Characterization of Some Graph Classes Using Excluded Minors

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March 28, 2007

Abstract

In this article we present a structural characterization of graphs without K_5 and the octahedron as a minor. We introduce semiplanar graphs as arbitrary sums of planar graphs, and give their characterization in terms of excluded minors. Some other excluded minor theorems for 3-connected minors are shown.

Keywords: excluded minors, $\leq k$-sum of graphs, tree-width of a graph
Mathematics Subject Classification. 05C75, 05C83

1 Introduction

Excluded minor theorems characterize the graph classes containing no fixed graph (or graphs) as a minor. The basic excluded minor theorem is known Wagner's reformulation of Kuratowski's theorem: *A graph is planar if and only if it has no K_5 or $K_{3,3}$-minor.* The comprehensive overview about known excluded minor theorems can be found in [3] and [11].

The powerful tool for proving excluded minor theorems for 3-connected graphs is the well-known Tutte's Wheel Theorem [12] and its strengthening, Theorem 1.2 below, proved by Seymour [10].

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Theorem 1.1 (Wheel Theorem [12]) *Every 3-connected graph can be obtained from a wheel by repeatedly applying operations of adding an edge between two nonadjacent vertices and splitting a vertex.*

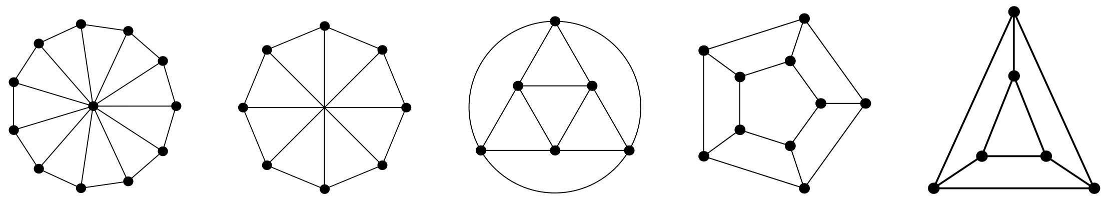
Theorem 1.2 [**10**] *Let H be a 3-connected minor of a 3-connected graph G such that if H is a wheel, then H is the largest wheel minor of G. Then there exists a sequence H_0, H_1, ..., H_k ($k \geq 0$) of 3-connected graphs such that H_0 is isomorphic to H, H_k is isomorphic to G, and for $i = 1, 2, \ldots, k$ the graph H_i is obtained from H_{i-1} either by adding an edge between two nonadjacent vertices or by splitting a vertex.*

Many excluded minor theorems can be reduced to a simple case checking using Theorems 1.1 and 1.2. This method is demonstrated by Thomas in [11]. Using a similar technique we present some new excluded minor theorems in Section 2. The main result of our research is a structural characterization of graphs without K_5 and the octahedron as a minor. We also introduce semiplanar graphs as arbitrary sums of planar graphs and give their characterization in terms of excluded minors. We notice that for planar graphs $\leq$ 3-sums and $\leq k$-sums ($k > 3$) correspond to the same class of graphs. The presented proof technique can be used to obtain some known results very effectively, which is shown on some examples.

Definitions

In this paper all graphs are finite and simple. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The operation of identifying the endvertices of an edge $e \in E(G)$ and deleting the resulting loop and parallel edges is called *contracting* the edge e. A graph H is a minor of G (or, equivalently said, G has an H-minor), if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges. We say that a graph G is obtained from a graph H by *splitting a vertex* if H is obtained from G by contracting an edge e, where e belongs to no triangle in G and both endvertices of e have degree at least three in G. By $G \setminus e$ (resp. $G \setminus v$) we denote a graph obtained from G by deleting the edge e (resp. the vertex v together with all edges adjacent to v).

Let G_1 and G_2 be graphs of order at least $k+1$. We say that G is a k-sum of graphs G_1 and G_2, if G is isomorphic to a graph, which can be obtained in the following way: choose cliques X_1 and X_2 of the same order k in G_1 and G_2, respectively, identify the cliques X_1 and X_2 in some way to a single clique and delete some edges (possibly none) of that clique. 0-sum corresponds to a disjoint union of G_1 and G_2. A k-sum will be also referred to a $\leq l$-sum for any $l \geq k$.



a) W_{11} b) V_8 c) the octahedron d) L_{10} e) L_6 (the prism)

Figure 1:

A graph G is internally 4-connected if it is 3-connected and for every two subgraphs G_1 and G_2 of G such that $G_1 \cup G_2 = G$, $|V(G_1) \cap V(G_2)| = 3$ either $|E(G_1)| \leq 3$ or $|E(G_2)| \leq 3$.

The *wheel* W_k $(k \geq 3)$ is a graph obtained from a circuit of order k by adding a new vertex joined to every vertex on the circuit (Fig. 1a). By V_8 we mean a graph obtained from a circuit of order eight by joining every pair of diagonally opposite vertices by an edge (Fig. 1b). The unique 4-connected planar triangulation of order 6 is called the *octahedron* (Fig. 1c). By L_{2k} we mean a cyclic planar *ladder* of order $2k$ consisting of two vertex-disjoint circuits with vertex sets $\{u_1, \ldots, u_k\}$ and $\{v_1, \ldots, v_k\}$ (in this order) and edges joining u_i and v_i for each $i = 1, 2, \ldots, k$ (Figs. 1d-e).

Notation. Let G be one of the graphs K_5, $K_{3,3}$, or L_6. We denote by G^* a unique graph (up to isomorphism) obtained from G by adding an edge between two nonadjacent vertices or splitting a vertex.

2 New Excluded Minors Theorems

Firstly, we prove a "meta-theorem", which allows us to focus only on 3-connected graphs:

Theorem 2.1 *Let $\mathcal{H}$ be a fixed set of 3-connected graphs and $\mathcal{M}_{\mathcal{H}}$ be the set of all 3-connected graphs with no $H \in \mathcal{H}$ as a minor. Then a graph G has no $H \in \mathcal{H}$ as a minor if and only if G can be obtained by means of $\leq$ 2-sums from copies of K_1, K_2, K_3, and graphs from $\mathcal{M}_{\mathcal{H}}$.*

Proof. The "if" part is easy. If G is a repeated $\leq$ 2-sum of graphs having no (3-connected) graph from $\mathcal{H}$ as a minor, then G also cannot have a minor from $\mathcal{H}$.

To prove "only if" part, assume for the contrary that there exists a graph with no minor from $\mathcal{H}$, which cannot be obtained by means of $\leq$ 2-sums from copies of K_1, K_2, K_3, and graphs from $\mathcal{M}_{\mathcal{H}}$. Let G be a such graph of the smallest order. Obviously, G is a 2-connected graph of order at least 4.

Now we prove by contradiction that G is 3-connected. If G is not 3-connected, then there exist two vertices u, $v \in V(G)$ such that $G \setminus \{u, v\}$ is a disconnected graph. Let G_1 and G_2 denote two subgraphs of G with the following properties: $|V(G_i)| < |V(G)|$ (for $i = 1, 2$), $G_1 \cup G_2 = G$ and $V(G_1) \cap V(G_2) = \{u, v\}$. For $i = 1, 2$ let G'_i be a graph obtained from G_i by adding an edge joining vertices u and v if there is no one (otherwise $G'_i := G_i$). The 2-connectivity of G implies that G'_1, G'_2 are minors of G. Thus, both graphs G'_1 and G'_2 can be constructed from given set of graphs and G is 2-sum of G'_1 and G'_2, a contradiction. Hence, G is necessarily the 3-connected graph. But $\mathcal{M}_{\mathcal{H}}$ is the set of all 3-connected graphs with no minor from $\mathcal{H}$. It means $G \in \mathcal{M}_{\mathcal{H}}$, a contradiction. $\Box$

Hence, to describe the structure of all graphs having no graph from a given set of 3-connected graphs as a minor, it is enough to characterize 3-connected graphs with such property. The following lemma contains some basic excluded minor results.

Lemma 2.2 *(i) Every 3-connected graph has K_4 as a minor.*

(ii) Every 3-connected graph except K_4 has W_4 as a minor.

(iii) Every 3-connected graph which has none of $K_{3,3}$, $K_5 \setminus e$, and L_6 as a minor is isomorphic to a wheel.

(iv) Let $k \geq 5$ be fixed. If a 3-connected graph G has none of W_k, $K_{3,3}$, $K_5 \setminus e$, and L_6 as a minor then G is isomorphic to a wheel W_l $(l < k)$.

(v) If a 3-connected graph G has L_{10} as a minor, but it contains neither K_5 nor the octahedron as a minor, then G is isomorphic to L_{10}.

(vi) If a 3-connected graph G has V_8 as a minor, but K_5 is not a minor of G, then G is isomorphic to V_8.

Proof. The both cases (i) and (ii) follow directly from Theorems 1.1 and 1.2. Furthermore, the case (iii) easily implies the case (iv).

(iii) As follows from Theorem 1.1 every 3-connected graph can be obtained from a wheel W_l $(l \geq 3)$ by repeatedly applying the operations of adding an edge and splitting a vertex. If $l \geq 4$, then every graph obtained from W_l by adding an edge and splitting a vertex has one of the graphs L_6, $K_{3,3}$, or $(K_5 \setminus e)$ as a minor. If $l = 3$, then the graph is isomorphic to a wheel.

(v) Suppose for the contrary that G is not isomorphic to L_{10}. As follows from Theorem 1.2, G can be constructed from L_{10} by applying operations of adding an edge or splitting a vertex. But every of three nonisomorphic graphs obtained from L_{10} by adding an edge has either the octahedron or K_5 as a minor, a contradiction.

(vi) It is easy to see that there are (up to isomorphism) exactly two graphs that can be obtained from V_8 by adding an edge between two nonadjacent vertices. However, both these graphs have K_5 as a minor. Hence, if G is not isomorphic to V_8, then K_5 is also a minor of G as follows from Theorem 1.2, a contradiction. □

In [11] it is proved, that an internally 4-connected nonplanar graph has K_5 or V_8-minor, or it is isomorphic to $K_{3,3}$. Due to Lemma 2.2(vi), we are able to give a stronger result:

Lemma 2.3 *Every internally* 4*-connected nonplanar graph has* K_5*-minor, or it is isomorphic to* $K_{3,3}$ *or* V_8*.*

Proof. Let G be an internally 4-connected nonplanar graph with no K_5-minor, which is not isomorphic to $K_{3,3}$. Then due to the minor properties of internally 4-connected graphs proved in [11], V_8 is a minor of G and hence G is isomorphic to V_8 by Lemma 2.2(vi). □

It is known that every nonplanar 3-connected graph with no $K_{3,3}$-minor is isomorphic to K_5 (see [11] or [13]). Applying Theorem 1.2 we obtain the structural characterization of 3-connected graphs with no $K^*_{3,3}$-minor.

Theorem 2.4 *Every* 3*-connected graph without* $K^*_{3,3}$ *as a minor is either a planar graph or isomorphic to* $K_{3,3}$ *or* K_5*.*

Proof. Let G be a nonplanar 3-connected graph without $K^*_{3,3}$ as a minor, which is not isomorphic to K_5 or $K_{3,3}$. A nonplanar 3-connected graph with no $K_{3,3}$-minor is isomorphic to K_5 ([11] or [13]), hence necessarily G has a $K_{3,3}$-minor. According Theorem 1.2 let H_0, H_1, $\ldots$, H_k be a sequence of graphs applied to $H_0 := K_{3,3}$ and $H_k := G$. Since G is not isomorphic to $K_{3,3}$, we see that $k > 0$. There is (up to isomorphism) only one possibility for H_1, namely $K^*_{3,3}$. Hence, $K^*_{3,3}$ is a minor of G, a contradiction. □

Dirac [4] characterizes graphs with no two disjoint circuits (or, in the case of 3-connected graphs equivalently with no prism as a minor), but his proof is rather complicated. The same result can be deduced easily using Theorem 1.2. We give here a sketch of the proof. Slightly different characterization of graphs with no prism as a minor can be found in [5].

Notation. By $K_{3,k}$ ($k \geq 3$) we denote the complete bipartite graph. Let $\{u_1, u_2, u_3\}$ denote nonadjacent vertices of the triple, which are connected with other k vertices. For $i = 1, 2, 3$ let $K^i_{3,k}$ denote the graph obtained from $K_{3,k}$ adding i edges between vertices $\{u_1, u_2, u_3\}$.

Theorem 2.5 *Every 3-connected graph without L_6 as a minor is isomorphic to the one of the following graphs: K_5, $K_5 \setminus e$, $K_{3,k}$, $K^1_{3,k}$, $K^2_{3,k}$ $K^3_{3,k}$, ($k \geq 3$), or a wheel.*

Proof. [Sketch of the proof] As follows from Theorem 1.1 every 3-connected graph G without L_6 as a minor can be obtained from a wheel W_l ($l \geq 3$) by repeatedly applying operations of adding an edge and splitting a vertex without introducing L_6 as a minor. For the graph G we introduce two classes of (nonisomorphic) graphs:

$$\mathcal{N}(G) = \{H, H \text{ is obtained from } G \text{ by adding an edge or splitting a vertex}\}, \tag{1}$$

$$\mathcal{N}_0(G) = \{H, H \in \mathcal{N}(G) \text{ and } L_6 \text{ is not a minor of } H\}. \tag{2}$$

Trivially, $\mathcal{N}(W_3) = \mathcal{N}_0(W_3) = \emptyset$. Furthermore, it is easy to check that $\mathcal{N}_0(W_l) = \emptyset$, if $l \geq 5$, and $\mathcal{N}_0(W_4) = \{K_{3,3}, K_5 \setminus e\}$, $\mathcal{N}_0(K_5 \setminus e) = \{K_5, K^1_{3,3}\}$, $\mathcal{N}_0(K_5) = \emptyset$. It can be also verified that for $k \geq 3$: $\mathcal{N}_0(K_{3,k}) = \{K^1_{3,k}\}$, $\mathcal{N}_0(K^1_{3,k}) = \{K^2_{3,k}\}$, $\mathcal{N}_0(K^2_{3,k}) = \{K^3_{3,k}, K_{3,k+1}\}$, and $\mathcal{N}_0(K_{3,k}) = \{K^1_{3,k+1}\}$.

Hence every graph which can be obtained from a wheel W_l ($l \geq 3$) by repeatedly applying operations of adding an edge and splitting a vertex without introducing L_6 as a minor, is isomorphic to the one of the following graphs: K_5, $K_5 \setminus e$, $K_{3,k}$, $K^1_{3,k}$, $K^2_{3,k}$ $K^3_{3,k}$, ($k \geq 3$). □

As easy consequence of the previous theorem and Theorem 1.2 we can prove

Theorem 2.6 *Every 3-connected graph without L^*_6 as a minor is isomorphic to the one of the following graphs: L_6, K_5, $K_5 \setminus e$, $K_{3,k}$, $K^1_{3,k}$, $K^2_{3,k}$ $K^3_{3,k}$, ($k \geq 3$), or a wheel.*

Proof. Let G be a 3-connected graph without L_6^* as a minor.

If L_6 is not a minor of G, then G is isomorphic to one of graphs listed in Theorem 2.5.

If G has an L_6-minor, then applying Theorem 1.2 to $H = L_6$ and G we obtain that L_6^* is a minor of G, a contradiction. Necessarily, G is isomorphic to L_6. □

In the following we introduce the class of semiplanar graphs and give their structural characterization.

Definition 2.7 *A graph is semiplanar if and only if it can be obtained from planar graphs by repeatedly applying arbitrary sums.*

Theorem 2.8 *For a graph G the following statements are equivalent:*

(i) G is a semiplanar graph,

(ii) G is a ≤ 3-sum of planar graphs,

(iii) G has none of graphs K_5 and V_8 as a minor.

Proof. (ii) $\Longrightarrow$ (i) It directly follows from Definition 2.7 of semiplanar graphs.

(i) $\Longrightarrow$ (iii) *No K_5-minor:* For the contradiction we assume that G is a semiplanar graph with K_5-minor such that all semiplanar graphs of smaller order than G have no K_5 as a minor. Obviously, G is not a planar graph. Due to the minimality of G, G_1 and G_2 have no K_5 as a minor and G is a ≤ 4- sum of graphs G_1 and G_2. Now it is easy to see, that G has no K_5 as a minor, a contradiction. *No V_8-minor:* Similarly as in the previous part we suppose that G is a semiplanar graph with V_8-minor and assume that all semiplanar graphs of smaller order than G have no V_8-minor. Obviously, G is a nonplanar 3-connected graph and hence, G is l-sum ($l \geq 3$) of some semiplanar graphs G_1 and G_2. Let G' denote the graph obtained from G by completing of subgraph $G_1 \cap G_2$ in G. G' is also the 3-connected semiplanar graph with no K_5-minor. But V_8 is a minor of G' and due to Lemma 2.2(vi), G' is isomorphic to V_8. But G' contains a triangle, a contradiction.

(iii) $\Longrightarrow$ (ii) Let G be a graph of the smallest order such that K_5 and V_8 are not minors of G and G is not a ≤ 3-sum of planar graphs. Obviously, G is a 3-connected graph. If G is not internally 4-connected then G is a 3-sum of graphs G_1 and G_2. Due to the minimality of G, the graphs G_1 and G_2 are ≤ 3-sums of planar graphs and the theorem holds.

If G is internally 4-connected then G is a planar graph or G is isomorphic to $K_{3,3}$ (Lemma 2.3). This is a contradiction with the assumption that G is not a ≤ 3-sum of planar graphs. □

Wagner [13] proved that a graph has no K_5-minor if and only if it can be obtained from planar graphs and V_8 by means of $\leq$ 3-sums. In [11] there is another proof of the same result based on the characterization of internally 4-connected graphs from Lemma 2.3. In the following lemma we give a short proof of the known characterization of graphs with no K_5^* as a minor (see [14]) based on the presented proof technique.

Lemma 2.9 *For every 3-connected graph G one of the following possibilities appears:*

(i) G is a ≤ 3-sum of planar graphs,

(ii) K_5^* is a minor of G,

(iii) G is isomorphic to K_5,

(iv) G is isomorphic to V_8.

Proof. Let G be a 3-connected graph and suppose that G is not a ≤ 3-sum of planar graphs (case (i)). In such case K_5 or V_8 is a minor of G by Theorem 2.8.

If V_8 is only a minor of G, then G is isomorphic to V_8 by Lemma 2.2 (case (iv)). Now suppose that K_5 is a minor of G and G is not isomorphic to K_5. But there is (up to isomorphism) only one graph, denoted K_5^*, which follows from K_5 in the sequence constructed according Theorem 1.2. It means, K_5^* is a minor of G (case (ii)). □

Theorem 2.10 *A graph has no K_5^*-minor if and only if it can be obtained from semiplanar graphs, K_5 and V_8 by means of ≤ 2-sums.*

Proof. It follows directly from Theorem 2.1 and Lemma 2.9. □

Halin and Jung [6] proved that every 4-connected graph has either K_5 or the octahedron as a minor. From this result it is easy to prove that every graph with minimum degree at least 4 has K_5 or the octahedron as a minor. Another self-contained proof of the previous result can be found in [2]. Maharry [7] presented a characterization of 4-connected graphs without the octahedron as a minor. In what follows we give a structural characterization of graphs which have neither K_5 nor the octahedron as a minor.

Definition 2.11 *Let G be a graph. Then G has tree-width k, $TW(G) = k$ if k is the smallest integer such that some supergraph of G is a $\leq k$-sum of graphs of order at most $k+1$.*

Another definition of tree-width based on a tree-decomposition was introduced by Robertson and Seymour (see [2], or [8] for the proof of its equivalence). Let TW_k denote the class of graphs with treewidth at most k. For any k, TW_k, can be characterized by a finite number of excluded (forbidden) minors. The full list of forbidden minors is known only for small values of k.

Lemma 2.12 *Let G be 3-connected graph which has neither K_5 nor the octahedron as a minor. Then one of the following holds:*

(i) $TW(G) \leq 3$,

(ii) G *is isomorphic to* V_8,

(iii) G *is isomorphic to* L_{10}.

Proof. Suppose that G is a 3-connected graph such that $TW(G) > 3$, G has neither K_5 nor the octahedron as a minor. Then necessarily V_8 or L_{10} is a minor of G, as follows from the known list of forbidden minors for TW_3 (see [1] or [9]). If V_8 and L_{10}, respectively, is a minor of G, then due to Lemma 2.2, G is isomorphic to V_8 and L_{10}, respectively. □

Theorem 2.13 *A graph has neither K_5 nor the octahedron as a minor if and only if it can be obtained from graphs with tree-width at most 3, V_8 and L_{10} by means of ≤ 2-sums.*

Proof. It easily follows from Theorem 2.1 and Lemma 2.12. □

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