Towards a wave-extraction method for numerical relativity. IV. Testing the quasi-Kinnersley method in the Bondi–Sachs framework

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We present a numerical study of the evolution of a non-linearly disturbed black hole described by the Bondi–Sachs metric, for which the outgoing gravitational waves can readily be found using the news function. We compare the gravitational wave output obtained with the use of the news function in the Bondi–Sachs framework, with that obtained from the Weyl scalars, where the latter are evaluated in a quasi-Kinnersley tetrad. The latter method has the advantage of being applicable to any formulation of Einstein's equations—including the ADM formulation and its various descendants—in addition to being robust. Using the non-linearly disturbed Bondi–Sachs black hole as a test-bed, we show that the two approaches give wave-extraction results which are in very good agreement. When wave extraction through the Weyl scalars is done in a non quasi-Kinnersley tetrad, the results are markedly different from those obtained using the news function.

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I. INTRODUCTION

Gravitational–wave detection has been gaining much interest over the last decades. Much effort has been correspondingly made in modelling possible sources. Interesting examples of sources of gravitational waves are, e.g., binary systems of merging black holes, spiralling systems of two neutron stars or coalescing black hole–neutron star binaries. Analytical tools able to investigate the dynamics of such sources when the merging takes place do not exist, due to the strong non-linearity of the problem. Numerical simulations are therefore invaluable in order to extract information about the gravitational-wave signal emitted. Typically, numerical relativity studies are done in three stages: First, one specifies initial data that correspond to the physical system of interest, and that satisfy certain constraint equations. Next, one evolves these initial data numerically, using the evolution equations (with or without enforcing the constraints), and finally, one needs to interpret the results of the simulation and extract the relevant physics thereof. This paper—like its prequels [1, 2, 3]—is concerned with this last stage, namely, with the problem of wave extraction.

In order to interpret the results of numerical relativity simulations—at least within the context of a specific formulation of Einstein's equations—a useful approach is that based on the characteristic initial value problem, originally introduced by Bondi and Sachs [4, 5]. The characteristic initial value problem has been extensively used in numerical relativity, for spherically symmetric systems [6, 7, 8, 9, 10, 11, 12, 13], axisymmetric systems [14, 15, 16] and 3D systems [17, 18, 19, 20]. This article is based on the results found in [16]—where the characteristic initial value problem, using Bondi coordinates, is used in axisymmetry to study a non-linearly disturbed non-rotating black hole metric. The non-linear response of the Schwarzschild black hole to the gravitational perturbation is embodied in a superposition of angular harmonics propagating outside the source. All the information concerning the angular harmonics and the energy radiated are easily derived by the evolved quantities. In fact, in this particular case one can identify a *news* function [4], i.e., a function which embodies the information about the gravitational–radiation energy emitted [17, 21, 22, 23, 24, 25, 26].

It is not currently possible to have at hand a quantity such as the news function when using other numerical approaches like, specifically, the widely used 3+1 decomposition of Einstein's equations [27]. Indeed, one of the outstanding problems of numerical relativity is that of *wave extraction*, i.e., the problem of how to extract the outgoing gravitational waves from the results of numerical simulations. In currently available methods, various approximations are applied to determine the gravitational-wave emission of isolated sources. One of the simplest approaches applies the quadrupole formula (that strictly speaking is valid for weak gravitational fields and slow motions) [28, 29]. This approach has been used effectively, e.g., in models of stellar collapse [30]. More sophisticated approaches use the Moncrief formalism [31, 32] to extract first-order gauge invariant variables from a spacetime which is assumed to be a perturbation of a Schwarzschild background at large distances [33, 34, 35, 36]. The strength of this approach is that it is gauge invariant, i.e., the information extracted is related to the physics of the system, and not to the coordinates used. Specifically, it avoids the pitfall of misidentifying gauge degrees of freedom as gravitational waves. [E.g., in the Lorenz gauge all degrees of freedom, including the (residual-) gauge ones, travel at the same speed—the speed of light—such that they might be confused with physical waves.] This procedure is usually performed under the assumption that the underlying gauge, i.e., the particular choice of the spacetime coordinate system, leads to a metric which is asymptotically Minkowski in its standard form, which is indeed the case for the most commonly used gauges in simulations of isolated systems. The gravitational waveform is determined by integrating metric components over a coordinate sphere at some appropriately large distance from the central source, and then subtracting the spherical part of the field (which is non-radiative). However, as we have already mentioned, such techniques are well defined when the background metric is assumed to be Schwarzschild, while their application to the more generic Kerr background metric can *at best* be intended as a very crude approximation. In addition, in the typical numerical relativity simulation, one does not usually have information about the mass and spin angular momentum of the eventual quiescent black hole, and no prescription is currently known to uniquely separate a background from a perturbation. Moreover, once the elimination of the spherical background is performed, what is left does not necessarily satisfy the perturbative field equations, and may, in fact, be quite large [37].

Another important approach is that of Cauchy–charactestic matching (CCM) schemes [10]. The peculiarity of such schemes is that they use a characteristic Bondi-Sachs approach to study the numerical space-time far from the source, where the fields are weak and the probability to form caustics, which would make the code crash, is limited. In this way, using the notion of the Bondi news function, it is possible to extract easily the gravitational wave information. In the strong field part of the computational domain, instead, a usual Cauchy foliation is used, so that the problem of caustic formation is irrelevant. The CCM schemes have been used successfully to simulate cylindrically symmetric vacuum space-times [11] or to study the Einstein–Klein–Gordon system with spherical symmetry [12]. A fully 3D application of the CCM scheme is, however, still unavailable.

A final approach worth mentioning here aimed at wave extraction is the one involving the Bel–Robinson vector [38], which can be considered a generalization to general relativity of the Poynting vector defined in electromagnetism. However its connection with the radiative degrees of freedom is still not entirely clear.

A novel approach has been suggested recently [1, 2, 39]: one extracts information about the gravitational radiation through quantities that are gauge and background-independent. One such quantity is the Beetle–Burko scalar, which is also tetrad independent. Specifically, no matter how one chooses to separate the perturbation from the background, the Beetle–Burko scalar remains unchanged. However, as pointed out in various contexts [3, 40, 41], the physical meaning of the Beetle–Burko scalar is non trivial. For example, in the stationary spacetime of a rotating neutron star its non-zero value is due to the deviation of the quadrupole from that of Kerr [41], while clearly no radiation is present. Thus the Beetle–Burko scalar awaits further study. At any case, the Beetle–Burko scalar—while including information only on the radiative degrees of freedom (when the notion of radiation is defined unambiguously)—describes the latter only partially. To obtain a full description of the radiative degrees of freedom, it is therefore desirable to consider an approach in which one calculates quantities, whose physical interpretation is more straightforward; however, the price to pay for these advantages is that it is harder to obtain such quantities uniquely: in fact, one still needs to break the spin/boost symmetry in a useful way.

This approach, which is the basis of this paper, is that of using the Weyl scalars—which, under certain assumptions, isolate the radiative degrees of freedom from the background and gauge ones [5, 42, 43]—for wave extraction. Teukolsky [44] showed that, choosing a particular tetrad, namely the *Kinnersley* tetrad [45], to calculate the Weyl scalars, it is possible to associate the Weyl scalar Ψ_4 with the outgoing gravitational radiation for a perturbed Kerr space-time. Other authors have suggested to use the Weyl scalars for wave extraction, most recently in [46], or to explore their relation with metric perturbations [47]. However, the Weyl scalars depend on the choice of tetrad. Specifically, performing null rotations on the basis vectors of the null tetrad one can change the values of the Weyl scalars. (Recall, that one of the advantages of the Beetle–Burko scalar is that it is tetrad *independent*.) In fact, extracting the Weyl scalar Ψ_4 is meaningless, unless one also describes how to construct the tetrad to which it corresponds. In most tetrads, the Weyl scalar Ψ_4 mixes the information of the outgoing radiation with other information, including gauge degrees of freedom. In our proposal the use of the Weyl scalars is intimately related to the construction of the tetrad in which they are to be calculated—the quasi-Kinnersley tetrad—and therein lies its strength.

With the aim of using Ψ_4 to extract information from simulations about the gravitational radiation output, starting with a transverse condition $\Psi_1 = \Psi_3 = 0$, recent work [1, 2, 3] has addressed the problem of computing Weyl scalars in a tetrad which will eventually converge to the Kinnersley tetrad. This tetrad has been dubbed the *quasi-Kinnersley tetrad*, and work is still in progress in order to uniquely identify it for a general metric. Up to now, it is possible, for a general spacetime, to identify a class of tetrads, namely the quasi-Kinnersley *frame* [1, 2], with the property that in this frame the radiative degrees of freedom (when and where the notion of radiation is unambiguous) are completely separated from the background ones. However, work is still in progress to identify the quasi-Kinnersley tetrad out of this frame. The difficuly in doing so is related to the following property of the tetrad members of the quasi-Kinnersley frame: they are all connected through type III ("spin/boost") rotations, and the spin/boost symmetry needs to be broken before the Weyl scalar Ψ_4 can be extracted in the quasi-Kinnersley tetrad. The quasi-Kinnersley frame is therefore a two-parameter family of tetrads, and the value of Ψ_4 (and that of Ψ_0) depends on the choice of the tetrad member of the frame. Notably, the Beetle–Burko scalar is invariant under type III rotations. That is, all the tetrad members of the quasi-Kinnersley frame share the same Beetle–Burko scalar. We avoid the spin/boost symmetry breaking difficulty in the present paper by applying an *ad hoc* technique to find the quasi-Kinnersley tetrad. This ad hoc technique allows us to obtain the Weyl scalar Ψ_4 in the quasi-Kinnersley tetrad, where its interpretation according to the gravitational compass is readily available.

Specifically, in this paper we use the Bondi–Sachs formalism [4], as it turns out that in this special case we can identify a (non-transverse) quasi-Kinnersley tetrad in a simple way, and to compute the Weyl scalars directly. The aim of this work is thus to demonstrate—in the context of this practical numerical example—the applicability, and necessity, of the quasi-Kinnersley tetrad method in using the Weyl scalars as wave extraction tools.

The article is organized as follows: Section II introduces our physical scenario, and Section III describes our Weyl scalars computation. In Section IV we present the expected result which links the Bondi *news function* to Ψ_4. Finally results and conclusions are presented in Section V.

II. THE BONDI PROBLEM

The numerical scenario we are studying is that of a non-linearly perturbed[1] Schwarzschild black hole using an ingoing null-cone foliation of the space-time.

We set up our system of coordinates as follows: a time-like geodesic is the origin of our coordinate system, photons are travelling from the origin in all directions, their trajectories forming null hypersurfaces. The hypersurface foliation is labelled by the coordinate v. As r coordinate we choose a luminosity distance, such that the two-surfaces of constant r and u have area $4\pi r^2$. Finally, each null geodesic in the hypersurface is labelled by the two angular variables θ and ϕ. We will restrict our attention to an axisymmetric space-time such that $\frac{\partial}{\partial\phi}$ is a Killing vector. Having chosen these variables, the Bondi metric in ingoing coordinates reads

$$\begin{aligned} ds^2 &= -\left[\left(1-2\frac{M}{r}\right)e^{2\beta} - U^2 r^2 e^{2\gamma}\right] dv^2 \\ &+ 2e^{2\beta} dv dr - 2U r^2 e^{2\gamma} dv d\theta \\ &+ r^2 (e^{2\gamma} d\theta^2 + e^{-2\gamma}\sin\theta^2 d\phi^2), \end{aligned} \tag{1}$$

where M, U, β, γ are unknown functions of the coordinates (v, r, θ). Within this framework, the Einstein equations decompose into three hypersurface equations and one evolution equation, as given below in symbolic notation

$$\Box^{(2)}\psi = \mathcal{H}_\gamma\left(M, \beta, U, \gamma\right), \tag{2a}$$
$$\beta_{,r} = \mathcal{H}_\beta\left(\gamma\right), \tag{2b}$$
$$U_{,rr} = \mathcal{H}_U\left(\beta, \gamma\right), \tag{2c}$$
$$M_{,r} = \mathcal{H}_M\left(U, \beta, \gamma\right), \tag{2d}$$

where $\Box^{(2)}$ is a 2-dimensional wave operator, $\psi = r\gamma$, and the various H symbols are functions of the Bondi variables. We will write here only the expression for $\mathcal{H}_\beta$, the simplest of all the functions, which is given by

$$\mathcal{H}_\beta = \frac{1}{2} r \gamma^2. \tag{3}$$

We will use this expression to describe the numerical algorithm. For exhaustive description of the system (2) we refer to [15] and [14].

The structure of Eq. (2) establishes a natural hierarchy in integrating them. By setting the initial value for the function γ on the initial hypersurface (in addition to four free parameters), it is possible to integrate Eq. (2b) to obtain β, then, having both β and γ, Eq. (2c) can be integrated to obtain U and finally M can be derived by integrating Eq. (2d). At this point we have all the metric functions on the initial hypersurface and we can integrate Eq. (2a) to obtain γ on the next hypersurface, and the procedure is iterated. The integration of Eq. (2) introduces constants of integration, which we set to zero (Bondi frame [14, 48]), which corresponds to having an asymptotically inertial frame.

The metric introduced in Eq. (1) describes a static Schwarzschild black hole if we set, in the Bondi frame, all the functions except M to zero everywhere in the domain. M is chosen to be the Schwarzschild mass M_0 of the black hole. Besides, outgoing gravitational radiation as a perturbation is introduced by the function γ; it turns out that γ is a spin-2 field and is actually related to the radiative degree of freedom. Choosing an initial shape for γ means in practice choosing the initial profile of outgoing gravitational waves. More specifically, the initial data are chosen in the following way:

- We recover the background Schwarzschild geometry by setting $\gamma = \beta = U = 0$ over the whole computational domain, while we set $M = M_0$

[1] We want to point out here that the expression "perturbed" could be misleading in this context, as it might suggest we are assuming some kind of approximation. Our numerical simulations are instead fully non-linear evolutions of Einstein's equations in the Bondi-Sachs formulation.

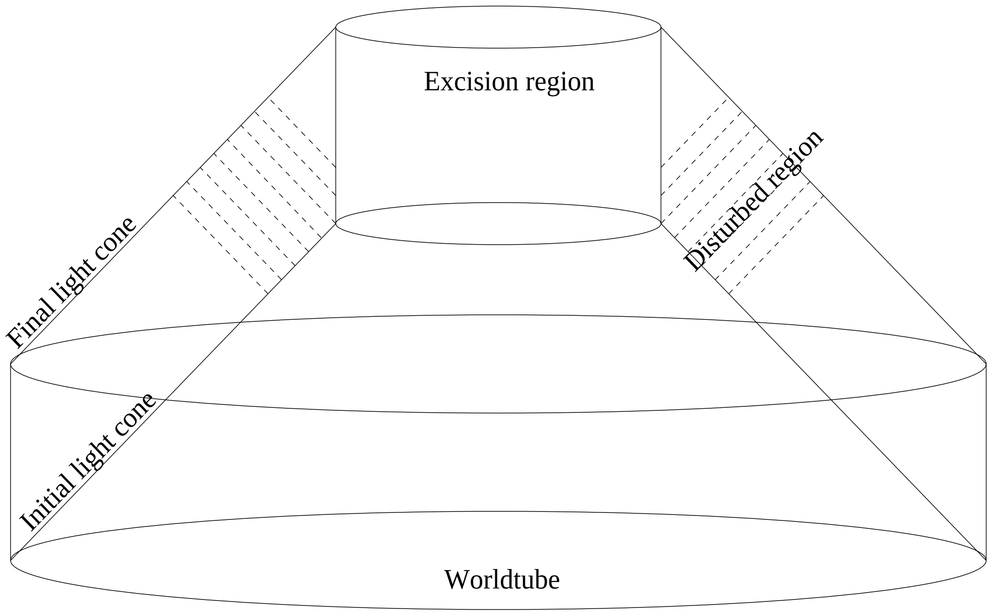

FIG. 1: Diagram of the Bondi algorithm: the foliation is built up using ingoing light cones that emanate from an external worldtube $\mathcal{W}$. We fix the geometry at the world tube to be that of a single Schwarzschild black hole. We non-linearly perturb the space-time by setting a non-vanishing value for the Bondi function γ. Such disturbance propagates outwards in the radiative region, after scattering off the black hole.

- We set up the initial data for a gravitational wave outgoing pulse by choosing a gaussian shape (with parameters r_c and σ) for the function γ (the choice of the gaussian shape is dictated by requiring the function to vanish at the outer boundary of the grid):

$$\gamma\left(r,\theta\right) = \frac{\lambda}{\sqrt{2\pi}\sigma} e^{-\frac{(r-r_c)^2}{\sigma^2}} Y_{2lm}\left(\theta\right), \tag{4}$$

  where λ is the amplitude of the perturbation, and Y_{lm} is the spherical harmonic of spin 2. In our numerical simulations we will set two types of initial data: the first one with $l = 2$ and $m = 0$ to get a pure quadrupole outgoing perturbation, while in the second one we will set $l = 3$ and $m = 0$.

The integration of the hypersurface equations leads to the problem of the gauge freedom in choosing the integration constants. This apparent freedom is fixed by choosing outer boundary conditions on our numerical grid. As depicted in Fig. (1) we in fact fix the metric at the outer boundary, i.e. on the worldtube $\mathcal{W}$, to be that of a Schwarzschild black hole. This automatically fixes the integration constants to be 0 for γ, U and β, and M_0 for M (Bondi frame). We point out here that such a boundary condition is well posed only for simulations which are limited in time, so that no relevant outgoing gravitational flow has crossed the worldtube. More information about the evolution routine can be found in [16, 49, 50].

III. WEYL SCALARS

Once we have the numerically computed metric for the evolved space-time we can derive the Newman-Penrose quantities we need. The Weyl scalars are

$$\Psi_0 = -C_{abcd} l^a m^b l^c m^d, \tag{5a}$$
$$\Psi_1 = -C_{abcd} l^a n^b l^c m^d, \tag{5b}$$
$$\Psi_2 = -C_{abcd} l^a m^b \bar{m}^c n^d, \tag{5c}$$
$$\Psi_3 = -C_{abcd} l^a n^b \bar{m}^c n^d, \tag{5d}$$
$$\Psi_4 = -C_{abcd} n^a \bar{m}^b n^c \bar{m}^d, \tag{5e}$$

where l^a, n^a, m^a and $\bar{m}^a$ are the Newman-Penrose null vectors. The five scalars defined in Eq. (5) are of course coordinate independent, but they do depend on the particular tetrad choice.

We calculate the scalars in a quasi-Kinnersley tetrad, i.e., in a tetrad that converges to the Kinnersley tetrad when space-time settles down to that of an unperturbed black hole. In [1, 2] a procedure to find the quasi-Kinnersley frame in a background independent way is given by looking at transverse frames, i.e., those frames where Ψ_1 and Ψ_3 vanish. In the following we use the word *frame* to indicate an equivalence class of tetrads which are connected by a type III spin/boost tetrad transformation. Fixing the right quasi-Kinnersley tetrad means choosing the tetrad in the quasi-Kinnersley frame which shows the right radial behavior for Ψ_0 and Ψ_4 according to the peeling-off theorem. That is, we seek the tetrad in which these two Weyl scalars peel off correctly. A background independent procedure to single out the tetrad out of the frame currently needs further investigation.

In the Bondi–Sachs framework the identification of a quasi-Kinnersley tetrad is simple, and does not need to use the notion of transverse frames. The main reason for this proprety is due to the asymptotic knowledge of the Bondi functions when spacetime approaches Schwarzschild: in fact γ, U and β tend to zero, while M tends to the Schwarzschild mass M_0 of the black hole (for further details see the next section).

This situation is much different from the typical situation in numerical relativity simulations, for which the wave-extraction methods needs to be background-independent. If we assume to be in the Schwarzschild limit, the background Kinnersley tetrad chosen by Teukolsky [44] in the perturbative scenario would look, using our coordinates (v, r, θ, ϕ), as

$$\ell^\mu = \left[\frac{2r}{r-2M}, 1, 0, 0\right], \tag{6a}$$

$$n^\mu = \left[0, -\frac{r-2M}{2r}, 0, 0\right], \tag{6b}$$

$$m^\mu = \left[0, 0, \frac{1}{\sqrt{2}r}, \frac{i}{\sqrt{2}r\sin\theta}\right]. \tag{6c}$$

This tetrad has been chosen by letting the ℓ^μ and n^μ vector coincide with the repeated principal null directions of the Schwarzschild space-time. Such a condition fixes a frame, i.e. a set of tetrads connected by a type III rotation. The type III rotation parameter is then fixed by setting the spin coefficient ϵ to be vanishing. Eq. (6) can be used to find the general expression for the tetrad in the full Bondi formalism. Using the asymptotic values of the Bondi functions, we can write down the expression for a general tetrad for the Bondi metric, whose vectors converge to the null vectors written in Eq. (6) in the Schwarzschild limit. The result is given by

$$\ell^\mu = \left[\frac{2}{\left[(1-2M/r)\,e^{4\beta} - U^2 r^2 e^{2(\gamma+\beta)}\right]}, e^{-4\beta}, 0, 0\right], \tag{7a}$$

$$n^\mu = \left[0, -\frac{\left[(1-2M/r)\,e^{2\beta} - U^2 r^2 e^{2\gamma}\right]}{2}, 0, 0\right], \tag{7b}$$

$$m^\mu = \left[0, \frac{rUe^{(\gamma-2\beta)}}{\sqrt{2}}, \frac{1}{\sqrt{2}re^{\gamma}}, \frac{i}{\sqrt{2}r\sin\theta e^{-\gamma}}\right]. \tag{7c}$$

Henceforth we will denote this first tetrad choice, which is supposed to be the successful one, as the tetrad $\mathcal{T}_1$. It is worth pointing out that the tetrad $\mathcal{T}_1$ is not transverse, i.e. Ψ_1 and Ψ_3 are not vanishing; Nevertheless it satisfies the requirements needed of a quasi-Kinnersley tetrad. (The quasi-Kinnersley tetrad does not have to be transverse, although it does need to be *asymptotically* transverse.)

As explained above, the choice of $\mathcal{T}_1$ has been driven by the form of the Kinnersley tetrad expressed in Eq. (6c) for the unperturbed black hole. However, this is not always possible in general where tetrad choices are dictated by different criteria. A straightforward example in this case would be that of basing the tetrad on null vectors directly derived from the metric. For instance, in the specific example of the metric (1), an apparently natural choice of the tetrad, obtained with the algebraic manipulation packages Maple and GRTensor, is the following:

$$\ell^\mu = \left[0, -e^{-4\beta}, 0, 0\right], \tag{8a}$$

$$n^\mu = \left[e^{2\beta}, \frac{\left[(1-2M/r)\,e^{2\beta} - U^2 r^2 e^{2\gamma}\right]}{2}, 0, 0\right], \tag{8b}$$

$$m^\mu = \left[0, \frac{rUe^{(\gamma-2\beta)}}{\sqrt{2}}, \frac{1}{\sqrt{2}re^{\gamma}}, \frac{i}{\sqrt{2}r\sin\theta e^{-\gamma}}\right]. \tag{8c}$$

The reason why this tetrad choice *looks* more natural than the first one is related to the fact that packages like GRTensor construct this tetrad starting from the ℓ^μ vector, which is assumed to be lying on the null foliation, leading to the expression $\ell_\mu = \delta_{\mu 0}$. The contravariant components are then given by Eq. (8a). Once ℓ^μ is fixed, the other tetrad vector expressions are found by imposing the normalization conditions between the vectors in the Newman-Penrose formalism.

The tetrad in Eq. (8) will be hereafter referred to as tetrad $\mathcal{T}_2$. We will show that this different tetrad choice leads to results which, although equivalent from a qualitative point of view, are different than those obtained in the quasi-Kinnersley tetrad $\mathcal{T}_1$.

In the numerical results that we are going to present in the next sections we have calculated the Weyl scalars in the two tetrads presented above, to show the reliability of quasi-Kinnersly tetrad method and an example, through a complete comparison, of the different results that we might obtain in a numerical simulation in doing wave extraction using Weyl scalars in different tetrads.

IV. THE LINEAR REGIME

In the linear regime both the Bondi and the Newman–Penrose formalisms define quantities which provide information about gravitational waves. We will discuss here briefly such definitions in order to get a correspondence between the two approaches, which will be tested numerically in the following section.

In the Bondi formalism the initial assumption is that the space-time is asymtotically flat, which leads to the following expansion for the function γ at null infinity:

$$\gamma = K + \frac{c}{r} + O\left(r^{-2}\right). \tag{9}$$

Here we assume to be in the Bondi frame [4], i.e. we set the integration constant K to be zero. By integrating the hypersurface equations for the other Bondi functions we can get their radial expansion at null infinity. Such integrations in general introduce other integration constants but, again, we assume that in our frame those constants are vanishing, ending up with the following expressions for the remaining Bondi functions [14]:

$$\beta = -\frac{c^2}{4r^2} + O\left(r^{-4}\right), \tag{10a}$$

$$U = -\frac{\left[c\sin^2\theta\right]_{,\theta}}{r^2\sin^2\theta} + O\left(r^{-3}\right), \tag{10b}$$

$$M = M_0 + O\left(r^{-1}\right). \tag{10c}$$

It is trivial to verify that, with this choice of integration constants, the space-time is asymptotically flat. It is possible to define at null infinity a notion of energy, which leads to the result found by Bondi

$$E = \frac{1}{4\pi}\oint M\sin\theta d\theta d\phi, \tag{11}$$

and in addition the energy flux per unit solid angle, which is given by

$$\frac{d^2E}{dvd\Omega} = -\frac{\left(c_{,v}\right)^2}{4\pi}. \tag{12}$$

It is clear that the information about the energy carried by gravitational waves is contained in the Bondi news function $c_{,v} \approx r\gamma_{,v}$, where the approximation is assumed to hold at large distances in the linear regime. Expressing Eq. (12) in terms of γ gives

$$\frac{d^2E}{dvd\Omega} \approx -\frac{r^2\left(\gamma_{,v}\right)^2}{4\pi}. \tag{13}$$

An analogous derivation can be achieved within the Newman-Penrose formalism. The key point is that Ψ_4 can be expressed, when computed in the *quasi-Kinnersley* tetrad, directly as a function of Riemann tensor components, i.e. [5, 44]

$$\left(\Psi_4\right)_{qKT} = -\left(R_{\hat{v}\hat{\theta}\hat{v}\hat{\theta}} - iR_{\hat{v}\hat{\theta}\hat{v}\hat{\phi}}\right). \tag{14}$$

The hatted symbols in Eq. (14) are indicating that the Riemann tensor components are contracted over a tetrad of vectors oriented along the coordinates. In the linear regime however, those vectors can be assumed to be the basis coordinate vectors, as the perturbation is already contained in the Riemann tensor. For this reason we will omit the hatted symbols from now on, and always talk about coordinate components.

The components of the Riemann tensor in Eq. (14) can then be related to the transverse-traceless gauge terms, using $R_{v\alpha v\beta} = -\frac{1}{2}\frac{\partial^2 h_{\alpha\beta}}{\partial v^2}$, which leads to the result

$$\left(\Psi_4\right)_{qKT} = -\frac{1}{2}\left(\frac{\partial^2 h_{\theta\theta}}{\partial v^2} - i\frac{\partial^2 h_{\theta\phi}}{\partial v^2}\right). \tag{15}$$

This relation between Ψ_4 in the *quasi-Kinnersley* tetrad and the transverse–traceless (TT) components of the perturbed metric leads us to a definition of the energy emitted by simply calculating the expression of the energy tensor for the gravitational wave defined by

$$T^{GW}_{\mu\nu} = \frac{1}{32\pi}\left[\partial_\mu\left(h^{TT}\right)^{\sigma\rho}\partial_\nu\left(h^{TT}\right)_{\sigma\rho}\right]. \tag{16}$$

The total energy flux is then given by the formula, which is assumed to hold at null infinity:

$$\frac{d^2E}{dvd\Omega} = -r^2\left(T^{GW}\right)^r{}_v = \frac{r^2}{16\pi}\left[\left(\frac{\partial h^{TT}_{\theta\theta}}{\partial v}\right)^2 + \left(\frac{\partial h^{TT}_{\theta\phi}}{\partial v}\right)^2\right], \tag{17}$$

and, by substituting our expression in terms of Ψ_4 we get the result

$$\frac{d^2E}{dvd\Omega} = -\frac{r^2}{4\pi}\left|\int_0^v\left(\Psi_4\right)_{qKT}dv\right|^2. \tag{18}$$

In our specific case of axisymmetry we expect only one polarization state to be present, which is made evident by the presence of a single news function. Correspondingly, we expect $\left(\Psi_4\right)_{qKT}$ to have only its real part non vanishing. Combining Eq. (18) with Eq. (13) and considering the presence of only one polarization state, we finally obtain that in the linearized regime the relation

$$\left(\Psi_4\right)_{qKT} = -\frac{\partial^2\gamma}{\partial v^2} \tag{19}$$

must hold. This is the relation we want to verify numerically. The minus sign comes from the negative sign given in Eq. (15). We want to stress again the attention to the fact that such relation is strictly true at null infinity, however, we expect it to be well satisfied provided we are at sufficiently large distances from the black hole. As it will be clear in the next section, this assumption turns out to be very well motivated.

V. NUMERICAL RESULTS

In this section we present numerical results for a standard simulation. We have written a code that solves the Bondi equations and calculates the Weyl scalars in the

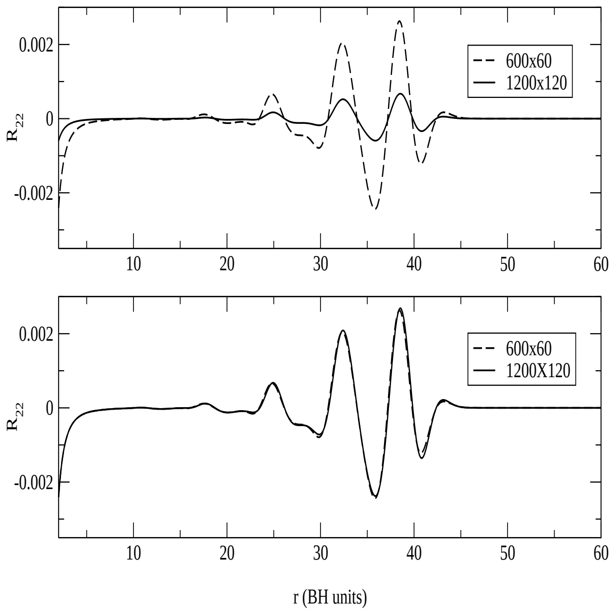

FIG. 2: Convergence test for the R_{22} component of the Ricci tensor. The top panel shows this component for two different resolutions for a radial slice on the equatorial plane. As expected the value is converging to zero. In the bottom panel we have tested the second order power law of convergence by multiplying the 1200x120 output by a factor of four. The two curves now overlap perfectly, thus proving second order convergence.

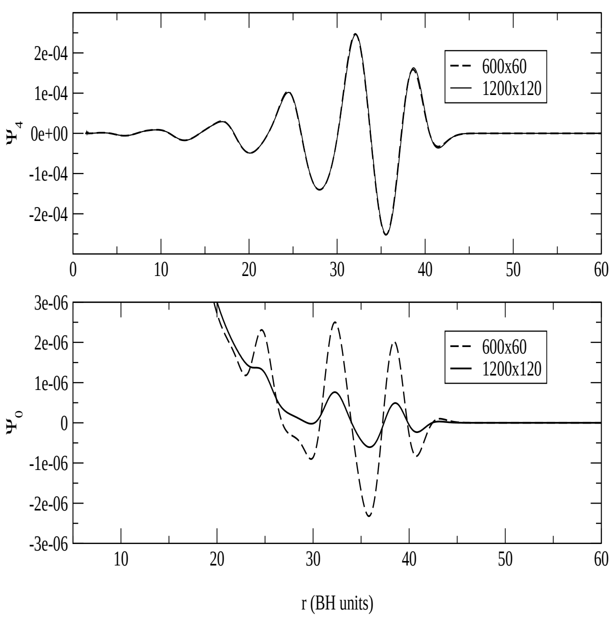

FIG. 3: (a) Top panel: the value for Ψ_4 for two different resolutions at time $v = 80$. (b) Bottom panel: the value for Ψ_0 for two different resolutions at $v = 80$. Both values are calculated on the equatorial plane.

two tetrads $\mathcal{T}_1$ and $\mathcal{T}_2$. Our code makes use of the Cactus infrastructure [51].

We set up an initial Schwarzschild black hole, and construct an initial quadrupole perturbation on γ using the expression indicated in Eq. (4). The values chosen in this case are $\lambda = 0.1$, $r_0 = 3$ and $\sigma = 1$, although various tries have been performed varying these parameters, all leading to the same physical results. We emphasize that λ represents the amplitude of the initial perturbation: the chosen value is such that the perturbation is somehow realistically small, yet large enough for the full non-linearity of the problem to appear clearly through the harmonic coupling, as we are going to show (see Figg. 10 and 11, cf. also [16]). All the results presented here are obtained using two different resolutions, the coarser one having 600 points in the radial dimension and 60 points in the angular direction, the finer one having those values doubled. The results which are not convergence test results are all obtained using the finer resolution of 1200 points in the radial direction and 120 points in the angular direction.

We will first present some tests in order to verify the robustness of our algorithm, and then we will proceed to a full comparison of our results in the two approaches presented here. The first two following subsections will deal with the calculation of the Weyl scalars in the tetrad $\mathcal{T}_1$ defined in Eq. (7); we don't expect the second tetrad to give different results for what concerns radial fall-offs and convergence. Section V C will instead deal with the relation of Ψ_4 with the news function and, within this context, it is very important to show a comparison of results in different tetrads, to have an evident demonstration of how important the choice of the right tetrad is, i.e. the quasi-Kinnersley tetrad, in the process of evaluating the outgoing gravitational wave contribution.

A. Convergence

The first thing we want to test in our code is of course convergence. In order to do so, once the numerical variables are computed, we have calculated independently the values of the Ricci tensor components, which should vanish in vacuum. Such components are suitable for doing convergence tests. In Fig. (2) we show the value of the Ricci component R_{22} for two resolutions, the picture shows a radial slice of our space-time on the equatorial plane, for the time value $v = 80$. The first figure simply superimposes the two values obtained for the two different resolutions, while in the second picture we have first multiplied the values for the finer resolution by a factor of four, as expected in a second order convergence code.

We have found similar results for the other components, which ensured us of the convergence of our algorithm.

B. Radial fall-offs of the scalars

Figg. (3a) and (3b) show the numerical output for Ψ_0 and Ψ_4 for two different numerical resolutions. The outputs show satisfactory convergence for Ψ_4, but not for Ψ_0. This is because the asymptotic radial behavior for Ψ_0 should be r^{-5}, as expected from the peeling-off conjecture (see e.g. [52]) and the linear perturbation analysis [44], and this gets completely embedded in the numerical error. We believe that this could constitute a serious numerical problem in situations where the initial tetrad chosen for the scalars computation is not the right one, and a tetrad rotation is needed. The numerical error found in Ψ_0 would then propagate when other quantities, like the curvature invariants I and J, are computed, thus leading to meaningless results. Recall, however, that the curvature invariants I, J, in addition to the Coulomb scalar χ and the Beetle–Burko scalar ξ, can be found invariantly and in a background-independent way which is also tetrad-independent, i.e., it does not require finding first the Weyl scalars to find I, J [1, 3].

Fig. (4a) and (4b) emphasize the radial dependence of Ψ_2 and Ψ_4, which is highlighted very well in our simulations. The two figures show that at late times Ψ_2 gets the background contribution with the superposition of a wave whose radial behaviour is r^{-3}. We have tested the convergence of such a wave to prove its physical meaning; this is itself a quite interesting result as we don't have a perturbation equation for Ψ_2, and it is entirely due to the full non-linear treatment of the problem. Of course, given its rapid fall-off, the wave contribution from Ψ_2 is negligible. Ψ_4 shows instead the well expected r^{-1} behaviour.

C. Relation of Ψ_4 with the Bondi news

In this section we want to show the comparison of Ψ_4 with the second time derivative of γ, where Ψ_4 will be calculated in the two tetrads shown in Eq. (7) and (8). We first start with the tetrad $\mathcal{T}_1$: Figg. (5) and (6) verify numerically the equivalence expressed by Eq. (19): it is clear that the two functions Ψ_4 and $-\gamma_{,vv}$ are different in the non-linear regime but converge in the linear regime. In particular Fig. (6) shows in logarithmic scale the absolute value of their difference at time $v = 80$, well in the linear regime. This numerical result proves the generic assumption that Ψ_4 is related to the outgoing gravitational radiation contribution.

As a counterexample, we show the same result when Ψ_4 is computed in tetrad $\mathcal{T}_2$, which would actually have been our simplest choice hadn't we applied the concept of a *quasi-Kinnesley tetrad*. The results for this calculation are shown in Fig. (7). It is evident that Ψ_4 does not get any contribution from the background, meaning that the tetrad we have chosen is part of the *quasi-Kinnersley frame*, however, it is evident that the result is rather different from that coming from the Bondi function γ.

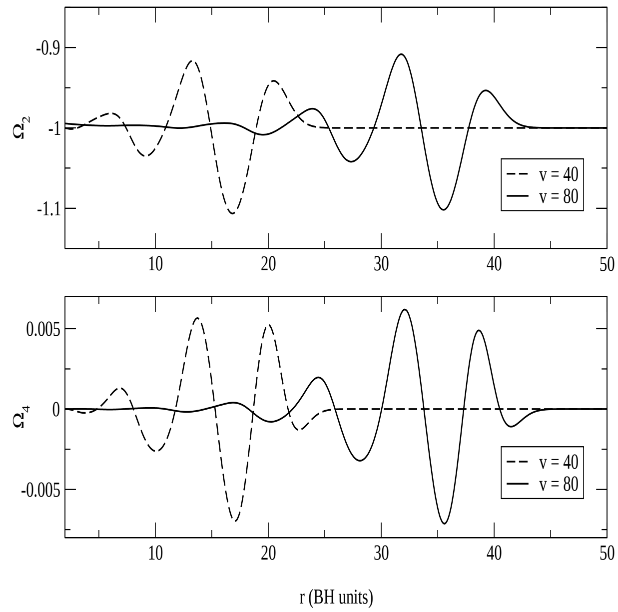

FIG. 4: (a) Top panel: the value for $\Omega_2 = r^3\Psi_2$ at two different times $v_1 = 40$ and $v_2 = 80$. (b) Bottom panel: the value for $\Omega_4 = r\Psi_4$ for the same couple of times $v_1 = 40$ and $v_2 = 80$.

In order to understand what is happening, we need to analyze further the tetrad $\mathcal{T}_2$, and in particular its limit when the space-time approaches a type D one. Using our well known asymptotic limits for the Bondi functions, it is easy to show that tetrad $\mathcal{T}_2$ converges, in the type D limit, to the tetrad

$$\ell^\mu = [0, -1, 0, 0], \tag{20a}$$

$$n^\mu = \left[1, \frac{r-2M}{2r}, 0, 0\right], \tag{20b}$$

$$m^\mu = \left[0, 0, \frac{1}{\sqrt{2}r}, \frac{i}{\sqrt{2}r\sin\theta}\right], \tag{20c}$$

which is different from the Kinnersly tetrad used by Teukolsky, Eq. (6). A simple analysis of the differences let us conclude that the original tetrad Eq. (6) can be obtained by first of all exchanging the two real null vectors ℓ and n, and then using a boost transformation of the type

$$\ell \to A\ell, \tag{21a}$$

$$n \to A^{-1}n, \tag{21b}$$

where A is a real parameter. It is easy to show that choosing $A = \frac{2r}{r-2M}$ we get that the new real null vectors coincide with the Kinnersley tetrad defined in Eq. (6). It is now straightforward to understand how these differences affect the values of the Weyl scalars. First of all, exchanging ℓ and n corresponds to exchanging Ψ_0 and Ψ_4; this means that if we use the tetrad $\mathcal{T}_2$ we will

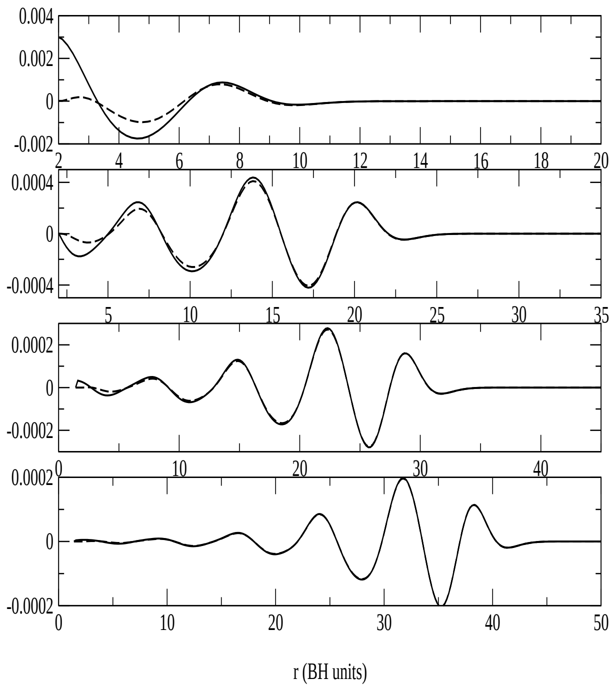

FIG. 5: From top to bottom: the comparison of Ψ_4 (dashed line) calculated in tetrad $\mathcal{T}_1$ and $-\gamma_{,vv}$ (solid line) for the values of $v_0 = 10$, $v_1 = 40$, $v_2 = 60$ and $v_3 = 80$.

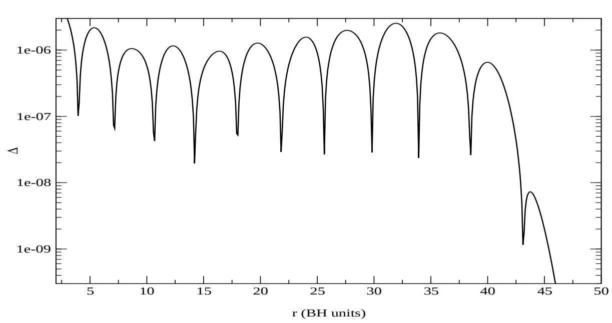

FIG. 6: The function $\Delta = |\Psi_4 + \gamma_{,vv}|$ at $v_3 = 80$.

find the outgoing radiative contribution in Ψ_0. This completely clarifies the result found in Fig. (7): it turns out that in this particular tetrad Ψ_4 is supposed to have a r^{-5} radial fall-off and, in practice, just like the result shown in Fig. (3a) for Ψ_0 in tetrad $\mathcal{T}_1$, we are not able to obtain this radial behaviour numerically, and we end up getting just numerical error.

In Fig. (8) we show the comparison of Ψ_0 with the news function; here the results are in better agreement but we still have no correspondence, the reason for this is to be found in the boost transformation, in fact a transformation like the one written in Eq. (21) changes the value of Ψ_0 according to

$$\Psi_0 \rightarrow A^{-2}\Psi_0. \tag{22}$$

This leads us to the final conclusion that, in the linearized regime, the following relation must hold:

$$(\Psi_4)_{\mathcal{T}_1} = \left(\frac{r-2M}{2r}\right)^2 (\Psi_0)_{\mathcal{T}_2} = -\frac{\partial^2\gamma}{\partial v^2} \tag{23}$$

We test this conclusion in Fig. (9) where we have plotted the value of $\left(\frac{r-2M}{2r}\right)^2 (\Psi_0)_{\mathcal{T}_2}$.

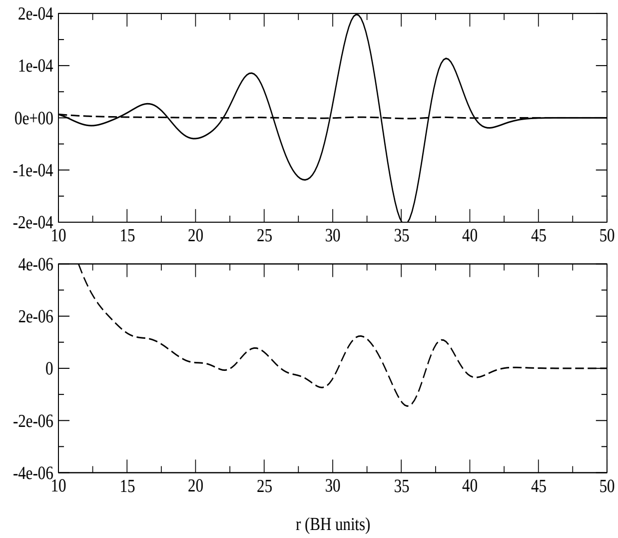

FIG. 7: (a) Top panel: the comparison of Ψ_4 (dashed line) calculated in tetrad $\mathcal{T}_2$ and $-\gamma_{,vv}$ (solid line) for $v = 80$. (b) Bottom panel: the value for Ψ_4 alone.

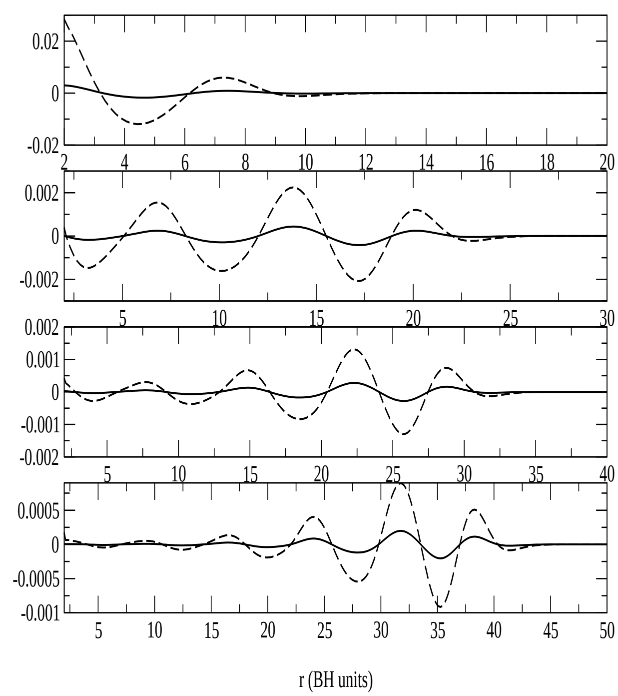

FIG. 8: From top to bottom: the comparison of Ψ_0 computed in the tetrad $\mathcal{T}_2$ and $-\gamma_{,vv}$ for the time values of $v_0 = 10$, $v_1 = 40$, $v_2 = 60$ and $v_3 = 80$. The dashed line is Ψ_0 while the solid line is $-\gamma_{,vv}$.

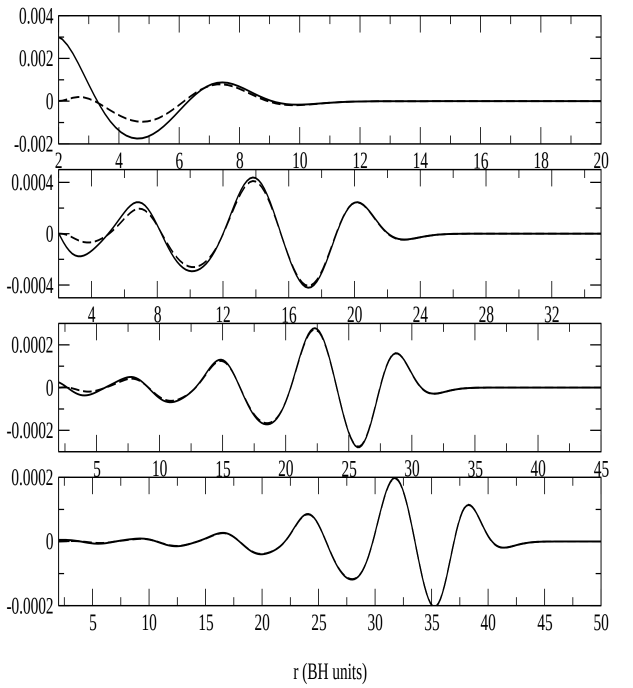

FIG. 9: From top to bottom: the comparison of $A^{-2}\Psi_0$ (dashed line) calculated in tetrad $\mathcal{T}_2$ and $-\gamma_{,vv}$ (solid line) for the values of $v_0 = 10$, $v_1 = 40$, $v_2 = 60$ and $v_3 = 80$.

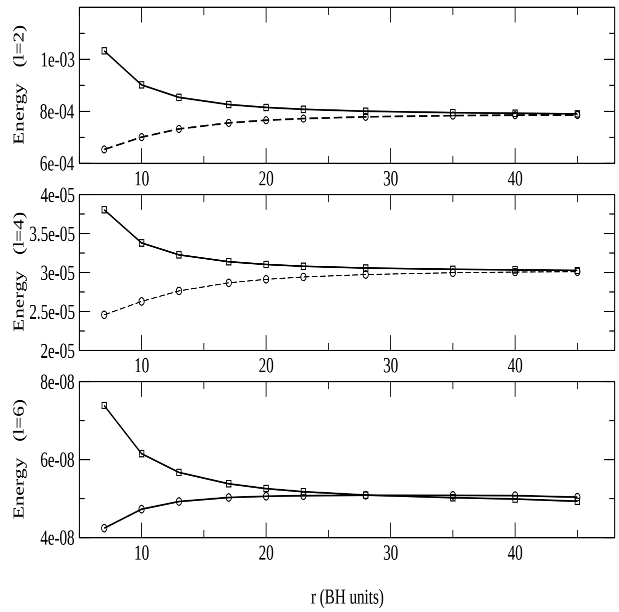

FIG. 10: From top to bottom: energy contribution of the $l = 2$ harmonic initial data. The three graphs represent the outgoing energy contribution for the values of $l = 2, 4, 6$. In each graph the two curves represent the value for the energy calculated using the Bondi news function (upper curve), while the lower curve uses the value of Ψ_4 in tetrad $\mathcal{T}_1$, as a function of the position of the observer. It is evident that at late times there is a convergence of the two values. This convergence seems to be less evident for the $l = 6$ case (lowest graph), but we expect this phenomenon to be purely numerical, because the numerical error on this multipole component is very high.

We emphasize that all the results that we have obtained in all the tetrads have wave-like profiles, although only one is the correct wave contribution. In practical numerical simulations one should really make sure that the tetrad in which the scalars are computed is a *quasi-Kinnersley* tetrad, otherwise the results, even if wave-like shaped, could be wrong.

D. Energy calculation

Having made sure that Ψ_4 calculated in tetrad $\mathcal{T}_1$ is related, in the linear regime, to the Bondi news function, we can use its expression to calculate the energy radiated from the black hole. In section IV we have shown that the expression of the energy flux per unit solid angle is given by

$$\frac{d^2E}{dvd\Omega} = -\frac{r^2\Phi^2}{4\pi}, \tag{24}$$

where we denote with Φ the generic expression for the news function, being it $\gamma_{,v}$ or $\int (\Psi_4)_{qKT}\, dv$. We can integrate the expression in Eq. (24) on a 2-sphere in order to obtain the energy flux. For the sake of simplicity we take a sphere of radius r_0, getting the result

$$\frac{dE}{dv} = -\frac{r_0^2}{4\pi}\oint \Phi^2 \sin\theta d\theta d\phi, \tag{25}$$

The computation of the energy flux was the goal of [16], where the news function was used to calculate the amount of energy which is carried away by each spin-weighted spherical harmonics of the outgoing radiation. We can perform a similar calculation using Ψ_4 and compare our results, in order to have a further demonstration of the validity of our approach. Given the results described in Section V C it is evident that also these results will be in good agreement, however we want to highlight their validity and to show their dependence on the position of the observer.

Since we are interested in the energy contribution of each spin weighted spherical harmonic, we first have to perform the decomposition of the signal into spin weighted spherical harmonics contributions. This is done by introducing the quantity Φ_l defined as

$$\Phi_l\,(v, r) = 2\pi \int_{-1}^{1} \Phi\,(v, r, y)\, Y_{2l0}\,(y)\, dy, \tag{26}$$

where $y = -\cos\theta$ and Y_{2l0} is the spin weighted spherical harmonic of spin 2.

Using Eqq. (25) and (26) we can get an expression for the total energy emitted in each angular mode after the

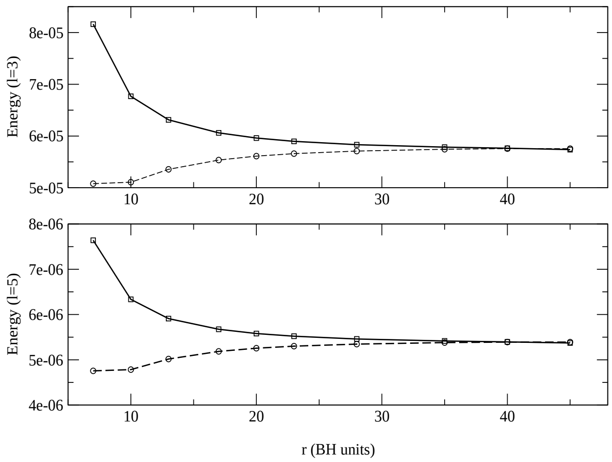

FIG. 11: From top to bottom: energy contribution of the $l = 3$ harmonic initial data. We show the two graphs corresponding to the dominant terms $l = 3, 5$ in the emitted gravitational signal. Again here we compare the result coming from the Bondi news function with the one using Ψ_4 in tetrad $\mathcal{T}_1$. The results are similar to the ones shown in Fig. (10).

evolution to a final time T, given by

$$E_l(T) = \frac{r_0}{4\pi}\int_0^T \left[\Phi_l(v, r = r_0)\right]^2 dv. \tag{27}$$

We have performed some numerical simulations where the energy, using both the Bondi news function and Ψ_4, has been calculated. The results are shown in Figg. (10) and (11).

Fig. (10) shows the result for a numerical simulation where the initial profile of the γ has been chosen to be quadrupolar, i.e. using the spin-weighted spherical harmonic with $l = 2$, $m = 0$. The non-linearity of the problem is translated into the fact that the evolution excites higher order multipolar terms. However, simmetry considerations allow only even multipolar terms to be excited. In the picture we show the energy at time $v = 80$ for the $l = 2, 4, 6$ terms. Such energy is calculated varying the position of the observer and it is evident that, as soon as we push the observer further from the source, the two energy calculations coincide. On the other hand, numerical errors become stronger when going higher order multipole terms, which explains the not-perfect convergence for the $l = 6$ terms.

Fig. (11) shows a similar simulation for an initial data with $l = 3$, $m = 0$. Here again we expect the non-linearity to excite the other harmonics. Differently from the $l = 2$ case, we don't expect to have forbidden modes, however our numerical results show that the highest amplitude modes are the odd ones, so we show only those modes. Anyway the results in this case are qualitatively equivalent to those obtained in the case of quadrupolar initial data.

VI. CONCLUSIONS

The problem of correctly extracting the gravitational signal in numerical simulations is of primary importance. We believe that the use of the Weyl scalars of the Newman-Penrose formalism offers a promising method for wave extraction, as it applies to any formulation of Einstein's equations and, even more important, to any kind of background we end up with, being it Schwarzschild or Kerr. However the problem of identifying the tetrad in which one is to compute the Weyl scalars still awaits a full solution. Recent work [1, 2] shows how to make the important first step of identifying an equivalence class of tetrads, the quasi-Kinnersley frame, of which the desired quasi-Kinnersley tetrad is a member. However, the problem of isolating the right tetrad out of this set is still under investigation.

In the present work we have considered a non-trivial numerical scenario, namely the evolution of a non-linearly perturbed black hole using Bondi coordinates, in order to show the importance of the tetrad choice for the calculation of wave related quantities. This particular scenario is well suited for a practical demonstration of the problems one would encounter if a careful choice of the tetrad for the Weyl scalar computation is not done. We have in fact shown that the computation of the Weyl scalars in an arbitrary tetrad, chosen by brute-force using mathematical packages like GrTensor, would lead to wrong results for Ψ_4, which is the quantity that typically is supposed to contain the (outgoing) gravitational wave degrees of freedom. This fact is evident in our case, where we have compared directly the results for the Weyl scalar in two different tetrads $\mathcal{T}_1$ and $\mathcal{T}_2$, using the Bondi news function in determining that the Weyl scalar corresponding to the quasi-Kinnersley tetrad is the right one. In Section V we have shown that only the quasi-Kinnersley tetrad $\mathcal{T}_1$ gives results in agreement with the news function.

Finally, we emphasize the importance of singling out an appropriate quasi-Kinnersley tetrad from the quasi-Kinnersley frame [1, 2]. For instance, the tetrad $\mathcal{T}_2$ after exchange of the ℓ and n null basis vectors is related to the tetrad $\mathcal{T}_1$ by a boost. This example indicates that every tetrad in the quasi-Kinnersley frame will give results for Ψ_0 and Ψ_4 that will show no contribution from the background, so that the wave-like shape of the scalars could lead us to the wrong conclusion of having the right outgoing gravitational signal. As we have shown, this conclusion could well be far from reality.

Acknowledgments

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