Boris Gremont, David Thomas and Ken Richardson for their work with me on fade mitigation techniques; Carron Wilson, Dr Chris Walden and Dr Charles Kilburn for their assistance with understanding the operation of meteorological radars; Judith Agnew for her preprocessing of the site diversity time series used in chapters six and seven; and Elizabeth Slack and Joe Waight for their hard work with the rain gauges and experimental equipment at Sparsholt and Chilbolton. I would also like to thank all the members of the COST 280 programme, who provided a vibrant and exciting forum in which to share my results.

Special thanks are due to the staff based at Chilbolton Observatory, who collected the radar data that I used for my analysis. Thanks are also due to the Met Office, for access to their Nimrod rain radar data analysed in chapter six. The data was kindly provided by the BADC (http://www.badc.rl.ac.uk), as was the HYREX raingauge network data analysed in chapter three. The AVHRR satellite pictures of the cloud cover over the British Isles were kindly provided by, and are copyright Dundee Satellite Receiving Station (http://www.sat.dundee.ac.uk).
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Table of Contents

ABSTRACT ii
Acknowledgements iv
Table of Contents vi
List of figures xiii
List of tables xxii
Glossary of acronyms xxiii
Glossary of symbols xxv

Part One: Verification of the fractal nature of rain fields

Chapter One: Objectives and Overview of Thesis 1

1.1 Introduction 1
1.2 The spatial variation of rain rate 1
1.3 Measuring devices and data 3
1.4 Fractals and rain 3
1.5 Thesis overview 4
1.6 References 6

Chapter Two: Overview of the Theory of Fractal Geometry 8

2.1 What is a fractal? 8
2.1.1 Fractal sets 8
2.1.2 Fractal surfaces 10
2.1 Methods of calculating the fractal dimension 12
2.2.1 Area-perimeter relationship 13
2.2.2 Box-counting dimension 15
2.2.3 Area distribution analysis (Korcak dimension) 15
2.2.4 Power spectral density function 17
2.2.5 Correlation dimension 17
2.3 The fractal dimension of embedded sets 18
2.4 Multifractality versus monofractality 19
2.4.1 Why multifractals 19
2.4.2 Multifractal measures and definitions of a multifractal 20
Chapter Three: Fractal Analysis of Rain Fields

3.1 Analysis of rain rate contours derived from meteorological radar scans
   3.1.1 Data description
   3.1.2 An illustration of statistical self-similarity in log rain fields measured by meteorological radar
   3.1.3 Software
   3.1.4 Area perimeter relationship
   3.1.5 Box counting dimension
   3.1.6 Korcak analysis (area distribution analysis)
   3.1.6 Two dimensional Fourier transforms and power spectral density functions
3.2 Fractal analysis of rain fields using a rain gauge network
   3.2.1 Data description
   3.2.2 Correlation dimension of the network
3.3 Multifractal analysis of radar derived rain fields
3.4 Other studies
3.5 References

Part Two: Production of a fractal rain field simulator

Chapter Four: Concepts Relating Physical Properties of Rain to its Fractal Description

4.1 Measuring device resolution and system errors
   4.1.1 Resolution of a measuring instrument
      4.1.1.1 Minimum resolution (pixel size)
      4.1.1.2 Maximum range (window size)
      4.1.1.3 Time resolution
   4.1.2 Measurement time
   4.1.3 System errors and noise
4.2 Assumptions on the nature of rain fields
Chapter Five: Cascade Processes and Simulations

5.1 Introduction

5.2 The simulation process

5.2.1 Measured parameters and simulator inputs

5.2.1.1 The Hurst exponent $H$

5.2.1.2 Lacunarity and its parameter $r_l$

5.2.1.2.1 Calculation of lacunarity for measured radar rain fields

5.2.1.3 The number of iterations $N_l$

5.2.2 The random additions algorithm

5.2.2.1 The Voss random additions algorithm for varying lacunarity

5.2.3 Simulation results for varying lacunarity

5.2.4 Simulation results and conversion to rain rate fields

5.2.5 Extension of the simulator results to include temporal variation

5.3 Supporting theory

5.3.1 Cascade processes and the simulation of rain fields

5.3.2 Fractional Brownian motion, fractional Gaussian noise and the physical implications of the Hurst exponent

5.3.3 Spectral densities for fractional Brownian motion, and the relationship between the spectral exponent $\beta$, the Hurst exponent $H$ and the fractal dimension $D$
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.3.4 Cross correlation between measured radar rain fields</td>
<td>111</td>
</tr>
<tr>
<td>5.4 References</td>
<td>114</td>
</tr>
<tr>
<td>Part Three: Application to a communications engineering case study</td>
<td></td>
</tr>
<tr>
<td>Chapter Six: The GBS Site Diversity Experiment</td>
<td>116</td>
</tr>
<tr>
<td>6.1 Introduction</td>
<td>116</td>
</tr>
<tr>
<td>6.2 Case Study</td>
<td>117</td>
</tr>
<tr>
<td>6.2.1 Why site diversity?</td>
<td>117</td>
</tr>
<tr>
<td>6.2.2 The GBS site diversity experiment</td>
<td>118</td>
</tr>
<tr>
<td>6.2.3 Measured data</td>
<td>122</td>
</tr>
<tr>
<td>6.2.4 Attenuation statistics</td>
<td>126</td>
</tr>
<tr>
<td>6.2.5 Diversity gain and improvement</td>
<td>128</td>
</tr>
<tr>
<td>6.2.6 Instantaneous diversity gain</td>
<td>129</td>
</tr>
<tr>
<td>6.3 Switching algorithm for a site diversity scheme</td>
<td>133</td>
</tr>
<tr>
<td>6.3.1 Fade detection mechanism</td>
<td>133</td>
</tr>
<tr>
<td>6.3.2 Short term prediction algorithm</td>
<td>135</td>
</tr>
<tr>
<td>6.3.2.1 Fade and inter-fade durations</td>
<td>135</td>
</tr>
<tr>
<td>6.3.2.2 Fade slope</td>
<td>138</td>
</tr>
<tr>
<td>6.3.2.3 A simple short-term rain attenuation prediction algorithm</td>
<td>139</td>
</tr>
<tr>
<td>6.4 Acknowledgements</td>
<td>140</td>
</tr>
<tr>
<td>6.5 References and bibliography</td>
<td>140</td>
</tr>
<tr>
<td>Chapter Seven: Application of a Fractal Rain Field Simulator to a</td>
<td>144</td>
</tr>
<tr>
<td>Communications System Switching Algorithm</td>
<td></td>
</tr>
<tr>
<td>7.1 Introduction</td>
<td>144</td>
</tr>
<tr>
<td>7.2 Description of switching algorithm</td>
<td>144</td>
</tr>
<tr>
<td>7.3 Use of rain maps in system design and operation</td>
<td>145</td>
</tr>
<tr>
<td>7.3.1 Conversion from rain rate fields $R(x,y,z,t)$ (mm/hr) to attenuation along a path $A_{dB}(p,t)$ (dB)</td>
<td>147</td>
</tr>
<tr>
<td>7.3.1.1 Met Office Nimrod data</td>
<td>147</td>
</tr>
<tr>
<td>7.3.1.2 Simulated rain fields</td>
<td>151</td>
</tr>
<tr>
<td>7.4 Comparison of measured attenuation data with simulated attenuation data</td>
<td>157</td>
</tr>
<tr>
<td>7.5 Diversity gain using measured and simulated data</td>
<td>163</td>
</tr>
</tbody>
</table>
### Chapter Eight: Conclusions and Recommendations for Further Work

8.1 Summary of conclusions
- 8.1.1 Part one: Verification of the fractal nature of rain fields
- 8.1.2 Part two: Production of a fractal rain field simulator
- 8.1.3 Part three: Application to a communications engineering case study

8.2 Novel contributions by the author

8.3 Recommendations for further work
- 8.3.1 Further fractal analysis of rain fields
- 8.3.2 Further modification of the rain field simulator
- 8.3.3 Further applications of the simulator
- 8.3.5 Investigation into instantaneous and statistical diversity gain

8.4 References

### Appendix A: Derivation of the Area Perimeter Relationship for Determining Fractal Dimension

### Appendix B: Derivation of the box counting method for determining fractal dimension

### Appendix C: The Hausdorff Dimension

C.1 References

### Appendix D: The Area Perimeter Relationship for Regular Shapes

- D.1 Square
- D.2 Circle
- D.3 Generalisation for N sided regular polygon
- D.4 Rectangle
Appendix E: Multifractals and Multifractal Measures

E.1 What is a measure? 194
E.2 Multifractals 194
E.3 References 196

Appendix F: Estimation of Rain Rate from Meteorological Radar

Reflectivity Measurements. 197
F.1 Drop size distribution 198
F.2 Weather radar equation 199
F.3 Rainfall measurement with radar 201
F.4 References 203

Appendix G: The Similarity Dimension 204

Appendix H: The Earth-space experimental history of the Radio Communications Research Unit 206
H.1 References 207

Appendix I: Software 208
I.1 Area perimeter dimension 208
I.2 Box counting dimension 210
I.3 Area distribution analysis (Korcak dimension) 212
I.4 Two-dimensional FFT 213
I.5 K(q) function (multifractal analysis) 214
I.6 Correlation dimension 216
I.7 Random additions algorithm for varying lacunarity 217
I.9 Switching algorithm 218
I.10 Function descriptions 218
   I.10.1 MATLAB defined functions 218
   I.10.2 Author defined functions 221
I.11 References 222
Appendix J: Publications and Internal Reports

1.1 Publications most relevant to the thesis
1.2 Contributions to COST 280
1.3 Other publications
1.4 Internal reports
1.5 Contractual reports
## List of figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>Comparison of attenuation statistics for different frequencies, measured on Earth-Space links</td>
<td>2</td>
</tr>
<tr>
<td>1.2</td>
<td>Schematic diagram of topics discussed in this thesis</td>
<td>7</td>
</tr>
<tr>
<td>2.1</td>
<td>Stages in the construction of a Koch curve</td>
<td>9</td>
</tr>
<tr>
<td>2.2</td>
<td>A completed Koch curve</td>
<td>9</td>
</tr>
<tr>
<td>2.3</td>
<td>From left to right, stages in the construction of a Koch Snowflake.</td>
<td>10</td>
</tr>
<tr>
<td>2.4</td>
<td>Schematic diagram of the three types of fractal surfaces. Left: a dense object with a fractal surface. Centre: a network or cluster (called a mass fractal). Right: a dense object containing pores (called a pore fractal).</td>
<td>11</td>
</tr>
<tr>
<td>2.5</td>
<td>Surface representation of a log rain field from the radar data set recorded on 1st May 2001</td>
<td>12</td>
</tr>
<tr>
<td>2.6</td>
<td>Common objects with their Euclidean and topological dimensions</td>
<td>14</td>
</tr>
<tr>
<td>2.7</td>
<td>Diagrammatic representation of the areas plotted for the area distribution (Korckak) analysis</td>
<td>16</td>
</tr>
<tr>
<td>2.8</td>
<td>Schematic diagram of rain intensity time series, with two thresholds</td>
<td>19</td>
</tr>
<tr>
<td>3.1</td>
<td>Chilbolton Advanced Meteorological Radar (CAMRa)</td>
<td>25</td>
</tr>
<tr>
<td>3.2</td>
<td>Example raster from radar scan taken on the 1st May 2001. The colours show the rain rate in mm/hr.</td>
<td>27</td>
</tr>
<tr>
<td>3.3</td>
<td>NOAA satellite picture showing the cloud cover over the British Isles at 06:00 on the 1st May 2001.</td>
<td>28</td>
</tr>
<tr>
<td>3.4</td>
<td>Example raster from radar scan taken on the 16th May 2001. The colours show the rain rate in mm/hr.</td>
<td>29</td>
</tr>
<tr>
<td>3.5</td>
<td>NOAA satellite picture showing the cloud cover over the British Isles at 12:58 on the 16th May 2001.</td>
<td>30</td>
</tr>
<tr>
<td>3.6</td>
<td>Example raster from radar scan taken on the 7th December 2000. The colours show the rain rate in mm/hr.</td>
<td>31</td>
</tr>
<tr>
<td>3.7</td>
<td>NOAA satellite picture showing the cloud cover over the British Isles at 16:37 on the 7th December 2000.</td>
<td>32</td>
</tr>
<tr>
<td>3.8</td>
<td>Example log rain rate field.</td>
<td>33</td>
</tr>
<tr>
<td>3.9</td>
<td>Zoom in to centre 94*94 pixels of log rain field shown in 3.8.</td>
<td>34</td>
</tr>
</tbody>
</table>
Figure 3.10  Entire rain field as shown in figure 3.8, with resolution degraded to the same number of pixels in figure 3.9................................. 34

Figure 3.11  Rain rate as recorded in a typical raster, 1st May 2001...................... 35

Figure 3.12  Contour lines drawn for areas of equal rain rate for the raster shown in figure 3.11................................................................. 36

Figure 3.13  Schematic example of a section of the rain rate contour array used in the calculations of fractal dimension............................... 37

Figure 3.14  Area-perimeter dimension for event recorded 1st May 2001. Rain rate threshold = 1mm/hr \( D_A = 1.19 \pm 0.01 \) ........................................... 39

Figure 3.15  Area-perimeter dimension for event recorded 1st May 2001. Rain rate threshold = 25 mm/hr \( D_A = 1.17 \pm 0.01 \) ........................................... 39

Figure 3.16  Box counting dimension against length of contour for event recorded on the 1st May 2001. Rain rate threshold=1mm/hr............ 41

Figure 3.17  Box counting dimension against length of contour for event recorded on the 1st May 2001. Rain rate threshold=25mm/hr........... 42

Figure 3.18  Number of contours with area greater than x-axis value for thresholds in mm/hr of 1 (○), 2(×), 5(＋), 10 (＊), 15 (□), 20 (◇), 25 (◇), 30(▲), 35(▲), 40(▲), 45 (＊), 50 (＊). Event recorded 1st May 2001.................................................. 43

Figure 3.19  Normalised number of contours with area greater than x-axis value for thresholds in mm/hr of 1 (○), 2(×), 5(＋), 10 (＊), 15 (□), 20 (◇), 25 (◇), 30(▲), 35(▲), 40(▲), 45 (＊), 50 (＊). Event recorded 1st May 2001.................................................. 43

Figure 3.20  2-D Spatial spectral density of rain rate for event on 1st May 2001, averaged over 230 scans......................................................... 47

Figure 3.21  2-D Spatial-temporal (x-t) spectral density of rain rate for event on 1st May 2001, averaged across the y direction............................ 48

Figure 3.22  2-D Spatial-temporal (y-t) spectral density of rain rate for event on 1st May 2001, averaged across the x direction............................ 48

Figure 3.23  Spectral density of log rain rate for event on 1st May 2001.
Exponent = -2.79............................................................... 50

Figure 3.24  Correlation dimension of Brue rain gauge network.......................... 52
Figure 3.25  Estimation of $K(q)$ values for different $q$. Frontal rain event, recorded on the 1st May 2001 ........................................................................ 53

Figure 3.26  $K(q)$ functions for different rain event types. Values averaged are rain rate, in mm/hr .............................................................. 54

Figure 3.26  $K(q)$ functions for different rain event types. Values averaged are log rain rate ................................................................. 56

Figure 4.1  Schematic diagram on calculating the fractal dimension of a time series ............................................................................... 62

Figure 4.2  Surface representation of a rain field from the radar data set recorded on 1st May 2001. Same radar “snapshot” as for figure 4.3 (1 pixel = 300m x 300m) .............................................................. 65

Figure 4.3  Surface representation of a log rain field from the radar data set recorded on 1st May 2001. Same radar “snapshot” as for figure 4.2 (1 pixel = 300m x 300m) ................................................. 66

Figure 4.4:  Example of the contour lines produced for one scan recorded on the 1st May 2001. Contour lines are at values of 1, 5, 10, 15, 20, 25, 30, 35 and 40 mm/hr .............................................................. 68

Figure 4.5  Changes in the distribution of rain after it is created ................................................................................................................. 71

Figure 4.6  PPI of a stratiform rain event .............................................................................................................................................. 73

Figure 4.7  RHI of the same stratiform rain event as in figure 4.6, with melting layer very clear at ~2.5 km ......................................................... 73

Figure 4.8  PPI scan of a convective rain event ...................................................................................................................................... 74

Figure 4.9  RHI of the same convective rain event as in figure 4.8 .......................................................................................................... 74

Figure 4.10 Schematic diagram of a cold front system ........................................................................................................................ 75

Figure 4.11 RHI scan of a frontal event showing the radar reflectivity factor dBZ ..................................................................................... 75

Figure 4.12 RHI scan of a frontal event showing the velocity of the raindrops ...................................................................................... 78

Figure 4.13 A schematic diagram to aid in the discussion of scaling limits applied to rain fields .............................................................................. 73

Figure 5.1  Example raster from the stratiform rain event recorded on the 7th December 2000. Contour lines are drawn at 1 and 5 mm/hr (1 pixel corresponds to an area of 300m*300m) .................. 83

Figure 5.2  Schematic diagram of a stratiform rain field ........................................................................................................................... 84
Figure 5.3  Example simulation of a stratiform event on the same length scale as figure 5.1. Contour lines are drawn at 1, 5 and 10 mm/hr (1 pixel corresponds to an area of 300m*300m).......................... 85

Figure 5.4  Example raster from the convective rain event recorded on the 16th May 2001. Contour lines are drawn at 1, 5, 10, 15, 20, 25, 30, 35 and 40 mm/hr (1 pixel corresponds to an area of 300m*300m) 86

Figure 5.5  Schematic diagram of a convective rain field.......................... 86

Figure 5.6  Example simulation of a convective event on the same length scale as figure 5.4. Contour lines are drawn at 1, 5, 10 and 15 mm/hr (1 pixel corresponds to an area of 300m*300m)........... 87

Figure 5.7  Flowchart of simulation process......................................... 89

Figure 5.8  Stages in the construction of a Cantor set $D_s = \log(2)/\log(3)$ ...... 91

Figure 5.9  Stages in the construction of a Cantor set. $D_s = 1/2$. Example 1... 91

Figure 5.10  Stages in the construction of a Cantor set. $D_s = 1/2$. Example 2... 92

Figure 5.11  Schematic diagram of the successive random addition method for simulating fractional Brownian motion in 2 dimensions. For simplicity, the stage n=1 is broken up into two steps in the diagram................................................................. 95

Figure 5.12  An example of a typical simulated field created using the Voss successive random additions algorithm................................. 96

Figure 5.13  The simulated field shown in figure 5.12, plotted as exponents to base $e$........................................................................ 97

Figure 5.14.  Schematic example of random addition algorithm for $r = 1/3$ $N_i = 2$. The number in each square refers to the iteration they were plotted in. The numbers plotted in previous generations are in italics............................................................... 98

Figure 5.15  Random addition surface $r_i = 1/2$ $N_i = 10$ .................................. 100

Figure 5.16  Random addition surface $r_i = 1/3$ $N_i = 6$ .............................. 100

Figure 5.17  Random addition surface $r_i = 1/4$ $N_i = 5$ .............................. 101

Figure 5.18  Random addition surface $r_i = 1/5$ $N_i = 4$ .............................. 101

Figure 5.19  The simulated field shown in figure 5.6, without the offset........ 103
Figure 5.20 Description of the three main types of procedure used to model the spatial distribution of rain................................. 105
Figure 5.21 Normalised cross-correlation function for stratiform event recorded 7th December 2000. 1 pixel corresponds to an area of 300m * 300m............................................................... 112
Figure 5.22 Normalised cross-correlation function for convective event recorded 16th May 2001. 1 pixel corresponds to an area of 300m * 300m......................................................................... 113
Figure 5.23 Normalised cross-correlation function for frontal event recorded 1st May 2001. 1 pixel corresponds to an area of 300m * 300m............................................................... 113
Figure 6.1 Schematic diagram of the inputs to a switching algorithm for a site diversity scheme............................................. 118
Figure 6.2 The GBS receiver in operation at Sparsholt .................................................. 119
Figure 6.3 The GBS receiver in operation at Chilbolton (opened to show the components inside.)........................................ 119
Figure 6.4 Schematic of the GBS receiver locations. The link between Sparsholt and South Wonston is a terrestrial link................. 121
Figure 6.5 Signal levels and attenuation time series for the 9th January 2004. Black: Sparsholt data, yellow: Chilbolton data. (a) Complete day. (b) Zoom on convective event........................... 123
Figure 6.6 Rain gauge measurements from the Chilbolton rain gauge for the 9th January 2004. .............................................. 124
Figure 6.7 Rain gauge measurements from the Sparsholt rain gauge for the 9th January 2004.............................................. 124
Figure 6.8 Radar reflectivity measurements of the convective rain event on the 9th January 2004, made with the 94 GHz cloud radar at Chilbolton .......................................................... 124
Figure 6.9 Signal levels and attenuation time series for the 12th January 2004. Black: Sparsholt data, yellow: Chilbolton data. (a) Complete day. (b) Zoom on stratiform event............... 125
Figure 6.10 Rain gauge measurements from the Chilbolton rain gauge for the 12th January 2004.............................................. 126
Figure 6.11 Rain gauge measurements from the Sparsholt rain gauge for the 12th January 2004 ................................................................. 126
Figure 6.12 Radar reflectivity measurements of the stratiform rain event on the 12th January 2004, made with the 94 GHz cloud radar at Chilbolton .................................................................................. 126
Figure 6.13 Monthly cumulative distributions of attenuation in comparison with the ITU-R prediction (from ITU-R Rec. 618-8) ............................................ 127
Figure 6.14 Graphical representation of the concepts of diversity gain and diversity improvement ..................................................... 128
Figure 6.15 Instantaneous diversity gain calculated for the convective event recorded on the 9th January 2004. (a) Entire day (b) Zoomed in to see the effects of filtering ......................................................... 131
Figure 6.16 Instantaneous diversity gain calculated for the stratiform event recorded on the 12th January 2004. (a) Entire day (b) Zoomed in to see the effects of filtering ......................................................... 132
Figure 6.17 Schematic representation of fade and inter-fade durations .................................................. 136
Figure 6.18 ITALSAT measurements at 49.5GHz. Absolute number of fades with duration equal or longer than x-axis value in comparison with RCRU (solid) and ITU-R (dashed) model. (Average year from April’97 to March’00) [Ventouras et al, 2000] ......................................................... 137
Figure 6.19 Fade slope values against excess attenuation at 49.5GHz as derived from measurements from April’97 to March ’00 [Ventouras et al, 2003] ......................................................................................... 138
Figure 6.20 Schematic representation of the time values used in the short term prediction algorithm .................................................. 139
Figure 6.21 The absolute error (dB) plotted against the probability of occurrence for the simple short-term prediction algorithm detailed in section 6.3.3.3 [Ventouras et al, 2003] ......................................................................................... 140
Figure 7.1 Schematic diagram of the switching algorithm for a site diversity scheme .................................................................................. 145
Figure 7.2 Example radar map of the UK derived from the Met Office Nimrod radar data .................................................. 147
Figure 7.3 Schematic diagram of a slant path link ................................................................................. 148
Figure 7.4  Attenuation time series for the stratiform event recorded at Chilbolton and Sparsholt on the 12th January 2004, in comparison with the attenuation calculated from the UK Met Office Nimrod radar maps.......................................................... 150

Figure 7.5  Attenuation time series for the convective event recorded at Chilbolton and Sparsholt on the 9th January 2004, in comparison with the attenuation calculated from the UK Met Office Nimrod radar maps.................................................................. 150

Figure 7.6  Schematic diagram of the process used to simulate the variation in time of the rain field........................................................................................................ 152

Figure 7.7  Full sized simulated rain field used to create a simulated time series of a **convective** event.......................................................... 152

Figure 7.8  Simulated instantaneous (1 minute integration time) rain attenuation calculated from the **convective** rain field in figure 7.7 154

Figure 7.9  Measured convective event (1 minute integration time) rain attenuation recorded on the 9th January 2004.................................................... 154

Figure 7.10  Full sized simulated rain field used to create a simulated time series of a **stratiform** event.......................................................... 155

Figure 7.11  Simulated instantaneous (1 minute integration time) rain attenuation calculated from the **stratiform** rain field in figure 7.9 156

Figure 7.12  Measured stratiform event (1 minute integration time) rain attenuation recorded on the 12th January 2004.................................................... 156

Figure 7.13  Power spectral density function calculated for data measured at Sparsholt.......................................................... 158

Figure 7.14  Power spectral density function calculated for data measured at Chilbolton.......................................................... 158

Figure 7.15  Power spectral density function calculated for simulated data at site 1.......................................................... 159

Figure 7.16  Power spectral density function calculated for simulated data at site 2.......................................................... 160

Figure 7.17  Cumulative distribution functions of measured and simulated stratiform events.......................................................... 161
Figure 7.18 Cumulative distribution functions of measured and simulated convective events................................................................. 162

Figure 7.19 Instantaneous diversity gain at a time $t$, plotted against the maximum of the measured attenuation values recorded at Sparsholt and Chilbolton for the same time $t$. Stratiform events.... 164

Figure 7.20 Schematic diagram of two attenuation time series, showing how at one instant the diversity gain is calculated and how it can never be greater than the maximum attenuation at that instant................. 164

Figure 7.21 Instantaneous diversity gain at a time $t$, plotted against the maximum of the measured attenuation values recorded at Sparsholt and Chilbolton for the same time $t$. Convective events... 165

Figure 7.22 Instantaneous diversity gain at a time $t$, plotted against the maximum of the simulated attenuation values recorded at the two simulated sites for the same time $t$. Stratiform events................... 166

Figure 7.23 Instantaneous diversity gain at a time $t$, plotted against the maximum of the simulated attenuation values recorded at the two simulated sites for the same time $t$. Convective events.............. 166

Figure 7.24 System attenuation at a time $t$ for varying $\Delta t_{ij}$ values plotted against the minimum of the measured attenuation values recorded at Sparsholt and Chilbolton for the same time $t$. Stratiform events. 168

Figure 7.25 System attenuation at a time $t$ for varying $\Delta t_{ij}$ values plotted against the minimum of the measured attenuation values recorded at Sparsholt and Chilbolton for the same time $t$. Convective events 168

Figure 7.26 System attenuation at a time $t$ for varying $\Delta t_{ij}$ values, plotted against the minimum of the simulated attenuation values recorded at the two simulated sites for the same time $t$. Stratiform events... 169

Figure 7.27 System attenuation at a time $t$ for varying $\Delta t_{ij}$ values, plotted against the minimum of the simulated attenuation values recorded at the two simulated sites for the same time $t$. Convective events 170

Figure 8.1 Overview of the topics studied in this thesis..................... 174

Figure A.1 Schematic plot of $\log(A)$ against $\log(P)$............................. 184

Figure B.1 Line segment (blue) covered by boxes of side length $\delta$........... 185

Figure B.2 Area (blue) covered by boxes of side length $\delta$.................... 185
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>F.1</td>
<td>Radar reflectivity factor, $Z$ plotted against the rainrate $R$ for different types of rain. [Skolnic, 1991]</td>
</tr>
<tr>
<td>I.1</td>
<td>Schematic diagram of the structure of the program used to calculate the area perimeter dimension of contour lines enclosing an area greater than or equal to a set rain rate threshold.</td>
</tr>
<tr>
<td>I.2</td>
<td>Schematic diagram of the structure of the program used to calculate the box counting dimension of contour lines enclosing an area greater than or equal to a set rain rate threshold.</td>
</tr>
<tr>
<td>I.3</td>
<td>Schematic diagram of the structure of the program used to analyse the radar data in terms of the distribution of the area enclosed by the contour lines (Korcak dimension).</td>
</tr>
<tr>
<td>I.4</td>
<td>Schematic diagram of the structure of the program used to perform two-dimensional Fourier analysis on the radar data.</td>
</tr>
<tr>
<td>I.5</td>
<td>Schematic diagram of the structure of the program used to perform multifractal (moment scaling function) analysis on the radar data.</td>
</tr>
<tr>
<td>I.6</td>
<td>Schematic diagram of the structure of the program used to calculate the correlation dimension of the Brue rain gauge network.</td>
</tr>
</tbody>
</table>
List of tables

Table 3.1  Details of the radar events studied. Events in bold were studied in
greater detail................................................................. 26

Table 3.2  Area-perimeter dimension calculated for a frontal, stratiform and
convective event at different rain rate thresholds........................ 38

Table 3.3  Kornak dimension calculated for a frontal, stratiform and
convective event at different rain rate thresholds ....................... 45

Table 3.4  Radial spectral density exponent calculated for a frontal,
stratiform and convective event at different rain rate thresholds..... 50

Table 3.5  Number of data points in each data set, and percentage of those
points with a rain rate value of less than 1 mm/hr..................... 55

Table 5.1  Prefactor values for given event types and rain rate thresholds,
calculated from the area perimeter relationship......................... 93

Table 6.1  Technical specifications for the receivers used in the GBS site
diversity experiment....................................................... 120
# Glossary of Acronyms

<table>
<thead>
<tr>
<th>Acronym</th>
<th>Full Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-D</td>
<td>Two-dimensional</td>
</tr>
<tr>
<td>3-D</td>
<td>Three-dimensional</td>
</tr>
<tr>
<td>AVHRR</td>
<td>Advanced Very High Resolution Radiometer</td>
</tr>
<tr>
<td>BADC</td>
<td>British Atmospheric Data Centre</td>
</tr>
<tr>
<td>B/W</td>
<td>Bandwidth</td>
</tr>
<tr>
<td>CAMRa</td>
<td>Chilbolton Advanced Meteorological Radar</td>
</tr>
<tr>
<td>CCLRC</td>
<td>Council for the Central Laboratory of the Research Councils</td>
</tr>
<tr>
<td>CDF</td>
<td>Cumulative Distribution Function</td>
</tr>
<tr>
<td>COST 280</td>
<td>COST (COperation européenne dans le domaine de la recherche Scientifique et Technique) - a European Union forum for cooperative scientific research. COST Action 280 considers Propagation Impairment Mitigation for Millimetre Wave Radio Systems.</td>
</tr>
<tr>
<td>CW</td>
<td>Continuous Wave</td>
</tr>
<tr>
<td>DoD</td>
<td>Department of Defense</td>
</tr>
<tr>
<td>EHF</td>
<td>Extremely High Frequencies</td>
</tr>
<tr>
<td>fBm</td>
<td>Fractional Brownian motion</td>
</tr>
<tr>
<td>FFT</td>
<td>Fast Fourier Transform</td>
</tr>
<tr>
<td>fGn</td>
<td>Fractional Gaussian noise</td>
</tr>
<tr>
<td>FCM</td>
<td>Fade Counter Measures</td>
</tr>
<tr>
<td>FMT</td>
<td>Fade Mitigation Technique</td>
</tr>
<tr>
<td>FRACTINT</td>
<td>Computer program to generate fractal images</td>
</tr>
<tr>
<td>GBS</td>
<td>Global Broadcast Service</td>
</tr>
<tr>
<td>GMT</td>
<td>Greenwich mean time</td>
</tr>
<tr>
<td>HYREX</td>
<td>Hydrological Radar Experiment</td>
</tr>
<tr>
<td>ICAP</td>
<td>International Conference on Antennas and Propagation</td>
</tr>
<tr>
<td>I.F.</td>
<td>Intermediate frequency</td>
</tr>
<tr>
<td>iid</td>
<td>Independent and indentically distributed</td>
</tr>
<tr>
<td>ITALSAT</td>
<td>Experimental communications satellite built by Alenia Spazio for the Italian Space Agency</td>
</tr>
<tr>
<td>Acronym</td>
<td>Description</td>
</tr>
<tr>
<td>---------</td>
<td>-------------</td>
</tr>
<tr>
<td>ITU-R</td>
<td>(Recommendations of the) International Telecommunications Union</td>
</tr>
<tr>
<td>MATLAB</td>
<td>Data analysis and programming language created by the MathWorks Inc.</td>
</tr>
<tr>
<td>NE</td>
<td>North East</td>
</tr>
<tr>
<td>NERC</td>
<td>Natural Environment Research Council</td>
</tr>
<tr>
<td>Nimrod</td>
<td>A fully automated system for weather analysis and nowcasting based around a network of C-band rainfall radars belonging to the UK Met Office.</td>
</tr>
<tr>
<td>NNW</td>
<td>North north west</td>
</tr>
<tr>
<td>NOAA</td>
<td>National Oceanic and Atmospheric Administration</td>
</tr>
<tr>
<td>Ofcom</td>
<td>Office of Communications (UK)</td>
</tr>
<tr>
<td>PLL</td>
<td>Phase Locked Loop</td>
</tr>
<tr>
<td>PPI</td>
<td>Plan Position Indicator</td>
</tr>
<tr>
<td>RA</td>
<td>Radiocommunications Agency (now Ofcom)</td>
</tr>
<tr>
<td>RAL</td>
<td>Rutherford Appleton Laboratory</td>
</tr>
<tr>
<td>RCRU</td>
<td>Radio Communications Research Unit</td>
</tr>
<tr>
<td>RHI</td>
<td>Range Height Indicator</td>
</tr>
<tr>
<td>SES</td>
<td>Spectrum Efficiency Scheme</td>
</tr>
<tr>
<td>SW</td>
<td>South west</td>
</tr>
<tr>
<td>UFO-9</td>
<td>UHF Follow-on 9, a US DoD satellite carrying the GBS payload</td>
</tr>
</tbody>
</table>
Glossary of Symbols

Latin

A
Area enclosed by a perimeter

$A_{dB}$
Attenuation (dB)

$A_{dB,\text{thresh}}$
Specified attenuation threshold

$A_{dB}(t)$
Attenuation at a time $t$ (dB)

$A_{dB,1}(t)$
Attenuation at site 1 at time $t$ (dB)

$A_{dB,2}(t)$
Attenuation at site 2 at time $t$ (dB)

$A_{dB,max}(t)$
Maximum attenuation for all sites at time $t$ (dB)

$A_{dB,min}(t)$
Minimum attenuation for all sites at time $t$ (dB)

$A_{dB,\text{adv}}(t)$
Joint attenuation from two or more sites using site diversity at time $t$ (dB)

$A_{dB}(p,t)$
Attenuation along a straight line radio path, $p$, at time $t$ (dB)

$b$
The base used in order to convert the fractally simulated array $X$ values from arbitrary intensities to rain rate $R_{sim}$

$B_r(x)$
closed disc of radius $r$ and centre $x$

$c$
Parameter of the $\alpha$ model for multifractal cascades

$C_t$
Co-dimension of the mean process (characterises multifractals)

$C(g)$
Number of points (rain gauges) within a region $g$

$D$
Fractal dimension by unspecified method

$D_0$
Fractal dimension of zerosets of $V_H(\bar{x})$

$D_A$
Fractal dimension from area-perimeter method

$D_B$
Fractal dimension from box-counting method

$D_{\text{boundary}}$
Fractal dimension of a contour line/boundary drawn on a surface

$D_C$
Correlation dimension

$D_E$
Euclidean dimension
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_K$</td>
<td>Fractal dimension from area distribution (Korcak) analysis</td>
</tr>
<tr>
<td>$D_{surface}$</td>
<td>Fractal dimension of a surface</td>
</tr>
<tr>
<td>$D_s$</td>
<td>Similarity dimension</td>
</tr>
<tr>
<td>$D_T$</td>
<td>Topological dimension</td>
</tr>
<tr>
<td>$D(\tau)$</td>
<td>Structure function (from turbulence theory)</td>
</tr>
<tr>
<td>$e$</td>
<td>Base of natural logarithms</td>
</tr>
<tr>
<td>$E_A$</td>
<td>Exponent of power law relationship between $A$ and $P$</td>
</tr>
<tr>
<td>$E_D$</td>
<td>Number of Euclidean dimensions</td>
</tr>
<tr>
<td>$f$</td>
<td>Frequency (units: $1/t$)</td>
</tr>
<tr>
<td>$F$</td>
<td>Fractal set</td>
</tr>
<tr>
<td>$G_i(t)$</td>
<td>Instantaneous diversity gain at time $t$ (dB)</td>
</tr>
<tr>
<td>$G_p(\tau)$</td>
<td>Two point autocorrelation function</td>
</tr>
<tr>
<td>$H$</td>
<td>Hurst exponent</td>
</tr>
<tr>
<td>$J$</td>
<td>Fade duration (seconds)</td>
</tr>
<tr>
<td>$k_{ITU-R}$</td>
<td>Frequency dependent coefficients calculated from ITU-R Recommendation 838-2</td>
</tr>
<tr>
<td>$K(q)$</td>
<td>Moment scaling function, the characteristic function of multifractal behaviour</td>
</tr>
<tr>
<td>$L_{k-1}$</td>
<td>Length of Koch curve at generation step $k-1$</td>
</tr>
<tr>
<td>$L_t$</td>
<td>Size of non-overlapping time segments used to divide rain time series</td>
</tr>
<tr>
<td>$m_1$</td>
<td>Mean value of $R_1$</td>
</tr>
<tr>
<td>$m_2$</td>
<td>Mean value of $R_2$</td>
</tr>
<tr>
<td>$M$</td>
<td>Dimension of a lower dimension space (can be fractal)</td>
</tr>
<tr>
<td>$n$</td>
<td>Generation number in simulation iteration process</td>
</tr>
<tr>
<td>$n_j(J; A_{dB,thresh})dJ$</td>
<td>Number of fades with fade duration $J$</td>
</tr>
<tr>
<td>$n_s(J; A_{dB,thresh})dJ$</td>
<td>Corresponding time of fades for attenuation threshold $A_{dB,thresh}$ (seconds)</td>
</tr>
<tr>
<td>$N_C$</td>
<td>Number of small self-similar parts of set remaining for each segment in each iteration in the generation of a Cantor set</td>
</tr>
</tbody>
</table>
$N_D$  
Number of dimensions in which the observable is measured  
(in spectral density function exponent)

$N_i$  
Total number of iterations of the simulation

$N(A \geq a)$  
Number of islands with area greater than $a$

$N(\delta)$  
Number of boxes of side length $\delta$

$N(L_l)$  
Number of boxes of length $L_l$

$N_j(J \geq j; A_{\text{dB,thresh}})$  
Absolute number of fades with duration $\geq j$ for an  
attenuation threshold $A_{\text{dB,thresh}}$

$N_s(J \geq j; A_{\text{dB,thresh}})$  
Accumulated time of fades with duration $\geq j$ for an  
attenuation threshold $A_{\text{dB,thresh}}$ (seconds)

$P$  
Length of a perimeter

$Pr(...)$  
Probability

$Q$  
Ratio between the perimeter $P$, and the square root of the  
enclosed area $A$, for a regular Euclidean shape

$r_i$  
Lacunarity parameter used in simulation, also the scaling  
parameter

$R$  
Rain rate (mm/hr)

$R_{0.01}$  
Point rainfall rate for a given location for 0.01% of an  
average year (mm/hr)

$R_1$  
Rain field $R(x,y,t=t_1)$ at time $t_1$ (mm/hr)

$R_2$  
Rain field $R(x,y,t=t_2)$ at time $t_2$ (mm/hr)

$R_{\text{sim}}$  
Simulated rain rate field (mm/hr)

$R_{\text{sim}}(x,y)$  
Simulated rain rate field in 2 spatial dimensions, $x$ and $y$  
(mm/hr)

$R(x,y)$  
Rain field in 2 spatial dimensions, $x$ and $y$ (mm/hr)

$R(x,y,t)$  
Rain field in 2 spatial dimensions, $x$ and $y$ and time $t$  
(mm/hr)

$R(x,y,z,t)$  
Rain field in 3 spatial dimensions, $x$, $y$, $z$ and time $t$  
(mm/hr)

$< R(x,y) >_{DC}$  
Discrete cascade rain field simulator in 2 dimensions.

$< R(x,y) >_{CC}$  
Continuous cascade rain field simulator in 2 dimensions.
\(< R(x, y) >_{SS} \) Synthetic storm rain field simulator in 2 dimensions.

\( S(\omega) \) Spectral density function

\( S_v(f) \) Spectral density function of random function \( V(t) \)

\( t, t' \) Specific times

\( V \) Velocity (m/s)

\( V(t) \) Random function in time

\( V(f) \) Mean square output of \( V(t) \) when put through a narrow bandpass filter of centre frequency \( f \) and bandwidth \( \Delta f \)

\( V(f,T) \) Fourier transform of a specific sample of \( V(t) \) for \( 0 < t < T \)

\( V_H(t) \) Random process used to indicate fractional Brownian motion time series in one dimension

\( V_H(\bar{x}) \) Statistically self-affine fractional Brownian function

\( W \) Independent and identically distributed random generators

\( \{W_b, t > 0\} \) Independent and identically distributed stochastic processes

\( x, x' \) Specific locations

\( X \) Result of simulation algorithm, a 2 dimensional array of values with intensity not mapped to rain rate

\( x,y,z \) Coordinate axes of a three dimensional space

\( Z \) Radar reflectivity factor (dBZ)

**Greek**

\( \alpha \) Variable differentiating between different sets in a multifractal measure

\( \alpha_{ITU-R} \) Frequency dependent coefficients calculated from ITU-R Recommendation 838-2

\( \alpha \) Lévy index (characterises multifractals)

\( \beta \) Spectral density function exponent

\( \beta_{fBm} \) Spectral density function exponent for fractional Brownian motion

\( \beta_{fGn} \) Spectral density function exponent for fractional Gaussian noise
\( \gamma^+ \) Parameter of the \( \alpha \) model for multifractal cascades

\( \gamma^+ \) Parameter of the \( \alpha \) model for multifractal cascades

\( \gamma_R \) Specific attenuation (dB/km)

\( \Gamma(R_1R_2) \) Covariance function

\( \delta \) Side length of boxes used to calculate box-counting dimension

\( \Delta_a \) Random element added during the simulation process for varying lacunarity

\( \Delta t_{fs} \) Separation time between samples for calculation of fade slope. Also used as the prediction interval for the short term attenuation predictor (seconds)

\( \varepsilon \) Scaling factor for Cantor set generation

\(< \varepsilon^q_\lambda >\) The ensemble average \( q \)th moment of the field studied at (i.e. averaged over) a scale specified by \( \lambda \)

\( \zeta(t) \) Rate of change of attenuation at time \( t \), also known as fade slope (dB/second)

\( \lambda \) Scale ratio

\( \mu \) Mass distribution

\( \mu(B,(x)) \) Mass concentration density

\( \mu_R \) Independent random factors governing the fraction of rain flux concentrated from a long period to one of its sub-intervals

\( \xi \) Gaussian random variable

\( \Xi \) Vector separation = \( x' - x \)

\( \rho(R_1R_2) \) Normalised two-dimensional cross-correlation

\( \sigma_n^2 \) Variance of \( \xi_n \)

\( \tau \) Time separation = \( t' - t \)

\( \omega \) Frequency (in 2-D has units of \( 1/(\text{unit of area}) \))
CHAPTER ONE
OBJECTIVES AND OVERVIEW OF THESIS

1.1 Introduction

Over the past ten years, the radio spectrum has become increasingly congested, mainly due to the proliferation of new applications such as mobile telephony and digital television and radio. These, and proposed next generation applications such as mobile internet and video on demand, have led to an increasing pressure to utilise the spectrum efficiently, and open up new, higher frequencies to commercial use.

Communications systems that operate at frequencies above 10 GHz (also known as EHF frequencies) are particularly susceptible to attenuation caused by rain, clouds and atmospheric gases. Systems currently in operation deal with these problems by allocating a fixed fade margin. At EHF, experiments have shown that fades of over 20dB can occur at frequencies of 49.5GHz, and are quite common, occurring 0.02% of the time [Ventouras et al, 2000, revised 2003]. Figure 1.1 shows the attenuation statistics for a slant path (satellite) system operating at 12.5, 18.7, 19.8, 39.6 and 49.5 GHz. For these frequencies the attenuation is unlikely to be compensated for by fade margin alone, so other methods are required for the design and optimum use of systems at those frequencies.

1.2 The spatial variation of rain rate

Systems can be designed to take advantage of the spatial and temporal inhomogeneity of rain in order to improve their operating efficiency. For example, an Earth-space system can use site diversity to switch to the receiving station least affected by rain attenuation, provided the stations are far enough apart so that the same rain event does not cover both stations at the same time. Similarly, terrestrial systems use route diversity. For those systems operating with only a single link, where spatial diversity is not an option, time diversity can be of use to improve signal availability on the link. This is done by sending multiple copies of the same data stream separated in time. Although different pieces will be lost due to the rain fades, the complete data...
stream may be reconstructed at the receiving station. Route, site and time diversity are techniques that are collectively known as fade mitigation techniques (FMTs) or fade countermeasures (FCM). Other FMTs have been proposed and are being studied, such as adaptive coding, but these are less dependant on the physical parameters that govern rain fading.

Figure 1.1 Comparison of attenuation statistics for different frequencies, measured on Earth-Space links.

To correctly configure such systems requires an extensive knowledge of how rain fields vary in time and space. Much work has been done in this area by hydrologists and meteorologists, but their studies are tailored to address their specific concerns. For example, hydrologists are chiefly concerned with extreme events which cause flooding, which occur at very low time percentages in an average year and so are of less concern to communications engineers. Meteorologists are often concerned with weather forecasts for the next day/week and so operate at a temporal resolution a lot lower than desired by systems engineers, who are interested in the temporal variations of rain on timescales of the order of minutes and seconds. A thorough understanding of the operations of a communications system is necessary to fully understand the impact of rain on the system. Studies and measurements of rain are easier to relate to the behaviour of communication systems if the study has been designed with systems in mind. Also, a good understanding of how a given system reacts to rain fading can be used in conjunction with a nowcasting technique to
automate the application of an FMT, resulting in less wastage of system resources such as power and bandwidth.

1.3 Measuring devices and data

It is common to use point rainfall rate (as measured by a rain gauge) to estimate the attenuation experienced by a radio link. However, this is only correct for the case of very short links, as the extrapolation from a point measurement to a line average measurement is not always accurate. Cases where a rain cell may occur along the radio path, resulting in attenuation, while at the same time no rain is measured at the rain gauge are common. Rain gauge networks are often used to give an indication of the spatial variation of the rain, though any fine structure of the rain field smaller than the distance between the gauges will be lost. Meteorological radar data can be used instead of, or in addition to, a rain gauge network to provide data with greater spatial resolution and fill in the gaps. However, radar data does have its own errors and issues which are discussed further in chapter four of this thesis.

Data on the spatial variation of rain can be difficult to obtain, as not everyone has access to high quality rain gauge array or meteorological radar data. For many climates, for instance tropical, rain gauge measurements at sufficient temporal and spatial resolution have not been made from which to extract rain rate statistics, and from them link attenuation statistics. Measurement campaigns are also very expensive, and difficult to arrange and implement, hence a realistic rain field simulator would be of great interest and use to system designers, especially one that could be customised to mimic the behaviour of their climatic region. Several simulators have been proposed in the literature, some producing more realistic output than others [Crane, 1996, Manning, 1984, Capsoni et al, 1987b, Feral, 2003]. A discussion of these simulators, their advantages and drawbacks is given in chapter five, where our own model and rain field simulator is presented and compared with those previously published.

1.4 Fractals and rain

The shapes of rain cells as they are recorded by meteorological radar can appear chaotic, especially the contour lines drawn enclosing areas equal to or greater than a specified rain rate. However, further study shows that these contours have a statistically self-similar behaviour, which is consistent with fractal concepts. This
therefore leads us to the study of rain fields using fractal methods, which is discussed in this thesis.

Fractals are most commonly known from TV and films, where they are either the subject of documentaries concerning chaotic systems and natural phenomena, or are used as part of the special effects to produce visually realistic, yet convincingly alien artificial landscapes. Some purely mathematical fractals, such as the Mandelbrot set, have achieved iconic status, and fractals in their pure mathematical form have been popularised with the users of home PCs through the spread and appeal of computer programs such as FRACTINT. James Gleick's book "Chaos" [Gleick, 1987] also helped to bring the appeal of fractals to a wider audience, making them popular with scientifically inclined members of the general public.

When the field of fractal geometry and chaos theory was first presented in the 1960s and 1970s, it was considered a revolution, on a par with quantum mechanics. Its application to such diverse topics as the pattern of formation of sunspots [Milovanov and Zelenyi, 1993, Makarenko, et al, 2001], the shape and boundaries of clouds [Lovejoy, 1982], and antenna design [Song et al, 2001] promised great things for our understanding of complex systems. Since then however, the situation has become less clear. Many of the basic principles of fractal geometry are easy to grasp, but are also correspondingly easy to misapply. And, researchers must be wary of losing themselves in the mathematics of fractals and drifting away from the basic physical properties of rain. Still, using fractal geometry as a tool opens up a whole new range of measurements that can be used to improve our understanding of rain fields, and help us compensate for the attenuation caused by them. Methods such as the area-perimeter method and the area-distribution (Korcak) analysis provide information on the structure of rain fields, which can also be used as inputs into synthetic storm models.

1.5 Thesis overview

This thesis serves to introduce fractal geometry into a communications engineering context concerned with radio propagation and rain fading effects, highlighting its usefulness as a tool in many situations. It also serves to emphasise the raw physicality of rain, and discusses the various pitfalls that can trap the unwary, both in terms of assumptions used to make the mathematics easier to deal with, and also the nature of the measuring instruments used to produce the analysed data.
The main aim of this project was to introduce fractal methods into the field of radio communications engineering, and use those methods as a tool with which to improve the methods and techniques already in use to design and optimally use communications systems. By necessity, the study evolved to cover a large number of topics, all of which are directly relevant to the study of rain, even if they are unnecessary from a communications engineering viewpoint.

Figure 1.2 gives an overview of the many topics discussed in this thesis, and also attempts to group them according to specific subject areas. Of course, the interconnectedness of all the topics is also of vital importance, this too is shown in the figure. There are six main groupings in the figure, A to F.

Group A is concerned with the theoretical basis of many fractal concepts. Its topics are discussed in chapter two (and the appendices), and can be used to provide important information about the spatial variation of rain fields. Chapter two also touches on more complex ideas relating to fractal geometry and chaotic systems, including multifractals. These concepts are then discussed further in chapter three, where the results from many different published studies are reviewed and cross-compared, along with the results of our own fractal analyses (group B). The results of this comparison and review lead us to chapter four, where we dive deeper into the underlying phenomenological and physical properties of rain, taking into account the design of experiments and instrumentation used in the previously mentioned studies. The topics outlined in this chapter are summarised in group C of figure 1.

Chapter five takes the methods described in chapters two and four to develop a model that can be used to synthesise the spatial variation of rain. It also discusses the other methods previously published in the literature, and weighs their advantages and disadvantages (group D). In order to determine the validity of the model, measured data from an on-going Earth-space site diversity experiment is presented in chapter six, along with the inputs required for chapter seven, the application of the rain field simulator to a communications engineering case study (group E).

The thesis concludes as shown in group F, with a discussion of the results presented, and our conclusions. Potential further work is outlined, along with its implications.

The author of the thesis appreciates that the prospect of reading the many pages of this thesis may appear daunting, but would like to draw the reader's attention
to the large number of graphs and diagrams that accompany the text, adding greatly to
the total page count.

1.6 References


Figure 1.2: Schematic diagram of topics discussed in this thesis.
2.1 What is a fractal?

The simplest way to describe a fractal is as an object that appears self-similar under varying degrees of magnification, with each small part of the object replicating the structure of the whole. This is a particularly loose definition, however it captures the essential defining characteristic of fractals, that of self-similarity. Some natural fractals, e.g. the boundary of clouds, cracks in a wall and a hillside silhouette possess statistical self-similarity, i.e. they possess the same statistical properties (the same degree of ruggedness) as we zoom in. Other natural fractals, for example, the fern, possess exact self-similarity. Each frond of the fern is a mini copy of the whole fern, and each frond branch is similar to the whole frond and so on. Also, as we move towards the top of the fern a smaller and smaller copy of the whole fern can be seen.

Fractal geometry is a tool that can use simple rules to construct realistically complex objects. In some cases, these rules can be related to the physical processes that operate to produce the real objects and surfaces. These processes always operate at many different dimensional scales and are typically iterative, classic signs of fractal behaviour.

2.1.1 Fractal sets

Detailing a precise definition of a fractal is very difficult. Instead, it is more convenient to have a list of properties that indicate fractal nature, thereby not ruling out any interesting cases that might not fit snugly into an exact definition. In this way, the definition of a fractal is similar to the definition of life, pointing us towards the unusual and extreme cases of the subject where more can be learned.

Falconer [1990] gives the following properties for a fractal set $F$:
i) $F$ has a fine structure, i.e. it is detailed on arbitrarily small scales.

ii) $F$ is too irregular to be described in traditional geometric language, both globally and locally.

iii) Often $F$ has some form of self-similarity that may be approximate, or statistical.

iv) Usually the 'fractal dimension' of $F$ (defined in some way) is greater than its topological dimension.

v) In most cases of interest $F$ can be defined in a very simple way, perhaps recursively.

The Koch curve is a well-known, simple fractal set that obeys all the above properties. It is constructed using an iterated procedure. Begin with a straight line (as seen in the top part of figure 2.1). Divide it into three equal segments and replace the middle segment by the two sides of an equilateral triangle of the same length as the segment being removed (the two red segments in the middle part of figure 2.1). Now repeat, taking each of the four resulting segments, dividing them into three equal parts and replacing each of the middle segments by two sides of an equilateral triangle (the red segments in the bottom part of figure 2.1). This process is repeated an infinite number of times to produce the Koch curve (see figure 2.2 for an example).

As can be seen, the self-similarity of the curve is evident: each sub segment is an exact replica of the original curve. The original curve has four sub segments and each sub segment is a $1/3$ reduction of the original curve.
Similarly, the sub segments in turn have their own sub segment, each of which is again scaled down by 1/3. As the processes that creates the Koch curve is infinite, there are an infinite number of sub segments, all of which are scaled from the previous generation by the same value. This scaling can be used to calculate the fractal dimension of the curve, through the use of the similarity dimension (appendix G).

A noticeable property of the curve is that it is seemingly infinite in length. At each step in the generation process, the length of the prefractal curve increases to $4/3 L_{K-1}$, where $L_{K-1}$ is the length of the curve in the preceding step. As the number of generations increases the length of the curve diverges, hence, over an infinite number of iterations is has an infinite length. It can also be shown that the curve is effectively constructed from corners, hence, no unique tangent occurs anywhere on it. The Koch curve is nowhere differentiable, and is not a smooth curve.

The Koch snowflake is made of three Koch curves rotated by suitable angles and fitted together, or it can also be generated as shown in figure 2.3. The perimeter of the snowflake is infinite, as described above. However, the area bounded by the perimeter is finite, as it will remain less than the area of a circle drawn around the original triangle. This area – perimeter relationship is investigated further in section 2.2.1, and has been shown to be of importance when using fractals to model rain and clouds [Lovejoy, 1982, and others].

Figure 2.3: from left to right, stages in the construction of a Koch Snowflake.

2.1.2 Fractal surfaces

There are at least three different types of fractal surface that can be distinguished [Russ, 1994]. The first is a dense object with a fractal surface, for example erosion surfaces such as mountains, fractures, machined surfaces and corrosion and wear surfaces. This type is a self-affine rather than a self-similar fractal.
The second type is a fractal object such as a network or cluster whose surface will also 
be fractal (called a mass fractal). The third is a dense object within which there exists 
a distribution of holes or pores with a fractal structure (called a pore fractal). 
Schematic examples of these can be seen in figure 2.4.

Figure 2.4: Schematic diagram of the three types of fractal surfaces. Left: a dense 
object with a fractal surface. Centre: a network or cluster (called a mass fractal). 
Right: a dense object containing pores (called a pore fractal)

Many surfaces of commercial importance are produced by the agglomeration 
of particles onto a substrate, for example, the formation of a layer of soot. The 
resulting material under the surface may be dense or porous, depending on the method 
of production. A common model for these processes is diffusion limited deposition, 
though others exist as the whole field is rich and well-studied.

Porous materials can be produced through chemical reaction and corrosive 
attack as well as particle agglomeration. Roads, power grids, communications 
networks and other man-made networks also share some of the same mathematical 
characteristics, which can generally be dealt with through the use of percolation 
theory. The important characteristics of any grid are the number of connections that 
meet at each node, and the fraction of the links that are open or occupied.

It is the first type of fractal surface that concerns us the most in this thesis, as 
we are concerned with radar measured rain (and log rain) fields, which to a certain 
extent can be considered a fractal surface of a dense object in three dimensions, two 
spatial (x and y) and one showing the log rain rate (z), or height. This assumption is 
not valid for the small spatial scales where individual raindrops can be resolved, but is 
useful for dealing with radar measurements, as the rain rate value in a given radar 
range gate is a function of all the individual drops in that gate.
In the case of a log rain-rate field, it can be seen that the field is self-similar along the x and y directions, but is self affine in the z,x or z,y directions. Figure 2.5 shows an example of a log rain-rate field, plotted as a surface. Other assumptions about the nature of rain fields are discussed in Chapter 4.

Figure 2.5: surface representation of a log rain field from the radar data set recorded on 1\textsuperscript{st} May 2001.

2.2 Methods of calculating the fractal dimension

Fractals are characterised by their fractal dimension, which is usually (but not always) a non-integer dimension greater than its topological dimension $D_T$ and less than its Euclidean dimension $D_E$. A good general test for a fractal object is the condition that its fractal dimension exceeds its topological dimension – whichever method of calculating the fractal dimension is employed. However, it should be noted that this test is not infallible; there are definite fractals that will fail it.

From general experience we are familiar with 0, 1, 2 and 3 dimensional objects, as they are present all around us. A point is 0 dimensional, a line is 1 dimensional, a plane is 2 dimensional and a volume is 3 dimensional.
However, fractals have fractional dimensions, and to understand that concept we need to revise what our normal perception of dimension really means.

The Euclidean dimension $D_E$ is easily defined, it is quite simply the number of co-ordinates required to specify the object. The topological dimension $D_T$ is more involved. Topology is concerned with the ways in which objects can be distorted from one shape into another without losing their essential features. For example, a straight line can be distorted into a smooth curve, or a jagged curve without changing its topological dimension. Figure 2.6 gives some common examples of objects and their Euclidean and topological dimensions.

In the following sections we will discuss the theory behind some methods of calculating fractal dimension, which has particular relevance and reference to (log) rain fields as measured by meteorological radar. More details about the measured data can be found in chapter three.

2.2.1 Area perimeter relationship

The power law relationship that governs the length of a fractal perimeter with respect to the area enclosed by it can be given by:

$$A \propto P^{E_A}$$

where the exponent $E_A = 2/D_A$ and $D_A$ is the fractal dimension determined by the area-perimeter method. For regular Euclidean shapes such as squares, circles etc., $D_A$ is equal to 1. For fractal contours, the contour line becomes more and more complicated and $D_A$ has a value greater than 1. A curve that completely fills the area it is drawn in has $D_A$ equal to 2.
Figure 2.6 Common objects with their Euclidean and topological dimensions
2.2.2 Box counting dimension

The Hausdorff dimension (appendix C) provides a rigorous mathematical definition of the fractal dimension $D$. However, it is practically impossible to calculate $D$ for real data. Therefore, it is common practice to use the box counting method to determine the fractal dimension for actual measurements. The fractal dimension derived using the box-counting algorithm is:

$$D_b = \lim_{\delta \to 0} \frac{\log N(\delta)}{\log(1/\delta)}$$  \hspace{1cm} (2.2)

where $N(\delta)$ is the number of boxes of side length $\delta$ which are required to completely cover the length of the curve under study (or area, or volume if applicable). By plotting $\log N(\delta)$ against $\log(1/\delta)$ on axes of equal scale it can be seen that, if there is a linear behaviour, the data displays a scaling behaviour, and the average slope of the plot gives an estimation of the box counting dimension, i.e.:

$$\log(N) = D_b \log(1/\delta) + \log(U)$$  \hspace{1cm} (2.3)

A more in-depth discussion of the box counting dimension and the Hausdorff dimension can be found in appendices B and C respectively.

2.2.3 Area distribution analysis (Korcak dimension)

Taking a rain rate field to be reasonably well approximated by a fractal surface, with the rain rate plotted on the $z$ axis, and the two spatial dimensions plotted on the $x$ and $y$ axes, we can use the concept of a zeroset to define a series of islands at a given rain rate threshold (i.e. contour lines drawn around areas where the rain rate is greater than or equal to a set threshold). Plotting the number of islands $N(A \geq a)$ whose area is greater than $a$ on a log-log graph will produce a straight line. The fractal dimension of the boundary lines is given by twice the slope of this graph (equation 2.4). $N$ and $a$ are then related by:
called the Korcak relationship [Russ, 1994, Hyslip and Vallejo, 1997]. $D_K$ is the fractal dimension found using the Korcak relationship.

For an isotropic self-similar surface (where the self-similarity is independent of which $x$ or $y$ direction is taken) the fractal dimension of the surface ($D_{\text{surface}}$) would be greater than the fractal dimension of the contour lines ($D_{\text{boundary}}$) by exactly 1. It should be noted that the use of a section plane to reduce the dimension by 1 is only applicable to exactly self-similar objects. Most surfaces of interest, including rain, are not self-similar but self-affine. This means that the vertical direction is not the same as the lateral directions parallel to the nominal surface and the scaling of magnitude with dimension is different vertically as compared with laterally. However, it is correct to reduce a self-affine surface using a section parallel to the normal surface orientation, as the boundary lines produced are self-similar, hence $D_{\text{boundary}} = D_{\text{surface}} - 1$.

Using any other orientation for the section plane does not produce this effect.

In the case of the radar data analysed in the next chapter (Chapter 3) using this method, a full radar event is analysed to ensure that there are enough contours of various sizes at the different rain rates to give an accurate estimation of the fractal dimension.
2.2.4 Power spectral density function

Two-dimensional Fourier transforms are often used to study the spatial variation in rain [Marsan et al, 1996, Paulson, 2002]. The characteristic parameter in this analysis is the spectral density function exponent, which is given by:

\[ S(\omega) \propto |\omega|^{-2H-N_D} \]  

(2.5)

where \( H \) is the Hurst exponent [Mandelbrot, 1985] and \( N_D \) is the number of dimensions in which the rain field is measured. Thus, a rain field measured on a two-dimensional plane (as is the case with radar data) will have an exponent \(-2H-2\). The spectral unit \( \omega \) would have dimensions of \( 1/(\text{unit of area}) \).

In practice, this method of analysis can be difficult to apply to real data, due to the physical nature of the rain field and the limitations of the radar as a measuring instrument. The boundary effects due to the finite region scanned by the radar can introduce low frequency artefacts. If the maxima of the rain rate is near the edge of the region scanned, this introduces a step function due to the nature of the Fourier analysis and its assumption that the data are periodic.

However, studies have also shown [Russ, 1994] that this method for calculating the fractal dimension is largely insensitive to the effects of added noise, unlike the sensitivity that is shown by other methods, like the area distribution (Korcak) method.

2.2.5 Correlation dimension

Most studies of the spatial variation of rain don’t have the luxury of radar data, instead relying on an irregularly spaced grid of rain gauges. This adds complications because these arrays have their own characteristic fractal dimension, which must be taken into account when using their data to study fractal aspects of rain. The fractal
dimension of a rain gauge network can be calculated using the correlation dimension [Hentschel and Procaccia, 1983]. This investigates whether the average number of points (rain gauges) within a region \( < C(g) > \) varies with the characteristic size of the region \( g \) as

\[
< C(g) > \propto g^{D_c}
\]  

(2.6)

where \( D_c \) is the correlation dimension. For each point in the set, the number of other points \( C(g) \) within a distance \( g \) is counted, and an average value \( < C(g) > \) is calculated. \( D_c \) is then found by plotting \( \log[C(g)] \) as a function of \( \log[g] \).

2.3 The fractal dimension of embedded sets

The first Mandelbrot conjecture [Russ, 1994] states that a fractal of dimension \( D \), when projected onto a lower dimension space (in this case a rain gauge network) with dimension \( M \), produces a fractal of dimension \( D \) provided that \( D \) is less than \( M \), in which case the dimension is \( M \). In other words, if the degree of self-similarity (fractal dimension \( D \)) is greater than the degree of self-similarity of the measuring instrument (fractal dimension \( M \), e.g. the rain gauge network), the instrument “blurs” the fractal dimension \( D \) with the dimension of the instrument \( M \). Thus, to observe and measure the “true” value of \( D \), one needs to ensure that \( D < M \).

For example, given a cluster of points in 3-D space with fractal dimension \( D < 3 \), its projection onto a plane, \( M = 2 \), will produce a fractal pattern with dimension \( = D \), unless \( D > 2 \). If that is so, the projected pattern will have a dimension of \( 2 (= M ) \) and it will appear as a solid smooth bounded region with no internal holes. The same principle holds if we replace the sampling plane \( (M = 2) \) with an array of irregularly spaced rain gauges \( (M < 2) \).

In other words, if the fractal dimension of the measuring network is smaller than the fractal dimension of the rain field, the network will “filter” the rain field, producing a measured rain field that has the same dimension as the network. This is discussed in greater detail in chapter four.

Radar data, because the measurements can be processed so that they are regularly sampled across a plane \( (M = 2) \), allows us to determine the value of \( D \) provided that \( D < 2 \).
Falconer [1990] discusses the projection (or shadows) of fractal sets onto lower dimensional subspaces in more detail, and with more mathematical rigour, but comes to the same conclusion as given in the first Mandelbrot conjecture.

2.4 Multifractality versus monofractality

2.4.1 Why multifractals?

The reasons why rain is believed to be multifractal are not clear cut or obvious. One reason is given as follows: for any given time period there is more time spent with the rain rate above a lower rain rate threshold than at a higher rain rate threshold (see figure 2.8). In this discussion a rain occurrence is defined as the time values (seconds, minutes) where the rain rate exceeds a specified intensity threshold. If we look at the fractal dimension of rain occurrences for different thresholds, we will see that the fractal dimension changes as the threshold changes, i.e. time occurrences at higher rain thresholds happen less often, and so therefore have a lower fractal dimension. This ties in with the use of multiplicative processes, which have multiple fractal dimensions that decrease as the threshold increases.

![Figure 2.8 Schematic diagram of rain intensity time series, with two thresholds](image)

However, looking at rain time series this way only takes into account scaling in the y-axis - the rain intensity, as the same sample length of time series is used for all thresholds. Rain is scaling in both intensity and time; therefore any fractal analysis must look at scaling in both the x and y directions. A rain event with a maximum rain rate of 50 mm/hr may occur only once or twice a year, but is statistically similar to a rain event with a maximum rain rate of 5 mm/hr that can occur once or twice a week. This is demonstrated further in chapter three.
Early fractal models of rain relied on a monofractal approach, where rain was simulated by the scaling sum of a large number of random increments or "pulses" of different sizes [Lovejoy and Mandelbrot, 1985]. In this model, commonly known as "simple scaling", a single scaling exponent describes the behaviour of the statistical moments at different scales.

The linear structure of such additive processes comes into conflict with the actual non-linear dynamics that produce rain. Instead multiplicative models, stemming from the phenomenological cascade models studied in turbulence, were proposed. These require multiple exponents and are therefore more general. If structures in the field are defined by those regions that exceed a fixed threshold, then additive processes have a single fractal dimension, which is independent of the threshold, while multiplicative processes have multiple fractal dimensions that decrease as the threshold increases [Schertzer and Lovejoy, 1987].

The proposal that rain variation can be directly modelled as a turbulent cascade process has a number of implications for the physical basis behind stochastic rain modelling. Seeing them as such, cascade processes generically yield multifractals [Lovejoy and Schertzer, 1995]. This added weight to the study of rain fields in the terms of multifractals.

2.4.2 Multifractal measures and definitions of a multifractal

Multifractals represent a move from the geometry of sets to the geometric properties of measures. A mass distribution $\mu$ may be spread over a region in such a way that the concentration of mass varies widely. Sets where the mass concentration has a given density, where $\mu(B_r(x)) \approx r^\alpha$ for small $r$ (where $B_r(x)$ is the closed disc of radius $r$ and centre $x$), often display fractal-like features, with different sets corresponding to different $\alpha$. A measure $\mu$, or mass distribution with this sort of property is called a multifractal measure.

As with fractals, an exact definition of multifractal measures tends to be avoided [Falconer, 1990]. More detail on multifractal measures can be found in appendix E.
2.4.3 Multifractal cascades

Another way of looking at multifractals comes from cascade processes and is described in detail in [Lovejoy and Schertzer, 1995]. Multifractals arise when cascade processes concentrate energy, water or other fluxes into smaller and smaller regions. Any process that goes from low to high resolution can be termed a cascade. In the atmosphere, because the lifetime of storms depends on their spatial scale, the actual cascade is a space-time process with analogous mechanisms concentrating water fluxes into smaller and smaller regions of space, resulting in the observed high spatial variability.

For example, if we divide a time period with an initial rain rate $R=1$ into sub periods of scale $\lambda^{-1}$ where $\lambda$ is the scale ratio (and in this case =2). The fraction of the rain flux concentrated from a long period into one of its sub-intervals is given by independent random factors $\mu R$ given by the Bernoulli law:

\[
\begin{align*}
\Pr(\mu R = \lambda^{r+}) &= \lambda^{-c} \\
\Pr(\mu R = \lambda^{r-}) &= 1 - \lambda^{-c}
\end{align*}
\]

(2.10) \hspace{1cm} (2.11)

$\gamma^+, \gamma^-$ and $c$ are usually constrained so that $<\mu R> = 1$, $\lambda^{r+} > 1$, $\gamma^+ > 0$, which corresponds to wet sub-intervals and $\lambda^{r-} < 1$, $\gamma^- < 0$, which corresponds to dry sub-intervals. This is known as the “$\alpha$ model”.

Multifractal processes were first developed as phenomenological models of turbulent cascades, with the $\alpha$ model being the simplest. They are designed to respect the basic properties governing non-linear dynamical equations and have the following three properties:

1) scaling symmetry (i.e. invariance under dilations, “zooms”)  
2) a quantity conserved by the cascade (energy fluxes from large to small scales)  
3) localness in Fourier space (the dynamics are most effective between neighbouring scales as direct transfer of energy from large scale to small scale structures is inefficient)

Even though the full non-linear partial differential equations governing the atmosphere are more complex than those given by hydrodynamic turbulence, they still follow those three properties and so cascade models are relevant to them.
More on cascade processes and their application in the simulation of rain fields is given in chapter five.

2.4.4 The \( K(q) \) function for characterisation of multifractals

It is commonly taken that the \( K(q) \) function, sometimes known as the moment scaling function, can be viewed as a characteristic function of multifractal behaviour. The shape of the \( K(q) \) function specifies the type of scaling involved for a given dataset. A curved \( K(q) \) function indicates a multifractal structure, whereas if the \( K(q) \) function is straight, this is indicative of a monofractal structure.

Schertzer and Lovejoy [1987] present a method of calculating \( K(q) \) through investigation of the variation of statistical moments with scale. Fields produced by multiplicative cascade processes may have scaling behaviour that can be expressed by different scale independent scaling relationships. One of these relationships is given by:

\[
\langle \epsilon^q \rangle \approx \lambda^{K(q)}
\]

(2.12)

where \( \langle \epsilon^q \rangle \) is the ensemble average \( q \)th moment of the field studied at (i.e. averaged over) a scale specified by \( \lambda \). \( \lambda \), sometimes called the scale ratio, is defined as the ratio of the outer, maximum scale of the field to the averaging scale.

The following general forms have been proposed for the \( K(q) \) function [Lovejoy and Schertzer, 1990, Olsson and Niemczynowicz, 1996]:

\[
K(q) = \frac{C_1}{\alpha_L - 1} (q^a - q) \quad 0 \leq \alpha_L < 1, 1 < \alpha_L \leq 2
\]

(2.13)

\[
K(q) = C_1 \log(q) \quad \alpha_L = 1
\]

(2.14)

where \( C_1 \) is the co-dimension of the mean process and \( \alpha_L \), also known as the Lévy index, is related to the type of multifractal process involved. It should be mentioned that the generality of these forms has been questioned [Gupta and Waymire, 1993], as they are based on certain assumptions about the cascade structure of the fields.

The moment scaling function was used to characterise the multifractal nature of the measured radar data (chapter three) in preference to other methods of multifractal analysis, as this enabled us to directly compare our results with others.

2.5 References

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CHAPTER THREE:
FRACTAL ANALYSIS OF RAIN FIELDS

3.1 Analysis of rain rate contours derived from meteorological radar scans

3.1.1 Data description

The rainfall rate contours analysed in this research have been obtained by means of the Chilbolton Advanced Meteorological Radar (CAMRa), which is located in Hampshire in the south of England, at the latitude 51° 9' North and the longitude 1° 26' West. The climate is temperate maritime, with an average annual rain rate exceeded 0.01% of the time of approximately 22.5 mm/hr. The radar is a 25 m steerable antenna, equipped with a 3GHz Doppler-Polarization radar, and has an operational range of 100 km, and a beam width of 0.25°. To avoid reflections from ground clutter, maps of the rain rate field near the ground were produced by scanning with an inclination of 1.2°. These maps are produced on a polar grid, with a range resolution of 300m and an angular resolution of 0.3°. The number of maps produced in a given time period is dependent on the total angle scanned. The radar has a maximum angular velocity of 1°/second.

The radar scans were interpolated onto a square Cartesian grid, with a grid spacing of 300m and a side length of 56.2km. Each grid contains 35344 data points (188²) covering more than 3100 km². The grids are separated in time by 2 minutes.

This interpolation from the polar coordinates of the radar measurements to the analysed data sampled evenly across the Cartesian plane is commonly used in fractal analysis of rain fields [Lovejoy, 1982, Rhys and Waldvogel, 1986, Deidda, 1999, Feral and Sauvageout, 2002]. The smoothing effects of this interpolation at the smallest scales observed by the radar must be taken into account, which may have an impact on the fractal dimension measurements of the rain fields. Paulson [2002] compares statistics calculated from two-dimensional radar derived rain fields (using the
same interpolation procedure as used in this thesis) with statistics from a rain time series measured by an optical rain gauge. In this paper, he shows that the one-dimensional temporal spectral density exponent of log rain rate calculated from the interpolated radar fields is similarly valued to that calculated from the rain gauge measurements, and that both are consistent with the results predicted by fluid dynamical models.

The reflectivity – rain rate relationship [Skolnic, 1990] used to convert from radar reflectivity to rain rate values (appendix F) was:

\[ Z = 280R^{1.48} \]  

The exponents in this relationship are derived from the drop size distribution and therefore cannot be universally applied to all conditions and geographical locations. Further details on the process of converting radar reflectivity to rain rate is given in appendix F.

The satellite cloud pictures used in this research were all taken in the infra-red, 11.5-12.5\( \mu \)m, and are copyright Dundee Satellite Receiving Station (http://www.sat.dundee.ac.uk/). They maintain an up-to-date archive of images from NOAA, SeaStar and Terra polar orbiting satellites and the images are free and are funded by NERC. All the images shown are from the NOAA series of satellites and were taken by the AVHRR - Advanced Very High Resolution Radiometer carried by the satellites.
Three particular events were studied in detail as they are representative of the main types of rain event: stratiform, convective and frontal. Stratiform rain is usually widespread, with low rain intensity, and covers a large geographic area. It is also categorized by the melting process of ice particles to rain drops (the so-called bright band) being very obvious in the radar echoes. Convective rain is of the showery type usually encountered during the summer and autumn months. It is characterized by intense rain for relatively short periods of time. Radar echoes show that there is a great deal of turbulence inside the body of the convective cell, hence the lack of a bright band. Frontal rain occurs when a band of stratiform rain, often containing convective cells, is pushed across the area of interest by a strong wind.

Other events were also studied in lesser detail (see table 3.1). For each of the three main events the meteorological conditions in the wider area are described, a sample raster (radar snapshot) of the rain field is plotted and the satellite cloud pictures are shown.
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<th>Type</th>
<th>Duration</th>
<th>Range</th>
<th>Intensity</th>
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<tr>
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<td>1/5/2001</td>
<td>Stratiform/cold front</td>
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<td>16/5/2001</td>
<td>Convective</td>
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<tr>
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<td>26/9/2001</td>
<td>Stratiform</td>
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<td>229</td>
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<td>6</td>
<td>8/10/2001</td>
<td>Convective</td>
<td>7 hours</td>
<td>207</td>
<td>56.2x56.2Km</td>
</tr>
</tbody>
</table>

Table 3.1: Details of the radar events studied. Events in bold were studied in greater detail.

The frontal event studied was recorded on the 1st May 2001 (fig 3.2). A slow moving cold front system (see figure 3.3) associated with a low pressure area over central France moved across southern UK, bringing with it widespread stratiform rain, interspersed with cells of very heavy rain. From 8am to noon GMT, the radar recorded 231 scans.
Figure 3.2: Example raster from radar scan taken on the 1st May 2001. The colours show the rain rate in mm/hr.
Figure 3.3: NOAA satellite picture showing the cloud cover over the British Isles at 06:00 on the 1st May 2001.
The convective event (fig. 3.4) was recorded on the 16\(^{th}\) May 2001, from 8:00 to 14:30 GMT. It was an unsettled day with heavy showers and some local thunderstorms in the south of England, (see figure 3.5). The Met Office Frontiers Database shows large numbers of convective cells moving in a northeasterly direction. The radar recorded 134 scans for this day. As can be seen from figure 3.4, convective cells are a lot more compact, and cover less area than the frontal and stratiform rain. These physical properties of the rain field will be discussed further in chapter four.

Figure 3.4: Example raster from radar scan taken on the 16\(^{th}\) May 2001. The colours show the rain rate in mm/hr.
Figure 3.5: NOAA satellite picture showing the cloud cover over the British Isles at 12:58 on the 16\textsuperscript{th} May 2001.
The stratiform event (figure 3.6) was recorded on the 7th December 2000, from 8:20 to 17:00 GMT. A slow moving low pressure area directly to the south of England (see figure 3.7) moved north along the west coast of France, bringing with it widespread stratiform rain to the south and west of the country. The radar recorded 259 scans, mainly of stratiform rain with a few intermittent showers around the edges of the main rain area.

Figure 3.6: Example raster from radar scan taken on the 7th December 2000. The colours show the rain rate in mm/hr.
Figure 3.7: NOAA satellite picture showing the cloud cover over the British Isles at 16:37 on the 7th December 2000.
3.1.2 An illustration of statistical self-similarity in log rain fields measured by meteorological radar

The following sections provide formal demonstrations of the self-similarity of log rain fields as measured by the radar. This section offers an illustration of this self-similarity based on perceived visual resemblances.

Figure 3.8 shows an example log rain field as recorded by CAMRa. A simple test for self-similarity, hence indicating that the log rain field has a fractal nature, would be if a "zoomed in" image resembled the original. As can be seen in figure 3.9, which shows a zoom in to the centre of figure 3.8, on the innermost 94*94 pixels, this is not the case. However, this is due to the fixed resolution of the radar, and that zooming in increases the perceived pixel sizes, making the zoomed image appear more "blocky". When the original image is degraded in such a way as to have the same number and size of pixels as the zoomed in image (fig. 3.10), the self-similarity is more striking. Figure 3.10 was created by taking regular blocks of four pixels from the original image in figure 3.8, and averaging the rain values in each pixel to provide a new pixel value for the degraded (filtered) image.

Figure 3.8: Example log rain rate field
Figure 3.9: Zoom in to centre 94*94 pixels of log rain field shown in 3.8

Figure 3.10: Entire rain field as shown in figure 3.8, with resolution degraded to the same number of pixels in figure 3.9
3.1.3 Software

The software written as part of this program of research was mostly written in MATLAB, with IDL used primarily for the processing of the GBS data discussed in chapters six and seven. Appendix I gives details of each program, describing its functions, structure and methods by which the software was validated. In the case of the programs that calculated the fractal dimension of contour lines, test data in the form of arrays of rectangles of various sizes and Koch snowflakes at various sizes and resolutions were used to confirm the correct operation of the programs.

Three methods of calculating the fractal dimension of contours of equal rain rate from meteorological radar data have been used, the box counting method, the area perimeter method and Korcak analysis. All methods require the contour lines to be drawn on the radar image, each contour line enclosing an area experiencing rain rate above a fixed threshold. Figure 3.11 gives an example of a typical raster (from the data recorded on the 1\textsuperscript{st} May 2001) and figure 3.12 shows its corresponding contour plot. The contour lines were drawn using MATLAB's predefined contour function.

![Rain rate raster and contour plot](image)

Figure 3.11: Rain rate as recorded in a typical raster, 1\textsuperscript{st} May 2001
Figure 3.12: Contour lines drawn for areas of equal rain rate for the raster shown in figure 3.11

The resulting contour array was used to create a numerical array with the same number of points as the original raster (i.e. 188*188). Each element in the array was assigned a value according to whether it was on a specific contour line or not. Elements that were not on a contour line were assigned a zero value. Figure 3.13 shows an example of a contour line in a section of the array. For clarity only two rain rate thresholds are shown, 1 and 5 mm/hr. The resulting array could then be used for a number of different calculations.
3.1.4 Area perimeter relationship

The theory behind the area-perimeter relationship is given in chapter 2. Figures 3.14 and 3.15 show the area plotted against the perimeter for rain rate threshold of 1 and 25 mm/hr. This data comes from the frontal event recorded on the 1st May 2001. Similar results were seen for the other events studied.

As can be seen, there is a definite straight line behaviour, the slope of the line $E_A$ being related to the area perimeter dimension $D_A$ by $E_A = 2 / D_A$. It can also be seen that the absolute number of contours at the higher rain rate threshold (25 mm/hr) is less than at the lower rain rate threshold (1 mm/hr). This adds to the uncertainty in the estimation of the fractal dimension.

Table 3.2 gives the calculated area perimeter dimension values for three different types of event, convective, stratiform and frontal, along with the one sigma errors [Press et al, 1993]. As can be seen, the errors in the estimation increase with increasing rain intensity, due to the physical behaviour of the rain. Areas of higher
rain rate are generally likely to be smaller in size than areas of lower rain rate, making their area and perimeter correspondingly harder to calculate due to the limited resolution and pixel size of the radar data. There is a saturation of the curve at high perimeter and area values due to the limited window size of the data used.

Figures 3.14 and 3.15 show that the data is scaling over a range of two decades (for a rain rate threshold of 1 mm/hr) and one decade (for a threshold of 10 mm/hr). A minimum data range of at least one decade is needed to have any confidence in the estimation of the area perimeter dimension, though, naturally, evidence of scaling over a wider range would be more reassuring.

<table>
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<th>Rain event date and type</th>
<th>Rain rate threshold (mm/hr)</th>
<th>Area-perimeter dimension</th>
<th>Error</th>
<th>Rain event date and type</th>
<th>Rain rate threshold (mm/hr)</th>
<th>Area-perimeter dimension</th>
<th>Error</th>
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<td>0.01</td>
<td>7th May 2001 Convective event</td>
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Table 3.2: Area-perimeter dimension calculated for a frontal, stratiform and convective event at different rain rate thresholds.
Figure 3.14 Area-perimeter dimension for event recorded 1st May 2001. Rain rate threshold = 1 mm/hr $D_A = 1.19 \pm 0.01$

Figure 3.15 Area-perimeter dimension for event recorded 1st May 2001. Rain rate threshold = 25 mm/hr $D_A = 1.17 \pm 0.01$
Our area perimeter results are contrasted with those published by Lovejoy [1982], Rhys and Waldvogel, [1986] and Feral and Sauvageot [2002]. Lovejoy used the area-perimeter relationship with radar and satellite data to determine that cloud and rain areas are for a range between $1 \text{ km}^2$ and $1.2 \times 10^6 \text{ km}^2$ and reported that the area-perimeter dimension had a value of 1.35 with correlation coefficient of 0.994. This value was confirmed by Feral and Sauvageot [2002]. Rhys and Waldvogel confirmed this with their analysis of radar reflectivities of clouds during hail storms, but also report a sharp crossover to a smooth behaviour ($D_a = 1$) at the centre of the hail clouds. This could be due to the resolution of their measuring radar, a topic which is discussed in the next section, but has also been explained as the influence of the dynamic organisation of the cloud on the cloud contours [Cahalan and Joseph, 1989]. For cumulus clouds with a diameter of less than around 1 km (the so-called "break diameter") the cloud appears to be made up a single cell, while those clouds with a greater diameter display a multi-cellular structure. A third possible explanation for this discrepancy in reported results is due to an observed "smoothing" effect of the rain rate contours by the wind, which is described further in chapter four (section 4.4.2), or the possible "filtering" of the rain field due to atmospheric turbulence (chapter four, section 4.4.3).

3.1.5 Box counting dimension

Unlike the area-perimeter method that requires a significant number of contours to calculate the fractal dimension, the box-counting method can provide an estimation of the fractal dimension for a single contour line. However, this comes at the cost of accuracy.

In the cases where the area of rain of interest approaches the pixel size, because of the limits of the measuring instrument's resolution, it is no longer possible to accurately determine the area, perimeter or shape of that section of rain field. In the case of the box-counting and area-perimeter methods, areas that are of the order of the pixel size are often categorised as being either points or irregular polygons, leading to an inaccurate fractal dimension result. This is particularly important to the method of calculating the box counting dimension, as the contour line has to be covered in boxes that vary in size from the minimum pixel size, to a number of factors of 2 larger,
whereas the area perimeter method only uses boxes of the minimum pixel size to perform the calculation.

Figure 3.16 shows an example of this situation. The box counting dimension for each contour (threshold level = 1 mm/hr) in the data set recorded on the 1st May 2001 is plotted against the length of that contour. For contour lengths up to 20 pixels long there is a definite spread in the value calculated for the fractal dimension, whereas for contour lengths beyond this level, the fractal dimension is mostly constant at ~1.2. Figure 3.17 shows the same graph, but at a threshold level of 25 mm/hr. Once again it can be seen that there are less contour lines occurring at the higher rain rate threshold and that they are correspondingly smaller, making it more difficult to estimate the fractal dimension.

The same behaviour was observed when test arrays consisting of rectangles and Koch snowflakes at different sizes were used as inputs into the box-counting program.

Figure 3.16 Box counting dimension against length of contour for event recorded on the 1st May 2001. Rain rate threshold=1mm/hr
Figure 3.17 Box counting dimension against length of contour for event recorded on the 1st May 2001. Rain rate threshold=25mm/hr

Feral and Sauvageot [2002] also use the box counting method to calculate the fractal dimension of the perimeter and area of contour lines drawn around areas of radar reflectivity greater than a given threshold. They present results showing that the box counting dimension of the perimeter of the contour is ~0.9, while the box counting dimension of the area varies between 1.66 and 1.39. This implies that the perimeter of the contours has gaps in it, something that should not be the case for a continuous contour line, and also that the area covered by the rain has holes. It should be noted that it is unclear from their paper whether they looked at contour lines individually, or as a group of contours in the measured area of interest. If the latter, then this makes it difficult to directly compare their results with the results published here.

Several other studies published in the literature deal with the use of the box counting method for calculating the fractal dimension [Olsson and Niemczynowicz, 1994, Lavergnat and Gole, 1998, among others]. However, they tend to use rain gauge time series data, rather than radar measurements and so it is not possible to directly compare them with our results.
3.1.6 Korcak analysis (area distribution analysis)

Figures 3.18 and 3.19 show the area distribution plots as used to calculate the Korcak dimension. Both graphs show distinctive straight line behaviour, indicative of scaling. As can be seen in figure 3.18, the number of rain cells occurring at the higher rain rates are less, giving less data points and making the estimation of the fractal dimension more error-prone. However, when the numbers are normalized to the percentage of total number of contour lines, it can be seen (figure 3.19) that there is very little difference between the behaviour of the rain cells at the different thresholds. The only difference between the rain rate thresholds is the location of the point where the straight line behaviour drops off, due to the smaller numbers of contour lines recorded with those measured areas.

Figure 3.18: Number of contours with area greater than x-axis value for thresholds in mm/hr of 1 (o), 2 (X), 5 (+), 10 (*), 15 (□), 20 (◇), 25 (▼), 30 (▲), 35 (▲), 40 (▲), 45 (▲), 50 (★). Event recorded 1st May 2001
Figure 3.19: Normalised number of contours with area greater than x-axis value for thresholds in mm/hr of 1 (○), 2 (×), 5 (+), 10 (★), 15 (□), 20 (◇), 25 (▽), 30 (▲), 35 (◀), 40 (▶), 45 (∗), 50 (★). Event recorded 1st May 2001

The values of the fractal dimension using the Korcak method, $D_K$, for each rain rate threshold can be seen in table 3.3. As can be seen, there is substantial variation in $D_K$ calculated for each rain rate threshold, however the mean value is 1.13. This is due to the method's susceptibility to noise [Russ, 1994], and therefore indicates that the Korcak dimension is not as suitable for estimation of the fractal dimension of rain fields as the two dimensional Fourier transform method (section 3.1.7). A further discussion on the effects of noise can be found in section 4.2.3. The variation in $D_K$ is not correlated to the rain rate threshold. The corresponding slope for the lines plotted in figure 3.19 is approximately $-0.5 \pm 0.05$. 
<table>
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<th>Korcak dimension</th>
<th>Errors</th>
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</tr>
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<td>16th May 2001 Convective event</td>
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</table>

Table 3.3: Korcak dimension calculated for a frontal, stratiform and convective event at different rain rate thresholds.

The value of the fractal dimension of the contour lines given by the Korcak analysis is generally less than that provided by the area-perimeter and
box-counting methods, which was \( \sim 1.2 \). This is not unexpected. As reported in
the literature [Russ, 1994, Rangarajan and Ding, 2000] it is often assumed
that the numerical values obtained from different methods of calculating the
fractal dimension are the same, and this may be the case for some generated
mathematical fractals carried out to a fixed level of detail. However this is a
dangerous assumption for real data due to the finite size of the pixels used by
the measuring instrument and the discrete sampling of the data. This is
corroborated by Peitgen et al [1992, page 215] where it states, “there are
several different dimensions which give different answers”. Mandelbrot [1985]
also states:

“It is known that there are several ways of measuring fractal dimension.
A better statement of the same fact is that several alternative definitions exist,
but in the extensively studied case of strictly self-similar fractals all these
definitions yield the same value.”

This disagreement between the fractal dimension results produced
using various different methods can be a source of confusion, but prompts us
to a better understanding of the methodology used and the nature of the data
being studied. This is addressed in more detail in the next chapter.

The author was unable to find any examples of other studies in the literature
where the Korcak method was used to analyse meteorological radar data. However,
the technique has been used to analyse the roughness and size distribution of granular
materials [Hyslip and Vallejo, 1997].

3.1.7 Two-dimensional Fourier transforms and power spectral density functions

If each near-horizontal radar scan is treated as an instantaneous snapshot of the
rain rate field then the spatial spectral density may be calculated via 2-D Fourier
transform. Figure 3.20 illustrates the two-sided spectral density of spatial (log) rain
rate variation, averaged over the 230 scans recorded for the event that occurred on the
1st May 2001. The near circular contours are consistent with a rotationally
symmetric, and hence quadrant symmetric, spectral density and spatial
autocorrelation. This shows that the (log) rain field is isotropic, i.e. that there is no
preferred direction or spatial scale. The units of 1/km define the spatial frequency.
Figure 3.20: 2-D Spatial spectral density of rain rate for event on 1st May 2001, averaged over 230 scans.

Figures 3.21 and 3.22 show the corresponding spatio-temporal isocorrelation contours in the Fourier space for the x-t and y-t sections respectively. We can see from these that the contours have undergone a rotation from the circularly symmetric as a result of the general overall advection present during the event. Our observed “squashing” of the contour lines by the wind (as demonstrated in the next chapter) could also be partially responsible for the change from circular contours to elliptical ones in the 2-D FFT. In figure 3.20, the cross in the centre of the graph is due to the edge effects of the 2-D FFT; a discontinuity is created at the edges where a tiling is performed in order to turn out discrete sample into a continuous function suitable for analysis. The peaks in the contour lines approximately 20 degrees anti-clockwise from the main “north-south” edge effect is due to ground clutter, in this case a barn which partially blocks one ray of the radar during scans. In figures 3.21 and 3.22 the edge effect is only seen in the space direction (along the y axis), as there were more samples recorded in time, and there was less of a discontinuity in the time samples.
In figures 3.20 to 3.22 the contours have been shifted so that the centre of the figure is the origin. Similar results are presented in [Marsan et al. 1996].

Figure 3.21. 2-D Spatial-temporal (x-t) spectral density of rain rate for event on 1\textsuperscript{st} May 2001, averaged across the y direction.

Figure 3.22. 2-D Spatial-temporal (y-t) spectral density of rain rate for event on 1\textsuperscript{st} May 2001, averaged across the x direction.
The spectral density function for a two dimensional isotropic random field [Paulson, 2002, and section 2.2.4] is given by:

\[ S(\omega) \propto \omega^{-2H-2} \]  

(3.2)

where \( H \) is the Hurst exponent, and is calculated as equal to 1/3 for the measured radar data. For surfaces, the fractal dimension \( D \) is related to the Hurst exponent by \( D_{\text{surface}} = 3 - H \) and \( D_{\text{boundary}} = D_{\text{surface}} - 1 \), where \( D_{\text{surface}} \) is the fractal dimension of the surface and \( D_{\text{boundary}} \) is the fractal dimension of a contour line drawn on the surface.

Therefore, in the case of the area-perimeter relationship:

\[ A \propto P^{E_A} \]  

(3.3)

the slope of the best fit line for \( \log(A) \) against \( \log(P) \) is \( E_A = \frac{2}{D_{\text{boundary}}} = \frac{5}{3} \), related to the exponent of \( \omega \) in the spectral density function.

Figure 3.23 shows the averaged radial spectral density of the log rain rate recorded during the event. As can be seen, it is a straight line with a slope \(-2.79 \pm 0.02\). Table 3.4 gives the spectral density exponent calculated for the three different types of events studied. The theory [Kolmogrov, 1941] predicts a slope of \(-8/3 \ (-2.66)\) in certain regions of the power spectrum. Our results are consistent with this, even though they tend to overestimate slightly.

The unit of spectral density, km, is derived as follows: The rain rate we use in analysis is rain rate relative to 1 mm/hr, which changes our rain rate into a dimensionless unit.

\[ \text{spectral density} = (\text{power})^2 / \text{frequency} \]  

(3.4)

where the relative rain rate is equivalent to power. A dimensionless unit squared is still dimensionless, giving the spectral density unit = \( 1/\text{frequency} = 1/(1/\text{km}) = \text{km} \).
Fig. 3.23. Spectral density of log rain rate for event on 1st May 2001. Exponent = -2.79 ± 0.02

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<th>Radial Spectral Density Exponent</th>
<th>Error</th>
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<td>Stratiform event</td>
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<td>16th May 2001</td>
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<td>Convective event</td>
<td></td>
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</tr>
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Table 3.4: Radial spectral density exponent calculated for a frontal, stratiform and convective event at different rain rate thresholds.

Two-dimensional Fourier analysis of rain fields is quite common in the literature. Studies published include those by Olsson et al, [1993], Tessier et al,
[1993], Marsan et al [1996], Harris et al [1996] and Purdy et al [2001]. The actual values for the power spectral density exponent varied from study to study. Chapter four discusses this disagreement in reported values within the context of the physical and phenomenological behaviour of rain.

3.2 Fractal analysis of rain fields using a rain gauge network

3.2.1 Data description

The Hydrological Radar Experiment (HYREX) was a UK Natural Environment Research Council (NERC) Special Topic that ran from May 1993 to April 1997. The experiment was centred on the Brue catchment in Southwest England where a dense rain gauge network data was sited. The dense rain gauge network comprised 49 Cassella 0.2mm tipping bucket rain gauges, each recording time of tip to a time resolution of 10 seconds. The network provided at least one rain gauge in each of the 2km radar grid squares that lay entirely within the catchment. In addition, there were two parallel lines of greater gauge density extending SW to NE across the catchment, aligned with the prevailing wind direction and running from lowland to upland. Within each line there was one 2km grid square containing a super-dense sub-network of 8 rain gauges arranged in a square-within-a-diamond configuration, with one sub-network in a lowland area and the other in an upland area.

The data used in this section of the study were supplied by the British Atmospheric Data Centre from the NERC Hydrological Radar Experiment Dataset (http://www.badc.rl.ac.uk/data/hyrex/).

3.2.2 Correlation dimension of the network

Figure 3.24 shows the correlation dimension as calculated for the “Brue” rain gauge network. As can be seen, for a range of distances between 0.5km and 20 km the network is itself scaling, with a dimension of $1.37 \pm 0.09$. All of the rain gauges are within a distance of 20 km of each other, hence the graph saturating after this point. Details of the theory behind this calculation can be found in chapter two, section 2.2.5.

This indicates that it is possible to use the data recorded by this rain gauge network to measure the fractal dimension of rain rate contours (which have a fractal dimension of 1.2) though this has not yet been attempted.
3.3 Multifractal analysis of radar derived rain fields

The multifractal behaviour of the radar derived rain fields was investigated through the variation of statistical moments with scale, as described in chapter two, section 2.4.4.

The equation used in the analysis was:

\[ \langle \epsilon^q \rangle \approx \lambda^k(q) \]  

(3.5)

where \( \langle \epsilon^q \rangle \) is the ensemble average \( q \)th moment of the field averaged over a scale specified by \( \lambda \). \( \lambda \) is defined as the ratio of the outer, maximum scale of the field to the averaging scale.

The fact that \( \langle \epsilon^q \rangle \) is an ensemble average means that it is not certain that an individual field (i.e. radar raster) is properly described by the above equation. It is only after averaging over a number of fields that the expected multifractal behaviour is likely to appear. We also assume that our data has temporal stationarity for this analysis, as temporal stationarity is required to
get accurate ensemble averages from multiple fields. This temporal stationarity, however, has not been confirmed by measurements.

To analyse the radar data, the area of the field is successively divided into non-overlapping squares of side length $\lambda$. For each $\lambda$ the average rainfall volume in each square $\varepsilon_\lambda$ is obtained as the average rain rate in the square. To obtain $\varepsilon_\lambda^q$, the average values are raised to the power of $q$, and $\langle \varepsilon_\lambda^q \rangle$ is obtained by averaging $\varepsilon_\lambda^q$ firstly over all squares, and secondly over all fields (radar rasters).

The validity of (3.5) is tested by plotting the average moments $\langle \varepsilon_\lambda^q \rangle$ as a function of $\lambda$ on a log-log diagram. If the points fall on a straight line, then the value of $K(q)$ can be estimated as the slope of line. The entire $K(q)$ function can be estimated by performing the above procedure for different values of $q$.

Figure 3.25 shows a plot of the average moments $\langle \varepsilon_\lambda^q \rangle$ against $\lambda$ for values of $q$ between 0.5 and 3.5. As can be seen, (3.5) holds, as plots for various $q$ are straight lines.
Figure 3.25. Estimation of $K(q)$ values for different $q$. Frontal rain event, recorded on the 1\textsuperscript{st} May 2001.

Figures 3.26 shows the full $K(q)$ functions for the three different rain event types studied. All rain types show a curving $K(q)$ function, indicating multifractal behaviour, though there is dramatic variation between the rain types, with the frontal event being the most curved and the stratiform the closest to a straight line. This corroborates with other results presented by Olsson and Niemczynowicz [1996], though their results show a convex curve. Our results curve in the opposite direction because when the rain is averaged across the radar rasters, the resulting ensemble average is generally very small. When these average values are raised to exponents greater than 1, their value decreases, resulting in the concave curve seen here. Table 3.5 shows the number of points in each data set for the three event types, and gives the percentage of those points with a value greater than 1 mm/hr.
Figure 3.26. $K(q)$ functions for different rain event types. Values averaged are rain rate, in mm/hr

<table>
<thead>
<tr>
<th>Data set</th>
<th>Total number of points in data set</th>
<th>% of points in data set with values greater than 1 mm/hr</th>
</tr>
</thead>
<tbody>
<tr>
<td>7th December 2000</td>
<td>9,224,784</td>
<td>90.4%</td>
</tr>
<tr>
<td>(stratiform event)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1st May 2001</td>
<td>8,164,464</td>
<td>66.6%</td>
</tr>
<tr>
<td>(frontal event)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16th May 2001</td>
<td>4,630,064</td>
<td>96.1%</td>
</tr>
<tr>
<td>(convective event)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3.5: Number of data points in each data set, and percentage of those points with a rain rate value of less than 1 mm/hr.

Figures 3.27, similarly to figure 3.26 shows the full $K(q)$ functions for the three different rain event types studied. In this case however, it is the log rain rate values that are analysed, rather than the rain rate values. This is a crucial distinction, as the $K(q)$ functions for the log rain rate values are not curved, and may be considered to be straight lines, indicating a monofractal structure.

From this demonstration we can see that even though rain fields can be considered multifractal, log rain rate fields can equally validly be considered to be monofractal. This provides a justification for our use of a method of simulating rain fields in chapter five, which is based on a monofractal, additive process in the logarithmic domain.
3.4 Other studies

The vast majority of other studies published in the literature tend to be split into different categories. There are many cloud studies published, but they are not concerned with rain, instead concentrating more on satellite pictures of cloud cover [Ali et al, 1998, Cahalan and Joseph, 1989, Gotoh and Fujii, 1998, Kwo-Sen-Kuo et al, 1993]. Most fractal analysis of rain has moved from the simpler monofractal analysis and description to the more complex multifractal analysis and synthesis [Lovejoy and Schertzer, 1985, 1992, Deidda, 1999, Marsan et al, 1996]. However, fractal methods are not the only ones in use to study rain, there have been a wide range of studies published which deal with rain in complex, non-fractal ways, such as the use of synthetic storms for modelling [Onof et al, 1996]. These will be further discussed in chapter 5, within the context of our own objectives, the synthesis of rain fields.
There is a pronounced tendency in the published multifractal studies of rain fields to rely more on the mathematics and modelling of the theory than on the actual physical measurements and results provided from the rain gauge or radar data. This is further addressed in chapter four, where we discuss in detail the phenomenological behaviour of rain fields.

3.6 References


Lovejoy, S., Area-Perimeter Relation for Rain and Cloud Areas, Science, Vol 216, 185-187, April 1982


CHAPTER FOUR:
CONCEPTS RELATING PHYSICAL PROPERTIES OF RAIN TO ITS FRACTAL DESCRIPTION

4.1 Introduction

This chapter discusses the physical and phenomenological processes essential to the study of rain fields, from the point of view of fractal analysis and synthesis. It is also concerned with the potential pitfalls that can cause confusion and ambiguity in the interpretation of results derived from fractal methods, including the effects of the resolution and dimension of the measuring instrument.

4.2 Measuring device resolution and system errors

4.2.1 Resolution of a measuring instrument

When studying the spatial and temporal variation of rain to determine the fractal dimension, it is essential to have a good understanding of the effects of the measuring instrument's resolution on the methods being employed.

The following three sections are focussed particularly on the problems encountered using radar measurements. However, rain gauges (both singly and in arrays) also have their associated problems.

Rain gauges are, in essence, point rainfall devices, in that they average and measure only the rain falling on a particularly small area, with a set integration time. Networks of rain gauges can be set up to give an indication of the spatial distribution of rain. However, any fine scale structure that appears on lengths smaller than the distance between the gauges is lost.

Two-dimensional rain gauge networks also introduce the complication that the measuring network itself has its own fractal dimension, which must be taken into account when using the network to calculate the fractal dimension of the rain field. The fractal dimension of a rain gauge network can be calculated using the correlation dimension [Hentschel and Procaccia, 1983] and this was described in more detail in chapter 2 (section 2.2.5) and chapter 3 (section 3.2.2).
Section 2.3 details the theory behind the fractal dimension of embedded sets, introducing the fact that if the fractal dimension of the measuring network is smaller than the fractal dimension of the rain field, the network will “filter” the rain field, producing a measured rain field that has the same dimension as the network. This can be thought of analogously, in terms of a band pass filter. The filter corresponds to the network, and the rain field is a signal with a given bandwidth. If the band pass filter is not large enough, then it will clip the edges of the rain field signal bandwidth, resulting in an inaccurate measurement. If the filter is wide enough, then the entire signal will pass through, but the measured signal will still be an intersection of the original with the filter.

Radar data, because the measurements can be processed so that they are regularly sampled across a plane \((M=2)\), allows us to determine the value of \(D\) provided that \(D<2\).

4.2.1.1 Minimum resolution (pixel size)

As was reported in chapter three and our previous studies [Callaghan, 2002a, Callaghan and Vilar, 2002 and 2003], in the cases where the area of rain of interest approaches the pixel size it is no longer possible to accurately determine the area, perimeter or shape of that section of rain field, due to the lack of resolution. To calculate fractal dimension using the box-counting and area-perimeter methods, the areas that are covered by a small number of pixels are often categorised as being either points or irregular polygons. This leads to an inaccurate fractal dimension result. A demonstration of this has been presented in chapter three, section 3.2.4.

4.2.1.2 Maximum range (window size)

As the area of rain grows to the size of the radar window and beyond, this situation can lead to errors in the calculation of its fractal dimension, because of the straight line introduced by the window's edge. Also, the maximum range of the window can determine the maximum scale up to which the fractal dimension results are valid. This ties in with the discussion of scaling limits presented in section 5 of this chapter. To a certain extent these problems are not dissimilar to those seen in Fourier analysis (as mentioned in section 3.1.7) where edge effects must be taken into account.
4.2.1.3 Time resolution

As stated before, rain gauges have set integration times, which serve to average the total rain that falls on the gauge over that time. The majority of continuous long rain gauge time series are derived using quite high integration times, using either daily or hourly accumulations. Radar data on the other hand can have as little as seconds between consecutive scans, depending on the scan type and radar hardware. Nevertheless radar often has only continuous data accumulated over the space of hours, maybe even months, whereas rain rate records, recorded using rain gauge networks, can have an observation time spanning to decades.

4.2.2 Measurement time

A key feature of rain is its intermittency, that is, the decreasing number of occurrences of rain events as the rainfall rate intensity increases. To collect enough data to accurately statistically describe these high rainfall events requires a very long measurement time, unfortunately such data is not always available.

A common method reported in the literature [Olsson and Niemczynowicz, 1994] for calculating the fractal dimension of rain time series is to divide the time series into non-overlapping time segments (boxes) of size $L_t$. $N(L_t)$ is the number of boxes where the rainfall intensity at some point inside the box exceeds a specified rain rate threshold (shaded boxes in figure 4.1).
The box counting dimension $D_B$ is given by the equation [Olsson and Niemczynowicz, 1994]:

$$N(L_i) \propto L_i^{-D_B}$$  \hspace{1cm} (4.1)

which can be evaluated by plotting, on a log-log scale, the number of boxes $N(L_i)$ above the set threshold, as a function of the size $L_i$ of the boxes, for different values of $L_i$. This is done between the lower limit of the resolution and the upper limit of the length of the series.

The resulting curves show breaks in the scaling which corresponds to the average length of a rain event, and an inter-rain event, which is not entirely unexpected. The slope of the line is also different for different rain rate thresholds, which would appear to show a fractal dimension decreasing with increasing rain rate. This is due to the limited length of the time series under investigation (approximately 2 years in length).

The nature of rain is such that larger rain events (events with high rain rate $R$ (mm/hr)) occur less often than smaller ones. Hence in any bounded time series there will be less larger events with progressively longer time gaps occurring between them as $R$ increases. To properly determine the scaling between different rain thresholds $R$, it is necessary to scale the corresponding observation time as well, so that similar numbers of rainfall events at the lower rain rate can be compared with similar numbers of more intense events.

### 4.2.3 System errors and noise

It is generally assumed that during measurement collection any system errors are constant over the whole period being recorded. For the most part this is a valid assumption, but as always, data must be recorded in such a way as to minimise errors and keep those errors that do occur from changing.
The presence of noise can also alter the fractal dimension of a field by virtue of changing the proportion of higher frequencies to lower frequencies present. In Russ [1994], the effect of noise on the correct estimation of the fractal dimension of a generated fractal surface was demonstrated. For methods including the perimeter area relationship and FFT, Russ showed that in most cases the fractal dimension estimate was thrown off by the addition of noise. However the FFT method was the most robust in the presence of noise, due to the “spectral window” filtering process.

4.3 Assumptions on the nature of rain fields

4.3.1 Definitions

When the field is stationary, in the wide sense, the time averaged results are independent of the specific times t and t' and are a function only of their time separation \( \tau = t' - t \). These results still depend on the specific locations \( x, x' \). If the field is such that the space averaged results are independent of the space locations \( x, x' \) and are a function only of their vector separation \( \mathbf{z} = x' - x \) then the field is said to be homogeneous. If the space averaged results further rely only on the magnitude \( |\mathbf{z}| \), independent of direction, then the field is said to be isotropic.

For homogeneous fields, the results can still depend on the specific times t, t'. Thus, to obtain time averaged and space averaged results that are not dependent on specific times and specific locations, the fields must be stationary and homogeneous. [Bendat and Piersol, 1971]

Curves that are invariant with respect to translation and scaling are described as self-similar. However, curves that are self-similar only when different directions (e.g. time and distance) are scaled by different factors are known as self-affine.

In the case of a log rain field, which to a certain extent can be considered a fractal surface in three dimensions, two spatial (x and y) and one showing the log rain rate (z), or height (see figure 4.3), it can be seen that the field is self-similar along the x and y directions, but is self affine in the z,x or z,y directions. For more detail on fractal surfaces, see chapter 2, section 2.2.2.

4.3.2 Differences in behaviour between rain rate fields and log rain rate fields.

Figures 4.2 and 4.3 show the contrast in shape and roughness between a rain field that is displayed as rain rate, and a rain field displayed as log rain rate. Of the
two, the log rain rate appears much rougher, with a lot more peaks. The log rain rate surface also looks a lot more like other fractal surfaces, especially those created to simulate fractional Brownian motion.

Taking sections through both surfaces produces contour lines with exactly the same fractal dimension, though the numbers and positioning of the contour lines are not the same.

Some problems that occur when dealing with the fractal nature of rain, e.g. its intermittency, can be dealt with by taking rain in the form of a log rain field, and therefore only studying its scaling nature when actually raining. This would remove breaks in the temporal scaling regime due to inter-event durations, but would also limit the maximum length of temporal scaling due to the limited time associated with rain events.

Figure 4.2: surface representation of a rain field from the radar data set recorded on 1st May 2001. Same radar “snapshot” as for figure 4.3 (1 pixel = 300m x 300m)

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1 Ordinary Brownian motion is a random process $V_H(t)$ with Gaussian increments and $\text{var}(V_H(t_2) - V_H(t_1)) \propto |t_2 - t_1|^{2H}$ where $H=1/2$. The generalisation to parameters $0 < H < 1$ is called fractional Brownian motion (fBm).
Figure 4.3: surface representation of a log rain field from the radar data set recorded on 1st May 2001. Same radar “snapshot” as for figure 4.2 (1 pixel = 300m x 300m)

The disadvantage with using log rain rate instead of rain rate is that it then becomes necessary to determine where and when it is and is not raining. Again, this is another resolution and definition problem. The atmosphere is full of water vapour at all times, it is only when certain physical conditions are met that rain begins to fall. However, determining whether it is raining or not is largely dependent on the resolution of the measuring instrument, especially for the cases when the rainfall is very light. There have been cases recorded in the literature [Ventouras et al, 1997] where a meteorological radar has recorded clouds that have droplets falling from them that evaporate again before hitting the ground (known as virga in the USA). This is obviously precipitation, and will affect earth-space communications links, but cannot be accurately described as rain.

The Federal Meteorological Handbook No. 1 defines precipitation as follows:

**Precipitation.** Precipitation is any of the forms of water particles, whether liquid or solid, that fall from the atmosphere and reach the ground.

This is further subdivided into other categories including:
**Drizzle.** Fairly uniform precipitation composed exclusively of fine drops with diameters of less than 0.02 inch (0.5 mm) very close together. Drizzle appears to float while following air currents, although unlike fog droplets, it falls to the ground.

**Rain.** Precipitation, either in the form of drops larger than 0.02 inch (0.5 mm), or smaller drops which, in contrast to drizzle, are widely separated.

However, these definitions are not easily converted into specific rain rate thresholds, and in many cases are below the resolving power of the instrument used to measure the rain. A clarification of these definitions with regards to specific rain rate thresholds would reduce the uncertainty in determining the areas in a radar image where it is and is not raining.

Recent assumptions made while dealing with log rain fields consider them to be homogeneous, isotropic, Gaussian, random fields [Paulson, 2002]. This appears to be confirmed by our calculation of the two-dimensional Fourier transform, as described in chapter three, section 3.1.7

However, other studies consider rain fields to be anisotropic [Lovejoy and Schertzer, 1985] due to the aforementioned turbulent intermittency of the rain field. The anisotropy arises when the scale-changing operator acts differently in different directions. The simplest case corresponds to self-affinity.

The issue of the added complication due to the intermittency of the rain field can be partially addressed through the use of fractal methods to characterize log rain fields, rather than simply rain rate fields. A combination of a monofractal log rain rate model, and a model that gives accurate estimation of the areas where it is not raining, could prove to be more useful and less complicated than a multifractal rain rate model that has to deal with both issues at once.

### 4.4 Atmospheric turbulence and fractal rain fields

#### 4.4.1 Basic principles of turbulence

Rain can be considered to be a passive tracer suspended in a turbulent medium, implying that its temporal and spatial dynamics can be fully addressed through the theory of fluid dynamics. These theoretical models of rain as a passive tracer in a turbulent, two-dimensional flow, [Kraichnan and Montgomery, 1980, Lovejoy and Schertzer, 1995], predict that the temporal and spatial spectral density
of log rain rate follow a segmented power law form. These fluid dynamical models often lead to spectral density power laws with exponents expressed as ratios of small integers. Over ranges of scales where the spectral density follows a simple power law, there is no special scale and the random variable exhibits self-similarity.

4.4.2 Smoothing effect of wind on rain contours

The data set recorded on the 1st May 2001 was that of a strong frontal event. Therefore the wind direction was remarkably constant over the period the radar was scanning. Figure 4.4 shows an example of the contour lines recorded for one scan during that event. The snapshot presented in figure 4.4 is part of a wider radar scan, which took 80 seconds to complete. Hence it can be considered to be a near-instantaneous snapshot of the rain field.

Figure 4.4: Example of the contour lines produced for one scan recorded on the 1st May 2001. Contour lines are at values of 1, 5, 10, 15, 20, 25, 30, 35 and 40 mm/hr.

The wind direction recorded at Chilbolton during the event shown in figure 4.4 was due North. Because of the way the radar data was mapped...
onto a Cartesian plane, this translates as the wind blowing along the direction of the negative x-axis. The average wind speed recorded at Chilbolton during this frontal event was approximately 8 m/s, with maximum gusts of 15 m/s.

It appears that the wind “compresses” the rain cells along the line parallel to the direction of the wind, and the rain cells are elongated along a line perpendicular to the direction of the wind. It is unknown whether or not the amount of compression and elongation is related to the wind speed. This would seem sensible but could be a topic for further study.

4.4.3 Relationship between the fractal dimension of clouds and the fractal dimension of rain fields

There have been a number of studies done on the fractal shape of clouds, [Lovejoy, 1982, Rys and Waldvogel, 1986, Kwo-Sen-Ko et al, 1993, Ali, et al, 1998, Cahalan and Joseph, 1989, Gotoh and Fujii, 1998]. Only the first mentioned however seeks to deal with rain areas as having a similar fractal dimension to the cloud areas.

In general, the studies report a change in fractal dimension for clouds greater than a certain threshold, smaller clouds having a fractal dimension in the region of $-1.35$ (which was found using the area-perimeter method). This is somewhat at odds with the results we have presented, though this discrepancy could be as a result of the smoothing effect of wind on the rain rate contours.

Figure 4.5 shows a schematic diagram of the basic processes involved during rainfall. At the top we have the rain bearing cloud, which is characterised by its own statistics and fractal dimension. Immediately below (or perhaps inside) the cloud is the area where the rain is produced (known as the melting layer, in cases of stratiform rain). A section through this area would give an accurate view of the spatial distribution of the rain as it formed, and before it had the chance to fall and get pushed around by wind and other atmospheric effects.

Clouds are extremely variable in time and space and individual cloud elements are usually very ephemeral. General observation can show that
cumulonimbus clouds can build up from nothing, and die away into nothing in the space of mere hours. The cloud droplets that form the cloud have an even shorter lifespan; the tendrils that emerge from the edges of clouds can evaporate into nothing in mere tens of seconds [McIlveen, 1992, page 140].

The difference between rain and cloud droplets is quite simply a matter of size. It can be shown quite easily that as soon as a droplet falls out of its parent cloud into subsaturated air, it begins to evaporate. The rate of this evaporation and fall speed correspond closely to the droplet size, meaning that the distance fallen by the drop before total evaporation increases very rapidly with initial droplet size. Falling cloud droplets (with a radius ~10μm) survive for only a few centimetres, whereas raindrops (with a radius ~0.1cm) can fall tens of kilometres.

The section just above the ground layer is the section that meteorological radar records. That section can be considered as a plane section through the three dimensional structure of the rain event. As can be seen from the diagram, the meteorological radar section is not perfectly parallel to the ground, due to the blocking effects of ground clutter (and also due to the curvature of the Earth). In the case of our data, the radar elevation angle used for the PPI scans was 1.2°.

In between these two sections is an area where we can’t be certain exactly what is happening to the raindrops as they fall and are moved about by the wind and atmospheric turbulence. However, turbulence models, where rain is modelled as a passive tracer in a three dimensional turbulent flow, give good results, and provide a method of understanding of the dynamical nature of rain, which can then be expanded and refined.

4.4.4 Orographic effects on measurements

As shown in figure 4.5, below the meteorological radar section is the ground, complete with hills and valleys and a rain gauge array. As mentioned before, the rain gauge array can provide time series of point rainfall, but is also affected by its position on the ground. Orographic features, such as hills and valleys can dramatically affect the amount of rain recorded at various positions in the geographical area of interest. For example, studies in
Bavaria at the Hohenpeissenberg Observatory show that standard rain gauges may overestimate the amounts of rain falling on them by about 10% on the lee slopes, and underestimate them by 14% on windward slopes. [Barry and Chorly, 1998, page 70].

Figure 4.5: changes in the distribution of rain after it is created.
Also, experimental work performed at CCLRC-Chilbolton Observatory, compared different makes of rain gauges on the ground, and also on the roof of a neighbouring building [Brand, 2000]. It was shown that the rain gauges on the roof recorded less rain than the same gauges on the ground, and that this variation increased with wind speed.

Studies of the fractal dimension of rain fields using data from rain gauge networks spread across areas with dramatically different orography [Harris et. al., 1996, Purdy et. al 2001] have shown that the measured fractal dimension is also strongly affected by the topography of the area of interest.

As of the date of this thesis, the author has yet to discover any studies that deal with the fractal dimension or spatial variation of rain in particularly flat areas, though there is a proposed experimental set-up in Serbia which hopes to address that issue [Nadj and Vucinic, 2002].

4.4.5 Differences between stratiform, convective and frontal rain

Figures 4.6 through 4.12 show examples of meteorological radar measurements of convective, stratiform and frontal rain recorded by CAMRa on different dates. These three categorizations are commonly used by radar meteorologists to classify the type and properties of rain events. In the following plots, PPIs are near-horizontal sections through the 3-dimensional rain field, whereas RHIs are vertical sections. In all of the plots, the radar itself is located at coordinates (0,0).

Stratiform rain (figures 4.6 and 4.7) is usually widespread, with low rain intensity, and covers a large geographic area. It is also categorized by the melting process of ice particles to rain drops (the so-called bright band) being very obvious in the radar echoes.
Convective rain (figures 4.8 and 4.9) is of the showery type usually encountered during the summer and autumn months. It is characterized by intense rain for relative short periods of time. The radar echoes show that there is a great deal of turbulence inside the body of the convective cell, hence the lack of a bright band.
Frontal rain occurs when a band of stratiform rain, often containing convective cells, is pushed across the area of interest by a strong wind. The schematic diagram in figure 4.10 shows the frontal boundary in a cold front system with the cold air passing under the warmer air.
Figures 4.11 and 4.12 show actual measurements of the radar reflectivity (dBZ) and the velocity, \( V \) (m/s) taken during the passage of a cold front covering a distance out to 90km from the radar. The velocity measurement shows the direction in which the drops are being moved by the wind.

Figure 4.10: schematic diagram of a cold front system.

Figure 4.11: RHI scan of a frontal event showing the radar reflectivity factor dBZ.
Figure 4.12: RHI scan of a frontal event showing the velocity of the raindrops.

As can be seen from the previous pictures, in a spatial variation sense, convective events are more intermittent, with more turbulent boundaries and higher reflectivity, whereas stratiform rain tends to have smoother variation, lower reflectivity and can cover the area under investigation almost completely. Frontal rain combines aspects of the other two. In the vertical plane the stratiform rain is more layered, whereas the convective rain (by its nature) is turbulent. We have not looked at the implications of this variation in the vertical plane in any great detail as we mainly consider the spread of rain across the horizontal plane, but the vertical variation will have implications for slant path communications systems.

Bundling stratiform events and convective events into the same data sets in order to perform spatial analysis on them could result in murky and difficult results to interpret. However this seems to be a common procedure in the studies so far published, with the exception of [Olsson and Niemczynowicz, 1996]. And, to be fair, there can be some difficulty determining from a rain gauge time series the classification of a single event. Coincident radar data gives the best surety for event type classification.
4.5 Scaling and scaling limits

Any discussion of scaling and scaling limits regarding rain fields must first take into account the medium in which the rain falls, i.e. the nature and scale of the Earth’s atmosphere. The Earth’s atmosphere has two very different scales. The order of magnitude of the horizontal scale of the atmosphere is that of the Earth itself, ~10,000 km. By contrast, the vertical scale of the atmosphere is very much smaller than the radius of the Earth. This is also complicated by the fact that the atmosphere has no definite upper surface, as its density falls continuously with increasing height.

As most of the material of the atmosphere is squeezed into a shallow layer overlying the surface of the Earth, the distributions of atmospheric properties such as temperature, humidity, etc. are strongly anisotropic – meaning that their gross vertical and horizontal distributions are very different. However, this anisotropy is well marked only on scales not insignificant in comparison with the scale height of the atmosphere. On much smaller scales, ~1m, atmospheric disturbances are much more isotropic. The rain fields studied in this thesis occur at these smaller scales.

Further discussion on the concept of the “stratified atmosphere” can be found in chapter 1 of [McIlveen, 1992].

Figure 4.13 shows a conceptual diagram of the scaling limits involved in treating rain as a fractal, and also discusses some of the limitations of the range of data available at this time.

Points are plotted on a set of axes, the x-axis representing the range of distances and the y-axis the range of time-scales. The limits involved in the treatment of rain as a fractal are more evident in the spatial, rather than temporal scale. For instance, the lower limit of the spatial scale, is the level where the rain can no longer be considered to be a field of varying intensity, but is quantized down to its component drops [Lovejoy and Schertzer, 1990]. This takes place at very short timescales, and needs very high resolving power to determine the fractal nature correctly.

At the other end of the distance scale is the limit of the size of the Earth. Rain fields can never be bigger than the Earth, and in most cases, often only cover a small percentage of the surface area of the planet. There are some data sources (such as the Landsat satellites), which have data covering a particularly large window size, but
have very low resolution in both time and space, as well as limited numbers of consecutive measurements.
Figure 4.13: A schematic diagram to aid in the discussion of scaling limits applied to rain fields.
As mentioned before, a single rain gauge gives results of the rainfall rate at a point, often with very high resolution (small integration times). This is useful for studying the temporal statistics of rain, but the length of the observation time has to be taken into account. The data sets with very long observation times, in the region of decades, often have very low temporal resolution, i.e. daily or hourly rainfall accumulations.

A rain gauge network has the same issues as a single rain gauge, and it is possible to use the array to investigate the spatial patterns in rain. However, the spacing between the gauges is crucial, as any fine structure in the rain field smaller than the spacing between the gauges will be lost.

Finally we have the meteorological radar data, similar to what is recorded by the radar at Chilbolton and also in the Met Office’s Frontiers Database. This type of data gives continuous coverage over a region of interest, the window, at a reasonable resolution, but there are issues with the fact that it takes a radar time to scan, therefore the snapshots taken are not instantaneous. Also length of observation time can be a problem, as radars have not been in general use for nearly as long as rain gauges and the highest quality radar data is collected on an event by event basis.

The Met Office’s Nimrod radar network provides a database of composite radar measured rainfall data at a time resolution of 15 minutes, and a spatial resolution of 5km*5km. Data are available from late 2002 from the BADC (http://www.badc.rl.ac.uk) and are updated on a regular basis. This database is an extremely valuable source of raw data for the further analysis of rain fields.

Rain gauges can be considered to be passive rain sensors, once installed they will keep recording until they break down or someone stops them. Radars are more active, they have to be deliberately targeted and require a lot more maintenance and are more expensive in terms of resources such as power.

Any discussion of the scaling involved in rain events must take into account the scaling limits mentioned above.
4.6 References


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5.1 Introduction

When developing a rain field simulator it is useful to consider the physical and phenomenological behaviour of rain, as well as the mathematics involved in the simulator. For this reason, we first of all consider figure 5.1, which is an example raster from a stratiform rain event recorded on the 7th December 2000. A number of key features can be seen immediately:

1. The rain is spread out over a large geographical area (as confirmed by figure 5.21, the normalised cross-correlation of two radar rasters separated in time by approximately 20 minutes, where a large, low area of rain is seen).
2. The rain is of low overall intensity, with a peak value of 7 mm/hr in this case.
3. There are few neighbouring “islands” of rain; most of the rain is gathered in the same large area.

![Figure 5.1 Example raster from the stratiform rain event recorded on the 7th December 2000. Contour lines are drawn at 1 and 5 mm/hr (1 pixel corresponds to an area of 300m*300m)
Figure 5.2 gives a schematic example of a stratiform rain field with much the same features. The rain covers a wide geographical area and changes gradually in intensity. As can be seen, there are few “gaps” between different sized regions of rain, as most of the rain is present in one large area.

![Figure 5.2 Schematic diagram of a stratiform rain field.](image)

A simulation of a stratiform rain field is shown in figure 5.3. It too covers a wide geographical area with low overall intensity and most of the rain occurring within one large mass (though there are a few “islands” of rain outside the main area). The simulation is plotted on the same length scale as the real example in figure 5.1, with one pixel corresponding to an area of 300m*300m.
Figure 5.3 Example simulation of a stratiform event on the same length scale as figure 5.1. Contour lines are drawn at 1, 5 and 10 mm/hr (1 pixel corresponds to an area of 300m*300m)

Similarly, an example of a convective rain field is shown in figure 5.4. It was recorded on the 16th of May 2001, and demonstrates key features of a convective rain field:

1. The areas of rain cover smaller geographical areas than those in a stratiform event. (This is confirmed by figure 5.22, the normalised cross-correlation of two radar rasters separated in time by approximately 20 minutes, where a number of distinct peaks are visible)
2. Unsurprisingly, the rain in those smaller areas (commonly known as “rain cells”) is quite intense, with peak values of up to 40mm/hr in this case.
3. The rain is not limited to one or two locations, but instead there are a sizeable number of cells, though some have merged to form a larger cell with two or more peaks. If we look at the field in terms of the total area covered by rain, that area is not condensed into one large widespread area, but instead is fragmented, with gaps between the rain cells.
Figure 5.4 Example raster from the convective rain event recorded on the 16th May 2001. Contour lines are drawn at 1, 5, 10, 15, 20, 25, 30, 35 and 40 mm/hr (1 pixel corresponds to an area of 300m*300m).

Figure 5.5 shows a schematic diagram of a convective event, emphasising all the aforementioned features.

Figure 5.5 Schematic diagram of a convective rain field.
Similarly to figure 5.3, figure 5.6 shows a simulation of a convective rain field, demonstrating the key features of convective rain.

![Simulation of convective rain field](image)

Figure 5.6 Example simulation of a convective event on the same length scale as figure 5.4. Contour lines are drawn at 1, 5, 10 and 15 mm/hr (1 pixel corresponds to an area of 300m*300m).

The simulator used to produce the simulated rain fields shown above is based on the Voss random additions algorithm [Voss, 1985] for discrete cascades. This is one of a family of methods used to model the spatial variation in rain. There are three main areas, discrete cascades, synthetic storms and random space-time function generators (also known as continuous cascades). Of the three, the group that seems to have the most potential when dealing with situations encountered by communication systems is the discrete cascades. The discrete cascade used in the above simulations is discussed in greater detail in sections 5.2.2 and 5.2.3. An overview of synthetic storm and continuous cascade models is given in section 5.3.1, along with a brief discussion of other discrete cascade models presented in the literature.
For clarity, this chapter will be subdivided into two main parts. The first will step through the simulation algorithm, discussing the methods and inputs used in the process.

The second part of the chapter will then discuss the other types of models published in the literature to simulate rain fields, concentrating on the three main types identified above. Section 5.3.2 goes into greater detail into the mathematics and justification behind the use of fractional Brownian motion to simulate (log) rain fields. In the rain field simulator described in this chapter, the parameter that controls the fractal dimension of the resulting fields, as well as the spectral density exponent, is the Hurst exponent $H$. The value of $H$ for a given system also has physical implications for the system memory and correlation in time. For this reason, its relationship with the spectral density exponent, and hence, fractal dimension, is discussed with more mathematical rigour in section 5.3.3.

Further information on the spatial behaviour of rain fields is given in section 5.3.4, where the cross correlation of radar rain fields is shown and discussed.

5.2 The simulation process

Figure 5.7 gives a flowchart representation of the simulation process. Moving from top to bottom, we see how the physically measured parameters tie in with the first three of the four simulator inputs, $H$, $r$, and $N$. The simulator itself is an iterative process, which produces an array of numbers that have to be further processed in order to correspond directly to rain rate. This conversion process introduces $b$, the final parameter used, and results in a simulated rain field with appropriate spectral density exponent, fractal dimension, and behaviour that is visually consistent with convective or stratiform type events (according to what is desired).
5.2.1 Measured parameters and simulator inputs.

5.2.1.1 The Hurst exponent $H$

The Hurst exponent is related to the spectral density exponent of the measured rain fields, and is also related to the fractal dimension of the contour lines enclosing
areas of equal rain rate. It controls the fractal dimension of the resulting simulated rain field.

As described in chapter two, section 2.2.4, and [Paulson, 2002], the spectral density function for an two-dimensional isotropic random field is given by:

\[ S(\omega) \propto \omega^{-2H-2} \]  

(5.1)

where our measured radar rain fields have \( H = 1/3 \).

In all the following simulations \( H = 1/3 \).

5.2.1.2 Lacunarity and its parameter \( r_i \)

The differences in visual behaviour between stratiform and convective rain fields are controlled by the parameter \( r_i \). This parameter controls whether or not all the rain occurs in one large area (as is the case for stratiform rain) or instead is broken up into convective type rain cells. It is related to the concept of “lacunarity” discussed in [Mandelbrot, 1983] and [Feder, 1988].

“Lacuna” (related to lake) is the Latin word for gap. Hence a fractal is called lacunar if its gaps tend to be large, in the sense that they contain large intervals. In the case of our data and simulations, presented above, stratiform rain fields are more lacunar, i.e., the gaps between areas of stratiform rain tend to be larger than the gaps between the convective rain cells, which are more fragmented, but closer together. In some cases the gaps between areas of stratiform rain are so large that we only see one area of rain at a time, this is the case in the examples given in section 5.1.

If we look at the power laws that govern fractals and scaling, we can see that the expressions tend to the form [Mandelbrot, 1983]:

\[ \text{prefactor} \times (\text{quantity})^{\text{exponent}} \]  

(5.2)

where the exponent is related to the fractal dimension, and the prefactor is related to the lacunarity.

A better understanding of the concept of lacunarity can be gained through the following example:

Consider a Cantor set, over the unit interval \([0,1]\) (figure 5.8). At each successive generation we divide the interval (black segment) into three parts and
delete the open middle part, leaving its endpoints. This procedure very quickly produces very short segments, and after an infinite number of generations what remains is an infinite number of points scattered over the interval. This is not yet a completely self-similar set, however, enlarging it is simply a matter of extrapolation so that it covers the region \([0,3]\) by two Cantor sets covering the intervals \([0,1]\) and \([2,3]\). Repeating this process ad infinitum gives us a self-similar set on the half line \([0,\infty]\).

![Figure 5.8 Stages in the construction of a Cantor set](image)

We can calculate the fractal dimension of this Cantor set by using the similarity dimension \(D_s\) (appendix G) where:

\[
D_s = \frac{\log(N_C)}{\log(1/\varepsilon)}
\]

and \(N_C = 2\) and \(\varepsilon = 1/3\). Hence the similarity dimension for this set is \(D_s = \log(2)/\log(3)\).

The following two Cantor sets both have similarity dimension \(D_s = 1/2\).

In example 1 (figure 5.9), instead of taking out the middle 1/3 of the line, we take out the middle 1/2. The process is repeated as described above with \(N_C = 2\) and \(\varepsilon = 1/4\). Hence, \(D_s = 1/2\).

![Figure 5.9 Stages in the construction of a Cantor set. \(D_s = 1/2\). Example 1.](image)
In example 2 (figure 5.10), instead of taking out the middle 1/3 of the line, we take out 2 sections of 3/9, leaving a middle section. The process is repeated with $N_c = 3$ and $\varepsilon = 1/9$. Hence, $D_s = 1/2$ also.

![Figure 5.10 Stages in the construction of a Cantor set. $D_s = 1/2$. Example 2.](image)

Example 1 and 2 are both Cantor sets with the same fractal dimension, yet they look different. This is because they have different lacunarity. Example 2 is more lacunar than example 1, as a greater proportion of the unit interval is removed in example 2 (6/9 as compared to 1/2).

Lacunarity is distinct from the fractal dimension in the fact that varying the lacunarity does not change the fractal dimension.

### 5.2.1.2.1 Calculation of lacunarity from measured radar rain fields

Unfortunately, the concept of lacunarity has not had the mathematical treatment that the concept of fractal dimension has, hence there is very little in the literature about how to calculate it for real data.

However, we can use the area-perimeter method to gain an estimation of the lacunarity of rain fields according to the equations given in appendix A and above.

From appendix A:

$$\log(A) = \frac{2}{D_A} \log(P) - 2 \log(Q) \quad (5.4)$$

hence the intercept of the line, corresponds to the prefactor in (5.2), which determines the lacunarity. Table 5.1 gives the calculated values of the prefactor for the different
types of events and at different rain rate thresholds, calculated from the area-perimeter relationship.

<table>
<thead>
<tr>
<th>Event date and type</th>
<th>RAIN RATE THRESHOLD</th>
<th>1 mm/hr</th>
<th>10 mm/hr</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st May 2001, frontal</td>
<td>0.082 ± 0.01</td>
<td>0.087 ± 0.01</td>
<td></td>
</tr>
<tr>
<td>16th May 2001, convective</td>
<td>0.098 ± 0.01</td>
<td>0.077 ± 0.03</td>
<td></td>
</tr>
<tr>
<td>7th December 2000, stratiiform</td>
<td>0.086 ± 0.01</td>
<td>0.96 ± 0.05</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.1 Prefactor values for given event types and rain rate thresholds, calculated from the area-perimeter relationship.

As can be seen, there is very little variation in prefactor values for either the event type or the rain rate threshold, though a deeper look into the mathematics of lacunarity would be advised before drawing any firm conclusions. Similarly, a more rigorous method of characterising the differences between stratiform events and convective events, without simply relying on their visual differences, could make a great impact on the study of rain.

For the simulations presented in section 5.1 and elsewhere in the chapter, a value of $r_i = 1/2$ is used for stratiform rain, and $r_i = 1/3$ is used for convective rain. These values were chosen empirically to produce simulated "convective-like" and "stratiform-like" rain fields with the same visual characteristics as described for each type of event in section 5.1. Obviously, the simulator would be improved if the values for $r_i$ were derived from physical measurements.

5.2.1.3 Number of iterations $N_i$

The total number of iterations, $N_i$, determines the size of the square array of numbers that is the result of the simulator. The side length of the array is given by:

$$\left(\frac{1}{r_i}\right)^{N_i} + 1$$  \hspace{1cm} (5.5)
In the case of our measured data, the rain rates are plotted on a grid of 188*188 pixels, with one pixel covering an area of 300m*300m. The simulated fields are plotted on a much larger grid, 1025*1025 pixels for the case of $r_t = 1/2$, and then are cut down to the required 188*188 pixel grids for comparison with the measured data. This performs a dual purpose: the resulting cut down simulations are at the right resolution for a direct comparison, as well as having enough iterations occur to ensure that the simulations are sufficiently detailed. $N_I$ is also limited by the speed and memory capabilities of the computer used for the simulation.

### 5.2.2 The random additions algorithm

To simulate the rain field we contemplate the use of the successive random addition algorithm introduced by Voss [1985] to generate fractional Brownian motion. The algorithm is easily extended to higher dimensions and can produce surfaces with coastlines that are self similar fractals with a fractal dimension given by $D = 2 - H$, where $H$ is the exponent used in the generation of the landscapes (and is also mathematically the Hurst exponent).

The algorithm uses an iterative additive process to generate the rain field, resulting in a monofractal field that closely resembles measured log rain rate fields. This is in contrast to the multiplicative methods proposed elsewhere in the literature, which result in a multifractal field. However, it has been shown in chapter 3, section 3.4, that log rain rate fields have a straight line $K(q)$ function, and may therefore be appropriately represented by monofractals.

The surfaces are generated on a lattice in an iterative manner, extrapolated from the method of successive midpoint displacement used for simulating Brownian motion in one dimension (see figure 5.11). On a square grid, the generation of a midpoint displacement surface has two stages for each step.

In the first generation an independent Gaussian variable $\xi$ is generated with zero mean and unit variance. This value is used as the level at the central point on the lattice (point A). The four corner points (points B) of the lattice are given a value equal to zero. The values at the midpoints (points C) of each of the four lines on the outside of the lattice are the average of the two end points and the centre point, i.e. the value at the midpoint of the line is given by the average of the values of its nearest neighbours. Then points inside the lattice (points D) are given values according to the
average of their diagonal neighbours. All the points plotted then have independent values of \( \xi_{n=1} \) added to them, where the Gaussian random variable now has the variance given by:

\[
\langle \xi_n^2 \rangle = \sigma_n^2 = r^{2nH}
\]  

(5.6)

with

\[
r_i = 1/ \sqrt{2}
\]

(5.7)

and

\[
n = 1.
\]

(5.8)

This determines a new square lattice at 45° to the original with lattice size \( 1/ \sqrt{2} \).

The procedure is continued for the next generation (points E) where the values at the new points are given by the average of the nearest neighbour locations, i.e., the neighbours in directions parallel to the axes. The points on the rim will have values given by the average of their three nearest neighbours, rather than sites inside the lattice, which have four. All the points plotted then have independent values of \( \xi_{n=2} \) added to them, with the variance given by the equation above. This produces the new square lattice with a scale 1/2 the original.

![Diagram of successive random addition method](image)

Figure 5.11: schematic diagram of the successive random addition method for simulating fractional Brownian motion in 2 dimensions. For simplicity, the stage \( n=1 \) is broken up into two steps in the diagram.
Each generation has the variance changed according to \( n \) and the process continues until all the points on the lattice are filled. The number of points on the lattice determines the computation time required to complete the process and the resolution of the resulting simulation. In the case of the simulations presented in 5.12 and 5.13, \( n \rightarrow N_i = 10 \).

The result of the algorithm is a two-dimensional array of values, \( X \), with a side length of \( 2^N_i + 1 \).

Figure 5.12 shows an example of a typical field generated using the successive random additions algorithm. Unlike other (multiplicative) models put forward in the literature, [Deidda, 1999, Lovejoy and Schertzer, 1995] the additive method of generation produces a monofractal field, which is characterized by one single fractal dimension. By contrast multiplicative cascades produce multifractal fields, which are characterized by a whole spectrum of fractal dimensions. A comprehensive discussion of multifractals can be found in chapter two, with further detail in appendix E.

![Figure 5.12: an example of a typical simulated field created using the Voss successive random additions algorithm.](image)

As expected, figure 5.12 is more similar to a log rain rate field than a rain rate field. Hence, to convert to a rain rate field it is necessary to use the values of the
simulated array, \( X \) as the exponents for an appropriate base. Figure 5.13 gives an example of this, where the base is \( e \).

Figure 5.13: The simulated field shown in figure 5.12, plotted as exponents to base \( e \)

Further discussion of the base used in the simulation is in section 5.3.4, where the method of conversion to rain rate is discussed in greater detail.

The spectral density exponent is explicitly stated in the algorithm to generate the rain fields and this is confirmed by 2D FFT, where the calculated exponent is found to be 1.66 (giving \( H = 1/3 \)).

5.2.2.1 The Voss random additions algorithm for varying lacunarity

Differentiating between simulations of stratiform events and convective events requires the use of the parameter \( r_i \), as described in section 5.3.1.2. The successive random addition algorithm described above uses the same algorithmic procedure as for the midpoint displacement algorithm [Voss, 1985], in which \( r_i \) is fixed with a value of 1/2 (though for each step in the simulation there are two stages, with \( r_i \) being reduced by \( 1/\sqrt{2} \) at each stage). However, because with successive random additions all points are treated equivalently at each stage, the resolution at the next stage can change by any factor \( r_i < 1 \).
Given a sample of $N_n$ points at stage $n$ with resolution $\lambda$, stage $n+1$ with resolution $r_l\lambda$ is determined by first interpolating the values

$$N_{n+1} = \left( \frac{1}{r_l} + 1 \right)^2$$  \hspace{1cm} (5.9)

at the new resolution from the old $N_n$ values. A random element $\Delta_n$ is then added to all $N_{n+1}$ points, where at stage $n$, with scaling ratio $r_l < 1$, $\Delta_n$ will have a variance:

$$\Delta_n^2 \propto (r_l^n)^{2H}$$  \hspace{1cm} (5.10)

Once again, the result of this algorithm is a two-dimensional array of values, $X$, with a side length of $1/r_{i_{n+1}} + 1$. Figure 5.14 shows a schematic example of this for $r = 1/3$ and $N = 2$.

$$n = 1$$

$$\begin{array}{cccc}
0 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 \\
\end{array}$$

$$n = N_l = 2$$

$$\begin{array}{cccccccc}
0 & 2 & 2 & 1 & 2 & 2 & 1 & 2 & 2 & 0 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 2 & 2 & 1 & 2 & 2 & 1 & 2 & 2 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 2 & 2 & 1 & 2 & 2 & 1 & 2 & 2 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
0 & 2 & 2 & 1 & 2 & 2 & 1 & 2 & 2 & 0 \\
\end{array}$$

Figure 5.14. Schematic example of random addition algorithm for $r_l = 1/3$ $N_l = 2$.

The number in each square refers to the iteration they were plotted in. The numbers plotted in previous generations are in italics.
5.2.3 Simulation results with varying lacunarity

Changing the value of $r_j$ (and hence the lacunarity) helps us simulate the differences in the observed patterns of behaviour between stratiform and convective events. This is possible because varying the lacunarity does not influence the fractal dimension, nor does it affect the spectral density function or its exponent. However, as mentioned before, the concept of lacunarity is not as mathematically well developed as the concept of fractal dimension, so even though the resulting fields with varying $r_j$ are visually convincing, a more thorough study of the lacunarity of rain fields is advised.

Figures 5.15 to 5.18 show example realisations of the random additions algorithm with varying values for $r_j$. The realisations are plotted as contour lines at intensity values of 1, 5, 10, 15 and 20 (arbitrary units). All four plots were generated using the same start state for the random number generator involved in their creation and the generated arrays $X$ provided the exponents for the same base, $e$. The side length of the plots varies as $(1/r_j^N) + 1$, where $N_j$ is the total number of generations calculated.

As $r_j$ decreases, the areas enclosed by the contour lines and the gaps between the contour lines (which correspond the lacunarity) also decrease. This is because in the realisation with $r_j = 1/2$, at $n = 1$, only 9 points have random additions, while for $r_j = 1/5$, at $n = 1$ there are 36 points with random additions. This means that for $r_j = 1/5$ the value at any given point in the last generation will have had a random number with larger variance added to it, than a similar point from the last generation with $r_j = 1/2$. 
Figure 5.15 Random addition surface $r_i = 1/2 \ N_i = 10$

Figure 5.16 Random addition surface $r_i = 1/3 \ N_i = 6$
Figure 5.17 Random addition surface $r_i = 1/4 \ N_i = 5$

Figure 5.18 Random addition surface $r_i = 1/5 \ N_i = 4$
5.2.4 Simulator results and conversion to rain rate field

The Voss algorithm produces synthetic rain fields that have the same spectral density exponent and area-perimeter and box counting dimensions as real rain fields, as well as being visually similar. These fields are equivalent to log rain rate fields, and therefore have to be treated as the exponent of an appropriate base to convert them to rain rate fields. As can be seen from figure 5.13, if the base used is \( e \), mapping the resulting intensity (arbitrary units) directly to rain rate (mm/hr) gives us values which are exceedingly high for a temperate climate such as the south of England.

It is for this reason that the parameter \( b \) is introduced, the base to which the simulated array \( X \), provides exponents. In mathematical terms, the field \( X \) is produced by the Voss algorithm, and is analogous to a log (rain rate) field. Conversion to a simulated rain rate field \( R_{\text{sim}} \) can be achieved as follows:

\[
R_{\text{sim}} = b^X
\]  

(5.11)

In the previous examples (figures 5.13, 5.15 – 5.18) the value of \( b = e \).

It is also necessary to take into account the variations in intensity of stratiform and convective rain types. In the UK, stratiform rain generally has peak rain rates of around 10 mm/hr (as shown in the example measured stratiform rain field, figure 5.1), while convective rain fields can have rain rates of 40 mm/hr and more (figure 5.4). \( b \) can therefore also be used as a climatological parameter that can vary according to the required maximum rain rate value. This enables the simulator to be used in other climatic regions where the stratiform and convective nature of rain remains similar, but the maximum rain rate for each type of event varies. In figures 5.3 and 5.6 \( b \) was chosen empirically, with a value of 1.9 for the stratiform simulation, and 2.3 for the convective simulation.

The maximum value of the field \( X \) can be used to give an indication of the value of \( b \) needed to produce a rain field with a specified maximum rain rate. In figure 5.12 the maximum value of \( X \) is 5. Hence, if we wished to have a simulated rain field with a maximum rain rate of 50mm/hr, then an appropriate choice for \( b \) would be 2.1867, as \( 2.1867^5 = 49.9972 \).

At the moment, \( b \) can be set by the operator to whatever value desired. A potential improvement to the simulator, would be to produce \( b \) as a function of measured rain rates for given geographical areas. The ITU-R rain maps are a potential source of these rain rate values, as is \( R_{0.01} \), the rain rate occurring for 0.01% of an
average year, which is a parameter commonly used by communications engineers in link budget calculations.

As a final, and optional, step, a small offset of 1-5mm/hr can be applied to the simulated rain field. This is particularly useful in the cases where a large proportion of the simulated rain field is covered by uniform rain of low intensity, as shown below in figure 5.19. This is the same data used as in figure 5.6, showing a simulated convective rain event, except the offset of 5mm/hr has not yet been applied.

![Figure 5.19: The simulated field shown in figure 5.6, without the offset](image)

As can be seen from figure 5.19, the shapes of the contour lines are consistent with a convective event, however, almost the entirety of the field has rain rate values of greater than 1 mm/hr, which is not consistent with a real convective event. Applying the offset of 5mm/hr results in the simulation shown in figure 5.6, which is far more realistic.
5.2.5 Extension of the simulator results to include temporal variation

The discrete cascade process described in section 5.2.2 produces realisations of rain fields that are statistically independent of each other, though there will be space correlation within each realisation. Creating a series of rain field realisations by this method will not result in a convincing synthetic event, because the method does allow for any "system memory" of the previous realisation to be carried over to the next. However, as the fields can be generated over far wider areas than measurements can be taken, it is possible that Taylor's frozen storm hypothesis [Taylor, 1938] could be of use to give some indication of the temporal variation.

The frozen field hypothesis postulates the equivalence between the spatial autocorrelation at a fixed point in time and the temporal autocorrelation at a fixed position in space. For this to hold, the spatial argument of the former must be interpreted as a time lag of the latter and the spatio-temporal field must be a fixed spatial field moving with a constant velocity. It has been shown [Zawadski, 1973] that this holds approximately for time lags under about 40 minutes, which is a timescale much longer than is needed to implement route or site diversity switching in microwave communication systems. Hence, generating a rain field over an area much wider than the area of direct interest can serve to give some indication of the temporal evolution of the field in the area of interest, provided that a constant velocity is used to move the larger field. Enjamio et al [2002] have studied the movement of convective cells in the Mediterranean region, showing that in some cases the rain field does indeed move with a constant velocity, while in others the rain field velocity is more random. These findings could also be used to provide an indication of the temporal movement of rain fields, given a purely spatial simulator.

5.3 Supporting theory
5.3.1 Cascade processes and the simulation of rain fields

Figure 5.20 shows a summary of the main methods for simulating rain fields discussed in the literature. As can be seen, there are three main areas, discrete cascades, synthetic storms and random space-time function generators (also known as continuous cascades). What follows is a brief discussion and review of these three methods.
Discrete Cascades

Iterative process:
Take isotropic area – divide into segments
For each segment assign a random variable with set probability distribution and scaled according to the step number.
Repeat, subdividing further.

Examples:
Over and Gupta, 1996
Marsan et al, 1996 (multiplicative)
Voss, 1985, (additive, using logs)
Deidda, 1999 (field obtained as a wavelet expansion coefficients extracted by stochastic cascade - multiplicative)

Produces fractal fields:
monofractal if the cascade is additive
Multifractal if the cascade is multiplicative
\(< R(x, y) >_{DC} \)

Synthetic Storm Models

Based on stochastic models of physical processes.
Describes arrival and movement of storms and rain cells.
Need to specify probability distributions and parameters of:
storm interarrival and duration
rain cell interarrival and duration
point intensity distributions
correlation structure etc.

Examples:
Cowpertwait, 1995
Cox and Isham, 1988
\(< R(x, y) >_{SS} \)

Continuous Cascades (also known as Random Space Time Function Generators)

Start with a random field (Gaussian or otherwise)
Fractionally integrate (filter in the Fourier domain) to get field with \( k^{-H} \) spectrum (H is Hurst exponent).
Exponentiate result to give the simulated field

Examples:
Tessier et al, 1993
Boris Gremont, 2002
Kevin Paulson, 2002

Produces multifractal fields
\(< R(x, y) >_{CC} \)

Figure 5.20: Description of the three main types of procedure used to model the spatial distribution of rain

Synthetic storm models [Cowpertwait, 1995, Cox and Isham, 1988, Wheater et al, 2000] do not take into account the scaling nature of rain, instead choosing to model
storms through the use of stochastic models of physical processes such as: probability
distributions and parameters of storm interarrival and duration, rain cell interarrival
and duration, correlation structure, point intensity distributions etc. The resulting
models describe the arrival and movement of storms and rain cells, but require large
numbers of parameters.

Continuous cascades make use of generators of random space-time functions to
generate fields with specified spatial and temporal covariance structures. Examples of
such cascades are given in [Gremont, 2002, Paulson, 2002 Tessier et al., 1993]. These
models are quite flexible, and can incorporate constraints to observed values at
appropriate positions. But they also require detailed specifications of the correlation
structure of the random function in both space and time, which can be hard to define,
and as yet no consensus has been reached on the general structure. Work is currently
in progress on the correlation structure of rain fields at the microscale level [Enjamio,
2002], which may help rectify this situation.

A discussion of the Voss algorithm [Voss, 1985] for discrete cascades has been
given in section 5.3.2. Other discrete cascade models are described in [Deidda, 1999,
Over and Gupta, 1996]. The iterative process involved in creating a discrete cascade
follows the same basic steps. Start with an isotropic area and divide it into a number of
segments. For each segment a value is assigned which takes into account the value of
the area at the previous step, of which it was a part, and scaled according to the step
number, with some randomness added by means of addition or multiplication by a
random variable with a set probability distribution. This process is then repeated,
subdividing further until the required level of resolution is reached.

Discrete cascades exploit self-affinity and self-similarity relationships to
produce rain rate fields through an iterative random cascade procedure. These
cascades are better able to incorporate non-rainy regions than continuous cascades,
and have concepts and ideas in common with disaggregation and downscaling studies*
done in hydrology [Onof et al., 1996, Charles et al, 1999, Koutsoyiannis and Onof,

* Disaggregation and downscaling involve taking rain data at coarse resolutions (for example, rain
gauge time series recorded at daily intervals) and process them to provide realisations at higher
resolutions (e.g. producing rain gauge time series at hourly intervals from the daily time series).
Downscaling indicates that the resulting realisations are only restricted by the required statistics, while
in disaggregation, the resulting realisations must have the required statistics as well as adding up to the
observed high resolution data (i.e. hourly rate).
2001]. On the other hand it is more difficult to describe the temporal evolution of rain fields using discrete cascades, though it is by no means impossible. Over and Gupta [1996] demonstrate a method of calculating a discrete random cascade, which distributes mass on successive sub-divisions of a d-dimensional cube, to which independent and identically distributed (iid) random variables called generators are applied. This theory is expanded from space to space-time by replacing the iid generators $W$, with iid stochastic processes $\{W_t, t > 0\}$.

Both types of cascade model are more algorithmic schemes for generating space-time fields than attempts to provide a valid physical representation of a rain event within a given framework. It is for this reason that model parameters were introduced into the simulator presented here, which relate to observed climatological parameters.

5.3.2 Fractional Brownian motion, fractional Gaussian noise and the physical implications of the Hurst exponent

The Voss [1985] algorithm generates fractional Brownian motion (fBm) in 2D, which is modified to produce realistic simulations of rain fields. In this section we will go deeper into the mathematical theory involved, providing further justification for its use in a rain field simulator, using the notation used by Voss [1985]. Further details can also be found in [West, 2001, Koutsoyiannis, 2002].

In one dimension, a fractional Brownian motion time series $V_H(t)$ is a single valued function of one variable $t$. Its increments have a Gaussian distribution with variance given by:

$$\langle |V_H(t_2) - V_H(t_1)|^2 \rangle \propto |t_2 - t_1|^{2H} \quad (5.12)$$

where the angle brackets $\langle \rangle$ denote the average over many samples, and $0 < H < 1$, where $H$ is the Hurst exponent discussed in previous chapters.

This incremental function is stationary and isotropic, and the mean square increments depend only on the time difference $t_2 - t_1$ and all $t$'s are statistically equivalent. In the case where $H = 1/2$ we have familiar Brownian motion, with $\Delta V^2 \propto \Delta t$.

Although $V_H(t)$ is continuous, it is nowhere differentiable, but constructs have been developed to give meaning to the term "derivative of fBm". These are commonly
known as fractional Gaussian noises (fGn) and are usually based on averages of $V_H(t)$ over decreasing scales.

The derivative of normal Brownian motion, with $H = 1/2$ corresponds to the uncorrelated white Gaussian noise and the Brownian motion is said to have independent increments. Formally, for any three times such that $t_1 > t > t_2$,

$$\Delta V_1 = V_H(t) - V_H(t_1)$$

is statistically independent of

$$\Delta V_2 = V_H(t_2) - V_H(t)$$

for $H = 1/2$.

For $H > 1/2$ there is a positive correlation both for the increments of $V_H(t)$ and its derivative fGn. For $H < 1/2$ the increments are negatively correlated.

In terms of the time series generated by a random walk model, if $H = 1/2$ successive steps are statistically independent of one another in the random walk and the mean square displacement increases linearly with time. If $H > 1/2$, having taken a step in a given direction, the walker is more likely to continue in that direction rather than reversing direction. In this case the mean square displacement increases faster than linearly and the random walk has positive long range correlation, or is persistent. If $H < 1/2$, the walker is more likely to reverse directions, having taken a step in a given direction, rather than to continue in the same direction. This means that the mean square displacement increases more slowly than linearly and the walk has long range anti-correlation, or is antipersistent.

Our calculated value for $H = 1/3$ shows that log rain rate is antipersistent, and the process has long range anti-correlation [Rangarajan and Ding, 2000]. This value for $H$ was calculated for the range of time that the radar data was recorded, i.e. from 2 minutes to 520 minutes (8hrs 40 minutes), though similar results have been published for optical rain gauge data operating at a measurement integration time of seconds [Paulson, 2002]. A more in-depth study of the antipersistence of log rain rate is recommended as it has implications for the study of the physical processes that produce rain.

Further detail on Brownian motion and its derivatives (also known as functionals) can be found in [Hida, 1980].
5.3.3 Spectral densities for fractional Brownian motion, and the relationship between the spectral exponent $\beta$, the Hurst exponent $H$ and the fractal dimension $D$

Random functions in time $V(t)$ are often characterised by their spectral densities, $S_V(f)$ (corresponding to $S(\omega)$, as described earlier). If $V(t)$ is input into a narrow bandpass filter of centre frequency $f$ and bandwidth $\Delta f$ then $S_V(f)$ is the mean square output $V(f)$ divided by $\Delta f$:

$$S_V(f) = \frac{|V(f)|^2}{\Delta f}$$

(5.15)

$S_V(f)$ also gives us information about the time correlations of $V(t)$, when $S_V(f)$ increases steeply at low $f$, $V(t)$ varies more slowly.

If $V(f,T)$ is the Fourier transform for a specific sample of $V(t)$ for $0 < t < T$:

$$V(f,T) = \frac{1}{T} \int_0^T V(t) e^{2i\pi f t} dt$$

(5.16)

then

$$S_V(f) \propto T |V(f,T)|^2$$

(5.17)

as $T$ approaches infinity.

Alternate characterisation of the time correlation $V(t)$ is given by the two point autocorrelation function:

$$G_V(\tau) = \langle V(t)V(t+\tau) \rangle - \langle V(t) \rangle^2$$

(5.18)

which gives a measure of how the fluctuations at two times separated by $\tau$ are related. $G_V(\tau)$ and $S_V(f)$ are not independent, and in many cases are related by the Wiener-Khintchine relationship.

$$G_V(\tau) = \int_{-\infty}^{\infty} S_V(f) \cos(2\pi f \tau) df$$

(5.19)

For a (Gaussian) white noise, $S_V(f)$ is constant and $G_V(\tau) = \Delta V^2 \delta(\tau)$ is completely uncorrelated. For certain simple power laws for $S_V(f)$, $G_V(\tau)$ can be calculated exactly.
For $S_v(f) \propto 1/f^\beta$, with $0 < \beta < 1$, $G_v(\tau) \propto \tau^{\beta-1}$ and $G_v(\tau)$ is directly related to the mean square increments of the fractional Brownian motion:

$$\langle [V(t+\tau) - V(t)]^2 \rangle = 2[\langle V^2 \rangle - \langle V \rangle^2] - 2G_v(\tau)$$  \hspace{1cm} (5.20)

This equation is also known as the structure function $D(\tau)$, as described in turbulence theory.

Roughly, $S_v(f) \propto 1/f^\beta$ corresponds to $G_v(\tau) \propto \tau^{\beta-1}$ with:

$$2H = \beta - 1$$ \hspace{1cm} (5.21)

A statistically self affine fractional Brownian function, $V_H(x) = (x_1, ..., x_E)$ in $E_D + 1$ Euclidean dimensions satisfies:

$$\langle [V_H(\bar{x}_2) - V(\bar{x}_1)]^2 \rangle \propto |\bar{x}_2 - \bar{x}_1|^{2H}$$ \hspace{1cm} (5.22)

and has fractal dimension $D = E_D + 1 - H$.

The zerosets of $V_H(\bar{x})$ form a statistically self-similar fractal with dimension $D_0 = E_D - H$.

Thus, $V_H(\bar{x})$ has a fractal dimension $D$ and spectral density $S_v(f) \propto 1/f^\beta$ for the fluctuations along a straight line path in any direction in $E_D$-space, with:

$$D = E_D + 1 - H = E_D + \frac{3-\beta}{2}$$ \hspace{1cm} (5.23)

This result agrees with other "extensions" of the concepts of spectral density to non-stationary noises where some moments may be undefined. It also provides a useful connection between $D, H, \beta$ for finite simulations, where $0 < H < 1$, $E < D < E + 1$ and $1 < \beta < 3$.

Although formal definition of fractional Brownian motion restricts $0 < H < 1$, the integration and appropriate definition of "differentiation" of fBm can extend the range of $H$.

Integration of fBm produces a new fBm with $H$ increased by 1, while "differentiation" reduces $H$ by 1. When $H \rightarrow 1$ the "derivative" of fBm looks like a
fBm with $H \rightarrow 0$. In terms of spectral density, if $V(t)$ has $S_V(f) \propto 1/f^\beta$ then $dV/dt$ has spectral density:

$$\frac{f^2}{f^\beta} = \frac{1}{f^{\beta-2}}$$

(5.24)

i.e. "differentiation" of fractional Brownian motion decreases $\beta$ by 2 and decreases $H$ by 1.

In terms of the data we have studied from the radar measurements of rain fields, $\beta = 8/3$ for our averaged radial spectral density function (chapter 2, section 2.2.6). This corresponds to $H = 1/3$, using $\beta = 2H + 2$ (the equation has $2H + 2$ rather than $2H + 1$ as given above because we are working in the two dimensional space of rain fields, rather than in the single dimensional space of a time series).

Other studies [Rangarajan and Ding, 2000] show graphs of the spectral density function with a slope the inverse of our results. From these they derive the equation:

$$\beta = 2H - 1$$

(5.25)

which is not compatible with our data. However, Rangarajan and Ding use fractional Gaussian noise in their work, rather than fractional Brownian motion, and the discrepancy can be resolved as:

$$\beta_{\text{fGn}} = \beta_{\text{fBm}} - 2$$

(5.26)

as is described above.

5.3.4 Cross-correlation between measured radar rain fields

The differences in spatial structure between stratiform and convective events are not obvious when looking at them in terms of monofractal analysis such as the area-perimeter relationship, the box-counting dimension, or power spectral density function analysis, however the cross-correlation function does show some dramatic differences between the types.

The normalised two-dimensional cross-correlation can be obtained between two radar rasters using the following equation:

$$\rho(R_1,R_2) = \frac{\Gamma(R_1,R_2)}{\sqrt{\max(\Gamma(R_1))} \ast \sqrt{\max(\Gamma(R_2))}}$$

(5.27)

where:
\[ \Gamma(R_1, R_2) \] is the covariance function obtained from the expression:

\[
\Gamma(R_1, R_2) = \langle (R_1 - m_1)(R_2 - m_2) \rangle
\]  

\[ R_1 = R(x, y, t = t_1), \] the rain field at time \( t_1 \)

\[ R_2 = R(x, y, t = t_2), \] the rain field at time \( t_2 \)

and \( m_1, m_2 \) are the mean values of the field \( R_1 \) and \( R_2 \) respectively.

Figures 5.21 to 5.23 show the normalised cross-correlation between two rasters (snapshots) of each rain event, separated in time by 10 rasters (approximately 20 minutes). The patterns for the different rain events are quite revealing. The stratiform event has a cross-correlation which has a very broad, almost elliptical, base, indicating that the event itself is widespread with little variation. The convective event has a series of low, but quite sharp peaks, indicating the more cell-like structure of convective rain. And the frontal event seems to be a mixture of the two, which is not surprising given the nature of the event itself (a band of stratiform rain studded with convective cells).

![Normalised cross-correlation function for stratiform event recorded 7th December 2000.](image)

Figure 5.21 Normalised cross-correlation function for stratiform event recorded 7th December 2000. 1 pixel corresponds to an area of 300m * 300m.
Figure 5.22 Normalised cross-correlation function for convective event recorded 16th May 2001. 1 pixel corresponds to an area of 300m × 300m.

Figure 5.23 Normalised cross-correlation function for frontal event recorded 1st May 2001. 1 pixel corresponds to an area of 300m × 300m.
5.4 References


Feder, J., “Fractals”, Plenum Press, 1988


6.1 Introduction

The previous chapters in this thesis have presented various fractal methods for studying and characterising the fractal nature of rain fields. Chapter seven presents the fundamental justification for this work, at least in terms of a radio communications engineering context: the application of fractal methods and a fractal rain simulator to a communications case study. But before this can be demonstrated, it is necessary to discuss some fundamental concepts relating to the system specific case study in the next chapter. Concepts such as diversity gain and improvement are well known in the communications system engineering community, and their definitions are included in this chapter for the sake of completeness.

The radio spectrum is a finite resource, one that is coming under increasing pressure as the range of applications requiring large bandwidths (such as third-generation mobile, video-conferencing etc.) become more and more prevalent. In the radio frequency bands already in everyday use, the UK’s Ofcom has instituted a yearly Spectrum Efficiency Scheme (SES), with the aim of encouraging more spectrally efficient methods of using, and potentially re-farming those bands. The need for more bandwidth has also fuelled the push into higher frequencies (20 GHz and above).

To make efficient use of frequencies above 20 GHz, new techniques to compensate for rain and cloud fades are required. Previous systems simply allocated a fixed fade margin to compensate for fading, however, as previously seen in figure 1.1, this technique would not be practical or economical for the high attenuation experienced by the systems using higher frequencies.

These fade mitigation techniques (FMTs), also known as fade countermeasures (FCM) are many and varied, including: adaptive modulation and coding, time diversity, adaptive power control etc. A comprehensive review of the FMTs presented in the literature is presented in [Richardson et al, 2004].
6.2 Case Study

6.2.1 Why site diversity?

The fade mitigation technique investigated in this and the following chapter is satellite communications ground station site diversity. It is commonly used with Earth-space systems operating at frequencies of 12 GHz and above [Dissanayake and Lin, 2000, Rogers, 2000, Goldhirsh et al, 1997, Bosisio and Riva, 1996, Cardoso et al, 1993, Bostian et al, 1990], as the effects of propagation factors such as cloud and light rain are likely to degrade system performance at these frequencies. This degradation cannot be economically compensated for by fade margin alone.

Obviously, rain and clouds change in time and space. As shown previously, intense areas of rain that cause large amounts of attenuation on earth space links are limited in the horizontal plane and often have dimensions of the order of 1~10km. Site diversity employs two or more ground stations receiving the same satellite signal with a separation distance such that the rain attenuation at both sites is de-correlated. The sites in a properly configured arrangement encounter intense rainfall at different times, and switching to the site experiencing the least fading improves system performance considerably.

Figure 6.1 shows a block diagram of the potential and required inputs necessary for the design and optimum use of a switching algorithm for a ground station site diversity scheme. A more in-depth discussion of these inputs is carried out later in the chapter.
Figure 6.1: Schematic diagram of the inputs to a switching algorithm for a site diversity scheme.

6.2.2 The GBS site diversity experiment

At the date of this thesis, the RCRU are making site diversity measurements at 20.7 GHz using the beacon carried on the US Department of Defense (DoD) satellite UFO-9. The US DoD delivers information to American armed forces via the Global Broadcast Service (GBS) network of satellites. Three satellites containing a Ka-band payload provide near-global broadcast coverage - they are located at ~23°W, 72°E and 172°E. Part of the payload is the earth-coverage Continuous Wave (CW) beacon that provides uplink level control and uplink-transmit terminal automated tracking.

The UFO-9 satellite at ~23°W is in a slightly inclined (~ +/-3.5 degrees) geostationary orbit so that a ground receiver would need to track the satellite to ensure that it pointed correctly. Since UFO-9 is a military satellite, ephemeris information is not easily available.

Due to the cost, it was not practical to implement a tracking system for the receivers, instead wider beamwidth antennas are used so that the complete movement of the satellite can be observed without a need to track it. There is a consequent
reduction in receiver gain (6dB) but there is still sufficient signal to enable a full range of measurements to be made. This also has the effect of introducing a sinusoidal diurnal variation into the recorded data.

Our measurements have shown that the GBS receivers have a dynamic range of \(~13\) dB, which is sufficient to measure attenuation at 20.7 GHz down to time percentages of 0.02\% (roughly corresponding to two hours in a year). Table 6.1 gives the technical specifications for the GBS receivers [Waight, 2004, private communication].

Figures 6.2 and 6.3 show the beacon receivers in operation at the two sites. Both receivers were designed and custom-built in-house by members of the RCRU, and are situated in specially designed cabins, looking along the path to the satellite through radomes of woven PTFE acquired from Gore Associates.

Figure 6.2: The GBS receiver in operation at Sparsholt

Figure 6.3: The GBS receiver in operation at Chilbolton (opened to show the components inside).
<table>
<thead>
<tr>
<th><strong>Receiver Type</strong></th>
<th>Single down-conversion, superheterodyne</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Antenna type</strong></td>
<td>Flann CW820-FA, 150mm diameter high performance horn lens antenna</td>
</tr>
<tr>
<td><strong>External antenna polariser transition</strong></td>
<td>Left Hand Circular to linear (in WG20/WR42)</td>
</tr>
<tr>
<td><strong>Antenna gain</strong></td>
<td>29dBi</td>
</tr>
<tr>
<td><strong>Beamwidth</strong></td>
<td>11°</td>
</tr>
<tr>
<td><strong>Local oscillator (two-stage)</strong></td>
<td>10.315GHz PLO (internal XTAL)→X2 multiplier→20.63GHz</td>
</tr>
<tr>
<td><strong>I.F. Frequency</strong></td>
<td>70.0MHz</td>
</tr>
<tr>
<td><strong>LNA mixer/preamplifier noise figure, F</strong></td>
<td>2.4dB</td>
</tr>
<tr>
<td><strong>I.F. bandpass filter bandwidth (3dB)</strong></td>
<td>3% (2.1MHz max.)</td>
</tr>
<tr>
<td><strong>Overall R.F. to I.F. gain at Mixer output</strong></td>
<td>&gt;47dB</td>
</tr>
<tr>
<td><strong>Overall noise figure (pre-beacon I.F. tracking)</strong></td>
<td>3.5dB max.</td>
</tr>
<tr>
<td><strong>I.F. amplifier gain</strong></td>
<td>56dB</td>
</tr>
<tr>
<td><strong>Intermediate Frequency (I.F.)</strong></td>
<td>70.0MHz</td>
</tr>
</tbody>
</table>

**Front-end I.F. processing by Satellite Beacon Receiver, type:** Matra Marconi Space SBR100 or Novella Satcoms B150 with preset 70MHz channel input, 50Ω impedance

<table>
<thead>
<tr>
<th><strong>Input frequency range for tracking (search range)</strong></th>
<th>70MHz +/- 200KHz (B150),</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Input signal range</strong></td>
<td>-60dBm to −20dBm (saturation level)</td>
</tr>
<tr>
<td><strong>Phase locked loop bandwidth</strong></td>
<td>300Hz</td>
</tr>
<tr>
<td><strong>Threshold for reacquisition of lock (300Hz PLL B/W)</strong></td>
<td>&lt;35dBHz</td>
</tr>
<tr>
<td><strong>Output range (for specified input range)</strong></td>
<td>-10V d.c. to +10V d.c.</td>
</tr>
<tr>
<td><strong>Slope</strong></td>
<td>2.0 dB/ V d.c.</td>
</tr>
</tbody>
</table>

**Datalogger details** | Microlink 3040 A/D converter, 12 bit resolution

| **Satellite beacon receiver d.c. output sampling rate** | One sample per second, continuous |

Table 6.1. Technical specifications for the receivers used in the GBS site diversity experiment.
Measurements are currently being taken at two different sites in the south of England, both located in Hampshire. One is located at Sparsholt (51° 04' N, 01° 26'W), the other at Chilbolton (51° 08' N, 01° 26' W), 7.8 km NNW of Sparsholt. Figure 6.4 gives a schematic view of the receiver sites and their relative positions. The site diversity measurements were begun in October 2000. A third receiver is located in Dundee and was installed in February 2004.

Each of the two sites in the south of England has also a range of supporting meteorological instruments, such as a drop counting rain gauge and instruments to measure wind speed and direction, as well as pressure and temperature. The meteorological measurements provided by these instruments are augmented by the multi-parameter 3 GHz radar, CAMRa, located at Chilbolton.

Figure 6.4: Schematic of the GBS receiver locations. The link between Sparsholt and South Wonston is a terrestrial link.

The radar provides detailed information on the presence of rain, cloud and ice along the path to the satellite as well as the area bounded by the melting layer and the slant path from the Sparsholt receiving station. It is not possible for CAMRa to scan
along the path from Chilbolton to the satellite, due to the near field effects associated with radars. This data provides concurrent information on the spatio-temporal characteristics of the rain fields surrounding the links, leading to a broader understanding of the physical mechanisms that lead to rain fades.

6.2.3 Measured data

The data recorded at the two sites is sampled at a frequency of 1 Hz, and the signal strength in dB is recorded. Due to the diurnal variation of the satellite, the recorded data must be pre-processed to produce the required attenuation time series. The pre-processing follows the same methodology as was used for the ITALSAT data, and complete details of the method can be found in [Ventouras et al, 2000, updated 2003]. The author has extensive experience gained from processing and pre-processing the ITALSAT data, however, the two days data presented here were pre-processed by Judith Agnew, one of the author’s colleagues in the RCRU.

Figure 6.5 shows the signal levels and attenuation time series for the whole day of the 9th January 2004. The x-axis gives the time values as minutes after midnight. The upper plot shows the signal levels as recorded by the beacon receivers. The blue dotted lines indicate the variation that the signal would have if the day were completely free of attenuation. These lines are fitted during the data pre-processing to remove the daily sinusoidal variation of the satellite relative to the beacon receiver, as well as adjusting for any sudden change in beacon signal power or receiver sensitivity. (The upper of the dotted lines indicates the attenuation relative to vacuum, whereas the lower dotted line indicates the attenuation relative to clear sky, i.e. attenuation including the gaseous attenuation due to atmospheric water vapour and oxygen, but not due to clouds or rain.) The lower plot in figure 6.6 shows the processed attenuation time series for the day.

As can be seen, between 800 and 900 minutes after midnight (13:20-15:00) a strong convective rain event was experienced by the two beacons. Rain gauge measurements with a sampling frequency of 0.1 Hz from the two sites for that day are shown in figures 6.6 and 6.7. These show rain events that are characteristic of convective rain, being short in duration, with high intensities, ~27 mm/hr at Chilbolton, and ~18 mm/hr at Sparsholt. This categorisation as a convective events is confirmed by Met Office radar data and also data from the 94 GHz cloud radar.
situated at Chilbolton (figure 6.8), which clearly shows the turbulent behaviour of a convective rain cell.

Figure 6.5 Signal levels and attenuation time series for the 9th January 2004. Black: Sparsholt data, yellow: Chilbolton data. (a) Complete day. (b) Zoom on convective event.
Similarly figure 6.9 shows the signal levels and attenuation time series for the whole day of the 12th January 2004, with figures 6.10 and 6.11 showing the concurrent rain gauge measurements for the day. The rain event between 200 and 700 minutes after midnight (3:20-11:40) is classified as a stratiform event, even though the peak rain rates at the two sites were the same as for the event on the 9th of January. This is indicated by the length of the rain event, and is confirmed by the Met Office radar data, and also the 94 GHz cloud radar at Chilbolton (figure 6.12). In this plot, the
melting layer (also known as the bright band) can be clearly seen at ~ 1km above the
ground, confirming that the rain is stratiform in type. The low rain rates around 400
minutes after midnight, compared with the lack of attenuation experienced by the
beacons at that time is also indicative of light rain with small raindrops, which is
another characteristic of stratiform events.

Figure 6.9 Signal levels and attenuation time series for the 12th January 2004. Black:
Sparsholt data, yellow: Chilbolton data. (a) Complete day. (b) Zoom on stratiform
event.
Figure 6.10 Rain gauge measurements from the Chilbolton rain gauge for the 12th January 2004.

Figure 6.11 Rain gauge measurements from the Sparsholt rain gauge for the 12th January 2004.

Figure 6.12: Radar reflectivity measurements of the stratiform rain event on the 12th January 2004, made with the 94 GHz cloud radar at Chilbolton.

6.2.4 Attenuation statistics

Earth-space systems, along with other radio communications systems, require appropriate propagation data and prediction techniques in order to properly plan and optimise them. The ITU-R presents methods that allow system designers to predict these propagation parameters, and has tested them against available data in order to check their accuracy. However, the database of measurements used by the ITU-R is
not complete, and measurements still must be made in order to confirm the behaviour of propagation parameters at higher frequencies.

One of these propagation parameters is the attenuation statistics for attenuation caused by rain and cloud. Figure 6.13 shows the monthly cumulative distributions of attenuation measured at the two sites, plotted in comparison with the predicted values given by ITU-R Rec. 618-8. As can be seen, even on a monthly basis, the variability of attenuation between the two sites is obvious.

The ITU-R prediction is based on the average annual cumulative distribution of attenuation, hence its over-estimation when compared with monthly distributions. Also, it is well known that precipitation attenuation distributions measured on the same path, and at the same frequency and polarization may show marked year-to-year variations [ITU-R Rec. 618-8].

Figure 6.13: Monthly cumulative distributions of attenuation in comparison with the ITU-R prediction (from ITU-R Rec. 618-8)
6.2.5 Diversity gain and improvement

The advantage gained by employing site diversity as a fade mitigation technique can be quantified by the calculation of diversity gain and improvement. First introduced by Hodge [1976], the ITU-R define the diversity improvement factor as being the ratio of the single-site time percentage and the diversity percentage at the same attenuation level. Diversity gain is defined as the difference in dB between the single site and diversity attenuation values for the same percentage.

Figure 6.14 gives a graphical representation of these concepts.

The joint site distribution is calculated from the following:

$$A_{dB, sdv}(t) = \min(A_{dB,1}(t), A_{dB,2}(t))$$ (6.1)

where:

- $A_{dB, sdv}(t)$ is the joint site distribution (i.e. the system attenuation experienced by a system using site diversity) at time $t$.
- $A_{dB,1}(t)$ is the attenuation at site 1 at time $t$.
- $A_{dB,2}(t)$ is the attenuation at site 2 at time $t$.

Figure 6.14: Graphical representation of the concepts of diversity gain and diversity improvement.

ITU-R Rec. 618-8 describes a complete method to estimate the diversity gain and diversity improvement factor, based on the measurements stored in the ITU-R databanks. A key parameter in the calculation of diversity gain and improvement is the distance between the two sites in the site diversity system. This is hardly
surprising, as the technique relies on rain varying in time and space, and it is well known that the rain at two points in space becomes de-correlated with increasing distance between them.

Statistical diversity gain and improvement for a small number of sites can be easily calculated using mathematical models. Hence, the use of simulators to produce long-term time series of attenuation for these, with which to calculate diversity statistics, is redundant, over-complicated and unnecessary.

However, rain field and attenuation simulators become useful when dealing with the short-term dynamic behaviour of a communications channel, as statistics based on long-term measurements do not give the necessary information about the potential range of short-term behaviour. In particular, the optimisation of a FMT switching algorithm is highly dependent on the dynamic behaviour of the channel.

Rain and attenuation simulators are also useful for producing events-on-demand, whereas with measured data it may take many weeks of measurement to record a single appropriate type of event. Also, most long-term measurement databases do not categorise the type and time of events experienced, making it difficult and time consuming to trawl through years of time series in search of an appropriate event.

In cases where attenuation and diversity statistics need to be calculated for many locations, for example in the case of a terrestrial mesh network with hundreds of sites, rain field simulators once again become useful. To calculate site diversity statistics for multiple sites it is not necessary to take into account the dynamical behaviour of the rain fields, provided that each field has the required spatial properties. A sufficiently large number of rain fields can be simulated with the required spatial properties and can then be used to calculate the diversity statistics between the sites without, as already stated, requiring the production of time series for each site.

### 6.2.6 Instantaneous diversity gain

Diversity gain is a statistical quantity and therefore provides no information about the instantaneous behaviour of the single-site and joint attenuations in an operating diversity system. For this reason, it is useful to use the concept of instantaneous diversity gain $G_i$. This is defined by Towner et al \[1984\] as the
difference between the maximum and minimum attenuations observed at the sites of a
diversity system at the same instant of time. Thus, for our two-site experimental
system described above:

\[ G_i(t) = A_{dB,max}(t) - A_{dB,min}(t) \]  \hspace{1cm} (6.2)

where:

\[ A_{dB,max}(t) = \max\{A_{dB,1}(t), A_{dB,2}(t)\} \]  \hspace{1cm} (6.3)

\[ A_{dB,min}(t) = \min\{A_{dB,1}(t), A_{dB,2}(t)\} \]  \hspace{1cm} (6.4)

This can easily be extended to systems operating with more than 2 sites.

A comparison of instantaneous diversity gain with statistical (long-term)
diversity gain published by Towner et al [1984] states “the statistical diversity gain
approximates the median value of the instantaneous diversity gain”. This very
interesting statement can be verified through the data produced as a result of the GBS
experiment, and is a topic for future work.

Figures 6.15 and 6.16 show the instantaneous diversity gain calculated using
the time series from the two events described above, for the raw (unfiltered) data, and
the same data filtered using a 30 second, 1 minute and 5 minute moving average filter.
Part (b) of these figures zooms in to give a close-up view of the behaviour of the filter.
It is worth noting that the integration time of the filters has different effects on the
time series, according to the type of event. In the convective event, figure 6.15(b), it is
obvious that the smoothing effect of the 5 minute moving average filter is so dramatic
that there is a difference of \(-6dB\) between the peak unfiltered attenuation and the 5
minute filter at the same time. This difference is not nearly as marked during the
stratiform event, figure 6.16(b), though the peak of the 5 minute filtered data does lag
significantly behind the peak of the unfiltered data.

This variation of behaviour of the filtered data with event type comes about as
the result of the characteristics of the event itself. Convective rain events are generally
intense and short-lived, hence the attenuation changes rapidly, meaning the longer
integration time filtering has more of a smoothing effect on the dynamic behaviour.
Figure 6.15: Instantaneous diversity gain calculated for the convective event recorded on the 9th January 2004. (a) Entire day (b) Zoomed in to see the effects of filtering.
Figure 6.17: Instantaneous diversity gain calculated for the stratiform event recorded on the 12th January 2004. (a) Entire day (b) Zoomed in to see the effects of filtering.
6.3 Switching algorithm for a site diversity scheme

As can be seen in figure 6.1, there are two primary of inputs into the development of an optimally designed switching algorithm for a site diversity system. Both of these are dependent on other sources of information about the propagation channel, and this information can be achieved through a number of ways: ITU-R models, measured data and simulated data.

The two inputs, the fade detection mechanism, and the short term attenuation prediction mechanism will be discussed in turn, before a description of the switching algorithm itself is presented.

The author of this thesis also participated in another case study of a satellite system using site diversity, which is published in [Richardson et al, 2004].

6.3.1 Fade detection mechanism

The fade detection mechanism is a key component of any system using a FMT, as it is necessary to know when the propagation channel is impaired, in order to use the FMT effectively. If we consider the situation where site diversity is used as a FMT on a downlink from a satellite (as is the case in the GBS experiment described above, where the receiving sites are measuring beacon attenuation), then ideally, both sites need to be able to measure the actual attenuation on their respective downlinks.

Two open loop fade detection methods can be envisaged, i.e. ones that estimate the downlink fade by independent means:

- Attenuation measurement of a satellite beacon:

  This method requires the transmission of a beacon from the satellite at a known EIRP so that the attenuation due to rain fades can be monitored on the ground. If the satellite is already equipped with a beacon as part of its propagation payload, then fades can be detected at each Earth station when the received signal level of this beacon varies from the expected values. However, the beacon frequency must be close enough to the receiver frequency to register the same fading effects. For example, if the beacon is at 40 GHz and the receiver frequency is at 20 GHz, then fades may be detected on the beacon
channel when none exist on the receiver channel, leading to inappropriate switching between Earth stations.

Similarly, if the situation is reversed, then fades on the receiver channel won’t be detected by the beacon, and switching between the sites won’t occur when it is needed. If it is not possible to use a beacon frequency in the same band as the downlink then an instantaneous frequency scaling factor between the receive channel frequency and the beacon frequency can be applied. However, at this time, data and models for instantaneous frequency scaling factors are limited.

- Co-sited radiometer:

  If the satellite does not have a beacon as part of its propagation payload it is possible to use a radiometer, co-sited with each site, pointing along the path to the satellite. (Catalan and Vilar [2002] have studied this method extensively.) Provided the radiometer operates at a similar frequency to the receive channel, it can register when the sky noise temperature increases, indicating rain or cloud along the path. (This sky noise temperature can also be converted to attenuation, for ease of comparison). The radiometer is a passive instrument, so its inclusion at the earth station would not affect interference calculations. Radiometers have the tendency to saturate when the sky noise temperature rises above a certain value. However, this would not be a problem since if the radiometer at one site has saturated, it is a clear indication that the path to the satellite is very attenuated, and that the system should switch to the other Earth station.

  Of the two open-loop fade detection methods described, it is preferable to use the beacon measurement method (assuming of course that the satellite carries one at an appropriate frequency), as radiometers can be very expensive. An estimate given by one of the author’s colleagues for the price of a 19.7 GHz radiometer was £36,500 for hardware, in addition to 3-4 months off staff time to build it. However, building a radiometer as part of the receiver, rather than having it as an additional co-located instrument, would reduce this price considerably.
Before the system is deployed in the field, the fade detection mechanism can be tested through the use of either measured or simulated attenuation time series. The fade detection mechanism also provides a key input into the behaviour of the short term attenuation predictor.

6.3.2 Short term prediction algorithm

As can be seen in figure 6.1, the accuracy of a short-term attenuation prediction algorithm is dependent on the real time instantaneous inputs of the fade detection mechanism. However, in order to implement the prediction algorithm in the first place, key inputs are needed in the form of fade slope and fade and inter-fade duration statistics. These inputs are important because they provide system designers with an understanding of the dynamic variation of atmospherically induced signal loss.

Current ITU-R recommendations do not yet have officially sanctioned models with which to calculate fade slope and fade and inter-fade duration statistics, though they do acknowledge their importance [ITU-R Rec. 618-8] and state that the distribution of rain fades that exceed a specified attenuation level are approximately log-normally distributed. Similarly, it is stated that there is broad agreement that the distributions of positive and negative fade rates are log-normally distributed and are very similar to each other.

Due to the lack of consensus regarding models which accurately describe the characteristics of propagation events, the main source of information with which to build an accurate and useful short term attenuation predictor must come from measured data.

Recent and extremely comprehensive reviews of the currently available fade slope and fade duration models have been presented by M. van de Kamp at the meetings of the COST 280 action [van de Kamp, 2003, a and b]. These reviews will become part of the COST 280 final report, but are unfortunately not yet freely available to non-COST 280 members.

6.3.2.1 Fade and inter-fade durations

Fade duration is defined as the period of time between two consecutive crossings of the received signal on the same attenuation threshold (i.e. the interval for which the attenuation continuously exceeds a specified attenuation threshold). Inter-
fade duration is the interval between two successive fades. (A graphic representation of these concepts can be seen in figure 6.17.)

![Figure 6.17: Schematic representation of fade and inter-fade durations.](image)

Fade duration statistics can be given in terms of the absolute number, \( N_J (J \geq j; A_{dB,thresh}) \), or the accumulated time, \( N_s (J \geq j; A_{dB,thresh}) \), of fades with duration equal or longer than \( j \) for a specified attenuation threshold \( A_{dB,thresh} \). The one distribution can be derived from the other as

\[
J \cdot n_J (J; A_{dB,thresh})dJ = n_s (J; A_{dB,thresh})dJ
\]

where \( n_J (J; A_{dB,thresh})dJ \) is the number of fades with fade duration \( J \) and \( n_s (J; A_{dB,thresh})dJ \) is the corresponding time for the attenuation threshold \( A_{dB,thresh} \).

Ventouras et al [2001] provide a comprehensive analysis of fade, inter-fade and joint fade/inter-fade durations deriving from the ITALSAT measurements at 18.7, 39.6 and 49.5 GHz, recorded in the south of England. Analysis of the whole database showed that the absolute number of fades/inter-fades with duration longer than \( \sim 20 \) seconds can be modelled as a log-normal distribution. When the distribution of the absolute number of fades/inter-fades includes fades/inter-fades shorter than \( \sim 20 \) seconds it seems to depart significantly from log-normal, approximating to a power law. Ventouras et al [2001] state that the behaviour of fades/inter-fades shorter than
~20 seconds is due to the effects of high frequency components of attenuation such as scintillation and receiver noise.

However, another factor to be considered is the uncertainty associated with threshold crossings. The time series signal can be particularly noisy, and can jump from an attenuation value just below the threshold, to just above it, to just below it, and back up again within the space of four consecutive measurements. This leads to uncertainty as to exactly when the threshold was crossed, hence most analysis of fade and inter-fade duration statistics filter the data in order to avoid this problem.

Systems operating in real time combat this uncertainty in threshold crossing by adding hysteresis, thereby delaying the decision whether the threshold has been crossed until the continuing attenuation confirms the crossing beyond doubt. More details on this subject from a systems context are given in [Banjo and Vilar, 1986].

Figure 6.18 shows the effectiveness of the recently proposed ITUR-model [ITU-R Document 3J/TEMP/42-E, 6 June 2001] in comparison with the RCRU model which is applicable for fades with duration greater than ~20sec [Ventouras et al., 2000]. The attenuation threshold used is 12 dB.

Figure 6.18: ITALSAT measurements at 49.5GHz. Absolute number of fades with duration equal or longer than x-axis value in comparison with RCRU (solid) and ITU-R (dashed) model. (Average year from April ’97 to March ’00). [Ventouras et al, 2000]
6.3.2.2 Fade slope

Fade slope is defined as the time rate of change of attenuation and is measured in dB/sec. It is a function of two rain attenuation samples as given below:

\[
\zeta(t) = \frac{A_{dB}(t+\Delta t)_{fs} - A_{dB}(t)}{\Delta t_{fs}}
\]

where \( \zeta(t) \) is the rate of change of attenuation at sample time \( t \), \( \Delta t_{fs} \) is the separation time between the two samples and \( A_{dB}(t) \) is the attenuation at sample time \( t \). This is not the only way of calculating rain fade slope, another method based on the discrete Fourier transform is published in [Baxter et al, 2001].

As rain attenuation is generally sampled at 1 Hz, if \( \Delta t_{fs} = 1 \) second then we may refer to \( \zeta(t) \) as being the instantaneous fade slope.

![Fade slope values against excess attenuation at 49.5GHz as derived from measurements from April '97 to March '00 [Ventouras et al, 2003].](image)

Analysis of the ITALSAT database has shown that there is a statistical increase of fade slope with increasing attenuation, and a statistical symmetry between the positive and negative fade slope values (figure 6.19). However for a given attenuation threshold, fade slope values exhibit considerable spread which widens as the attenuation threshold increases [Ventouras et al, 2003].
6.3.2.3 A simple short-term rain attenuation prediction algorithm

A very simple short-term rain attenuation prediction algorithm can be generated using the concept of fade slope as defined above. Quite simply, the attenuation at a time \((t + \Delta t_{fs})\) can be estimated from the attenuations two samples previously, at time \((t)\) and \((t - \Delta t_{fs})\), which corresponds to the fade slope.

\[
A_{db}(t + \Delta t_{fs}) = A_{db}(t) + (\zeta(t) \cdot \Delta t_{fs})
\]  \hspace{1cm} (6.7)

\[
A_{db}(t + \Delta t_{fs}) = A_{db}(t) + \frac{A_{db}(t - \Delta t_{fs}) - A_{db}(t)}{\Delta t_{fs}}
\]  \hspace{1cm} (6.8)

Figure 6.20 gives a schematic diagram of the times mentioned above.

![Schematic representation of time values used in the short term prediction algorithm](image)

**Figure 6.20**: Schematic representation of the time values used in the short term prediction algorithm

Figure 6.21 shows the percentage probability of error for different values of \(\Delta t_{fs}\) [Ventouras, 2003, presented at ICAP 2003, but not included in the conference proceedings]. As can be seen, the probability of error increases with increasing \(\Delta t_{fs}\).
Figure 6.21: the absolute error (dB) plotted against the probability of occurrence for the simple short-term prediction algorithm detailed in section 6.3.3.3. [Ventouras et al, 2003].

Kastamonitis et al [2003] also present a method of predicting rain attenuation on a short term basis using fade slope, though in their predictor they use an average fade slope calculated from the last three samples of fade slope.

6.4 Acknowledgements

The author would like to thank Joe Waight for the technical specifications of the GBS receivers and all his hard work building and installing them. The author would also like to express her appreciation to Judith Agnew for her work preprocessing the GBS beacon data.

6.5 References and bibliography


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ITU-R Document 3J/TEMP/42-E, 6 June 2001

ITU-R P.618-8 Propagation data and prediction methods required for the design of Earth-space telecommunication systems


Van de Kamp, M., “Fade duration models put to the test”, COST 280 meeting, Abingdon, October 2003 (a)
Van de Kamp, M., "Fade slope models under examination", COST 280 meeting, Abingdon, October 2003 (b)


CHAPTER SEVEN:
APPLICATION OF A FRACTAL RAIN FIELD SIMULATOR TO A COMMUNICATIONS SYSTEM SWITCHING ALGORITHM

7.1 Introduction

The aim of this chapter is the application of fractal methods and a fractal rain simulator to a communications case study. This is the fundamental justification for the fractal analysis and synthesis of rain fields as performed in previous chapters of the thesis.

This application makes use of a site diversity switching algorithm, whose inputs have been outlined in the previous chapter. The algorithm is presented in greater detail, along with its behaviour when tested with real and simulated attenuation time series.

7.2 Description of switching algorithm

For the purposes of this case study, it is assumed that there is a perfect terrestrial link (whether radio or fibre) between the earth stations so that communication between sites will not fail. One site is designated as Control, where extra processing power is located in order to make the switching decision.

The switching algorithm will work on the basis of always selecting the link with the least attenuation on it.

All stations must be equipped with a fade detection mechanism and be continuously monitoring the rain attenuation on their respective links. All of the stations must also have their own clocks, which are regularly synchronised with the one at Control.

Two measurements are made for each link in each iteration of the switching loop. The measurements are separated in time by $\Delta t$, and are used to predict the attenuation on each link at time $t+2*\Delta t$. This prediction is sent to Control, where the predictions from all the links are compared and a switching decision is made.
If propagation conditions require a site switch, Control informs the other stations through their dedicated terrestrial link. This command assumes that the switchover will take place at the time $t+2\Delta t$. The choice of $\Delta t$ should therefore include sufficient time for the other stations to send an acknowledgement and for this to be received at Control. At the designated time the switchover occurs.

Once the switchover is complete, the switching loop starts again, and the procedure cycles through, switching as necessary.

In order to optimise this algorithm, its behaviour must be tested in a dynamic context. To do this, attenuation time series data (either measured or simulated) is needed. If measured attenuation time series data is not available, then data derived from radar rain maps may be substituted. The following section deals with the procedure to derive attenuation time series data from rain maps, for real radar data and simulated rain fields.
7.3 Use of rain maps in system design and operation

There have been a few studies published in the literature which deal with radar observations of site diversity measurements and site diversity performance prediction using radar modelling techniques [Goldhirsh, 1982, Bostian et al, 1990], however, in each case the radar data has been specifically collected with the view of analysis in terms of space diversity measurements. Meteorological radar data is a valuable resource, and not all system designers have the necessary access to it in order to take advantage of it. It is for this reason that the rain field simulator was created.

A potential source of radar rain fields for systems designers in the UK is the Met Office's Nimrod data. Available from the British Atmospheric Data Centre (BADC - http://www.badc.rl.ac.uk), Nimrod is a fully automated system for weather analysis and nowcasting based around a network of C-band (4-8 GHz) rainfall radars. The BADC holds the analyses as a time resolution of 15 minutes, along with composite pictures of the rain rate as measured by the radar network. Data are available from late 2002, and are updated on a regular basis.

Figure 7.2 shows an example gif image created from the Nimrod data and available on the BADC. The radar blocks are 5 km*5km in area and the images produced are the same data scaled to fit a 512*512 square.
7.3.1 Conversion from rain rate fields $R(x,y,z,t)(\text{mm/hr})$ to attenuation along a path $A_{db}(p,t)$ (dB)

7.3.1.1 Met Office Nimrod data

In general, communications engineers are only interested in rain as a result of how it affects the attenuation experienced by a radio path. For this reason it is necessary to convert the rain rates as given by the rain fields (measured or simulated) into attenuation along the link.

Calculating the attenuation along a slant path $A_{db}(p,t)$ that results from the rain rate measured in a field $R(x,y,z,t)$ requires a number of steps. Figure 7.3 gives a schematic diagram of some of the concepts introduced in these steps.
1. It is assumed that the variation in the vertical direction is constant, i.e. that the rain rate measured in each pixel is the same whether one takes a horizontal slice through the field at height 0km or 5km. (This assumption could be removed and a more realistic model of the vertical variation in rain substituted as an area of future work).

2. The effective path length of the slant path should be calculated. The variation of temperature with height above ground level is well documented, and has implications for the calculation of rain attenuation from radar measurements. As height above ground level increases, the atmospheric temperature also decreases, until a point is reached called the zero-degree isotherm. Above this point, all of the non-gaseous water in the atmosphere takes the form of ice crystals, while below it, liquid water droplets form. The height of the zero-degree isotherm varies with the variation of ground temperature. A good rule of thumb to calculate the height for a given time is to assume that the temperature falls off at a rate of 6 degrees Celsius for every kilometre above ground level.

3. The location of the sites should be determined, and from the effective path length, the number and identity of the pixels intersecting the path can be found.

4. The rain rate for each kilometre block along the slant path up to the effective path length should be calculated. This can be done by converting the radar rain fields (which are essentially the rain rates on
the ground) into 1km blocks, then using simple geometry to calculate the rain rate along the slant path in blocks of length $= 1/\cos(\text{elevation angle})$. These rain rates along the path can then be reapportioned to give rain rate along the path in 1km segments.

5. Finally, the total attenuation along the path can be calculated according to ITU-R Recommendation 618-8, as the sum of the specific attenuation (dB/km) given by the formula:

$$\gamma_R = k_{\text{ITU-R}} R^{\alpha_{\text{ITU-R}}}$$  \hspace{1cm} (6.6)

where $\gamma_R$ is the specific attenuation (dB/km), $R$ is the rain rate (mm/hr) and $k_{\text{ITU-R}}$ and $\alpha_{\text{ITU-R}}$ are the frequency dependent coefficients calculated using ITU-R Rec. 838-2.

Figures 7.4 and 7.5 show the results of this procedure, using the Met Office Nimrod data. As can be seen, for both events, the radar derived attenuation is in good agreement with the measured beacon attenuation data. However, because the radar resolution (5km*5km) was so large in comparison to the separation distance between the two sites (~7.5 km), the radar derived attenuation does not show any variation due to site diversity.

The Nimrod database is an extremely useful resource for systems designers, however, in this case, the spatial resolution is not high enough to be able to accurately used the data as a substitute for the beacon measurements made in the GBS site diversity experiment.

The benefits of a fractal rain field simulator in this case is that the simulated data can be scaled in space to any size resolution required, without adversely affecting the statistics and spatial behaviour of the rain field.
Figure 7.4: Attenuation time series for the stratiform event recorded at Chilbolton and Sparsholt on the 12th January 2004, in comparison with the attenuation calculated from the UK Met Office Nimrod radar maps.

Figure 7.5: Attenuation time series for the convective event recorded at Chilbolton and Sparsholt on the 9th January 2004, in comparison with the attenuation calculated from the UK Met Office Nimrod radar maps.
7.3.1.2 Simulated rain fields

A number of assumptions were made in the conversion from the simulated rain fields described in chapter five, into simulated attenuation time series. These are detailed as follows:

1) Each pixel in the simulated field covers an area of 250*250m (similar to the measured radar rain fields described in earlier chapters, where each pixel measured 300*300m).
2) There is no variation of the rain rate with height.
3) The array is aligned so that each path falls along a straight line of pixels.
4) The array is also aligned so that the pixels fall evenly along the slant path.
5) The elevation angle of the slant path is assumed to be 30°, and the zero degree isotherm is 1km above the ground. (This height was chosen for ease of comparison, as the same zero degree isotherm height was measured for the two events recorded by the GBS beacon receivers studied in the previous chapter.) This gives an effective path length of 2km.
6) The prevailing winds blow along a direction parallel to the links (i.e. the rain gets blown along the radio paths, rather than across)
7) The wind speed and direction of movement is constant, and was set to a value of 8m/s (equivalent to 2 pixels/minute). This value was chosen because it is of the same order as the average wind speeds, ~6 m/s, measured during the two events studied in chapter six.
8) The rain field moves at the same speed as the wind.
9) The two sites were set at a distance of 7.5km (30 pixels) apart, and site 2 was located at an angle of 55° with respect to due North from site 1.

The time variation in these simulations was achieved through the use of Taylor's frozen storm hypothesis [Taylor, 1938]. Essentially, each simulated rain field was cut down into a number of smaller "snapshots" of 200*200 pixels. Within these snapshots, the locations of the sites and paths were fixed. The variation in time was simulated by moving the position of the snapshots in the full-size simulated array by a small amount Δx and Δy, then saving the resulting new snapshot to give the rain field at time t=1. The process was repeated to give snapshots at time t=t...n. Δx and Δy
were chosen to give similar wind velocities as were experienced by the rain events studied above. Figure 7.6 gives a schematic example of this process.

![Diagram](image)

Figure 7.6: Schematic diagram of the process used to simulate the variation in time of the rain field.

Due to the size that the pixels were set to, and the average speed of the rain field, the time elapsed between each snapshot was 1 minute. In order to compare the simulated data with the measured data, the measured time series was averaged to the same sampling rate, 0.01667 Hz. This averaging of the measured data is acceptable, as it filters out the fast fluctuations in the signal due to receiver noise and scintillations, without risking aliasing of the slower signal variations due to rain fading. (Rain fades generally last of the order of tens to hundreds of minutes.) Rain gauge time series measurements are also commonly recorded at an integration time of one minute allowing future comparisons of attenuation derived from rain gauge measurements to be compared with attenuation derived from this simulator.

The time resolution could be improved by making each simulated pixel cover a smaller area, thereby introducing more variability. However, this would then reduce
the number of snapshots that could be extracted from the full-size array, leading to less measurements in time. It is possible to make the movement of the snapshots loop back on itself, however, this would lead to repeating patterns in the attenuation time series, which are not physically realistic.

Figure 7.7 shows the simulated rain field used to create the simulated attenuation time series shown in figure 7.8. As can be seen in the simulated rain field, the rain is concentrated in a number of small “rain cells”. This has its impact on the simulated attenuation time series, where one can see three peaks of varying sizes that occur at different times for each site. Figure 7.9 shows the attenuation measured during a convective event on the 9th January 2004, which has been block averaged to a one minute integration time. As can be seen, the measured event has a lower peak attenuation value (6dB as compared to 18dB), and appears “spikier”.

Figure 7.7: Full sized simulated rain field used to create a simulated time series of a convective event.
Figure 7.8: Simulated instantaneous (1 minute integration time) rain attenuation calculated from the *convective* rain field in figure 7.7.

Figure 7.9: Measured convective event (1 minute integration time) rain attenuation recorded on the 9th January 2004.
Similarly, figure 7.10 shows the full-size simulated array used to create the simulated attenuation time series shown in figure 7.11. The resulting simulated attenuation has the time variability at the two sites associated with site diversity, and is also longer lasting (at 2 ½ hours, compared with ~50 minutes for one of the convective cells), and of lower peak attenuation than the simulated convective attenuation. Figure 7.12 shows the attenuation measured during a stratiform event on the 12th January 2004, which has been block averaged to a one minute integration time. Once again, the simulated data has a higher peak attenuation than the measured event, however, this can be adjusted through the variation of the parameter $b$ in the rain field simulator. As in the convective event, the simulated data appears smoother than the measured data. This is possibly due to the overly simplified method of introducing time variation, as it ignores any possible evolution of the rain field on small scales from one minute to the next.

Figure 7.10: Full sized simulated rain field used to create a simulated time series of a stratiform event.
Figure 7.11: Simulated instantaneous (1 minute integration time) rain attenuation calculated from the stratiform rain field in figure 7.9

Figure 7.12: Measured stratiform event (1 minute integration time) rain attenuation recorded on the 12th January 2004.
7.4 Statistical comparison of measured attenuation data with simulated attenuation data

It is very difficult to draw any quantitative conclusions about the comparison between measured and simulated data working solely on an event-by-event basis, as any observations are limited to what is visually perceived. Measured attenuation events are extremely variable and there is no guarantee that any one event is representative of its type. It is for this reason that a statistical comparison is required. For the statistics to be significant there must be a sufficiently large amount of data available for analysis. Unfortunately, due to the fact that the GBS experiment was only begun in October 2003, there is a shortage of measured site diversity attenuation data available at this time.

In the month of January 2004, ten stratiform events and four convective events were recorded. The convective events were supplemented with another three convective events recorded in the beginning of February 2004. These seventeen events formed the measured database, which was statistically compared to a database of eighteen simulated events (nine convective and nine stratiform). The measured events were characterised as stratiform and convective according to the presence or absence of the melting layer as recorded by the 94GHz cloud radar at Chilbolton (more details in section 6.2.3).

The different events were concatenated together to form a continuous time series. In order to test the validity of the temporal variation in the simulated data given by the method described in section 7.3.1.2, the continuous time series for the two measured sites and the two simulated sites have had their power spectra calculated. Figure 7.13 shows the power spectrum for the time series data measured at Sparsholt, while figure 7.14 shows the power spectrum for the time series measured at Chilbolton. The straight dashed line in the plots is the best fit line for the power spectrum. The slope for the power spectrum of the Sparsholt data was calculated to be $-1.78 \pm 22.57$, while the slope for the Chilbolton data was $-1.80 \pm 21.21$. The high errors in calculating the slope are a result of the wide spread of the spectral density values at high frequencies.
Figure 7.13: Power spectral density function calculated for data measured at Sparsholt.

Figure 7.14: Power spectral density function calculated for data measured at Chilbolton.
Figures 7.15 and 7.16 show the power spectral density functions for the simulated attenuation time series data at the two simulated sites. Once again, the straight dashed line in the plots is the best fit line for the power spectrum. The slope for the power spectrum of the site 1 data was calculated to be $-2.51 \pm 23.11$, while the slope for the site 2 data was $-2.39 \pm 21.58$. These plots clearly show that, as was seen in section 7.3.1.2 above, the simulated time series data have less high frequency components than the measured data.

This power spectral density function analysis also indicates that as a method of introducing temporal variation, use of Taylor’s frozen storm hypothesis [Taylor, 1938] is only a first approximation and does not produce realistic attenuation time series behaviour. A more sophisticated method of introducing temporal variation is therefore required.

![Power spectral density function for simulated data at site 1](image)

Figure 7.15: Power spectral density function calculated for simulated data at site 1.
As a further statistical comparison between the measured and simulated attenuation time series, the cumulative distribution functions for the bundled stratiform events (figure 7.17) and convective events (figure 7.18) were plotted. Cumulative distribution functions (CDFs) were chosen for the comparison as they are commonly used in the communications engineering field, and are well modelled in the ITU-R recommendations. Monthly cumulative distributions of measured attenuation were shown in section 6.2.4, figure 6.13, as were annual statistics in section 1.2, figure 1.1.

In figures 7.17 and 7.18, the attenuation is plotted on the x-axis, while the proportion of the total number of observations that the attenuation is less than or equal to the x-axis value is given on the y-axis. In the case of the stratiform events recorded at Sparsholt, we can see over 90% of the observations were of attenuation values less than 1 dB. Similar behaviour was shown by the data measured at Chilbolton, though the two measured curves do not have exactly the same values, indicating that the attenuation at the two sites is not completely correlated.
In comparison, the simulated data has a higher proportion of observations at higher attenuation levels resulting in a discrepancy from the measured curves. The simulated data curves have similar shape, and their closeness indicates that the simulated attenuation at the sites is highly correlated.

![Empirical CDF - Stratiform](image)

Figure 7.17: Cumulative distribution functions of measured and simulated stratiform events.

Figure 7.18 shows the same CDFs for the measured and simulated convective events. At the lower attenuation thresholds the simulated data deviates from the measured curves in a similar way as to presented in figure 7.17. However, for the higher attenuation thresholds, the simulated data curves are very close to the Chilbolton measurements curve. There is a greater discrepancy between the measured curves than in the stratiform case, indicating that for convective events there is less correlation of attenuation between the two sites than there is with stratiform events. This is not surprising, given the physical differences between stratiform and convective events described in chapter four, section 4.4.5.
Figure 7.18: Cumulative distribution functions of measured and simulated convective events.

The CDF curves seem to indicate that the simulated events spend more time above a given attenuation threshold than the measured events for the same threshold. Combining this with the observation in section 7.4.1.2 that the simulated time series appear smoother than the measured counterparts suggests that the simplified method of introducing temporal variation is affecting the simulator's ability to create realistic event time series.

However, it should be pointed out again that the measurement database used for this comparison is very limited. The winter months tend to be dominated by long periods of rain with low intensities, while most high intensity rain events occur during the summer. Ideally, the simulator should be compared against a full year's events, in order to determine if it can accurately simulate the wide range of attenuation behaviour measured in the real world. If a full year's worth of data is not available, it would also be possible to test the simulator in comparison with the ITU-R models of attenuation CDFs. This was not done in this thesis, as it would require simulating many hundreds of events of different types, with varying event durations and peak

162
attenuation values, and unfortunately time is limited. However, the author would strongly recommend it as a topic for future work.

7.5 Diversity gain using measured and simulated data

Figure 7.19 plots the instantaneous diversity gain at a time $t$, against the maximum of the measured attenuation values recorded at Sparsholt and Chilbolton for the same time $t$, limited to stratiform events. As expected, the diversity gain at a given instant can never be more than the maximum attenuation at the two sites for that instant. (This corresponds to the case where at one instant in time, the attenuation at one site is zero, while the attenuation at the other is greater than zero. Figure 7.20 shows a schematic diagram of this case. Section 6.2.6 gives more information on how to calculate the instantaneous diversity gain.)

We compare the instantaneous diversity gain with the maximum attenuation of the two time series at a given instant because the measured data has shown that the two sites are well correlated, and hence are statistically equivalent. If the sites were further apart, and hence less correlated, then we would take the average of the two attenuations as our comparison.

Figure 7.21 plots the same parameters, but instead is limited to convective events. If we compare figures 7.19 and 7.21 we can see that the convective points are less spread out than the stratiform, and are closer to the one-to-one best case line between the maximum instantaneous diversity gain and the maximum attenuation. This is because convective events tend to be smaller in geographical size than stratiform events, meaning that there is more of a chance that there will be little or no attenuation on one site, while the other is heavily attenuated by a single convective rain cell. As stratiform rain events are more widespread, this deviation from the best case is less likely to happen.

In both the figures, however, it should be noted that as the attenuation threshold decreases, the spread from the best case line increases.
Figure 7.19: Instantaneous diversity gain at a time $t$, plotted against the maximum of the measured attenuation values recorded at Sparsholt and Chilbolton for the same time $t$. Stratiform events.

Figure 7.20: Schematic diagram of two attenuation time series, showing how at one instant the diversity gain is calculated and how it can never be greater than the maximum attenuation at that instant.
Figure 7.21: Instantaneous diversity gain at a time \( t \), plotted against the maximum of the measured attenuation values recorded at Sparsholt and Chilbolton for the same time \( t \). Convective events.

Figures 7.22 and 7.23 show plots of the same parameters, but instead display data points for the simulated stratiform and convective events respectively. As with the measured data, the simulated stratiform events are widely spread beneath the one-to-one best case line, while the simulated convective events are more concentrated next to the line.
Figure 7.22: Instantaneous diversity gain at a time $t$, plotted against the maximum of the simulated attenuation values recorded at the two simulated sites for the same time $t$. Stratiform events.

Figure 7.23: Instantaneous diversity gain at a time $t$, plotted against the maximum of the simulated attenuation values recorded at the two simulated sites for the same time $t$. Convective events.
We can establish the probability percentage of time that the instantaneous diversity gain is less than or equal to a set threshold value. However, as before, we would require more simulated and measured data to get accurate statistics. Also, the conditional distributions of instantaneous diversity gain are conditioned to the site location, meaning that for accurate comparisons, we would need to be sure that the simulator could accurately represent the statistics of rain in a given climatic region. Hence, further work is recommended.

Another potential for further work is to use the simulator to produce diversity gain statistics at varying site separation distances. The resulting curves could then be compared with the proposed ITU-R models to model the behaviour of diversity gain with varying site separation distance. However, as before, this would require substantially more simulations to be produced.

7.6 Performance of the switching algorithm with the test use of real and simulated data

The switching algorithm described in section 7.2 was coded up using MATLAB, and the resulting system attenuation was derived from the real and simulated attenuation time series data for differing values of $\Delta t_\phi$. The events were concatenated with a period of 30 minutes of zero attenuation between them, in order to avoid any sudden changes between one event and the next, which could cause errors in the switching algorithm's short-term prediction method.

The data was presented in a way similar to that shown in section 7.5, but for these plots the system attenuation was plotted against the instantaneous minimum attenuation at the two sites, as it is the ideal case. If the switching occurred instantaneously and with no errors, the resulting system attenuation would be the instantaneous minimum attenuation at the two sites.

As can be seen from the measured stratiform and convective events plotted in figures 7.24 and 7.25 respectively, at low $\Delta t_\phi$ (1 minute) the system attenuation agrees extremely closely with the minimum attenuation at both sites. It is for values of $\Delta t_\phi$ above 1 minute that the system attenuation starts deviating from the ideal. It is still obvious that there are gains to be made as a result of using site diversity as a FMT, even when the switching algorithm is not optimised.
Figure 7.24: System attenuation at a time $t$ for varying $\Delta t_f$ values plotted against the minimum of the measured attenuation values recorded at Sparsholt and Chilbolton for the same time $t$. Stratiform events.

Figure 7.25: System attenuation at a time $t$ for varying $\Delta t_f$ values plotted against the minimum of the measured attenuation values recorded at Sparsholt and Chilbolton for the same time $t$. Convective events.
Figures 7.26 and 7.27 show plots of the same parameters, but instead of plotting the values for the measured data, they plot the simulated values. As with the measured data the system attenuation agrees closely with the minimum attenuation at both sites, for $\Delta t_5 = 1$ minute. Again, the deviation from the ideal case increases as $\Delta t_5$ increases.

This comparison of the performance of the switching algorithm with measured and simulated data is not intended to offer any definite statistical results. The switching algorithm itself is very simplistic and any results are unlikely to have implications for real communications systems operation. Instead, the comparison serves as a proof-of-concept, which can be expanded on in the future to perform case studies testing the architecture of operational systems.

Figure 7.26: System attenuation at a time $t$ for varying $\Delta t_5$ values, plotted against the minimum of the simulated attenuation values recorded at the two simulated sites for the same time $t$. Stratiform events.
Figure 7.27: System attenuation at a time $t$ for varying $\Delta t_\beta$ values, plotted against the minimum of the simulated attenuation values recorded at the two simulated sites for the same time $t$. Convective events.

7.7 Acknowledgements

The author would like to acknowledge the use of the Met Office's Nimrod radar data in this chapter. This data is available from [http://www.badc.rl.ac.uk](http://www.badc.rl.ac.uk) for bona fide academic purposes.

7.8 References


ITU-R P.618-8 Propagation data and prediction methods required for the
design of Earth-space telecommunication systems

ITU-R P.838-2 Specific attenuation model for rain for use in prediction
methods

Atmospheric Fade Compensation for Spacecraft Antennas Part 1 – Rain Fade

476-490, 1938
The overall aim of this thesis was quite simple: to introduce fractal methods to the radio communications systems engineering community for analysing, characterising and synthesising the rain fields that cause excessive attenuation for systems operating at frequencies above 10 GHz.

However, even through the aim was simple, the path taken to get to it was not. This thesis therefore covers a wide range of topics, from basic (and not-so-basic) fractal and multifractal analysis, to consideration of the physical behaviour and properties of rain fields, to synthesis of the spatial variation of a rain field using fractal methods, and finally to the application of this work, a demonstration of its use in a communications engineering case study.

Figure 8.1 gives an overview of the topics studied in this thesis. As can be seen, the thesis can be split into three main sections. Part 1 deals with the verification of the fractal nature of rain fields. Part 2 considers the methods and inputs involved in production of a fractal rain field simulator. And finally, part 3 applies the work done in the previous two parts to a communications engineering case study, in this case a satellite communications system using site diversity as a fade mitigation technique (also known as fade countermeasures, FCM).

8.1 Summary of conclusions
8.1.1 Part 1 – Verification of the fractal nature of rain fields

Before any useful conclusions can be drawn about the applicability of fractal methods to rain field analysis and synthesis, it must be verified that rain fields are indeed fractal, and that useful results can be obtained by this analysis. A thorough literature search was performed in order to determine what analysis and fractal
methods were in use, and data recorded by the RCRU's meteorological radar was then analysed using these methods.

The results from this analysis confirmed the fractal nature of rain fields, with the following specific results:

- From the radar data it was found that contour lines enclosing areas of rain greater than or equal to a specified threshold (mm/hr) have an area-perimeter dimension and box-counting dimension that agree at \(-1.2\). This is contrasted with other results published in the literature, which give a fractal dimension of \(-1.35\) [Lovejoy, 1982].

- The 2-D FFT of the log rain rate fields shows a spectral density function that is a straight line with a constant slope, also indicative of scaling. The exponent of the spectral density function is related to, and agrees with, the value of the fractal dimension of the contour lines given by the area-perimeter and box-counting methods.

- The distribution of the number of these contours with respect to their enclosed area (also known as the Korcak distribution) was also shown to be a power law, another sign indicative of scaling and fractal behaviour.

- The correlation dimension of a rain gauge network gives an indication as to the effectiveness of the network, i.e. a rain gauge network with too many gauges packed too closely is inefficient and expensive, whereas if they are too far apart any small-scale variation in the rain less than the spacing between the gauges will be lost.

- Multifractal analysis of the rain fields distinguishes between the different types of rain, however, multifractal analysis of log rain fields shows that they have a straight line \(K(q)\) function, indicating that they can be approximated by monofractal fields. \(K(q)\) represents the moment scaling function and is commonly taken to be the characteristic function of multifractal behaviour.
Figure 8.1 Overview of the topics studied in this thesis.
It is the author's view that the final specific conclusion above has the most impact on the use of fractal methods by communications engineers. Most fractal analysis of rain fields to date has been carried out by meteorologists, hydrologists and applied physicists, who are generally very interested in the extreme rainfall events that occur, and have the (very laudable) aim of using fractal methods to characterise all of the extremely variable behaviour of rainfall.

Communications systems engineers are more pragmatic, with their fundamental concern being the behaviour of their communications system. Most systems are designed to be operational within a certain range of attenuations (given by the fixed fade margin chosen, or the limitations of the fade mitigation technique used). Once this range is exceeded, the system is not operational, and it is of little concern to the engineer whether the range is exceeded by 1 dB or 100 dB. For this reason, engineers are not interested in extreme rainfall events, instead preferring to ensure that their system is operational for the 99.99% of the time required to provide the specified level of quality of service to their customers.

The transformation of the representation of the observable, from rain rate, to log rain rate, is also consistent with engineers' "world-view", accustomed as they are to dealing with parameters with naturally logarithmic scales, such as transmit and receive power (dBm), antenna gain (dBi) and attenuation (dB). Granted, this transformation does mean the loss of some information about the rain fields, as log rain rate can only be determined when it is raining, and cannot deal with mixtures of rainy and non-rainy times or areas. This does not limit the validity of the model from a communications engineering context, as a method which can produce synthetic events is extremely useful to systems engineers.

8.1.2 Part 2 - Production of a fractal rain field simulator

The fractal rain field simulator presented in this thesis was originally a mathematical construct first proposed by Voss [1985] to simulate fractional Brownian motion in two dimensions. It is a discrete additive process that produces a monofractal field as an output. However, as mentioned above, multifractal analysis of log rain fields has shown that they may be assumed to be monofractal field (for the range of rain rates that interest systems engineers).

In order to tailor this model to better fit the needs of communications engineers, and in order to further investigate some issues that had been identified as a
result of the fractal analyses performed in part one of the thesis, a study of the phenomenological and physical properties of rain was undertaken. The topics investigated included: measuring device resolution in time and space, the effects of noise on measurement of fractal dimension, assumptions on the nature of rain fields, the impact of atmospheric turbulence on the measured fractal dimension of rain fields, orographic effects on rain measurements, the differences between stratiform, convective and frontal rain, and finally, scaling and scaling limits.

The results of this study, particularly the differences between different rain event types, then fed into the revised parameters required for the fractal simulator, resulting in simulated rain fields which are visually convincing and have the same fractal dimension and spectral density function exponent as measured rain fields. The simulator is capable of producing stratiform-like and convective-like synthetic rain fields, with fractal dimension of contour lines enclosing areas of rain greater than or equal to a threshold and spectral density function exponent consistent with measured data. The simulator also employs a climatological parameter, which enables its application to climatic regions other than the South of England.

Each realisation of the rain fields produced by the simulator is statistically independent of the others, though there is space correlation within each realisation. The method used does allow for any "system memory" of the previous realisation to be carried over to the next, hence a series of rain field realisations will not result in a convincing synthetic event. For this reason, an important piece of future work would be in the addition of a time dimension to the simulator. A first approximation of temporal variation was introduced in chapter seven in order to successfully convert the simulated rain fields to attenuation time series.

Chapter five also presents a more rigorous mathematical description of fractional Brownian motion and the relationships between the spectral density exponent $\beta$, the Hurst exponent $H$ and the fractal dimension $D$. This discussion serves to add depth to the simulator, and justify the use of a discrete cascade method to simulate the rain fields. The calculated value for $H = 1/3$ shows that log rain rate is antipersistent, and the process has long range anti-correlation.

Also presented was a review of other methods for simulating rain fields, subdividing the main methods into three types: discrete cascades, continuous cascades and synthetic storm models. Finally, in order to show that non-fractal methods can
also give important information on the structure of rain fields, a brief analysis of the cross-correlation between measured radar rain fields was presented.

8.1.3 Part 3 – Application to a communications engineering case study

Once again, in order to accurately apply the outputs of the fractal simulator to a communications engineering case study, some necessary ground work and theory had to be investigated. It is for the sake of completeness that definitions of concepts such as diversity gain and improvement are included in the thesis, even though they are well understood by the communications engineering community. (It was for the same reason that the definitions of similarly well understood fractal concepts were presented in the first part of the thesis).

There has been substantial work presented and published on simulators capable of producing attenuation time series data for various systems, a lot of which has been presented at the COST 280 meetings concerned with propagation impairment mitigation for millimetre wave radio systems. However, this thesis takes the simulator a step further, and applies it to a case study, investigating how a simple site diversity switching algorithm for an Earth-space radio communications system behaves when tested with measured and simulated attenuation time series.

Before this could be done, it was necessary to define and discuss the inputs needed in order to optimise such a switching algorithm. This was done, using as examples fade and inter-fade duration and fade slope statistics as measured during the RCRU’s ITALSAT experiment and from ITU-R models. A simple short-term attenuation predictor was also presented as a key input into the switching algorithm.

The procedure required to convert from rain rate fields \( R(x,y,z,t) \) (mm/hr) to attenuation along a path \( A_{db}(p,t) \) (dB) was detailed, first using the radar rain fields measured by the Met Office’s Nimrod radar network. It was found that even though the Nimrods database is an extremely useful resource for systems designers, in the case of interest, the spatial resolution of the recorded data was not high enough to be able to accurately use the data as a substitute for the beacon measurements made in the RCRU’s Earth-space site diversity experiment (which uses the 20.7 GHz GBS beacon on board the satellite UFO-9). The benefits of a fractal rain field simulator are that the simulated data can be scaled in space to any size resolution required, without adversely affecting the statistics and spatial behaviour of the rain field.
A similar procedure was used to convert the simulated rain fields into simulated attenuation along a radio path, with the temporal variation given as a first approximation by the use of Taylor's frozen storm hypothesis [Taylor, 1938]. (This hypothesis presupposes that the spatio-temporal rain field may be approximated to a fixed spatial field moving with a constant velocity).

A short description of the GBS site diversity experiment was presented, as measured attenuation time series data from this experiment was used to provide a comparison with the simulated time series data produced by the rain field simulator. Cumulative distribution functions of the attenuation time series were calculated for a database of ten measured stratiform events, seven measured convective events, nine simulated stratiform events and nine simulated convective events. This statistical analysis gives an indication of how realistic the simulator outputs are, however, as there was very little measured data available, more measured data is required to give a complete and thorough statistical analysis.

8.2 Novel contributions by the author

Over the course of this research work, the author has produced a number of novel contributions to the various research areas studied.

Of these, probably the one with the most potential impact is the introduction and publication of fractal methods for characterising rain to the communications engineering community.

The author has also:

- Used multifractal methods for analysing rain rate and log rain rate fields, demonstrating that log rain rate fields have a straight line $K(q)$ function, and therefore monofractal methods and fields can be used to analyse and synthesise rain fields for the rain rate ranges of interest to communications engineers.
- Demonstrated the power law that exists in the distribution of the number of rain rate contours with respect to their enclosed area (also known as the Korcak distribution). These and other relationships (such as the area-perimeter relationship) are not limited to fractal models of rain alone, but may also be used as inputs into other models, such as synthetic storms.
- Extended the Voss algorithm for simulating fractional Brownian motion in order to simulate the spatial variation of rain fields, by including parameters
that deal with the distinction between stratiform and convective rain, and climatological factors such as the required maximum rain rate in the simulated field. The choice of these parameters was made empirically, based on the visual characteristics of the measured rain fields.

- Used the rain field simulator to produce simulated attenuation time series, though a number of simplifications and assumptions were made in order to convert two-dimensional rain fields into time series of attenuation along a slant path. The simulated event time series were statistically compared with a limited number of measured events. The simulated time series were then used as an input to investigate the behaviour of a simple site diversity switching algorithm. The author was unable to find any references to any papers or reports that deal with an application of a rain field simulator to a specific communications engineering case study, such as the one presented in this thesis.

8.3 Recommendations for further work

It is a truism that scientific research is never truly finished, there are always interesting topics identified over the course of any research that cannot be fully investigated due to time and/or financial constraints. This section contains a few of these topics identified in the course of this research project which the author feels are worth investigating in the future.

8.3.1 Further fractal analysis of rain fields

This thesis considered only the spatial variation of rain in the horizontal plane, which is perfectly adequate for terrestrial communications systems such as point to point links or mesh networks. However, the vertical variation of rain is not insignificant, and is highly influenced by the rain type (as described in chapter four). It has also serious implications for those slant path systems in operation, such as Earth to satellite communications. Experimental data from meteorological radars is available with which to study this vertical variation.

Another topic of interest is the relationship between the fractal nature of clouds, and the fractal nature of the rain produced by those clouds. The genesis of rain from clouds is not clearly understood, and there is a marked difference reported for the fractal dimensions of the two. Also the smoothing effect of the wind on the rain
contours raises interesting questions about the effect of the turbulent atmosphere on rain fall. Does the atmosphere have a filtering effect on the fractal dimension of the rain field as it is formed at the cloud level and falls to Earth? Or is the disparity simply due to the change in size from a cloud droplet (diameter ~1μm) to a rain drop (diameter ~1mm). A program of experimental work designed to answer such questions could vastly improve the state of knowledge in this area.

Numerically quantifying the visual and physical differences between the different rain types would improve the realism of future rain field simulators. If the concept of lacunarity was used for this, the mathematical basis of the concept would need to be explored in greater detail, and measurements made of the lacunarity of the rain fields.

The physical implications of the value of the Hurst exponent, $H = 1/3$, reported in this thesis, deserve further investigation as this value has interesting implications for the physical processes underlying the production of rain.

Finally, the issue of interpolating the radar data from polar coordinates to a Cartesian grid and the impact on the value of the fractal dimension of the rain fields calculated from this interpolated data should be investigated. This could be achieved through modification of standard fractal analysis methods such as the box counting method to deal with polar coordinate data.

### 8.3.2 Further modification of the rain field simulator

As stated earlier, the rain field simulator as presented in this thesis deals with the temporal variation of rain fields only as a first approximation, using Taylor's frozen storm hypothesis [Taylor, 1938]. Rain variation in time is a product of the velocity, advection and evolution of the rain field, all of which need to be taken into account to accurately describe the temporal variation. The simulator would therefore be improved by the addition of more realistic temporal variation.

The vertical variation of rain is another dimension that could be added to the simulator, especially if the fractal analysis of this variation was performed, as suggested above.

In the present version of the simulator the operator empirically chooses the parameter $b$. A future improvement would be producing $b$ as a function of a rain
rate parameter that can be calculated from measured rain rate statistics or ITU-R recommendations.

8.3.3 Further applications of the simulator

The simulator as presented here is not limited in application only to the single communications case study described in this thesis. It can also be used to deal with other communications systems and proposed fade mitigation techniques.

Even without the addition of improved temporal variation, the simulator can be used to calculate diversity gain and improvement statistics for systems utilising a large number of sites, where conventional (ITU-R) models would be overly time consuming. Mesh networks, for example, with hundreds of nodes, could benefit from this work. Validation of these statistics, in comparison with measured or modelled statistics (from other methods) would provide a valuable tool to the communications engineering community.

The attenuation time series calculated from the rain field simulator are compared with measured data in chapter seven. However, only a single month’s worth of measured data was compared with the simulated data. As there is substantial variation in attenuation cumulative distribution statistics on a month by month (and year by year) basis, further measurements are needed to ensure that the simulator can produce the wide variation in rain rates and duration times required.

8.3.4 Investigation into instantaneous and statistical diversity gain

As stated in chapter 6, a comparison of instantaneous diversity gain with statistical (long-term) diversity gain published by Towner et al [1984] states “the statistical diversity gain approximates the median value of the instantaneous diversity gain”. This statement has very interesting implications and can be verified through further analysis of the data produced as a result of the GBS experiment, or any other site diversity dataset.

8.4 References


APPENDIX A: DERIVATION OF THE AREA PERIMETER RELATIONSHIP FOR DETERMINING FRACTAL DIMENSION

For regular Euclidean shapes, like circles, squares, triangles, hexagons, etc, the ratio, Q, between the perimeter, P, and the square root of the enclosed area, A, is constant, regardless of the size of the shape

\[ Q = \frac{P}{\sqrt{A}} \]  \hspace{1cm} (A.1)

One can generalise this rule for areas bound by fractal curves, where the length of the perimeter diverges as one uses smaller and smaller unitary lengths of measurement, or sticks, to measure that perimeter. It can be shown that the equation becomes:

\[ Q = \frac{P^{1/D_A}}{\sqrt{A}} \]  \hspace{1cm} (A.2)

where P and A are now the measured perimeter and area of the boundary, using a length scale \( \delta \) and \( D_A \) is the fractal dimension. For example, measuring a line segment 10 cm long using a scale length of \( \delta = 2 \text{cm} \) will give a measured length \( P = 5 \).

In practice we can investigate a whole series of fractal island shapes by covering them with a grid of boxes with side length \( \delta \). The perimeter P is then \( P = N_p \delta \) and the area \( A = N_A \delta^2 \) where \( N_p \) and \( N_A \) are the number of boxes required to cover the perimeter and area respectively.

As \( \delta \) becomes small enough to accurately measure the perimeter and areas of the islands, i.e. as \( \delta \to 1 \), \( N_p \to P \) and \( N_A \to A \). One can then rearrange equation A.2 as follows:

\[ \log(A) = \frac{2}{D_A} \log(P) - 2 \log(Q) \]  \hspace{1cm} (A.3)

which is in the form of the equation for a straight line with slope of \( 2/D_A \) (figure A.1).
Figure A.1: Schematic plot of log(A) against log(P)
APPENDIX B: DERIVATION OF THE BOX COUNTING METHOD FOR DETERMINING FRACTAL DIMENSION

For a line segment of length 1 we require N cubes of side length $\delta$ to cover the line (figure B.1). This is also true for any regular shape, such as a rectangle drawn in the plane.

Therefore $N\delta=1$ and $N = \frac{1}{\delta^1}$

Figure B.1 Line segment (blue) covered by boxes of side length $\delta$

Similarly, for a unit area, we need N cubes of side length $\delta$ to cover the area (figure B.2).

Therefore $N\delta^2=1$ and $N = \frac{1}{\delta^2}$

Figure B.2, Area (blue) covered by boxes of side length $\delta$
Generalising for unit volumes and unit hypervolumes in three and more dimensions, it can be seen that the exponent of $\delta$ is the dimension of the object, also called the box counting dimension $D_B$. Thus, for objects of unit length, area, volume etc.:

$$N = \frac{1}{\delta^{D_B}}$$ \hspace{1cm} (B.1)

and

$$D_B = \lim_{\delta \to 0} \left( \frac{\log(N)}{\log(1/\delta)} \right)$$ \hspace{1cm} (B.2)

For an area $= U$ (or a line segment with length $= U$ or a volume $= U$ etc.) equation B.1 becomes:

$$N(\delta) = \frac{U}{\delta^{D_B}}$$ \hspace{1cm} (B.3)

and equation B.2 becomes:

$$D_B = \frac{\log(N) - \log(U)}{\log(1/\delta)}$$ \hspace{1cm} (B.4)

We arrange B.4 to set:

$$\log(N) = D_B \log(1/\delta) + \log(U) + \epsilon(\delta)$$ \hspace{1cm} (B.5)

where $\epsilon$ is the error and $\epsilon \to 0$ as $\delta \to 0$.

Equation B.5 is in the form of a straight line with $D_B$ as the slope of the line.
Definitions of the Hausdorff dimension can also be found in [Hausdorff, 1918 and Peitgen et al, 1992].

We restrict ourselves to a definition of the Hausdorff dimension for sets $A$ which are embedded in Euclidean space.

\[ \mathbb{R}^n = \{ x \mid x = (x_1, \ldots, x_n), x_i \in \mathbb{R} \} \]  

for some natural number $n$. To arrive at a definition we need to introduce some mathematical notation.

Firstly there is the distance function $d(x, y)$ which is the Euclidean distance between $x$ and $y$ in $\mathbb{R}^n$

\[ d(x, y) = \sqrt{\sum_{i=0}^{n} (x_i - y_i)^2} \]  

Secondly there is the infimum and supremum of a subset $X$ of real numbers,

\[ \inf\{ x \in X \} = \text{largest lower bound of } X. \]

\[ \sup\{ x \in X \} = \text{smallest upper bound of } X. \]

This means that $a = \inf\{ x \in X \}$ provided $a \leq x$ for all $x \in X$ and for any $\epsilon > 0$ there is $x \in X$ such that $x - a < \epsilon$. Similarly $b = \sup\{ x \in X \}$ provided $b \geq x$ for all $x \in X$ and for any $\epsilon > 0$ there is $x \in X$ such that $b - x < \epsilon$.

Using these notions we can now define the diameter of a subset $U$ of $\mathbb{R}^n$.

\[ \text{diam}(U) = \sup\{ d(x, y) \mid x, y \in U \} \]  

The final piece of notation we need is that of the open cover of a subset $A$ of $\mathbb{R}^n$. A subset $U$ of $\mathbb{R}^n$ is called open provided for any $x \in U$ there is a small ball $B_\epsilon(x) = \{ y \in \mathbb{R}^n \mid d(x, y) < \epsilon \}$ of radius $\epsilon > 0$ centred at $x$ which is entirely in $U$. A family of open subsets $\{ U_1, U_2, U_3, \ldots \}$ is called an open cover (countable) of $A$ provided:

\[ A \subseteq \bigcup_{i=1}^{\infty} U_i \]  

We are now ready to define the Hausdorff dimension of $A$.

Let $s$ and $\epsilon$ be positive real numbers. Then define
\[ h'_\varepsilon(A) = \inf \sum_{i=0}^{\infty} \text{diam}(U_i)^s \mid \{U_1, U_2, \ldots\} \text{open cover of } A \text{ with } \text{diam}(U_i) < \varepsilon \]  \hspace{1cm} (C.5)

Thus, the infimum is extended over all open covers of \( A \) for which the covering sets \( U_i \) have diameter less than \( \varepsilon \). For each such cover we take the diameters of the open sets of the cover, raise them to the \( s \)-th power and take the sum, which may be finite or infinite. As we decrease \( \varepsilon \) the class of permissible covers of \( A \) is reduced. Therefore the infimum increases and so approaches a limit as \( \varepsilon \to 0 \), which can be infinite or a real number. We write

\[ h' (A) = \lim_{\varepsilon \to 0} h'_\varepsilon(A) \]  \hspace{1cm} (C.6)

The limit \( h'(A) \) is called the \( s \)-dimensional Hausdorff measure of \( A \). It follows that the \( s \)-dimensional Hausdorff measure of the empty set is 0 and \( h'(A) \leq h'(B) \) if \( A \subset B \). Moreover, \( h'(A) \) is the length of a smooth curve \( A \); \( h^2(A) \) is the area of a smooth surface \( A \) up to a factor of \( \pi/4 \); \( h^3(A) \) is the volume of a smooth 3-dimensional manifold \( A \) up to a factor of \( 4\pi/3 \).

Another important property is: If \( f: A \to \mathbb{R}^s \) satisfies a Hölder condition for all pairs \( x, y \in A \), i.e.

\[ d(f(x), f(y)) \leq c(d(x, y))^a \]  \hspace{1cm} (C.7)

for some constants \( c>0 \) and \( a>0 \), then

\[ h^{s+a}(f(A)) \leq c^{s+a} h^s(A) \]  \hspace{1cm} (C.8)

For example if \( f \) is a similarity transform with contraction factor \( 0 \leq \lambda < 1 \) then \( f \) satisfies a Hölder condition with \( a=1 \) and \( h^s(f(A)) \leq c^s h^s(A) \).

Hausdorff also proved that for any set \( A \) there is a number \( D_H(A) \) such that

\[ h^s(A) = \begin{cases} \infty \text{ for } s < D_H(A) \\ 0 \text{ for } s > D_H(A) \end{cases} \]  \hspace{1cm} (C.9)

The number \( D_H(A) \) is defined as the Hausdorff dimension

\[ D_H(A) = \inf \{s \mid h^s(A) = 0\} = \sup \{s \mid h^s(A) = \infty\} \]  \hspace{1cm} (C.10)

if \( s = D_H(A) \) then \( h^s(A) \) may be zero, infinite or some positive real number.

Finally, some fundamental properties of the Hausdorff dimension are:

1) If \( A \subset \mathbb{R}^s \) then \( D_H(A) \leq n \)

188
2) If \( A \subset B \) then \( D_H(A) \leq D_H(B) \)

3) If \( A \) is a countable set then \( D_H(A) = 0 \)

4) If \( A \subset \mathbb{R}^n \) and \( D_H(A) < 1 \) then \( A \) is totally disconnected.

C.1 References


APPENDIX D: THE AREA PERIMETER RELATIONSHIP FOR REGULAR SHAPES

D.1 Square
For a square, side length L
Perimeter $P = 4L$
Area $A = L^2$
Ratio, $Q$, of perimeter to the square root of the area is constant.

$$Q = \frac{P}{\sqrt{A}} = \frac{4L}{\sqrt{L^2}} = 4$$

D.2 Circle
Similarly for a circle of radius $L$
Perimeter $P = 2\pi L$ and Area $A = \pi L^2$

$$Q = \frac{P}{\sqrt{A}} = \frac{2\pi L}{\sqrt{\pi L^2}} = 2\sqrt{\pi}$$

D.3 Generalisation for N sided regular polygon
Generalising for an N sided polygon ($N \geq 3$):
Inscribe an N sided polygon in the unit circle.
Perimeter:

\[ P = NL \]

\[ \frac{L}{2} = \sin\left(\frac{\pi}{N}\right) \]

\[ P = 2N \sin\left(\frac{\pi}{N}\right) \]  \hspace{1cm} (D.3.1)

Area:

Divide the polygon up into N identical isosceles triangles with their points at the centre of the unit circle.

\[ A = NA_T \] where \( A_T \) is the area of one of the triangles.

\[ A_T = 2 \cdot \frac{1}{2} \cdot \cos\left(\frac{\pi}{N}\right) \cdot \sin\left(\frac{\pi}{N}\right) \]

\[ A = N \cos\left(\frac{\pi}{N}\right) \cdot \sin\left(\frac{\pi}{N}\right) \]  \hspace{1cm} (D.3.2)

Substituting into the formula for the ratio, \( R \) gives:

\[ R = \frac{P}{\sqrt{A}} = \frac{2N \sin(\pi / N)}{\sqrt{N \cos(\pi / N) \sin(\pi / N)}} \]

Rearranging, and using the trigonometric identity \( \sin(2x) = 2 \sin x \cos x \) gives:

\[ R = 2\sqrt{N} \cdot \frac{\sin(\pi / N)}{\sqrt{1/2 \sin(2\pi / N)}} \]

Using Taylor's Theorem: \( \sin(x) = x - (1/3!)x^3 + (1/5!)x^5 - (1/7!)x^7 \ldots \)

\[ R = 2\sqrt{N} \cdot \frac{\left(\frac{\pi}{N}\right) - \left(\frac{1}{3!}\right)\left(\frac{\pi}{N}\right)^3 + \ldots}{\left(\frac{2\pi}{N}\right) - \left(\frac{1}{3!}\right)\left(\frac{2\pi}{N}\right)^3 + \ldots}^{1/2} \]

\[ = 2\sqrt{2N} \cdot \frac{\frac{\pi}{N} \left(1 - \frac{1}{3!}\right)\left(\frac{\pi}{N}\right)^2 + \ldots}{\left(\frac{2\pi}{N}\right)^{1/2} \left(1 - \frac{1}{3!}\right)\left(\frac{2\pi}{N}\right)^2 + \ldots}^{1/2} \]

Using the binomial theorem \( (1 + x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \ldots \)

the segment \( 1 - \frac{1}{3!}\left(\frac{2\pi}{N}\right)^2 + \ldots \)\(^{1/2} \) in the denominator becomes \( 1 - \frac{1}{2} \cdot \frac{1}{3!}\left(\frac{2\pi}{N}\right)^2 + \ldots \)

In the limit as \( N \) approaches infinity:
and the two series rapidly converge to 1 giving:

\[
\lim_{N \to \infty} R = \lim_{N \to \infty} 2\sqrt{N} \frac{\frac{\pi}{N}}{(\frac{2\pi}{N})^{1/2}} \frac{1}{(1 - \frac{1}{3!} \frac{2\pi}{N}^2 + \ldots)} = \lim_{N \to \infty} 2\sqrt{\pi} = 2\sqrt{\pi}
\]

which is as expected the same ratio as for the circle.

For the area perimeter relationship, equation A1.3:

\[
\log(A) = \frac{2}{D_A} \log(P) - 2 \log(Q)
\]

becomes:

\[
\log(\pi r^2) = \frac{2}{D_A} \log(2\pi r) - 2 \log(2\sqrt{\pi})
\]

Exponentiating gives:

\[
\pi r^2 = \frac{(2\pi r)^{2/D_A}}{(2\sqrt{\pi})^2}
\]

which simplifies to:

\[
4\pi^2 r^2 = (2\pi r)^{2/D_A}
\]

which is only true if \( D_A \) is equal to 1.

### D.4 Rectangle

For a rectangle with side lengths \( x \) and \( y \):

\[
P = 2x + 2y
\]

\[
A = xy
\]

\[
Q = \frac{P}{\sqrt{A}} = \frac{2x + 2y}{\sqrt{xy}}
\]

Assuming that \( x = cy \), where \( c \) is a constant

\[
Q = \frac{2cy + 2y}{\sqrt{c^2y^2}} = \frac{(2c + 2)y}{\sqrt{cy}} = \frac{2c + 2}{c}
\]
which is constant.

When \( c=1 \) we get \( Q=4 \), which, as expected, is the ratio for a square.
APPENDIX E: MULTIFRACTALS AND MULTIFRACTAL MEASURES

E.1 What is a measure?

A measure is just a way of ascribing a numerical "size" to sets in such a way that if a set is decomposed into a finite or countable number of pieces then the size of the whole is the sum of the sizes of the pieces. \( \mu(A) \) is called the measure of the set \( A \). We can think of \( \mu(A) \) as the size of \( A \) measured in some way.

\( \mu \) is called a measure on \( \mathbb{R}^n \) (where \( \mathbb{R}^n \) is n-dimensional Euclidean space) if \( \mu \) assigns a non-negative number, possibly \( \infty \) to each subset of \( \mathbb{R}^n \) such that:

i) \( \mu(\emptyset) = 0 \), i.e., the empty set has zero measure

ii) \( \mu(A) \leq \mu(B) \) if \( A \subset B \) i.e., the larger the set, the larger the measure. (\( A \) is a subset of \( B \))

iii) If \( A_1, A_2, \ldots \) is a countable (or finite) sequence of sets then:

\[
\mu \left( \bigcup_{i=1}^{\infty} A_i \right) \leq \sum_{i=1}^{\infty} \mu(A_i)
\]

i.e. that if the set is a union of a countable number of pieces (which may overlap) then the sum of the measure of the pieces is at least equal to the measure of the whole.

We think of the support of a measure as the set on which the measure is concentrated. Measures are often thought of intuitively as a mass distribution: we take a finite mass and spread it in some way across a set \( X \) to get a mass distribution on \( X \). This satisfies the conditions for a measure.

E.2 Multifractals

The concept of multifractality is tied up with the concept of fractal measures rather than fractal sets [Peitgen et al, 1992] A set is defined by its indicator function \( I(P) \), which can only take two values: \( I(P)=1 \), or \( I(P)="true" \) if the point \( P \) belongs to the set \( S \); and \( I(P)=0 \), or \( I(P)="false" \) if \( P \) does not belong to \( S \).

In cases where facts cannot be expressed so conveniently in terms of yes and no, or black and white, measures are used to embody the idea of "shades of grey". If we consider a geographical map of a continent or island, an example of a measure \( \mu \)
on such a map is "the quantity of ground water". To each subset $S$ of the map, the measure attributes a quantity $\mu(S)$, which is the amount of ground water below $S$. If we divide the map into two equally sized pieces $S_1$ and $S_2$, we will not be surprised if the water contents $\mu(S_1)$ and $\mu(S_2)$ are unequal. Similarly if $S_1$ is divided up into two equally sized pieces $S_{11}$ and $S_{12}$, their ground water contents would also differ.

This subdivision could be carried out to the size of the pores in the rocks, where some pores would be found full of water, while others would be empty. This shows that the quantity $\mu = "the amount of ground water beneath S"$ is an example of a measure which is irregular at all scales. When the irregularity is the same at all scales, or at least statistically the same, it can be said that the measure is self-similar. Self-similar sets are described as fractal, whereas self-similar measures are commonly known as multifractals.

If we apply the concept of the box counting dimension to the set $S$ (i.e. the map of the continent mentioned above) supporting a measure $\mu$ ("the amount of ground water beneath S") we cover $S$ with a collection of boxes of size $\varepsilon$. The value of the dimension $D$ is found by evaluating the number $N(\varepsilon)$ of boxes needed to cover the set and using the scaling relation $N(\varepsilon) \sim \varepsilon^{-D}$. However, this method simply finds the fractal dimension of the set $S$, the support on which the measure is based, as every point on $S$ supports a value of the measure $\mu$.

To give a quantitative description of the self-similar measure, we need to give the measure contained in each box a weight. A priori, the obvious weight would be the average density of probability in each box. However, in the case of self-similar measures, this process loses all meaning, because the density itself loses all meaning. Instead, the notion that usually leads to a density gives us instead a different and more complicated quantity,

$$\alpha = \frac{\log \mu(box)}{\log \varepsilon}$$

(E.1)

called the coarse Hölder exponent.

This is the log of the measure of the box, divided by the log of the size of the box. For a large class of self-similar measures $\alpha$ is restricted to an interval $[\alpha_{\text{min}}, \alpha_{\text{max}}]$ where $0 < \alpha_{\text{min}} < \alpha_{\text{max}} < \infty$. Then we draw the frequency distribution of $\alpha$ as follows.

For each value of $\alpha$, we evaluate the number $N_\varepsilon(\alpha)$ of boxes of size $\varepsilon$ having a
coarse Hölder exponent equal to $\alpha$. If we suppose that a box of size $\varepsilon$ has been selected at random among boxes whose total number is proportional to $\varepsilon^{-E}$. The probability of hitting the value $\alpha$ of the coarse Hölder exponent is $p_\varepsilon(\alpha) = N_\varepsilon(\alpha)/\varepsilon^{-E}$. The first inclination is to draw the distribution of this probability, but this is not useful. In the case of interest to us the distribution no longer tends to a limit as $\varepsilon \to 0$, hence an intrinsic characteristic has to be found elsewhere. Instead we take the weighted logs and consider the functions

$$f_\varepsilon(\alpha) = \frac{-\log N_\varepsilon(\alpha)}{\log \varepsilon} \quad (E.2)$$

or

$$C_\varepsilon(\alpha) = \frac{-\log p_\varepsilon(\alpha)}{\log \varepsilon} \quad (E.3)$$

As $\varepsilon \to 0$ both $f_\varepsilon(\alpha)$ and $C_\varepsilon(\alpha)$ tend to well defined limits, $f(\alpha)$ and $C(\alpha)$. Of the pair $f(\alpha)$ is more widely known, whereas $C(\alpha)$ is more widely applicable. When $f(\alpha)$ exists we have: $C(\alpha) = f(\alpha) - E$.

The definition of $f(\alpha)$ means that for each $\alpha$, the number of boxes increases for decreasing $\varepsilon$ as $N_\varepsilon(\alpha) \sim \varepsilon^{-f(\alpha)}$. The exponent $f(\alpha)$ is a continuous function of $\alpha$ and in the simplest cases is usually shaped like the symbol $\cap$, usually leaning to one side. The values of $f(\alpha)$ could be loosely interpreted as a fractal dimension of the subsets of boxes of size $\varepsilon$ having coarse Hölder exponent $\alpha$ in the limit $\varepsilon \to 0$. As $\varepsilon \to 0$, the number of subsets increases to infinity, each characterized by its own $\alpha$ and a fractal dimension $f(\alpha)$, giving one of the many reasons for the term multifractal.

E.3 References

APPENDIX F: ESTIMATION OF RAIN RATE FROM METEOROLOGICAL RADAR REFLECTIVITY MEASUREMENTS.

Weather radars are incapable of directly measuring surface rainfall rate. It is necessary to relate the rainfall rate to one or more of the measured radar parameters. With conventional (reflectivity only) measurements, relations used are usually of the form:

\[ Z = aR^b \]  

where \( Z \) is the reflectivity factor (dBZ), \( R \) is the rainrate (mm/hr) and \( a \) and \( b \) are constants. Such relations are by no means unique, there have been over 300 different relationships between \( Z \) and \( R \) reported in the literature since work on this subject began [Goddard et al, 1994]. The above equation is dependent on the drop size distribution to the extent that it cannot be universally applied to all conditions.

Figure F.1 shows the radar reflectivity factor, \( Z \) plotted against the rainrate \( R \) for different types of rain.

![Figure F.1. Radar reflectivity factor, \( Z \) plotted against the rainrate \( R \) for different types of rain. [Skolnic, 1990]](image-url)
Table F.1 gives the $a$ and $b$ coefficients for the different rain types [Skolnic, 1990]:

<table>
<thead>
<tr>
<th>RAIN TYPE</th>
<th>$a$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stratiform</td>
<td>200</td>
<td>1.6</td>
</tr>
<tr>
<td>Orographic</td>
<td>31</td>
<td>1.71</td>
</tr>
<tr>
<td>Thunderstorm/convective</td>
<td>486</td>
<td>1.37</td>
</tr>
<tr>
<td>Snow</td>
<td>2000</td>
<td>2</td>
</tr>
</tbody>
</table>

Table F.1: $a$ and $b$ coefficients for radar reflectivity according to rain type [Skolnic, 1990].

Orographic rain is precipitation that is influenced or induced by hills or mountains.

F.1 Drop size distribution

The drop size distribution $N(D)$ is, by definition, the average number of raindrops per unit volume having (equivalent) diameter in the range $D$ to $D + dD$. The Marshall-Palmer drop size distribution is exponential, and was based on an empirical study of drop radii $a$ (drop diameter $D = 2a$) and rainfall rate $R$ in mm/hr, given as:

$$N(D) dD = N_0 \exp(-\Lambda D) dD \quad \text{(F.2)}$$

where $N_0 = 80000 \, (m^{-3} \, cm^{-1})$ and $\Lambda = 41 \times R^{-0.21} \, (cm^{-1})$

Ulbrich [1983] proposed a new model of which the Marshall-Palmer model was a specific case, working from observations of variations in drop size distributions performed by Joss et al, [1968] and others. This new model is as follows:

$$N(D) dD = N_0 D^\mu \exp(-\Lambda D) dD \quad \text{(F.3)}$$

where

$$N_0 = 60000 \exp(3.2\mu) \, (m^{-3} \, cm^{-1-\mu}) \quad \text{(F.4)}$$

The unitless parameter $\mu$ corresponds to different types of rainfall. It is generally accepted that $\mu < 0$ corresponds to orographic rain, small values of $\mu$ (0-1)
correspond to thunderstorm rain and larger values of $\mu$ (2-5) correspond to stratiform rain and showers.

$$\Lambda D_0 = 3.67 + \mu$$  \hspace{1cm} (F.5)

where $D_0$ is the median drop diameter, and can be defined as:

$$\int_0^{D_0} D^3 N(D) dD = \int_{D_0}^{D_{max}} D^3 N(D) dD$$  \hspace{1cm} (F.6)

$D_0$ is such that all drops with a diameter $\leq D_0$ contribute to exactly half of the total liquid water content.

Hence:

$$N(D)dD = 60000 \exp(3.2\mu) D^\mu \exp(-(3.67 + \mu) D / D_0) dD$$  \hspace{1cm} (F.7)

It should be noted that though the exponential distribution is generally accepted, it is not applicable under some conditions, especially those where wind shear is an important consideration. Battan [1977] has shown that, beneath the melting layer, vertical wind shear sorts drops into size order so that they fall in a bimodal distribution. Wind shear is also responsible for rain drops falling with their axis angled from the vertical. This is known as canting, and measurement has shown that generally canting angle is $< 10^\circ$ [McCormick and Hendry, 1974].

**F.2 Weather radar equation**

The average received power $P_r$ from a number of individual scatterers, also known as the radar equation is [Skolnik, 1990]:

$$\overline{P_r} = \frac{P g^2 \lambda^2}{L(4\pi)^3 r^4} \sum_{i=1}^{N} \sigma_{b(i)}$$  \hspace{1cm} (F.8)

where:

- $P_r$ is the transmitted power,
- $g$ is the antenna gain,
- $\lambda$ is the radar wavelength,
- $\sigma_b$ is the back scattering cross section,
- $r$ is the range,
$L$ is the total systems loss, including the attenuation due to atmospheric gases, radome attenuation etc.

For a weather radar, where we are observing a number of scatterers in the volume illuminated by the antenna, which is assumed to have a Gaussian-like radiation pattern, the radar equation can be written:

$$ \frac{\bar{P}_r}{g} = \frac{P_g^2 \lambda^2 \theta^2 h}{512(2 \ln 2) \pi^2 r^2 \eta} \quad (F.9) $$

where:

$h$ is the pulse length

$\eta$ is the reflectivity

$\eta$ is given by:

$$ \eta = \frac{1}{\Delta V} \sum_{\Delta V} \sigma_{b(i)} \quad (F.10) $$

where $\Delta V$ is the volume illuminated by the transmitted pulse.

If we assume Rayleigh scattering:

$$ \eta = \frac{\pi^5}{\lambda^6} |K|^2 Z \quad (F.11) $$

where:

$Z$ is the reflectivity factor

$$ Z = \frac{1}{\Delta V} \sum_{\Delta V} D_i^6 \quad (F.12) $$

and

$|K|^2$ is the dielectric factor related to the complex refractive index $n$ (or equivalently the complex relative permittivity $\varepsilon_r$)

$$ |K|^2 = \frac{\varepsilon_r - 1}{\varepsilon_r + 2} = \frac{n^2 - 1}{n^2 + 2} \quad (F.13) $$

When Rayleigh scattering does not apply, the reflectivity factor $Z$ is replaced by the effective reflectivity factor $Z_e$

$$ Z_e = \frac{\lambda^4}{\pi^5 |K|^2 \eta} \quad (F.14) $$

The units of $Z$ and $Z_e$ are $mm^6 m^{-3}$. 

200
Reflectivity can take a wide range of values so it is usually expressed relative to $1 \text{mm}^6 \text{m}^{-3}$ in dB, also called dBZ.

**F.3 Rainfall measurement with radar**

If we assume that all raindrops have the same diameter $D$, the rainfall rate $R$ (i.e. the rate at which the depth of water in a rain gauge of constant cross section increases with time) is the product of the total volume of water in a unit volume of air and the terminal velocity $V_t(D)$ of a raindrop:

$$R = N \frac{\pi}{6} D^3 V_t(D)$$  \hspace{1cm} (F.15)

where $N$ is the number of drops per unit volume of air.

It should be noted that $V_t$ is the velocity relative to the ground as other theoretical and empirical expressions are for terminal velocities relative to the air through which the droplet falls.

In principle $R$ is determined by the amount of water in the air and the rate at which it is falling to the ground. These two quantities are correlated to a certain extent, $V_t$ increasing with increasing $D$, as does the amount of water, for a given $N$. However, the correlation has not been established precisely, so it is best to assume two independent measurements.

Raindrop diameters range from less than 1 mm to 5 mm, which is less than the wavelength of weather radars (usually 3, 5, or 10 cm).

The backscattering cross-section of a sphere is proportional to the sixth power of its radius in the Rayleigh approximation. We can define the radar reflectivity factor:

$$Z = N D^6$$  \hspace{1cm} (F.16)

which determines the amount of power scattered by raindrops in a unit volume at a given distance back to the receiving antenna.

When a raindrop ceases to accelerate, its weight is balanced by the drag force, assuming that its buoyancy is negligible.

$$\rho_w \frac{\pi}{6} D^3 g = \frac{1}{2} \rho_a V_t^2 C_d \frac{\pi}{4} D^2$$  \hspace{1cm} (F.17)

where:

- $\rho_a$ and $\rho_w$ are the densities of air and water respectively,
- $g$ is the acceleration due to gravity and
\( C_d \) is the drag coefficient, which is nearly independent of the diameter of the drop. (The terminal velocity of the drop \( V_t \) is approximately proportional to the square root of the drop diameter, \( D \).)

Combining results, we get:

\[
Z = KN^{-0.71} R^{1.71}
\]

where \( K \) is a constant.

If we assume that raindrop diameters are distributed according to a continuous function \( N(D) \):

\[
Z = \int N(D)D^6 dD
\]

and

\[
R = \frac{\pi}{6} \int N(D)V_t(D)D^3 dD
\]

In general, most raindrops, especially the bigger ones, are no spherical, but oblate. Moreover drops can oscillate between oblate and prolate, which can make estimating the rain rate using directly measured radar reflectivity that much harder.

Drop shapes and drop size distributions can be investigated through the use of differential reflectivity \( Z_{dr} \) [Goddard, et al, 1994], where:

\[
Z_{dr} \propto D_0
\]

where \( Z_{dr} \) is the differential reflectivity in dB and \( D_0 \) is the average drop diameter in mm.

\[
Z_{dr} 10\log_{10}(Z_H/Z_V)
\]

where \( Z_H \) and \( Z_V \) are the horizontally and vertically polarised returns.

The general assumption is that rain is homogenous in a radar volume, however this is challenged by Lovejoy and Schertzer [1990]. They calculated the correlation dimension of raindrops which had fallen on large sheets of blotting paper, coming to the conclusion that the raindrops were fractally distributed, with a scaling that requires a correction to the weather radar equation.
F.4 References


APPENDIX G: THE SIMILARITY DIMENSION

The concept of similarity dimension is closely associated with that of scaling. To start off with, we'll look at the similarity dimension of some normal Euclidean objects, i.e. a line, a plane, and a volume, all of which have been divided up into self-similar sub-lengths, sub-areas and sub-volumes of side length \( a \). For simplicity, it is assumed that the length, \( L \), area, \( A \) and volume, \( V \) area all equal to unity.

![Diagram showing different objects with their Euclidean and topological dimensions](image)

Figure 2.5 Common objects with their Euclidean and topological dimensions

If the line is divided into \( N \) small self-similar parts, each \( \varepsilon \), in length, then \( \varepsilon \), is the scaling ratio as \( \varepsilon / L = \varepsilon \) as \( L=1 \). Therefore \( L = N \varepsilon = 1 \), i.e. unit line is composed of \( N \) self-similar parts scaled by \( \varepsilon = 1 / N \). For the area, divided into self-similar parts of area \( \varepsilon^2 \) then \( A = N \varepsilon^2 = 1 \) and the unit surface is composed of \( N \) self-similar parts scaled by \( \varepsilon = 1 / N^{1/2} \). Similarly, \( V = N \varepsilon^3 = 1 \), i.e. unit solid is \( N \) self-similar parts scaled by \( \varepsilon = 1 / N^{1/3} \).

Examining these equations we can see that the exponent of \( \varepsilon \), in each case is a measure of the similarity dimension of the object, and we have in general \( N \varepsilon^{D_r} = 1 \).
Hence: $D_s = \frac{\log(N)}{\log(1/\varepsilon)}$ (where the subscript S denotes the similarity dimension).

Applying this definition to the Koch curve (chapter two, section 2.2.1) we see that at each scale there are four subsections making up the curve, each one a $1/3$ reduction of the original curve. Thus $N=4$, $\varepsilon = 1/3$, and the similarity dimension

$$D_s = \frac{\log(4)}{\log(3)} = 1.2618...$$

This shows that the Koch curve has a dimension greater than the unit line ($D_E = D_T = 1$) and less than that of the unit area ($D_E = D_T = 2$). The Euclidean dimension of the Koch curve is 2 as we need two co-ordinate directions to specify all points on it, whereas the topological dimension is 1 as we can cover it in successively smaller discs intersecting in pairs.
APPENDIX H: THE EARTH-SPACE EXPERIMENTAL HISTORY
OF THE RADIO COMMUNICATIONS RESEARCH UNIT

The Radio Communications Research Unit at the Rutherford Appleton Laboratory started radiometric observations at the Appleton Laboratory in Slough through solar observations which were used to yield tropospheric rain fade data. Fading statistics for daylight hours were derived for 12, 19, 37 and 71 GHz, well before any satellite beacons were available. Later, from 1975 to 1981 those measurements were extended to include monitoring of beacon transmissions at 30 GHz from the ATS-6 and 12 GHz from the OTS and SIRIO satellites. Similarly, measurements were carried out using the Olympus beacons until 1993, the ITALSAT beacons at 18.7, 39.6 and 49.5 GHz until 2002, and more recently, with the GBS beacons carried on UFO-9 at 20.7 GHz.

ITALSAT F1 (owned and operated by the Italian Space Agency, ASI) was launched in January 1991 and was placed in a geostationary orbit at 13.2 degrees east. Over the course of its operational lifetime it was observed from the RCRU ground station at Sparsholt, Hampshire, at an elevation angle of \sim 30 degrees above the horizon.

The receivers for the 49.5 and 39.6 GHz beacons began operations in April 1997, while the 18.7 GHz was observed from December 1995, until the north - south movement of the satellite reduced the quality of the data significantly in 1998. However, in August of 1999 the 18.7 receiver was switch to monitor the 18.7 GHz beacon on ITALSAT F2, which is located close to ITALSAT F1. ITALSAT F1 ceased operations in January 2001, while ITALSAT F2 ceased operations in July 2002. All of the data from the receivers and meteorological instruments was sampled and stored once a second. Further experimental details for the ITALSAT experiment can be found in [Ventouras et al 2000 (updated 2003)].

These observations have provided the RCRU with a continuous database of measurements spanning more than five years, with which it has been possible to test an atmospheric model applicable to EHF. This model includes the losses caused by cloud at the beacon frequencies, as well as the losses caused by rain and the absorption by atmospheric gases. The RCRU is also carrying out focused studies on
the fade characteristics experienced by the beacons, and is addressing the development of fade mitigation techniques for low margin satellite systems at Ka and V bands.

H.1 References

APPENDIX I: SOFTWARE

I.1 Area perimeter dimension

Functions:

- load
- contour
- polyarea
- loglog
- fitline

Structure:

Figure I.1 shows a block diagram of the structure of the program used to calculate the area perimeter dimension. The theory and mathematics behind the program is described in chapter two, section 2.2.1.
For all the raster files in the event:
Load each file

Create contour array at a set level

Separate each contour line out individually. For each contour line will have 2 vector arrays, xdata, ydata

Use polyarea to calculate the area enclosed by the contour line
Save area value to an array

Use
\[ length = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]
to calculate the length of each line segment in the contour and sum them to get the contour perimeter
Save perimeter value to an array

Loop until no contours left in contour array

Loop until no files left in event.

Plot area values against perimeter values using loglog

Use fitline to calculate slope and errors

Area perimeter dimension = \( 2/slope \)

Figure I.1 Schematic diagram of the structure of the program used to calculate the area perimeter dimension of contour lines enclosing an area greater than or equal to a set rain rate threshold.

Method of validation:

Test data arrays containing rectangles at different sizes were used as inputs. The resulting output gave an area perimeter dimension of 1, as expected. Similarly, test data arrays containing Koch snowflakes at different sizes (and hence resolutions) were used. The area perimeter dimension calculated for the snowflakes was \( 1.27 \pm 0.01 \), comparing well with the similarity dimension of the snowflake = 1.26.
1.2 Box counting dimension

Functions:

- load
- contour
- box_count
- loglog
- fitline
- fopen
- fprintf
- fclose

Structure:

Figure 1.2 shows a schematic diagram of the structure of the program used to calculate the box counting dimension. Further information on the theory and equations used can be found in chapter 2, section 2.2.2.
For all the raster files in the event: 
Load each file

Create contour array at a set level

Separate each contour line out individually. For each contour line will have 2 vector arrays, xdata, ydata

Use polyarea to calculate the area enclosed by the contour line

Save area value to an array

Use box_count function to calculate the box counting dimension for each contour line

Save number of vertices in contour line to an array

Save box counting dimension value to an array

Save contour level (for each contour line) to an array

Loop until no contours left in contour array

Write a file to save the number of vertices, contour level, box counting dimension and area enclosed by the contour, for each contour line.

Loop until no files left in event.

Figure I.2 Schematic diagram of the structure of the program used to calculate the box counting dimension of contour lines enclosing an area greater than or equal to a set rain rate threshold.

Method of validation:

The same test arrays were used to test this software as were used to test the area perimeter dimension software. The resulting dimension outputs were as expected for the shapes used.
1.3 Area distribution analysis (Korckak dimension)

**Functions:**
- `read_contours_area`
- `sort`
- `fitline`

**Structure:**

Further details on the theory behind the area distribution analysis can be found in chapter two, section 2.2.3. A schematic diagram of the program structure is shown in figure I.3.

![Schematic diagram of the program structure](image)

Figure I.3 Schematic diagram of the structure of the program used to analyse the radar data in terms of the distribution of the area enclosed by the contour lines (Korckak dimension)
Method of validation:

The same test arrays were used to test this software as were used to test the area perimeter dimension software. The resulting dimension outputs were as expected for the shapes used.

1.4 Two-dimensional FFT

Functions:

- load
- fft2
- fftshift
- contour
- fitline

Structure:

Figure 1.4 shows a schematic diagram of the structure of the program used to calculate the two-dimensional FFT and spectral density function exponent described in chapter two, section 2.2.4 and in chapter three, section 3.1.7.
For all the raster files in the event:
Load each file

Save each file as (x,y) in a 3D array of values (x,y,t)
where t is the raster number

For 2D FFT (x,y) average 3D array across t to get a 2D array
For 2D FFT (x,t) average 3D array across y to get a 2D array
For 2D FFT (y,t) average 3D array across x to get a 2D array

Use fft2 to calculate the 2D FFT for the averaged array

Use ffsift to shift the origin to the centre of the figure

Use contour to plot the figure

Calculate average radial spectral density

Use fitline to calculate the slope of the average radial spectral density

Figure I.4 Schematic diagram of the structure of the program used to perform two-dimensional Fourier analysis on the radar data.

Method of validation:
Results were compared with the results from the same input data set used in an IDL program written by Dr. Kevin Paulson for his analysis published in Radio Science [Paulson, 2002].

I.5 $K(q)$ function (multifractal analysis)

Functions:
- load
- mean
- fitline
Details of the algorithm and mathematical theory behind this program can be found in chapters two and three, sections 2.4.4 and 3.3. Figure I.5 shows a schematic diagram of the program's structure.

Figure I.5 Schematic diagram of the structure of the program used to perform multifractal (moment scaling function) analysis on the radar data.
Method of validation:

Our results were compared directly with other results published in the literature [Olsson and Niemczynowicz, 1996] using exactly the same method of analysis and were found to be comparable.

1.6 Correlation dimension
Functions:
- textread
- loglog
- fitline
- mean

Structure:
The structure of the program used to calculate the correlation dimension of the Brue rain gauge network is given in figure 1.6. The theory behind the program is given in chapter two, section 2.2.5.
Read in the text file with the locations of all the rain gauges

For each rain gauge, calculate the distance between it and all the others

Save distances to an array.

For each gauge count the number of gauges $C(g)$ within a distance $g$ of them

Average $C(g)$ across all the gauges

plot $<C(g)>$ as a function of $g$ on a loglog plot

For a given range of distances the plot will be a straight line. Use fitline to calculate slope and errors

Correlation dimension = slope

Figure I.6 Schematic diagram of the structure of the program used to calculate the correlation dimension of the Brue rain gauge network.

Method of validation:

The program was tested with an array of 94*94 evenly spaced points, and returned a correlation dimension of 2±0.01 as expected.

1.7 Random additions algorithm for varying lacunarity

Functions:
- randn
- meshgrid
- interp2
- save

Structure:

The structure of this program is described in chapter five, figure 5.7.
Method of validation:

The "log rain rate" fields produced by the simulator were input into the area perimeter and two-dimensional FFT programs. The resulting spectral density function exponent value and area perimeter dimension were as expected.

1.8 Conversion from a rain field to attenuation along a path

Functions:
- load
- sum
- fopen
- fprintf
- fclose

Structure:

The procedure for converting from a rain field to attenuation along the path is given in a step by step fashion in chapter seven, section 7.3.1.1. The assumptions made to convert from the simulated rain fields to simulated attenuation time series are given in section 7.3.1.2.

Method of validation:

Radar derived attenuation from the Met Office's NIMROD database was compared with attenuation time series measured at Sparsholt and Chilbolton in the south of England. The radar derived attenuation was largely in agreement with the measured attenuation, though differences were observed due to the limited resolution of the radar data.

1.9 Switching algorithm

Functions:
- textread
- plot

Structure:

The structure of this program is described in chapter seven, figure 7.1

Method of validation:

The switching algorithm program was a software model of a simplified switching algorithm. Details of switching algorithms for communications systems
currently in use are not available, hence this program was not validated using external comparisons.

I.10 Function descriptions

I.10.1 MATLAB defined functions:

These functions are listed in alphabetical order.

CONTOUR Contour plot.

C=CONTOUR(Z,V) computes LENGTH(V) contour lines at the values specified in vector V. The contour matrix C is a two row matrix of contour lines. Each contiguous drawing segment contains the value of the contour, the number of (x,y) drawing pairs, and the pairs themselves. The segments are appended end-to-end as:

C = [level1 x1 x2 x3 ... level2 x2 x2 x3 ...;
    pairs1 y1 y2 y3 ... pairs2 y2 y2 y3 ...]

FCLOSE Close file.

ST = FCLOSE(FID) closes the file with file identifier FID, which is an integer obtained from an earlier FOPEN.

FFT2 Two-dimensional discrete Fourier Transform.

FFT2(X) returns the two-dimensional Fourier transform of matrix X. If X is a vector, the result will have the same orientation.

FFTSHIFT Shift zero-frequency component to centre of spectrum.

For vectors, FFTSHIFT(X) swaps the left and right halves of X. For matrices, FFTSHIFT(X) swaps the first and third quadrants and the second and fourth quadrants. For N-D arrays, FFTSHIFT(X) swaps "half-spaces" of X along each dimension.

FITLINE Fits a straight line through data points

[P,PERR] = FITLINE(X,Y) Fitting straight line through data points specified by vectors X and Y. Simplified and slightly changed version of POLYFIT Returns parameters of the fit and their estimated errors instead of Cholesky factor matrix.

FOPEN Open file.
FID = FOPEN(FILENAME) opens the file FILENAME for read access.

PRINTF Write formatted data to file.

COUNT =PRINTF(FID,FORMAT,A,...) formats the data in the real part of matrix A (and in any additional matrix arguments), under control of the specified FORMAT string, and writes it to the file associated with file identifier FID. COUNT is the number of bytes successfully written. FID is an integer file identifier obtained from FOPEN.

INTERP2 2-D interpolation (table lookup).

ZI = INTERP2(X,Y,Z,XI,YI) interpolates to find ZI, the values of the underlying 2-D function Z at the points in matrices XI and YI. Matrices X and Y specify the points at which the data Z is given. Out of range values are returned as NaN.

LOGLOG Log-log scale plot.

LOGLOG (X,Y) plots vector Y versus vector X. with logarithmic scales used for both the X- and Y- axes.

MEAN Average or mean value.

For vectors, MEAN(X) is the mean value of the elements in X. For matrices, MEAN(X) is a row vector containing the mean value of each column. For N-D arrays, MEAN(X) is the mean value of the elements along the first non-singleton dimension of X.

MESHGRID X and Y arrays for 3-D plots.

[X,Y] = MESHGRID(x,y) transforms the domain specified by vectors x and y into arrays X and Y that can be used for the evaluation of functions of two variables and 3-D surface plots.

PLOT Linear plot.

PLOT(X,Y) plots vector Y versus vector X.

POLYAREA Area of polygon.
POLYAREA(X,Y) returns the area of the polygon specified by the vertices in the vectors X and Y.

RANDN Normally distributed random numbers.

RANDN(N) is an N-by-N matrix with random entries, chosen from a normal distribution with mean zero, variance one and standard deviation one.

SORT Sort in ascending order.

For vectors, SORT(X) sorts the elements of X in ascending order.

SUM Sum of elements.

For vectors, SUM(X) is the sum of the elements of X. For matrices, SUM(X) is a row vector with the sum over each column. For N-D arrays, SUM(X) operates along the first non-singleton dimension.

TEXTREAD Read formatted data from text file.

[A,B,C,...] = TEXTREAD('FILENAME','FORMAT',N,param,value, ...) reads data from the fileFILENAME into the variables A,B,C, etc. The type of each return argument is given by the FORMAT string. The number of return arguments must match the number of conversion specifiers in the FORMAT string.

I.10.2 Author defined functions

BOX_COUNT

[no_boxes, box_size]=box_count(con_array, con_value, height, width)
calculates the box counting dimension for a given array of values

inputs: con_array - array containing the contour lines
con_value - value of each contour
height - height of the array i.e. number of rows
width - width of the array i.e. number of columns

outputs: no_boxes - number of boxes needed to cover contour
box_size - width of the box in cells

This function takes the con_array, (for a schematic example of the contour array see figure 3.13), and counts the number of pixels that have the given con_value
in them. The function then overlays a grid of varying box sizes on the con_array, and again counts the number of boxes with the con_value in them. (A box is defined as having the con_value in it, if any of the pixels in the box have that value.)

The box counting dimension is calculated from the slope of the line given by the number of boxes plotted against the box size on a loglog graph.

**READ CONTOURS AREA**

[num_vert, contour_value, dimension, area] = read_contours_area(file_name)

Reads in a file (identified as file_name) created by the box counting dimension program and returns arrays with the following data:

- num_vert – number of vertices for a given contour line
- contour_value – value of the level that the contour line was drawn at
- dimension – box counting dimension of the contour line
- area – area enclosed by the contour line

**I.11 References**


APPENDIX J: PUBLICATIONS AND INTERNAL REPORTS

Publications marked with an asterisk (*) have been reproduced at the back of this appendix.

I.1 Publications most relevant to the thesis


I.2 Contributions to COST 280


I.3 Other publications

These papers are mainly concerned with measurements and statistics derived from the RCRU’s Earth-space ITALSAT propagation experiment and so have relevance to the case study presented in chapters six and seven.


I.4 Internal reports


16. Callaghan, S.A, Vilar, E., “Considerations concerning the investigation and use of rain field fractals, in particular with applications to communications engineering.” Internal report 03/12 for the University of Portsmouth, May 2003

17. Callaghan, S.A, Vilar, E., “Dynamic modelling of rainfall rate over wide areas for adaptive satellite communications, Mphil to PhD transfer report” Internal report 02/10 for the University of Portsmouth, July 2002


I.5 Contractual reports


Analysis of the Fractal Dimension of Rain Rate Contours with Reference to Wide Area Coverage of Satellite Communications

Sarah Callaghan and Enric Vilar

Treating rain fields as a fractal surface raises interesting considerations for the characterization and calculation of the fractal dimension of the rain field. Data recorded at the Chilbolton Observatory in the south of England suggests that the shape of contour lines enclosing areas of rain with equal or greater than a set rain rate have a fractal dimension -1.2. It is shown that the accuracy of this fractal dimension is dependent on the size of the area enclosed by the contour line, due to the resolution of the radar, but is independent of the rain rate threshold. The background and application implications for communications systems at high frequencies are outlined.

Introduction and Background

At this time, the demand for accurate and reliable statistics and models for systems operating at Ka and V bands is very high, due to the increased need for more bandwidth demanded by multimedia (such as video conferencing), Internet applications and other high data rate transmissions. As the electromagnetic spectrum becomes increasingly more congested at lower frequencies it is necessary to turn to Ka and V bands to supply the necessary bandwidth. Satellite and terrestrial systems are not the only systems to be considered for use at these frequencies, also under active consideration are other types of high altitude platforms (HAPs) such as stratospheric balloons. These would deliver broadband services to major urban areas, with the added advantages of being less expensive and more manoeuvrable.

Rain is a major cause of attenuation on links operating above 10GHz, and one which is unlikely to be compensated for by available fade margin alone. It is for this reason that there is a need to turn to so-called fade mitigation techniques in order to ensure a sufficiently high availability for the service in question.

Many of the proposed fade mitigation techniques rely heavily on knowledge of the temporal and spatial characteristics of rain. For example, in satellite communications, site diversity employs two or more ground stations receiving the same satellite signal with a separation distance usually greater than the diameter of the rain cells. The sites in a properly configured arrangement encounter intense rainfall at different times, and switching to the site experiencing the least fading improves system performance considerably. A similar effect can be achieved using route diversity on a terrestrial network as intense rain cells that cause large amounts of attenuation on radio paths often have horizontal dimensions of no more than a few kilometres. Another fade mitigation technique, called adaptive power control, adjusts the transmit power of the link to compensate for the fading occurring along the link. To implement this technique correctly to minimise both the time that the signal is completely lost and the interference with other links in the same geographical area, it is necessary to have, at least, a good understanding of the distribution of fades in time, if not an accurate fade prediction method.

An important characteristic of rain fields is their spatial and temporal inhomogeneity. A number of studies have developed rain cell models from radar measurements but these are statistical in nature and do not enable the construction of typical two dimensional rain-rate fields [1]. Other models have disadvantages in that they only deal with the spatial variation of the rain-rate within a rain cell [2], or do not take into account the full range of rain rates that are significant for frequencies up to, and beyond Ka band (27 GHz to 40 GHz) [3]. These models assume regular shapes for the rain cells, such as ellipses, or Gaussian functions of position centred on the area of maximum rain rate.

Other studies [4, 5], suggest that fractal methods may be of use in characterising the shapes of rain cells and clouds. The use of fractal methods to study rain has become increasingly popular in fields such as meteorology and hydrology, but has yet to become prevalent in the radio propagation area. The concept of the fractal dimension gives new information on the shape of rain areas, which, as can be seen in figure 1, are not easily approximated to regular shapes such as ellipses.
In this paper certain methods of dealing with and analysing the fractal dimension are discussed. The authors would like to recommend the books by Gleick [6] (for a good general, non-rigorous introduction to the subject), Addison [7] and Peitgen et al [8] (for a more mathematically complete treatment) to any reader wishing to have a more complete understanding of the field of fractals and chaotic dynamics. Russ [9] is also recommended as it deals specifically with fractal surfaces and theories that are employed in this paper.

There have been a large number of papers published in the literature [e.g.: 10,11,12], which make the assumption that the spatial and temporal accumulations of rain can be considered to be a multifractal measure, instead of a monofractal set. However, this concept of multifractality adds complexity, which may not be necessary when using fractals in a communications context. The following analysis is based on the assumption that a rain field can be considered a fractal surface rather than a multifractal measure.

Description of Data and Methods of Analysis

In this work we present an analysis of the spatial structure of rain fields as recorded by the Chilbolton Advanced Meteorological Radar (CAMRa) based in the south of England. The radar data was recorded as a plan position indication (PPI) at an elevation angle of 1.2° to avoid ground clutter. The angular resolution was 0.3° and the range resolution was 300m. The data used in this analysis was obtained using an azimuth scan of 80° looking south-east from Chilbolton over Surrey and Hampshire. These radar data were interpolated onto a square Cartesian grid, 56.2 km along each edge and with a grid spacing of 300m. The radar reflectivities Z were transformed into rain rates. Each grid contains 188² data points covering more than 3100 km². The maximum rain rate observed during all the events was 60 mm/hr.

A typical grid is plotted in figure 1, showing contour lines of 1 and 10 mm/hr. The fractal behaviour of the contours lines is determined using two methods, the area-perimeter method and the box counting method. The contouring algorithm was used in order to separate out the rain cells to study individually, and the same contouring algorithm was used on all the rasters and events studied.

If we consider rain to be a fractal surface, with two spatial dimensions, x and y and the height of the surface, z, being given by the rain rate, then there are a number of fractal methods that can be used to determine the scaling and fractal dimension of the rain field. However it should be noted that other published studies [10,11,12] instead consider the rain field to be a multifractal measure, complicating the analysis greatly.

In general, the fractal dimension D characterises any self-similar system; if the linear dimension of a fractal observable is changed by a scale factor f, then, for any value of f the values of the fractal observable will be changed by the factor $f^D$. For surfaces, the value of the surface dimension lies in the range $2 \leq D_s \leq 3$. A smooth surface has $D_s = 2$. Similarly, for a contour line $1 \leq D_L \leq 2$, and $D_L = 1$ for smooth lines. The more twisted and "wriggily" the contour line is, the higher the value of $D_L$. If pathological cases are disregarded [13] a planar section of a fractal surface has

$$D_L = D_s - 1.$$ (1)

It should be noted that the use of this relationship to reduce the dimension by 1 is only applicable to exactly self-similar objects. Most surfaces of interest, including rain, are not self-similar but self-affine. This means that the vertical direction (i.e. rain rate) is not the same as the lateral directions (i.e. the two spatial directions) parallel to the nominal surface and the scaling of magnitude with dimension is different vertically as
compared with laterally. However, it is correct to reduce a self affine surface using a section parallel to the normal surface orientation, as the boundary lines produces are self-similar, hence \( D_L = D_S - 1 \). Using any other orientation for the section plane does not produce this effect.

**Area perimeter method**

The area, \( A \) within a contour depends fractally on the contour length or perimeter \( P \) following a law of the type:

\[
A \propto P^E,
\]

where the exponent \( E = 2/D_A \) and \( D_A \) is the fractal dimension determined by the area-perimeter method.

Figures 2a and 2b show the areas of various rain rate contours, plotted on a log-log grid, against their perimeters for fixed values of 1 and 10 mm/hr respectively. Similar results were found for other rain rate thresholds and different rain events.

The results obtained using the area perimeter method on six different rain events and over 1200 different data grids show that the fractal dimension of the rain rate contours is close to 1.2, with no slope discontinuity in the plots which would otherwise indicate the presence of a characteristic length.

**Box counting dimension**

If one uses the box counting algorithm, the fractal dimension \( D_B \) is given by the limit:

\[
D_B = \lim_{\delta \to 0} \frac{\log N(\delta)}{\log(1/\delta)},
\]

where \( \delta \) is an arbitrary length unit selected such that \( N(\delta) \) is then the number of boxes (or cubes) of side length \( \delta \) required to completely cover the curve under study. By plotting \( \log N(\delta) \) against \( \log(1/\delta) \) the average slope of the plot gives an estimation of the box counting dimension.

The box counting dimension was calculated individually for each contour. Figures 3a and 3b show the box counting dimension values for each contour in one of the rain events, against the length of each of the contours for rainfall rates of 1 mm/hr and 10 mm/hr respectively. Again, similar results were observed for different rainfall thresholds and events.

For contours with lengths less than 20 pixels long the box counting dimension could not be determined accurately. This is due to the inevitably limited resolution of the radar and the minimum pixel size in the raw data grids. For smaller perimeter lengths the contouring algorithm approximates the contour lines by irregular polygons.
which have a fractal dimension of 1) or by points (which have a fractal dimension of 0). However, for lengths above this threshold the fractal dimension does not vary with contour length, though there is a spread in the values around a value of ~1.2. It can also be seen that at the higher rain rate threshold (10 mm/hr), in general the perimeter lengths of the contour lines are smaller, making it harder to accurately determine the box counting dimension. In a temperate maritime climate, such as the south of England, where the data was recorded, it is rare to get areas of rain with rain rate greater than ~20 mm/hr covering areas greater than a few km². However, when large rain cells at those rain rates do occur, their contours show the same behaviour as those at lower rainfall thresholds.

A typical method of analysis for this type of data is to calculate the power spectral density function of the data, looking at it in a purely spatial and a spatio-temporal state. Calculation of the fractal dimension also can reveal some interesting and useful information about the structure of the rain field.

If each near-horizontal radar scan is treated as an instantaneous snapshot of the rain rate field then the spatial spectral density may be calculated via 2-D Fourier transform. Figure 4a illustrates the two-sided spectral density of spatial rain rate variation, averaged over the 230 scans recorded for the event that occurred on the 1st May 2001. The near circular contours are consistent with a rotationally symmetric, and hence quadrant symmetric, spectral density and spatial autocorrelation.

Figures 4b and 4c show the corresponding spatio-temporal isocorrelation contours in the Fourier space for the x-t and y-t sections respectively. We can see from these that the contours have undergone a rotation from the circularly symmetric as a result of the general overall advection present during the event.

In figures 4a to 4c the contours have been shifted so that the centre of the figure is the origin.

Figure 5a, similarly to figure 4a, shows the two-sided spectral density of spatial rain rate variation, averaged over the 260 scans recorded for the event that occurred on the 7th December 2000. Below it, in figure 5b is the averaged radial spectral density of the log rain rate recorded during the event. As can be seen, it is a straight line with a slope = -2.89. This is a bit higher than is expected from theory, which predicts a value of -7/3, but is still within an acceptable range. Similar results were seen for the other rain events studied.

The fractal dimension value of 1.2 ties in very well with fluid dynamical models leading to segmented spectral density power laws [10,14]. It has been experimentally verified [15] that point log rain rate variations are self-similar over scales ranging from 10 s to one day and the one-dimensional power spectrum

\[ S(\omega) \propto \omega^{-2H-1}, \]  

(4)

Figure 3. Length of various contours (in pixels) plotted against the corresponding box counting dimension for rainfall rates of 1 mm/hr (a); Length of various contours (in pixels) plotted against the corresponding box counting dimension for rainfall rates of 10 mm/hr (b)
where H is the Hurst exponent [16], and is equal to 1/3. For surfaces, the spectral density function for an isotropic random field is given by:

\[ S(\omega) \propto \omega^{-2H-2} \]  

(5)

and the fractal dimension D is related to the Hurst exponent by \( D_S = 3 - H \), and from equation 1, \( D_L = D_S - 1 \). Therefore, in the case of the perimeter area relationship (equation 2), the slope of the best fit line for \( \log(A) \) against \( \log(P) \) is \( E = 2/D_L = 5/3 \), corresponding to the exponent of \( \omega \) in the spectral density function.

**Discussion and Conclusions**

The conclusion of the analysis is that the methods used to calculate the fractal dimension of the contours agree with each other. For completeness we used the area-perimeter and box counting methods described above to determine the fractal dimension of a number of different sizes rectangles (fractal dimension = 1) and different sized Koch snowflakes (fractal dimension = 1.2618) in data grids of the same size and resolution as the recorded radar data. For both methods the correct dimension was estimated.

Previously published results [4,5] using these particular methods suggest that the fractal dimension, as measured by radar and satellite pictures, has a dimension of ~1.35. However, as has been shown here, the resolution of the instrument used to determine the size and shape of the rain areas is very important and needs to be taken into account during the analysis.

This resolution issue is one of a vast range of important issues that need to be taken into account when using fractal methods to study rain fields. For instance, in the real world, there are fixed spatial limits that cannot be avoided (it is impossible to have a rain field covering an area greater than the size of the earth) where the scaling nature of the fractal model breaks down. This also happens where the areas of interest are very small, and the rain field must be quantized into its individual rain drops.

A concept of “filtered fractals” should also be introduced and investigated. It is commonly known that orographic features, such as hills and valleys can dramatically affect the amount of rain recorded at various positions in the geographical area of interest. For example, studies in Bavaria at the Hohenpeissnberg Observatory show that standard rain gauges may overestimate the amounts of rain falling on them by about 10% on the lee slopes, and underestimate them by 14% on windward slopes [17]. Wind, especially the strongly directional wind experienced during the passage of fronts seems to have a smoothing effect on the shape of the rain rate contours. For example, the wind direction recorded at Chilbolton during the event shown in figure 1 was remarkably constant, with the wind direction being due North. Due to the way the radar data was mapped onto a Cartesian plane, this translates as the wind blowing along the direction of the negative x-axis.
It appears that the wind appears to "compress" the rain cells along the line parallel to the direction of the wind, and the rain cells are elongated along a line perpendicular to the direction of the wind. It is unknown whether or not the amount of compression and elongation is related to the wind speed. This would seem sensible but could be a topic for further study.

These would imply that the rain field has a "natural" fractal shape, at the point of its creation, which is then altered or "filtered" by the effects of wind. Investigating this concept more thoroughly could serve to end a large part of the uncertainty about the true nature of the fractality of the rain field.

The results presented here raise interesting implications for the current state of knowledge regarding the physical nature and shape of rain cells. The fractal nature and dimension value of the rain rate contours open a door into new methods of quantifying and eventually simulating the shape and distribution of rain areas. This has, in turn, implications into bringing in new ideas which will be of interest to a wide range of researchers including hydrologists, meteorologists and communications engineers.

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References


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ВНИМАНИЕ!

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FRACTAL ANALYSIS OF RAIN FIELDS FOR COMMUNICATIONS SYSTEMS DESIGN

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Rationale
Communications systems operating at frequencies above 10 GHz suffer from severe attenuation due to rain, clouds and atmospheric gases, though rain is the predominant factor responsible for most of the time-variant fading experienced. Systems currently in operation deal with the problem of rain fading by allocating a fixed fade margin. However, for higher frequencies the fade margin required to provide a set quality of service becomes too high, and therefore is economically unfeasible. For this reason, much attention is being paid to methods of dealing with fades on an adaptive and reactive basis. Such methods are collectively known as Fade Mitigation Techniques (FMTs). One of them, which concerns the (time-variant) two dimensional distribution of the precipitation rate $R(x,y,t)$ (mm/h), is called Space Diversity.

Figure 1 is an artist’s impression of a 3-station communications system link layout for broadcast applications superimposed on a real radar scan of rain-induced reflectivity $Z$. Site diversity in a satellite system employs two or more ground stations receiving the same satellite signal with a separation distance sufficiently large so that the rain at the two sites is de-correlated. The sites, in a properly configured arrangement, encounter intense rainfall at different times, and switching to the site experiencing the least fading improves system performance considerably. The radar scan shown in figure 1 is an example of the type of primary experimental result being used in this research and the scans are obtained at the Chilbolton Advanced Meteorological Radar (CAMRa) located in Chilton, South West of England, [1].

![Figure 1: Example layout of a 3 site communications system in the South of England with measured radar reflectivities of a frontal rain event. The grayscale values show the reflectivity strengths.](image)

The Research
Communications engineers often assume that rain fields can be adequately described as a field of individual rain cells. Moreover, rain cell models from radar measurements are often statistical in nature and do not enable the construction of typical two dimensional rain-rate fields [2]. Other excellent models have disadvantages in that they only deal with the spatial variation of the rain-rate within a rain cell [3].

The current telecommunications research summarized in this paper takes a different approach: it considers the contours of the rain rates $R$, or of the reflectivity $Z$, observed by the radar, which are then analyzed in terms of their fractal behavior. An example is shown in Figure 2. The rain contours encircle regions within which the rain rate, $R$, exceeds a specified threshold. It should be noted that this approach, in its abstract concept, has been considered by other authors [4,5], though not for engineering applications. Simple fractal analysis, such as the area-perimeter method to determine the fractal dimension of rain rate contours, has been used in this research and can also be used to determine some
of the quantities of the rain field which are of interest to systems designers. Similarly, the power laws that govern the distribution of the number of contours with respect to their enclosed area (also known as the Korcak distribution) are useful as inputs in rain variation models, including synthetic storm models, in their own right. Our analysis of the contours illustrated in Figure 2 has confirmed their fractal behavior [6,7] and is leading us now to the design aspects of fractal based rain field synthesis and ways of analyzing specific engineering case studies similar to that illustrated in Figure 1.

![Wind direction](image)

Figure 2: Example rain field. Rain rate contours plotted at 1mm/hr (black line) and 10 mm/hr (gray line)

The other key objective of the research work is the study of the phenomenological aspects of rain fields, and the instruments used to measure them. For instance, the optimization of rain gauge distribution in terms of a rain gauge network can be clearly and easily defined in terms of the networks correlation dimension, giving an objective value for the ranges of scales over which the rain field can be measured. This aspect is directly related to the fractal nature of the 2-D layout of the instruments and its fractal dimension as compared with the fractal dimension of the phenomenon being analyzed. In simple terms, a rain gauge network with too many gauges packed too closely is inefficient and expensive, whereas if they are too far apart any small-scale variation in the rain less than the spacing between the gauges will be lost.

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Synthesis of Two Dimensional Rain Fields for Systems Using Spatial Diversity

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ABSTRACT

Radio communications systems, operating at 10 GHz and above, suffer from severe attenuation due to rain and cloud, which is unlikely to be compensated for by available fade margin alone. An important feature of rain is its temporal and spatial inhomogeneity, which can be taken advantage of to improve the availability of a system that uses route or site diversity. To correctly configure such a system to optimise the availability and minimize the cost requires detailed knowledge of typical rain fields. Data is also required to test the proposed system during development.

In some cases it is sometimes more convenient to use simulated data for the testing and development of a system. Cascade models have been proposed as a computationally effective method of performing such simulations, and have also been shown to produce the same statistics as real rain fields.

In this paper, we will demonstrate a typical discrete cascade model, highlighting its parameters and their physical significance. We will also discuss the applications and implications arising from the use of such a model.

INTRODUCTION

Rain, by its very nature, is inhomogeneous, and intermittent. Communications systems operating at frequencies likely to be adversely affected by rain can use this to improve their availability through the use of fade mitigation techniques (FMTs) such as route or site diversity. Intense rain cells that cause large amounts of attenuation on radio paths often have horizontal dimensions of no more than a few kilometres. Site diversity employs two or more ground stations receiving the same satellite signal with a separation distance usually greater than the diameter of the rain cells. The sites in a properly configured arrangement encounter intense rainfall at different times, and switching to the site experiencing the least fading improves system performance considerably.

To properly configure such a system requires knowledge of the spatial variation of rain fields. A number of studies have developed rain cell models from radar measurements but these are statistical in nature and do not enable the construction of typical two dimensional rain-rate fields [1]. Other models have disadvantages in that they only deal with the spatial variation of the rain-rate within a rain cell [2], or do not take into account the full range of rain rates that are significant for frequencies up to, and beyond Ka band [3]. These models assume regular shapes to the rain cells, such as ellipses, or Gaussian functions of position centred on the area of maximum rain rate.

The field of fractals and chaotic dynamics had major implications and much promise when it was first introduced to meteorology in the 1960s, especially with regards to the subject of rain. However, the course of the development and utilization of fractal methods to deal with rain fields in a radio communications context has not been a smooth one. In fact, the majority of those scientists and engineers who find their systems heavily impacted by the effects of rain aren’t generally aware of the progress that has been made in other areas such as meteorology and hydrology to study and model rain.

Fractals are most commonly seen in the form of computer-generated images, such as the Mandelbrot and Julia sets. They are also often used in special effects in films and TV to provide realistic but also alien looking computer-generated landscapes.

The simplest way to describe a fractal is as an object that appears self-similar under varying degrees of magnification, with each small part of the object replicating the structure of the whole. This is a particularly loose definition, though it
captures the essential defining characteristic of fractals, that of self-similarity. Some natural fractals, e.g. the boundary of clouds, cracks in a wall and a hillside silhouette possess statistical self-similarity, i.e. they possess the same statistical properties (the same degree of ruggedness) as we zoom in. Other natural fractals, for example, the fern, possess exact self-similarity. Each frond of the fern is a mini copy of the whole fern, and each frond branch is similar to the whole frond and so on. Also, as we move towards the top of the fern a smaller and smaller copy of the whole fern can be seen.

Mathematically speaking, the fractal dimension $D$ characterizes any self-similar system; if the linear dimension of a fractal observable is changed by a scale factor $f$, then, for any value of $f$ the values of the fractal observable will be changed by the factor $f^D$. For surfaces, the value of the surface dimension lies in the range $2 \leq D_s \leq 3$. A smooth surface has $D_s=2$. Similarly, for a contour line $1 \leq D_L \leq 2$, and $D_L=1$ for smooth lines. The more twisted and "wriggly" the contour line is, the higher the value of $D_L$. If pathological cases are disregarded a planar section of a fractal surface has

$$D_L = D_s - 1.$$  \hspace{1cm} (1)

It has been previously shown [4], with the assumption that rain fields can accurately be described as a fractal surface, that the fractal dimension of rain rate contours calculated by the area-perimeter method has a nearly constant value of $\sim 1.2$.

**SPATIAL SPECTRAL DENSITY FUNCTION**

A common assumption made while dealing with (log) rain fields is that they can be considered to be homogeneous, isotropic Gaussian, random fields [5]. This can be confirmed by the calculation of the two-dimensional Fourier transform, as described below.

If each near-horizontal radar scan is treated as an instantaneous snapshot of the rain rate field then the spatial spectral density may be calculated via 2-D Fourier transform. The spectral density function for an isotropic random field is given by:

$$S(\omega) \propto \omega^{-2H-2}$$ \hspace{1cm} (2)

where $H$ is the Hurst exponent, and is equal to 1/3. For surfaces, the fractal dimension $D$ is related to the Hurst exponent by $D_s=3-H$ and from equation 1, $D_L = D_s - 1$.

Fig. 1 shows the two-sided spectral density of spatial rain rate variation, averaged over the 260 scans recorded for the stratiform rain event that occurred on the 7th December 2000. The near circular contours are consistent with a rotationally symmetric, and hence quadrant symmetric, spectral density and spatial autocorrelation. Below it in fig. 2 is the averaged radial spectral density of the log rain rate recorded during the event. As can be seen, it is a straight line with a slope $\sim -2.89$. This is a bit higher than is expected from theory, but is still within an acceptable range. Similar results were seen for the other events studied, including convective and frontal events.

Similarly, the corresponding spatio-temporal isocorrelation contours in the Fourier space for the x-t and y-t sections can be plotted. Other studies [6, 7] have shown that the contours undergo a rotation from the circularly symmetric as a result of the general overall advection present during the event.
Fig. 1. 2D spatial spectral density of rain rate for event on 7/12/00, averaged over 260 scans.

Fig. 2. Spectral density of log rain rate for event on 7/12/00. Exponent = -2.8951
CASCADE MODELS

The Voss algorithm, described below, is part of a family of models known as cascade models. These come in two distinct types, discrete cascades and continuous cascades (also known as random space-time function generators). The Voss algorithm is a discrete cascade.

Continuous cascades make use of generators of random space-time functions to generate fields with specified spatial and temporal covariance structures. Examples of such cascades are given in [8, 9]. These models are quite flexible, and can incorporate constraints to observed values at appropriate positions. But they also require detailed specifications of the correlation structure of the random function in both space and time, which can be hard to define, and as yet no consensus has been reached on the general structure.

Discrete cascades exploit self-affinity and self-similarity relationships to produce rain rate fields through an iterative random cascade procedure. These cascades are better able to incorporate non-rainy regions than continuous cascades, and have concepts and ideas in common with disaggregation studies done in hydrology [10]. On the other hand it is more difficult to describe the temporal evolution of rain fields using discrete cascades, though it is by no means impossible. Examples of discrete cascades are given in [7, 11].

Both types of cascade model are more algorithmic schemes for generating space-time fields than attempts to provide a valid physical representation of a rain event within a given framework.

VOSS ALGORITHM

To simulate the rain field we contemplate the use of the successive random addition algorithm introduced by Voss (1985) [12] to generate fractional Brownian motion. The algorithm is easily extended to higher dimensions and can produce surfaces with coastlines that are self similar fractals with a fractal dimension given by $D=2-H$, where $H$ is the exponent used in the generation of the landscapes (and is also mathematically the Hurst exponent mentioned earlier).

The surfaces are generated on a lattice in an iterative manner (see fig. 3). In the first generation an independent Gaussian variable $\xi$ is generated with zero mean and unit variance. This value is used as the level at the central point on the lattice (point A). The four corner points (points B) of the lattice are given a value equal to zero. The values at the midpoints (points C) of each of the four lines on the outside of the lattice are the average of the two end points and the centre point, i.e. the value at the midpoint of the line is given by the average of the values of its nearest neighbours. Then points inside the lattice (points D) are given values according to the average of their diagonal neighbours. All the points plotted then have independent values of $\xi_{n=1}$ added to them, where the Gaussian random variable now has the variance given by:

$$\langle \xi^2 \rangle = \sigma^2 = r^{2H}$$

with $r = 1/\sqrt{2}$ and $n=1$.

The procedure is continued for the next generation (points E) where the values at the new points are given by the average of the nearest neighbour locations, i.e., the neighbours in directions parallel to the axes. The points on the rim will have values given by the average of their three nearest neighbours, rather than sites inside the lattice, which have four. All the points plotted then have independent values of $\xi_{n=2}$ added to them, with the variance given by the equation above.

Each generation has the variance changed according to $n$ and the process continues until all the points on the lattice are filled. The number of points on the lattice determines the computation time required to complete the process.
Fig. 3: schematic diagram of the successive random addition method for simulating fractional Brownian motion in 2 dimensions.

Fig. 4 shows an example of a typical field generated using the successive random additions algorithm. Unlike the other models mentioned [7, 11] the additive method of generation produces a monofractal field, which is characterized by one single fractal dimension. By contrast multiplicative cascades produce multifractal fields, which are characterized by a whole spectrum of fractal dimensions. A comprehensive discussion of multifractals can be found in [13] Fig. 5 shows a close up of the field in fig. 4, which makes it easier to compare with a plot of a real rain field (fig. 6) as both figures now have the same spatial resolution.

Fig. 4: an example of a typical simulated field created using the Voss algorithm. Contours are drawn at rain rate values of 1, 10, 20, 30 and 40 mm/hr.
Fig. 5: close up of an example of a typical simulated field created using the Voss algorithm.

Fig. 6: example of a typical measured field for comparison with fig. 5
FURTHER WORK

The successive random additions model, as it stands, provides a convincing algorithm for the generation of fractal surfaces that can be used to simulate rain fields. However, it can be made even more realistic by a certain amount of fine-tuning of various parameters, the better to suit situations with different climatological behaviours.

The average rain rate across the entire simulated rain field is determined partially through the random process that generates the field, but can be influenced by the choice of value given to the four outer corners (points B in fig. 3) during the first iteration of the cascade process. Changing their values will affect the average rain rate across the whole field, but will not adversely affect the statistics and spectral density function of the field. In general, to raise the average value, corner values greater than 0 should be chosen, whereas to lower the average value the corner values should be less than 0. The fractal dimension D of the generated field is determined only by H; varying the value of r changes the lacunarity, or proportion of holes in the field.

The cascade process described above produces rain fields that are statistically independent of each other, i.e., creating a series of rain fields by this method will not result in a convincing synthetic event. However, as the fields can be generated over far wider areas than measurements can be taken, it is possible that Taylor’s frozen storm hypothesis [14] could be of use to give some indication of temporal variation.

The frozen field hypothesis postulates the equivalence of the spatial autocorrelation at a fixed point in time and the temporal autocorrelation at a fixed position in space, if the spatial argument of the former can be interpreted as a time lag of the latter. This holds if the spatio-temporal field is a fixed spatial field moving with a constant velocity. It has been shown [15] that this holds approximately for time lags under about 40 minutes, which is a timescale much longer than is needed to implement route or site diversity switching. Hence generating a rain field over an area much wider than the area of direct interest can serve to give some indication of the temporal evolution of the field in the area of interest, provided that a constant velocity is used to move the larger field.

CONCLUSIONS

A discrete cascade model that produces monofractal fields with the same statistics and spectral density function as real rain fields has been presented. Its applications in the design and optimum use of a communications system have been outlined and potential improvements to the model have been discussed.

References


INTRODUCTION
Rain is a major source of attenuation or radio paths operating above 10 GHz, one that is unlikely to be compensated for by available fade margin alone. A number of fade mitigation techniques have been proposed to increase the potential availability of a system; one of which, site diversity, will be investigated by the RCRU during the forthcoming STENTOR experiment.

Site diversity relies on the spatial and temporal inhomogeneity of rain fields. As it can be expensive to implement, it is advisable to study the spatial and temporal variation of rain using other methods such as rain gauge networks and radar. These studies provide valuable information on structures inside the rain fields, such as the size, shape and orientation of rain cells. This information is needed to accurately model and simulate the spatial variation of 2-D rain fields.

This paper discusses the implications of the results of one such analysis, completed using the data recorded from CAMRa (the Chilbolton Advanced Meteorological Radar), which is located in the South of England.

Using these results and other fractal techniques a model can be devised to simulate the spatial variation in rain rate fields. The use of this model for design and optimum use of communications systems (both slant path and terrestrial) will be outlined.

METHODS OF ANALYSIS
A typical method of analysis for this type of data is to calculate the power spectral density function of the data, looking at it in a purely spatial and a spatio-temporal state. Calculation of the fractal dimension also can reveal some interesting and useful information about the structure of the rain field.

In general, the fractal dimension D characterises any self-similar system; if the linear dimension of a fractal observable is changed by a scale factor f, then, for any value of f the values of the fractal observable will be changed by the factor \( f^D \). For surfaces, the value of the surface dimension lies in the range \( 2 \leq D_s \leq 3 \). A smooth surface has \( D_s = 2 \). Similarly, for a contour line \( 1 \leq D_L \leq 2 \), and \( D_L = 1 \) for smooth lines. The more twisted and "wriggly" the contour line is, the higher the value of \( D_L \). If pathological cases are disregarded a planar section of a fractal surface has \( D_L = D_s - 1 \).

\[
D_L = D_s - 1.
\]

Power spectral density function
If each near-horizontal radar scan is treated as an instantaneous snapshot of the rain rate field then the spatial spectral density may be calculated via 2-D Fourier transform. Figure 1 illustrates the two-sided spectral density of spatial rain rate variation, averaged over the 230 scans recorded for the event that occurred on the 1st May 2001. The near circular contours are consistent with a rotationally symmetric, and hence quadrant symmetric, spectral density and spatial autocorrelation.

The radar scans were interpolated onto a square Cartesian grid, with a grid spacing of 300m and a side length of 56.2km. Each grid contains 35344 data points (188²) covering more than 3100 km².

Data Format
The rainfall rate contours analysed in this research have been obtained by means of the Chilbolton Advanced Meteorological Radar (CAMRa), which is located in Hampshire in the south of England, at the latitude 51° 9' North and the longitude 1° 26' West. The climate is temperate maritime, with an average annual rain rate exceeded 0.01% of the time of approximately 22.5 mm/hr. The radar is a 25 m steerable antenna, equipped with a 3GHz Doppler-Polarization radar, and has an operational range of 100 km, and a beam width of 0.25°.

To avoid reflections from ground clutter, maps of the rain rate field near the ground are produced by scanning with an inclination of 1.2°. These maps are produced on a polar grid, with a range resolution of 300m and an angular resolution of 0.3°. The number of maps produced in a given time period is dependent on the total angle scanned. The radar has a maximum angular velocity of 1°/second. The time interval between scans is ~2 minutes.
Figures 2 and 3 show the corresponding spatio-temporal isocorrelation contours in the Fourier space for the x-t and y-t sections respectively. We can see from these that the contours have undergone a rotation from the circularly symmetric as a result of the general overall advection present during the event.

In figures 2 to 3 the contours have been shifted so that the centre of the figure is the origin.

Figure 2. 2-D Spatial-temporal (x-t) spectral density of rain rate for event on 1/5/01, averaged across the y direction.

Figure 3. 2-D Spatial-temporal (y-t) spectral density of rain rate for event on 1/5/01, averaged across the x direction.

Area – perimeter and box counting relationship
A full and detailed account of the area-perimeter and box counting relationships can be found in Callaghan and Vilar [1].

The area, $A$ within a contour depends fractally on the contour length or perimeter $P$ following a law of the type:

$$ A \propto P^E $$

where the exponent $E = 2/D_A$ and $D_A$ is the fractal dimension determined by the area-perimeter method.

Figures 4 and 5 show the areas of various rain rate contours, plotted on a log-log grid, against their perimeters for fixed values of 1 and 10 mm/hr respectively. Similar results were found for other rain rate thresholds and different rain events.

The results obtained using the area perimeter method on six different rain events and over 1200 different data grids show that the fractal dimension of the rain rate contours is very close to 1.2, with no slope discontinuity in the plots which would otherwise indicate the presence of a characteristic length.

If one uses the box counting algorithm, the fractal dimension $D_B$ is given by the limit:
\[ D_B = \lim_{\delta \to 0} \left[ \log N(\delta) / \log(1/\delta) \right] \] (3)

where \( \delta \) is an arbitrary length unit selected such that \( N(\delta) \) is then the number of boxes (or cubes) of side length \( \delta \) required to completely cover the curve under study. By plotting \( \log N(\delta) \) against \( \log(1/\delta) \) the average slope of the plot gives an estimation of the box counting dimension.

The box counting dimension was calculated individually for each contour. Figures 6 and 7 show the box counting dimension values for each contour in one of the rain events, against the area enclosed by each of the contours for rainfall rates of 1 mm/hr and 10 mm/hr respectively. Again, similar results were observed for different rainfall thresholds and events.

As can be seen, for contours enclosing areas less than 200 km\(^2\), or 14 km x 14 km, the box counting dimension could not be determined accurately. This is due to the inevitably limited resolution of the radar and the minimum pixel size in the raw data grids. For smaller areas the contouring algorithm approximates the contour lines by irregular polygons (which have a fractal dimension of 1) or by points (which have a fractal dimension of 0). However, for areas between 200 km\(^2\) and 1400 km\(^2\) the fractal dimension was constant at \(-1.2 \pm 0.05\), irrespective of the area. It can also be seen that at the higher rain rate threshold (10 mm/hr), in general the areas enclosed by the contour lines are smaller, making it harder to accurately determine the box counting dimension. In a temperate maritime climate, such as where the data was recorded, it is rare to get areas of rain with rain rate greater than \(-20\) mm/hr covering areas greater than a few km\(^2\). However, when large rain cells at those rain rates do occur, their contours show the same behaviour as those at lower rainfall thresholds.

The fractal dimension value of -1.2 ties in very well with fluid dynamical models leading to segmented spectral density power laws [2,3]. It has been experimentally verified [4] that point log rain rate variations are self-similar over scales ranging from 10 s to one day and the one-dimensional power spectrum

\[ S(\omega) \propto \omega^{-2H+1} \] (4)

where \( H \) is the Hurst exponent and is equal to 1/3. For surfaces, the fractal dimension \( D \) is related to the Hurst exponent by \( D_s = 3 - H \) and from equation 1, \( D_L = D_s - 1 \). Therefore, in the case of the perimeter area relationship (equation 2), the slope of the best fit line for \( \log(A) \) against \( \log(P) \) is \( E = 2 / D_L = 5 / 3 \), corresponding to the exponent of \( \omega \) in the spectral density function.

**SIMULATION OF RAIN FIELDS**

The Voss Algorithm

To simulate the rain field we contemplate the use of the successive random addition algorithm introduced by Voss (1985) [5] to generate fractional Brownian motion. The algorithm is easily extended to higher dimensions and can produce surfaces with coastlines that are self similar fractals with a fractal dimension given by \( D_s = 2 - H \), where \( H \) is the exponent used in the generation of the landscapes (and is also mathematically the Hurst exponent mentioned earlier).

The surfaces are generated on a lattice in an iterative manner (see figure 8). In the first generation an independent Gaussian variable \( \xi \) is generated with zero mean and unit variance. This value is used as the level at the central point on the lattice (black point). The four corner points (red) of the lattice are given a
value equal to one. The values at the midpoints (yellow) of each of the four lines on the outside of the lattice are the average of the two end points and the centre point, i.e. the value at the midpoint of the line is given by the average of the values of its nearest neighbours. Then points inside the lattice (green) are given values according to the average of their diagonal neighbours. All the points plotted then have independent values of $$\xi_{n=1}$$ added to them, where the Gaussian random variable now has the variance given by:

$$\left< \xi_n^2 \right> = \sigma_n^2 = r^{2nH}$$

with $$r = 1/\sqrt{2}$$ and $$n = 1$$.

The procedure is continued for the next generation (blue) where the values at the new points are given by the average of the nearest neighbour locations, i.e., the neighbours in directions parallel to the axes. The points on the rim will have values given by the average of their three nearest neighbours, rather than sites inside the lattice, which have four. All the points plotted then have independent values of $$\xi_{n=2}$$ added to them, with the variance given by the equation above.

Each generation has the variance changed according to $$n$$ and the process continues until all the points on the lattice are filled.

Figure 8: schematic diagram of the successive random addition method for simulating fractional Brownian motion in 2 dimensions.

![Figure 8](image)

Figure 8: schematic diagram of the successive random addition method for simulating fractional Brownian motion in 2 dimensions.

Figure 9: an example of a typical simulated field created using the Voss algorithm. Contours are drawn at rain rate values of 1, 10, 20, 30 and 40 mm/hr.

This procedure produces rain fields with the same fractal dimension and spectral density functions as observed in real rain fields and is computationally easy to implement and generate. It also shares similarities with the numerous discrete cascade models that are discussed in the literature [6,7] but whereas the cascade models start off with an average rain flux that is split up and subdivided on smaller and smaller scales using a multiplicative method, the Voss algorithm is an additive method and results in a log rain rate field.

DISCUSSION AND CONCLUSIONS

The results presented here raise interesting implications for the current state of knowledge regarding the physical nature and shape of rain cells. The fractal nature and dimension value of the rain rate contours open a door into new methods of quantifying and eventually simulating the shape and distribution of rain areas. This has, in turn, implications into bringing in new ideas, which will be of interest to a wide range of researchers including meteorologists and communications engineers.

The use of cascade methods to simulate rain fields is a relatively new concept which has yet to be applied in a communications engineering context, but does show a very promising potential, not only in terms of simulations, but also future prediction methods.

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