Towards a novel wave-extraction method for numerical relativity

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Abstract. We present the recent results of a research project aimed at constructing a robust wave extraction technique for numerical relativity. Our procedure makes use of Weyl scalars to achieve wave extraction. It is well known that, with a correct choice of null tetrad, Weyl scalars are directly associated to physical properties of the space-time under analysis in some well understood way. In particular it is possible to associate Ψ_4 with the outgoing gravitational radiation degrees of freedom, thus making it a promising tool for numerical wave–extraction. The right choice of the tetrad is, however, the problem to be addressed. We have made progress towards identifying a general procedure for choosing this tetrad, by looking at transverse tetrads where $\Psi_1 = \Psi_3 = 0$.

As a direct application of these concepts, we present a numerical study of the evolution of a non-linearly disturbed black hole described by the Bondi–Sachs metric. This particular scenario allows us to compare the results coming from Weyl scalars with the results coming from the news function which, in this particular case, is directly associated with the radiative degrees of freedom. We show that, if we did not take particular care in choosing the right tetrad, we would end up with incorrect results.

INTRODUCTION

With the rising interest in gravitational–wave detection, the importance of numerical simulations aimed at modelling possible sources of gravitational waves is growing. Possible sources of gravitational waves include binary systems of merging black holes, spiralling systems of two neutron stars or coalescing black hole–neutron star binaries. In order to simulate such physical events numerical relativity is required to study the strong field region of the coalescence. Most of the simulations which have been performed so far use formulation of Einstein's equations such as ADM [1, 2, 3] or BSSN [4, 5]. Using these formulations the problem of wave extraction arises, as none of the variables used for the numerical evolution are related directly to the radiative degrees of freedom.

In this paper we review the research progress we have done in the field of wave extraction using the Newman-Penrose formalism and, in particular, Weyl scalars [6, 7, 8, 9]. The final aim of this project is to have at hand a quantity which can be computed directly from the numerically evolved variables, and which is directly associated with the radiative degrees of freedom. The strength of this technique is that it is completely general, i.e. it does not depend upon the specific properties of the space-time under consideration. We apply these concepts to a specific numerical example, namely to the characteristic initial value problem [10, 11], in which we show the validity of our approach, and the problems that could arise if we were not careful enough in computing Weyl scalars.

WEYL SCALARS

When dealing with vacuum space-times, all the information about the curvature is encoded in the Weyl tensor C_{abcd}, which has ten independent components. All the ten degrees of freedom can be rewritten in a coordinate independent

way, through the introduction of the five complex Weyl scalars, defined as

$$\Psi_0 = C_{pqrs}\ell^p m^q \ell^r m^s, \tag{1}$$

$$\Psi_1 = C_{pqrs}\ell^p n^q \ell^r m^s, \tag{2}$$

$$\Psi_2 = C_{pqrs}\ell^p m^q \bar{m}^r n^s, \tag{3}$$

$$\Psi_3 = C_{pqrs}\ell^p n^q \bar{m}^r n^s, \tag{4}$$

$$\Psi_4 = C_{pqrs}\bar{m}^p n^q \bar{m}^r n^s, \tag{5}$$

where ℓ^p, n^p, m^p and $\bar{m}^p$ are a set of four null vectors, two real and two complex conjugates, satisfying the relations $\ell^p n_p = -1$ and $m^p \bar{m}_p = 1$. Of course the values of the scalars will depend on the particular tetrad choice, so it is our aim to identify the best tetrad choice for numerical relativity, so that Weyl scalars are directly associated with the relevant physics degrees of freedom [12, 11, 13, 14]. One approach is to calculate Weyl scalars in a fiducial tetrad, i.e. the easiest we can think of, and then perform tetrad transformations in order to get to the physically relevant tetrad. There are three sets of tetrad transformations which reflect, in tetrad language, the Lorentz group of degrees of freedom for the basis vectors; they are normally referred to as type I, type II and type III rotations; in particular type III rotations are also known as spin/boost transformations (for more details see [15, 7]). What we need is a general criterion, independent of the background space-time under study, for determining what is the best tetrad transformation to perform in order to get to the physically relevant tetrad. As we will see in the next section, this criterion is given by the notion of transverse frames.

TRANSVERSE FRAMES

We will show in this section and in the following that transverse frames are what we believe to be an attractive choice for performing wave extraction. Following [16, 6, 7], we define a transverse frame as a frame where $\Psi_1 = \Psi_3 = 0$. Again following the definitions given in those papers we call it frame instead of tetrad because it actually identifies a family of tetrads connected by a type III (spin/boost) transformation: the property $\Psi_1 = \Psi_3 = 0$ is in fact type III rotation invariant.

Finding a transverse frame is something that we can always achieve for a generic space-time. In fact, it is clearly shown in [16, 7] that there are three transverse frames for a generic Petrov Type I space-time. As a consequence of this, the definition of a transverse frame is general and does not depend at all on the properties of the background space-time we are dealing with.

Why transverse frames provide a general criterion for finding the right tetrad is explained in the next section. Here we just want to stress the attention on the generality of transverse frames: they always exist for a general space-time and can always be found using two tetrad rotations, namely a type I and a type II rotation (see [7] for further details).

THE QUASI-KINNERSLEY FRAME

The relevant property of a frame choice is that it has to converge to the Kinnersley frame when the space-time approaches a type D, i.e. that of an unperturbed black hole. If this is true, then the results found by Teukolsky ([14]) assure us that the Weyl scalars, in the linearized regime, are directly associated to the physical properties of the space-time, and in particular Ψ_4 is associated to the outgoing gravitational radiation degrees of freedom. To be more precise, in the linearized regime Ψ_4 can be directly related to the TT gauge expression of the perturbed metric:

$$\Psi_4 = \frac{\partial^2 h_+^{TT}}{\partial t^2} + i \frac{\partial^2 h_\times^{TT}}{\partial t^2}. \tag{6}$$

A *Kinnersley frame* for a type D space-time is a frame where the two real tetrad null vectors coincide with the two repeated principal null directions of the Weyl tensor. The fundamental property of a Kinnersley frame is that it has all Weyl scalars vanishing except Ψ_2. This makes it an ideal frame where to build perturbation theory, and in fact this is what was done by Teukolsky [14] for a perturbed Kerr space-time.

In his original article, Kinnersley [17] makes a second step with an additional condition that sets the spin coefficient ε to zero. This corresponds to fixing the additional degrees of freedom coming from a spin/boost transformation, i.e. to identifying a particular tetrad out of the Kinnersley frame.

The quasi-Kinnersley frame is then defined as a general frame that converges to the Kinnersley frame when the space-time approaches a type D. We have shown in [7] that one of the three transverse frames is a quasi-Kinnersley frame; this comes as a direct consequence of how a transverse frame sees the principal null directions of the space-time under study. Finding the transverse frames, and especially the one which is actually a quasi-Kinnersley frame, is for that reason a general criterion to compute Weyl scalars in the right tetrad, in a way that is completely independent from the properties of the background metric.

In our study of the quasi-Kinnersley frame we are also left with the problem of fixing the spin/boost degrees of freedom. It is unlikely that the best choice is to impose the condition $\varepsilon = 0$ in the quasi-Kinnersley frame, as this condition does not fix univoquely the spin/boost degeneracy of the frame, but only up to integration constants, so work is still in progress to find an equivalent condition which is also easier to implement.

A NUMERICAL APPLICATION: THE BONDI PROBLEM

We have applied the concepts of quasi-Kinnersly frame to a specific numerical simulation based on the characteristic initial value problem, originally introduced by Bondi and Sachs [10, 11]. This approach has been extensively studied in numerical relativity, for spherically symmetric systems [18, 19, 20, 21, 22, 23, 24], axisymmetric systems [25, 26, 27] and 3D systems [28, 29, 30, 31]. The results presented here are based on the procedure presented in [27], where Bondi coordinates are used in axisymmetry to study a non-linearly perturbed black hole.

In this particular case the criterion to find the quasi-Kinnersley frame does not follow from the introduction of transverse frames, but from simple considerations on the asymptotic behavior of the metric; it turns out, in fact, that the tetrad we choose here as quasi-Kinnerlsey tetrad, is not even transverse. The metric in Bondi coordinates has the following expression:

$$ds^2 = -\left[\left(1 - 2\frac{M}{r}\right)e^{2\beta} - U^2 r^2 e^{2\gamma}\right]dv^2 + 2e^{2\beta}dvdr - 2Ur^2 e^{2\gamma}dvd\theta + r^2(e^{2\gamma}d\theta^2 + e^{-2\gamma}\sin^2\theta d\phi^2), \quad (7)$$

where M, U, β, γ are unknown functions of the coordinates (v, r, θ). Within this framework, the Einstein equations decompose into three hypersurface equations and one evolution equation, as given below in symbolic notation

$$\Box^{(2)}\psi = \mathcal{H}_\gamma(M, \beta, U, \gamma), \quad (8a)$$
$$\beta_{,r} = \mathcal{H}_\beta(\gamma), \quad (8b)$$
$$U_{,rr} = \mathcal{H}_U(\beta, \gamma), \quad (8c)$$
$$M_{,r} = \mathcal{H}_M(U, \beta, \gamma), \quad (8d)$$

where $\Box^{(2)}$ is a 2-dimensional wave operator, $\psi = r\gamma$, and the various H symbols are functions of the Bondi variables. A more detailed description of the system defined in Eq. (8) can be found in [26, 25]. The algorithms for integrating that system of equations numerically are well known and tested.

The next step is to decide the initial conditions for our numerical evolution: the metric introduced in Eq. (7) describes a static Schwarzschild black hole if we set, in the Bondi frame, all the functions except M to zero everywhere in the domain. M is chosen to be the Schwarzschild mass M_0 of the black hole. Besides, outgoing gravitational radiation as a perturbation is introduced by the function γ: in fact γ is a spin-2 field and is actually related to the radiative degree of freedom. Choosing an initial shape for γ means in practice choosing the initial profile of outgoing gravitational waves. The typical profile that we set up for initial data has the following expression

$$\gamma(r, \theta) = \frac{\lambda}{\sqrt{2\pi}\sigma}e^{-\frac{(r-r_c)^2}{\sigma^2}}Y_{2lm}(\theta), \quad (9)$$

where λ is the amplitude of the perturbation, r_c is the center of the perturbation in r and σ its variance; finally Y_{lm} is the spherical harmonic of spin 2. For the numerical simulations we will consider here, we will set the initial data to be a pure quadrupole wave, i.e. $l = 2$ and $m = 0$ in $Y_{2lm}(\theta)$.

Tetrad Choices

Choosing the quasi–Kinnersley tetrad for our specific case is quite simple: the equations for the Bondi functions do not determine those completely, in fact we have of course the freedom coming from the integration constants. If we set those integration constants to be zero (Bondi frame) we know what is the asymptotic behavior of all the functions when we are far from the source, so γ, U and β tend to zero, while M tends to the Schwarzschild mass M_0 of the black hole.

In the Schwarzschild limit the Kinnersley tetrad in our coordinate system is given by

$$\ell^\mu = \left[\frac{2r}{r-2M}, 1, 0, 0\right], \tag{10a}$$

$$n^\mu = \left[0, -\frac{r-2M}{2r}, 0, 0\right], \tag{10b}$$

$$m^\mu = \left[0, 0, \frac{1}{\sqrt{2}r}, \frac{i}{\sqrt{2}r\sin\theta}\right]. \tag{10c}$$

From this we get that a general tetrad that converges to the Kinnersley tetrad, using our knowledge of the asymptotic behavior of the functions, is given by

$$\ell^\mu = \left[\frac{2}{\left[(1-2M/r)\,e^{4\beta} - U^2 r^2 e^{2(\gamma+\beta)}\right]}, e^{-4\beta}, 0, 0\right], \tag{11a}$$

$$n^\mu = \left[0, -\frac{\left[(1-2M/r)\,e^{2\beta} - U^2 r^2 e^{2\gamma}\right]}{2}, 0, 0\right], \tag{11b}$$

$$m^\mu = \left[0, \frac{rU e^{(\gamma-2\beta)}}{\sqrt{2}}, \frac{1}{\sqrt{2}r e^\gamma}, \frac{i}{\sqrt{2}r\sin\theta e^{-\gamma}}\right]. \tag{11c}$$

Another way to pick the tetrad for computing Weyl scalars is use algebraic manipulation packages like Maple and GRTensor. For the Bondi metric they would come out with the following expression for the tetrad:

$$\ell^\mu = \left[0, -e^{-4\beta}, 0, 0\right], \tag{12a}$$

$$n^\mu = \left[e^{2\beta}, \frac{\left[(1-2M/r)\,e^{2\beta} - U^2 r^2 e^{2\gamma}\right]}{2}, 0, 0\right], \tag{12b}$$

$$m^\mu = \left[0, \frac{rU e^{(\gamma-2\beta)}}{\sqrt{2}}, \frac{1}{\sqrt{2}r e^\gamma}, \frac{i}{\sqrt{2}r\sin\theta e^{-\gamma}}\right]. \tag{12c}$$

GRTensor gives this expression because it constructs the tetrad starting from the ℓ^μ vector, which is assumed to be lying on the null foliation, leading to the expression $\ell_\mu = \delta_{\mu 0}$. The contravariant components are then given by Eq. (12a). Once ℓ^μ is fixed, the other tetrad vector expressions are found by imposing the normalization conditions between the vectors in the Newman-Penrose formalism.

NUMERICAL RESULTS

We show here an example of the results for a numberical simulation of the Bondi metric, with wave extraction performed by using both news function and Weyl scalars. We set up an initial Schwarzschild black hole plus an
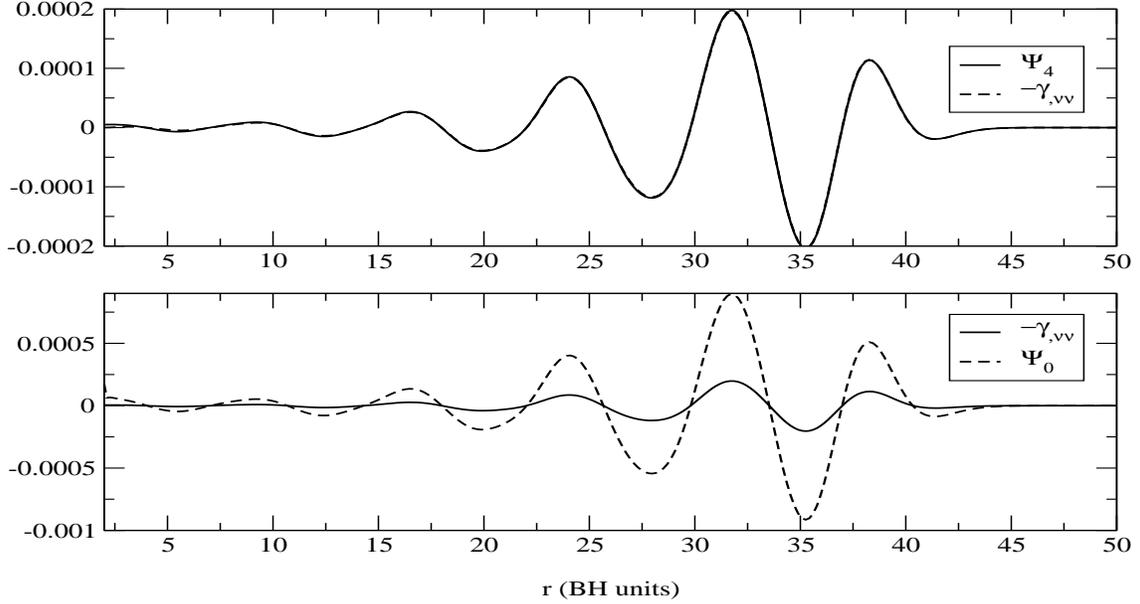
FIGURE 1. Comparison of different news functions. The upper figure shows the comparison of the second time derivative of the Bondi function γ with the value of Ψ_4 at time $v = 80$, i.e. in the linearized regime. It is clear that the two functions are in very good agreement. The same thing is done in the lower figure but this time using the second tetrad, in this case the outgoing contribution is contained in Ψ_0 but it is clear that the two news function do not agree, in fact we show in [9] that this second tetrad turns out to be boosted with respect to the quasi-Kinnersley tetrad.

initial quadrupole perturbation on γ using Eq. (9). In this particular case we set $\lambda = 0.1$, $r_0 = 3$ and $\sigma = 1$. The Bondi functions are then evolved and the wave is extracted using the news function, namely the time derivative of the Bondi function γ. We try to achieve the same results by using Weyl scalars. It is easy to see that, by means of equation 6, Ψ_4 has to be equal to the second time derivative of γ in the linearized regime provided it is computed in the right tetrad. So a very straightforward test to prove that we are in the right tetrad is to compare Ψ_4 with $\gamma_{,vv}$ in the linear regime, i.e. at late times and far from the source.

We have done this for our simulation, and a significant result is presented in Fig. (1). Here the news function is compared with Ψ_4 for the first tetrad, and with Ψ_0 in the second tetrad. The reason why we consider Ψ_0 for the second tetrad is due to the fact that in the second tetrad the ℓ null vector is ingoing instead of outgoing, and this changes the roles of the scalars in such a way that the outgoing contribution is in Ψ_0 instead of Ψ_4 (see [9] for further details). It is clear that the value for Ψ_0 in the second tetrad does not give the correct value for the gravitational radiation emitted by the source. Speaking in terms of tetrad rotations, this means that the second tetrad is actually boosted with respect to the quasi-Kinnersley tetrad. By comparing the two tetrads, we can compute the coefficient of the boost transformation that relates them, which is given by

$$A = \frac{2r}{r - 2M}. \tag{13}$$

This means that Ψ_0 in the second tetrad is related to Ψ_4 in the first tetrad by the relation

$$(\Psi_4)_{\mathscr{T}_1} = \left(\frac{r - 2M}{2r}\right)^2 (\Psi_0)_{\mathscr{T}_2}, \tag{14}$$

where $\mathscr{T}_1$ and $\mathscr{T}_2$ denote the two tetrads. It can be shown [9] that the results in the two tetrads coincide if we consider this additional boost factor. The derivation of this boost coefficient has been possible thanks to the determination of the quasi-Kinnersley tetrad: had we started directly from tetrad $\mathscr{T}_2$ we would have incorrectly associated the value of Ψ_0 to the gravitational radiation.

CONCLUSIONS

We have presented the recent progress made in the field of wave extraction using Weyl scalars. We believe that a promising technique for solving generally the problem of the tetrad choice is to look at transverse frames, i.e. those families of tetrads where $\Psi_1 = \Psi_3 = 0$. This approach has the advantage that for such families of tetrads it is possible to associate Ψ_4 with the outgoing radiative degrees of freedom in a background independent way. Our project still needs further work concerning how to break the spin/boost degeneracy in a background independent way. We have applied these concepts to a specific numerical example, using the Bondi-Sachs metric. In this particular case we can avoid the the spin/boost degeneracy problem and choose directly a quasi-Kinnersley tetrad for wave extraction, that is the one associated to the news function. Knowing this tetrad then allows us to explicitly show how choosing a different tetrad for wave extraction leads to wrong results.

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