Short Communication

"A note on using centroid weights in additive Multi-Criteria Decision Analysis"

Pierre L. Kunsch, *Corresponding Author*, Vrije Universiteit Brussel (VUB), BUTO Dept., Pleinlaan 2, Brussels, Belgium, email: pkunsch@vub.ac.be
Alessio Ishizaka, Portsmouth Business School, Centre for Operations Research and Logistics, University of Portsmouth, Portsmouth, PO1 3DE, United Kingdom,
email: alessio.ishizaka@port.ac.uk

Abstract

In this short communication, the authors challenge the comparison experiments made by several authors regarding the performances of surrogate weights frequently used in additive multicriteria decision analysis. Statistical tests compare the rankings obtained, either with random 'true weights' from a given weight simplex, or with the considered surrogate techniques. It is shown that the performance ranking of the surrogate techniques entirely depends on the way the weight simplex is defined. As an example, the rank-order-centroid weights, often found in this way to be the most performant ones, are only first-ranked when a point-allocation elicitation approach is adopted.

Keywords: Decision processes; Surrogate weights; Simplex Centroid; Comparison of performances

1. Introduction

Surrogate – meaning substitute - weights are often used in Multi-Criteria (or Multi-Attribute) Decision Analysis when only incomplete, purely ordinal information about the relative importance of the weights is available, often expressed as a ranking. Using surrogate weights thus means substituting default values to the weights. In this short communication we analyse the usefulness of a statistical procedure comparing different surrogate-weights techniques for obtaining a performance ranking. In section 2 this procedure is discussed in the light of existing studies, and of our own critical analysis. Conclusions are given in section 3.

2. Comparison experiments

The four 'basic' surrogates in the literature based on ordinal ranking properties are: Equal Weights (EW) (Dawes and Corrigan, 1974), Rank-Sum (RS) (Stillwell et al., 1981), Reciprocal of the weights (RR) (Stillwell et al., 1981), and Rank-Order Centroid (ROC) (Barron, 1992); (Barron & Barrett, 1996a; 1996b). For each surrogate Table 1 defines the corresponding weight domain with n criteria, $\left(w_i, i=1,2,\cdots,n\right)$, called in the following the 'simplex'. The last column on the right indicates which surrogate weights $\left(W_i, i=1,2,\cdots,n\right)$ are located at the simplex centroid, when assuming a uniform weight-vector distribution. In such cases $\left(W_i=\overline{w_i}, i=1,2,\cdots,n\right)$, where a bar on top of a variable indicates its average value; this property means that the simplex-centroid vector is the average weight vector of the simplex. This is well-known for EW, and it has been first recognised for the ROC weights in (Barron 1992); the RS centroid-property as being the average weight vector for the uniform distribution in the corresponding surrogate simplex has been established later in (Jia & al., 1998) based on order-statistics arguments (Arnold & al., 1992); (Kunsch & Ishizaka, 2018); (Kunsch & Brans, 2019) but it is not mentioned by later authors. The centroid vector is located at the barycentre or centre of gravity (COG) of each simplex. When weight vectors are uniformly distributed in the simplex, which is always assumed in the surrogate literature regarding RS, ROC, or EW, the COG minimises the squared deviations from the simplex vertices: the centroid vector is then given by the arithmetic averages of the simplex vertices (=extreme points) coordinates (Ahn, 2017). This explains the surrogate formulas in Table 1. Note that in Table 1 the EW and RS simplexes can be arbitrarily rescaled by any strictly positive factor; to obtain normalised EW and RS weights - i.e., such that $\sum_{i=1}^{n} W_i = 1$, as for the ROC surrogates where it automatically results from the simplex choice, the rescaling has been made with the factor $2/n$. This rescaling does not affect the ranking of alternatives obtained with the weighted sum.

Table 1. Properties of four ordinal surrogate weights: formulas and links with simplex centroid when assuming uniform weight-vector distributions.

Surrogate technique	**Simplex (rescaled for normalising EW&RS)**	**Surrogate formula**	**=Centroid vector?**
Rank-Sum (RS)	$2/n \geq w_1 \geq \cdots \geq w_n \geq 0$	$W_i = 2(n+1-i)/n(n+1)$	YES
Rank Order Centroid (ROC)	$1 \geq w_1 \geq \cdots \geq w_n \geq 0; \sum_{i=1}^{n} w_i = 1$	$W_i = 1/n \sum_{j=i}^{n} 1/j$	YES
Equal Weights (EW)	$2/n \geq w_i \geq 0$	$W_i = 1/n$	YES
Reciprocal of the weights (RR)	Same as ROC	$W_i = 1/i \Big/ \sum_{j=1}^{n} 1/j$	NO

Note also from Table 1 that RR weights are not the components of the centroid vector, therefore their formula is only empirical. To compare the performances of the four ordinal surrogate weights, (Barron & Barrett, 1996a; 1996b) have introduced a procedure, we call in the following B&B procedure for conciseness. The evaluations of alternatives and the resulting ranking are obtained in two different ways with two weighted sums. First a 'true weights' vector is drawn uniformly at random from the adopted simplex, and second, for each such vector, the surrogate-weight vector is found, sharing the same ordinal and normalisation properties. The rankings obtained from the two weighted sums are compared by means of statistical indicators to measure the surrogate efficacy. The mainly used indicator is the 'Hit Rate' expressing the percentage of "correct" first-ranks selection as assumed to be obtained by the 'true weights' vector. Kendall's tau measuring correlation in the range $[-1,+1]$ by counting the number of inversions between both rankings has also been used by some authors; it is highly correlated to the Hit Rate (Jia et al., 1998). The dominant performance-ranking pattern found in (Barrett & Barron, 1996a; 1996b) was ROC>RR>RS>EW, using the $(n-1)D$ 'ROC simplex' $w_1 > w_2 > \cdots > w_n \,; \sum_{i=1}^{n} w_i = 1$ of Table 1. A similar ranking has been obtained by other authors working in the same simplex: let us mention here in a non-exhaustive way (Butler et al., 1997); (Jia et al., 1998); (Ahn & Park, 2008); (Ahn, 2011; 2017); (Wang & Zionts, 2015) testing next to ordinal surrogates extensions incorporating additional preference information; (de Almeida Filho et al., 2018) testing ordinal-surrogates for use in the Promethee methodology (Brans & Vincke, 1985). These authors all placed ROC weights at the first rank among ordinal surrogates and recommended their use in additive multi-criteria or multi-attribute formulas. This result is of course not surprising because ROC weights are the components of the 'ROC simplex' centroid vector in Table 1 – i.e., the simplex of normalised and ranked weights. It is therefore strongly suspected that using the 'RS simplex' – i.e., the simplex of not-normalised ranked weights - we would observe RS>ROC; that using the 'EW simplex' – i.e. the simplex of not-normalised, not-ranked weights - we would observe EW>ROC. RR weights not being the simplex-centroid vector would never be first-ranked.

To confirm this, we have used the same setup as in the original B&B procedure comparing five values 3, 6, 9, 12, or 15 for n, the number of criteria/attributes, and five values 5, 10, 15, 20 or 25 for m, the number of alternatives. For each of the 25 n/m combinations, 100 alternative scenarios values were randomly generated in $[0,1]$, as well as 1,000 'true weights' vectors, which were drawn uniformly at random from the three different simplexes in Table 1. For each simplex we thus had 2,500,000 problems to compare with the four RS, ROC, EW, RR formulas in Table 1. We used Hit Rate, Kendall's tau, and Value Loss explained in the B&B procedure. We found for all 25 n/m situations – without any exception - that the centroid of each simplex was leading in performance, thus alternately RS, ROC, or EW for the three simplexes. These results are shown in Table 2 synthesising the results by indicating the maximum, mean and minimum indicator values of the 25 n/m situations. We show here only Kendall's tau results confirming the high correlation to Hit Rate results. The simulations within the 'ROC simplex' in Table 1 confirm the previous results of authors using the B&B procedure.

Table 2. Simulation of Kendall's tau for three different simplexes and four surrogate-weights methodologies. The best performances are indicated in bold.

Ordinal Surrogate	Not-normalised ranked			Normalised ranked			Not-normalised Not-ranked		
	MAX	MEAN	MIN	MAX	MEAN	MIN	MAX	MEAN	MIN
RS	**0.93**	**0.89**	**0.86**	0.84	0.80	0.77	0.66	0.58	0.54
ROC	0.86	0.80	0.76	**0.89**	**0.87**	**0.85**	0.58	0.49	0.43
EW	0.73	0.68	0.65	0.67	0.57	0.52	**0.74**	**0.69**	**0.66**
RR	*0.87*	*0.74*	*0.62*	*0.85*	*0.82*	*0.78*	*0.64*	*0.48*	*0.39*

The simulations in three different simplexes give two counterexamples contradicting the conclusion accepted by the above authors claiming that the ROC-weights best performs among ordinal surrogates. (Roberts & Goodwin, 2002); (Danielson & Ekenberg, 2014; 2016; 2017) also showed by simulations that ROC weights are only first in the performance ranking of ordinal surrogates when decision-makers are using a Point Allocation (PA) weight-elicitation approach. PA consists in considering a point budget, for example100, to be shared between the criteria corresponding to their relative importance. This is of course equivalent to including the constraint $\sum_{i=1}^{n} w_i = 1$ in the $(n-1)D$ 'ROC simplex' of Table 1. If on the contrary this constraint is removed and direct rating (DR) of the relative criteria importance is made on an arbitrary 0-1, or 0-100 scale, etc., they found that $RS > ROC > RR > EW$ - as observed in Table 2, for the nD 'RS simplex'. Note that (Roberts & Goodman, 2002) had formed so-called ROD weights (Rank-Order Distribution weights) obtained from the 'RS simplex' without normalisation, but subsequently normalising each individual 'true weights' vector. This subsequent normalising induces distribution non-uniformities, changing the position of the ROD-COG from the RS-COG. However, this operation only changes the scaling of the weighted sum, but not the alternatives ranking, therefore $ROD = RS$ regarding performances. (Danielson & Ekenberg, 2014; 2016; 2017) discuss the biases introduced by both elicitation techniques; they produced simulations for the ordinal case confirming the results in Table 2 for the ordinal surrogates of Table 1; they showed that the performances of cardinal-enriched surrogates are also linked to the choice of PA or DR, using respectively a $(n-1)$ or a n Degree-of-Freedom 'true weights' generator, or a mixture of both. However, none of the last-mentioned authors did provide the full explanation of this phenomenon to be searched in the simplex-centroid properties. Our short note fills a knowledge gap in this respect.

We now bring a stronger proof that, whatever the chosen simplex, the centroid vector of the simplex is so clearly privileged in the B&B procedure that no other surrogate formula is likely to bring a better performance, as measured by the rank indicators. This will prove in a rigorous way that simulations are superfluous for nominating a surrogate winner.

Let V be the multi-criteria (MC) random value of some alternative $V = \sum_{i=1}^{n} w_i v_i$ for some 'true weights' vector $\{w_i, i = 1, 2, ..., n\}$ in the adopted simplex, where $v_i, i = 1, 2, ..., n$ are the alternatives evaluations by criterion, utility values, Promethee flows, and so forth. Using again a bar above a random variable to indicate its mean value, we obtain the mean MC value $\overline{V} = \sum_{i=1}^{n} W_i . v_i$ where $\left\{ W_i = \overline{w_i}, i = 1, \cdots, n \right\}$ is the simplex-centroid vector.

Property: The least average squared deviation (LASD) from all random MC values V obtained with the 'true weights' vectors is achieved at the mean MC value $\overline{V}$ obtained with the simplex-centroid vector; the LASD is equal to the variance σ_V^2 of the random MC values V .

The direct proof of this well-known property of distributions in statistics is as follows – it is valid whatever the 'true weights' vector distribution, contrary to the least-square-deviation-from-the-vertices property mentioned in (Ahn, 2017), which is only valid for uniform distributions:

Let C be the deterministic MC value minimising the average squared deviation to all random V values in the simplex; we request $\overline{(V - C)^2} = \left(\overline{V^2} - 2\overline{V}.C + C^2 \right) \rightarrow \min$ by setting to zero the first derivative with respect to C . Thus $-2\overline{V} + 2C = 0$ and $C = \overline{V}$, $\overline{d^2} = \overline{V^2} - \overline{V}^2 = \sigma_V^2$, where $\overline{d^2}$ =LASD. The second-order condition is also verified as the second derivative=2 is positive. QED

The LASD is an important indicator of performance, although it has not been considered so far as we know in the surrogate-weights literature. It expresses the representability of the surrogate-weight formula with respect to all 'true weights' vectors in the given simplex; its minimum is obtained at the simplex centroid, yielding the best performance to the centroid vector.

3. Conclusion

The B&B procedure is unable to provide a generally valid performance ranking of surrogate weights. Each performance critically depends on the way 'true weights' are formed. The centroid vector of the 'true-weights' simplex presents the best performance, because its evaluation has on average the least squared deviation (LASD) to all 'true weights' evaluations. There is thus no need to use simulations to find the surrogate winner. Authors considering uniform distributions and a weight-elicitation technique based on Point Allocation, have found that ROC weights are the best ordinal performers, but this result is a special case among other elicitation possibilities, including non-uniform distributions in the simplex.

References

Ahn, B.S. (2011) Compatible weighting method with rank order centroid: Maximum entropy ordered weighted averaging approach, European Journal of Operational Research, 212,552-559.

Ahn, B.S. (2017). Approximate weighting method for multiattribute decision problems with imprecise parameters. Omega, 72, 87-95.

Ahn, B.S. & Park, K.S. (2008) Comparing methods for multi-attribute decision making with ordinal weights. Computers & Operations Research, 35, 1660-1670.
Arnold, B.C., Balakrishnan, N. & Nagaraja, N. (1992) A first course in order statistics, John Wiley & Sons, Inc., New York.
Barron, F. H. (1992). Selecting a best multiattribute alternative with partial information about attribute weights. Acta Psychologica, 80(1-3), 91-103.
Barron, F.H. & Barrett, B.E. (1996a). Decision quality using ranked attribute weights. Management Science 42(11), 1515-1523.
Barron, F.H. & Barrett, B.E. (1996b). The efficacy of SMARTER — Simple multi-attribute rating technique extended to ranking. Acta Psychologica 93(1–3), 23-36.
Brans, J.P., & Vincke, Ph. (1985). A preference ranking organization method. Management Science, 31(6), 647-656.
Butler, J., Jia, J. & Dyer, J (1997). Simulation techniques for the sensitivity analysis of multi-criteria decision models. European Journal of Operational Research, 103, 531-546.
Danielson M. & Ekenberg L. (2014) Rank ordering methods for multi-criteria decisions. In: Proceedings of the 14th Group Decision and Negotiation—GDN 2014. Springer
Danielson, M. & Ekenberg, L. (2016). The CAR Method for using preference strength in Multi-criteria Decision Making. Group Decision and Negotiation, 25, 775-797. https://doi.org/10.1007/s10726-015-9460-8
Danielson, M. & Ekenberg, L. (2017). A robustness study of state-of-the-art surrogate weights for MCDM. Group Decision and Negotiation 26, 677-691. https://doi.org/10.1007/s10726-016-9494-6
Dawes, R.M. & Corrigan, B. (1974). Linear models in decision-making. Psychological Bulletin, 81, 95-106.
de Almeida Filho, A.T., Clemente, T.R.N., Morais, D.C., & de Almeida, A.T. (2018). Preference modeling experiments with surrogate weighting procedures for the PROMETHEE method. European Journal of Operational Research, 264(2), 453-461.
Jia, J., Fisher, G.W., & Dyer, J.S. (1998) Attribute weighting methods and decision quality in the presence of response error: a simulation study. Journal of Behavioral Decision Making, 11, 85-105.
Kunsch, P.L., & Ishizaka, A. (2018) Multiple-criteria performance ranking based on profile distributions: An application to university research evaluations, Mathematics and Computers in Simulation, 154, 48-64.
Kunsch, P.L., & Brans, J.P. (2019) Visualising multi-criteria weight elicitation by multiple stakeholders in complex decision systems, Operational Research. An International Journal, DOI 10.1007/s12351-018-00446-0, Published on line 10 January 2019.
Roberts, R. & Goodwin, P. (2002) Weight approximations in Multi-Attribute Decision Models, Journal of Multi-Criteria Decision Analysis, 11, 291:303.
Stillwell, W.G., Seaver, D.A., & Edwards W. (1981). A comparison of weight approximation techniques in multiattribute utility decision making. Organizational Behavior and Human Performance, 28(1), 62-77
Wang, J., & Zionts, S. (2015). Using ordinal data to estimate cardinal values. Journal of Multicriteria Decision Analysis, 22, 185-196.