Complexity of approximating bounded variants of optimization problems

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Abstract

We study low degree graph problems such as Maximum Independent Set and Minimum Vertex Cover. The goal is to improve approximation lower bounds for them and for a number of related problems like Max-B-Set Packing, Min-B-Set Cover, and Max-B-Dimensional Matching, $B \geq 3$. We prove, for example, that it is NP-hard to achieve an approximation factor of $\frac{95}{94}$ for Max-3-DM, and a factor of $\frac{48}{47}$ for Max-4-DM. In both cases the hardness result applies even to instances with exactly two occurrences of each element.

1 Introduction

This paper deals with combinatorial optimization problems related to bounded variants of Maximum Independent Set (Max-IS) and Minimum Vertex Cover (Min-VC) in graphs. We improve approximation lower bounds for low degree variants of them and apply our results to highly restricted versions of set covering, packing, and matching problems, including Maximum-3-Dimensional Matching (Max-3-DM).

It has been well known, that Max-3-DM is APX-complete (or MAX SNP-complete) even when restricted to instances with the number of occurrences of any element bounded by 3. To the best of our knowledge, the first inapproximability result for bounded Max-3-DM with the bound 2 on the number of occurrences of each element in triples, appeared in our paper [5], where the first explicit approximation lower bound for Max-3-DM problem was

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given. For less restricted matching problem, Max-3-Set Packing, an inapproximability result for instances with 2 occurrences follows directly from hardness results for Max-IS problem on 3-regular graphs [2], [3]. For the B-Dimensional Matching problem with $B \geq 4$, the lower bounds on approximability were recently proven by Hazan, Safra, and Schwartz [12]. A limitation of their method, as they explicitly state, is that it does not provide an inapproximability factor for 3-Dimensional Matching. But just the 3-dimensional case is of major interest, as inapproximability results for it allow to improve on hardness of approximation factors for several problems of practical interest, e.g., scheduling problems, some (even highly restricted) cases of generalized assignment problem, and other packing problems.

This fact, and an important role of low degree variants of Maximum Independent Set and Minimum Vertex Cover as intermediate steps in reductions to many other problems of interest, motivated our attempt to push the current technique to its limits.

We build our reductions on a restricted version of Maximum Linear Equations over $\mathbb{Z}_2$ with 3 variables per equation and with the (large) constant number of occurrences of each variable. Recall that this method, based on the deep Håstad's version of PCP theorem, was also used to prove $(\frac{117}{116} - \varepsilon)$-approximability lower bound for the Traveling Salesman problem by Papadimitriou and Vempala [15], and our lower bound of $\frac{96}{95}$ for the Steiner Tree problem in graphs [6]. In this paper we optimize equation gadgets (Section 2) and their coupling via *consistency gadgets* (Section 3) that are suitable for problems studied in low degree graphs. The notion of a consistency gadget varies slightly from one problem to another one. Generally speaking, consistency gadgets are graphs with suitable expanding (or mixing) properties. Interesting quantities, in which the lower bounds on efficient approximability can be expressed, are parameters of consistency gadgets that provably exist.

The approximation hardness results for Max-3-DM and Max-4-DM nicely complement the recent results of [12] on Max-B-DM given for $B \geq 4$. To compare our results with their for $B = 4$, we have better lower bound ($\frac{48}{47}$ vs. $\frac{54}{53} - \varepsilon$) and our result applies even to highly restricted instances with exactly two occurrences of each element in quadruples. On the other hand, their NP-hard type result has *almost perfect completeness*. But we can prove that approximation hardness results with almost perfect completeness cannot be achieved in our case. We do not elaborate on this fact in the paper, but the main idea is easy: under our 2-occurrence restriction, instances with perfect matching can by solved exactly by a polynomial time algorithm, and such algorithm can be *robust* and provide a matching that is almost perfect for instances with almost perfect matching.

The main new explicit NP-hardness factors of this paper are summarized in the following theorem. In more precise parametric way they are expressed in Theorems 17, 19, and 20. Better upper estimates on parameters from these theorems would immediately improve lower bounds given below.

Theorem. It is NP-hard to approximate:

- MAX-3-DM and MAX-4-DM to within $\frac{95}{94}$ and $\frac{48}{47}$ respectively, both results apply to instances with exactly two occurrences of each element;
- MAX-3-IS (even on 3-regular graphs) and MAX TRIANGLE PACKING (even on 4-regular line graphs) to within $\frac{95}{94}$;
- MIN-3-VC (even on 3-regular graphs) and MIN-3-SET COVER (with exactly two occurrences of each element) to within $\frac{100}{99}$;
- MAX-4-IS (even on 4-regular graphs) to within $\frac{48}{47}$;
- MIN-4-VC (even on 4-regular graphs) and MIN-4-SET COVER (with exactly two occurrences) to within $\frac{53}{52}$;
- MIN-B-VC ($B \geq 3$) to within $\frac{7}{6} - 12\frac{\ln B}{B}$.

Preliminaries and definitions

For a simple graph $G = (V, E)$, an *independent set* is a subset of vertices of G that are pairwise nonadjacent by an edge. A *vertex cover* in G is a subset of vertices of G containing at least one vertex from each edge $e \in E$. The MAXIMUM INDEPENDENT SET problem, resp. MINIMUM VERTEX COVER problem, asks for an independent set of maximum cardinality, resp. a vertex cover of minimum cardinality. Let $\alpha(G)$, resp. $vc(G)$, denote the corresponding optima. We use acronym B in the notation of any graph problem restricted to graphs of degree at most B.

A triangle packing for a graph $G = (V, E)$ is a collection $\{V_i\}$ of pairwise disjoint 3-sets of V, such that every V_i induces a triangle in G. The goal of the MAXIMUM TRIANGLE PACKING problem is to find a triangle packing of maximum cardinality.

The MAXIMUM SET PACKING (resp., MINIMUM SET COVER) problem is the following: Given a collection $\mathcal{C}$ of subsets of a finite set S, find a maximum (resp., minimum) cardinality collection $\mathcal{C}' \subseteq \mathcal{C}$ such that each element in S is contained in at most one (resp., in at least one) set in $\mathcal{C}'$. If each set in $\mathcal{C}$ is of size at most B, we speak about B-SET PACKING (resp., B-SET COVER). The MAXIMUM B-DIMENSIONAL MATCHING problem (MAX-B-DM) is a variant of a B-SET PACKING problem where the set S is partitioned into B subsets $S_1, \ldots, S_B$, and each set in $\mathcal{C}$ contains exactly one element from each of sets $S_1, \ldots, S_B$.

Let us recall the definition of MAX-E3-LIN-2 and some known results for restricted versions of this problem, that will be used later on.

Definition 1 MAX-E3-LIN-2 *is the following optimization problem: Given a system I of linear equations over $\mathbb{Z}_2$ with exactly 3 (distinct) variables in each equation. The goal is to maximize, over all assignments φ to the variables, the ratio $\frac{sat(\varphi)}{|I|}$, where $sat(\varphi)$ is the number of equations of I satisfied by φ.*

We use the notation Ek-MAX-E3-LIN-2 for this problem restricted to instances such that each variable occurs exactly in k equations. The following theorem follows from Håstad's results [11] (one can see also [5] for more details)

Theorem 2 (Håstad) *For every $\varepsilon \in \left(0, \frac{1}{4}\right)$ there is an integer $k(\varepsilon)$ such that for every $k \geq k(\varepsilon)$ the following problem is* NP*-hard: given an instance of* Ek-MAX-E3-LIN-2*, decide whether the fraction of more than $(1-\varepsilon)$ or less than $(\frac{1}{2}+\varepsilon)$ of all equations is satisfied by an optimal (i.e., maximizing) assignment.*

To use properties of our equation gadgets in optimal way, an order of variables in equations will also play a role. We denote by E$[k,k,k]$-MAX-E3-LIN-2 the restriction of E3k-MAX-E3-LIN-2 to instances such that each variable occurs exactly k times as the first variable, k times as the second variable, and k times as the third variable in equations. (For an equation $x+y+z=j$, $j \in \{0,1\}$, the first variable is x, the second one is y, and the third one is z.) Given an instance I_0 of Ek-MAX-E3-LIN-2, we can easily transform it into an instance I of E$[k,k,k]$-MAX-E3-LIN-2 with the same optimum, as follows: for any equation $x+y+z=j$ of I_0 we put in I the triple of equations $x+y+z=j$, $y+z+x=j$, and $z+x+y=j$. Hence the same NP-hard gap as in Theorem 2 applies for E$[k,k,k]$-MAX-E3-LIN-2 as well. We describe several reductions from E$[k,k,k]$-MAX-E3-LIN-2 to bounded occurrence instances of NP-hard problems that preserve the hard gap of E$[k,k,k]$-MAX-E3-LIN-2.

2 Equation Gadgets

The important part of our reduction for MAX-3-DM, and MAXIMUM INDEPENDENT SET, MINIMUM VERTEX COVER in low degree graphs are parametrized equation gadgets. For each equation $x+y+z=j$ ($j \in \{0,1\}$) of MAX-E3-LIN-2 we use an equation gadget G_j. We use slightly modified equation gadgets for distinct values for B in MAX-B-IS problem (or MIN-B-VC problem, respectively) to obtain better inapproximability results. For $j \in \{0,1\}$ we define equation gadgets $G_j[3]$ for MAX-3-IS problem (Fig. 1), $G_j[4]$ for 4(5)-MAX-IS (Fig. 2(i)), and $G_j[6]$ for MAX-B-IS $B \geq 6$ (Fig. 2(ii)). The vertices $\boxed{000}$, $\boxed{110}$, $\boxed{101}$, and $\boxed{011}$ are called *special vertices*. In each case the gadget

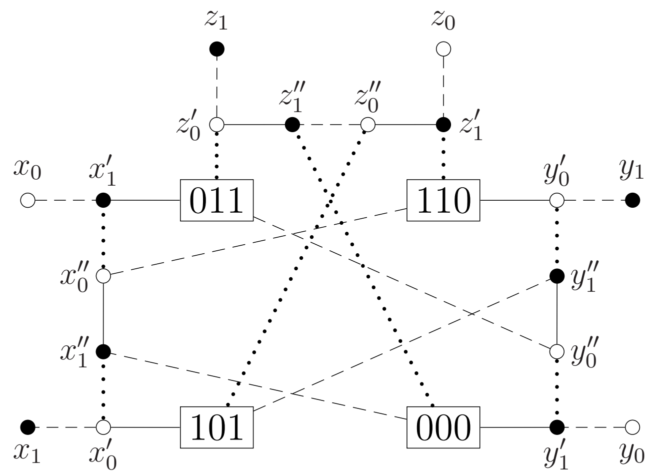

Fig. 1. The equation gadget $G_0 := G_0[3]$ for Max-3-IS and Max-3-DM.

$G_1[*]$ can be obtained from $G_0[*]$ replacing each $i \in \{0, 1\}$ in indices and labels by $1 - i$.

For each $u \in \{x, y, z\}$, we denote by F_u the set of all accented u-vertices from G_j (hence F_u is a subset of $\{u_0', u_1', u_0'', u_1''\}$), and $F_u := \emptyset$ if G_j does not contain any accented u vertex. Let further $T_u := F_u \cup \{u_0, u_1\}$. For a subset A of vertices of G_j and any independent set J in G_j, we will say that J *is pure in* A if all vertices of $A \cap J$ have the same lower index (0 or 1). If, moreover, $A \cap J$ consists exactly of all vertices of A of one index, we say that J *is full in* A.

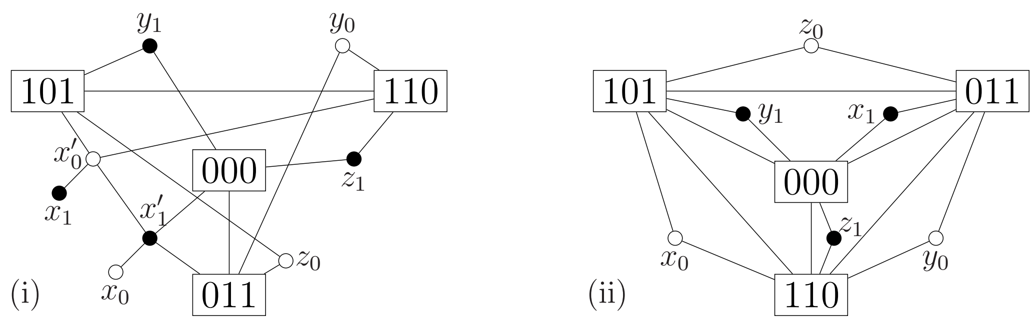

Fig. 2. The equation gadget (i) $G_0 := G_0[4]$ for Max-B-IS, $B \in \{4, 5\}$, (ii) $G_0 := G_0[6]$ for Max-B-IS ($B \geq 6$).

The following theorem describes basic properties of equation gadgets.

Theorem 3 *Let G_j ($j \in \{0, 1\}$) be one of the following gadgets: $G_j[3]$, $G_j[4]$, or $G_j[6]$, corresponding to an equation $x + y + z = j$. Let J be an independent set in G_j such that for each $u \in \{x, y\}$ at most one of two vertices u_0 and u_1 belongs to J. Then there is an independent set J' in G_j with the following properties:*

(I) $|J'| \geq |J|$,
(II) $J' \cap \{x_0, x_1, y_0, y_1\} = J \cap \{x_0, x_1, y_0, y_1\}$,

(III) $J' \cap \{z_0, z_1\} \subseteq J \cap \{z_0, z_1\}$ *and* $|J' \cap \{z_0, z_1\}| \leq 1$,

(IV) J' *contains exactly one of special vertices. Furthermore,* J' *is pure in* T_u *and full in* F_u *for each* $u \in \{x, y, z\}$.

PROOF. We prove the theorem for the gadgets of the form G_0, the modifications of proofs for G_1 are obvious. Let S stand for the set of four special vertices of a gadget G_0 under consideration.

A: The equation gadget for MAX-B-IS, $B \geq 6$ (Figure 2(ii)).
If J contains a special vertex, then clearly $|J \cap \{z_0, z_1\}| \leq 1$ and one can take $J' = J$. Assume now that J contains no special vertex. Let $\psi(x), \psi(y) \in \{0, 1\}$ be chosen in such way that $x_{1-\psi(x)} \notin J$ and $y_{1-\psi(y)} \notin J$. Let s be the special vertex in G_0 labeled by $\psi(x)\psi(y)\psi(z)$, where $\psi(z) = (\psi(x) + \psi(y)) \bmod 2$. If $z_{1-\psi(z)} \notin J$, then clearly one can take $J' = J \cup \{s\}$, otherwise one can obtain J' from J replacing $z_{1-\psi(z)}$ by s.

B: The equation gadget for MAX-B-IS, $B \in \{4, 5\}$ (Figure 2(i))

(a) Assume first, that J contains no special vertex. One can choose $\psi(x) \in \{0, 1\}$ such that $x_{1-\psi(x)} \notin J$, $x'_{1-\psi(x)} \notin J$, and $\psi(y) \in \{0, 1\}$ such that $y_{1-\psi(y)} \notin J$. Let s be the special vertex labeled by $\psi(x)\psi(y)\psi(z)$, where $\psi(z) = (\psi(x) + \psi(y)) \bmod 2$. If $z_{1-\psi(z)} \notin J$, then clearly one can take $J' = J \cup \{s, x'_{\psi(x)}\}$, otherwise $J' = (J \setminus \{z_{1-\psi(z)}\}) \cup \{s, x'_{\psi(x)}\}$.

(b) Assume now that J contains exactly one special vertex, say s, and let its label starts with $\psi(x) \in \{0, 1\}$. Then clearly $|J \cap \{z_0, z_1\}| \leq 1$. If $x_{1-\psi(x)} \notin J$, one can take $J' = J \cup \{x'_{\psi(x)}\}$. Otherwise one can modify J replacing s by $x'_{1-\psi(x)}$ to contain no special vertices, and to continue as in the case (a).

(c) If J contains 2 special vertices, then the label of one of them, say s_0, starts with 0, and the label of the other one, say s_1, starts with 1. From the structure of G_0 we can see that then $J \cap \{x'_0, x'_1\} = \emptyset$. Let further $\psi(x) \in \{0, 1\}$ be chosen such that $x_{1-\psi(x)} \notin J$. Now replacing $s_{1-\psi(x)}$ in J by $x'_{\psi(x)}$ will produce J' as required.

C: The equation gadget for MAX-3-IS (Figure 1)

(a) First we show that we can always modify J to J' satisfying (I), (II), and (III). For this purpose let J as above be fixed with both $z_0 \in J$ and $z_1 \in J$. Then clearly, $z'_0 \notin J$ and $z'_1 \notin J$. We can assume that either z''_0 or z''_1 is in J because otherwise we could either add z''_1 to J (if a special vertex $\boxed{000} \notin J$), or replace $\boxed{000}$ in J by z''_1, to ensure this property. Hence we will assume in what follows that $z_1'' \in J$ (the discussion for the case $z_0'' \in J$ is, due to symmetry, analogous).

So, we are in the situation $\{z_0, z_1, z_1''\} \subseteq J$, implying $z_0'' \notin J$, $\boxed{000} \notin J$. We can further assume that $\boxed{110} \in J$ (because otherwise replacing z_0 in J by z_1' we are done with this part of the proof).

(i) Assume first that $\boxed{101} \notin J$. Replacing z_1'' in J by z_0'' we reduce this to the case $\{z_0, z_1, z_0'', \boxed{110}\} \subseteq J$, $\boxed{101} \notin J$, $\boxed{000} \notin J$. We can further assume that $\boxed{011} \in J$ (because otherwise replacing z_1 in J by z_0' we are done). As both $\boxed{110}$ and $\boxed{011}$ belong to J, clearly $|F_x \cap J| \leq 1$. So we can modify J inside F_x, T_z and S to J' with $|J'| \geq |J|$ as follows. Let $j \in \{0, 1\}$ be fixed such that $x_{1-j} \notin J$. We take J' with $F_x \cap J' = \{x_j', x_j''\}$, $T_z \cap J' = \{z_{1-j}, z_{1-j}', z_{1-j}''\}$ and $S \cap J' = \{\boxed{j1(1-j)}\}$.

(ii) Assume now that $\boxed{101} \in J$. We can also assume that $\boxed{011} \notin J$ (because otherwise one could replacing $\boxed{101}$ in J by y_1'' obtain the situation already discussed in (i)). So, we have now $\{z_0, z_1, z_1'', \boxed{110}, \boxed{101}\} \subseteq J$, $\boxed{011} \notin J$, $\boxed{000} \notin J$. Clearly, $|F_y \cap J| \leq 1$. Now we can modify J inside F_y, T_z, and S to J' with $|J'| \geq |J|$ as follows. Let $j \in \{0, 1\}$ be fixed such that $y_{1-j} \notin J$. We take J' with $F_y \cap J' = \{y_j', y_j''\}$, $T_z \cap J' = \{z_{1-j}, z_{1-j}', z_{1-j}''\}$, and $S \cap J' = \{\boxed{1j(1-j)}\}$.

The proof of the part (a) is complete.

(b) After reduction from the part (a) we can assume that J is an independent set in G_0 such that for each $u \in \{x, y, z\}$ $|J \cap \{u_0, u_1\}| \leq 1$. Keep one such J fixed and denote by $\mathcal{J}$ the set of all independent sets J' in G_0 satisfying (II) and (III). Our aim is to prove that some of sets from $\mathcal{J}$ have to satisfy (I) and (IV) as well. In the following part we will prove that there is J' in $\mathcal{J}$ satisfying (I) and (IV′), where (IV′) is a slight relaxation of (IV), namely

(IV′) J' contains at most one special vertex and for each $u \in \{x, y, z\}$ the set J' is pure in T_u and full in F_u.

To prove that such J' exists, we will show that some extremal elements of $\mathcal{J}$ have this property. Choose $J' \in \mathcal{J}$ as follows: from all sets $J' \in \mathcal{J}$ with maximum cardinality consider those with the least number of special vertices, and from such sets the one which is pure in as many of T_x, T_y, T_z, as possible. Let us keep one such extremal $J' \in \mathcal{J}$ fixed. We will show that J' satisfies (IV$'$) ((I) being trivial). We will proceed in several steps.

Observation 1. If $u \in \{x, y, z\}$ and J' is pure in T_u, then it is full in F_u.

PROOF. Take $j \in \{0, 1\}$ such that $T_u \cap J'$ contains vertices with the lower index j only. Fix a vertex $v \in F_u$ with the lower index j, and show that $v \in J'$. Assume, on the contrary, that $v \notin J'$. As $J' \cup \{v\}$ is not an independent set, due to our choice of J', a neighbor of v (one of special vertices) belongs to J'. Replacing this special vertex in J' by v we obtain $J'' \in \mathcal{J}$ with $|J''| = |J'|$, but with less special vertices, a contradiction.

Observation 2. If $u \in \{x, y, z\}$ and J' is not pure in T_u, then one of the following possibilities occurs:

(i) $T_u \cap J' = \{u'_0, u'_1\}$ and both special vertices adjacent to u''_0 and u''_1 belong to J';
(ii) for some $j \in \{0, 1\}$: $T_u \cap J' = \{u_j, u''_{1-j}\}$ and both special vertices adjacent to u'_j and u''_j belong to J'.

PROOF. Assume first that $T_u \cap J' = \{u'_0, u'_1\}$. If for some $j \in \{0, 1\}$ the special vertex adjacent to u''_j does not belong to J', then replacing u'_{1-j} in J' by u''_j results in $J'' \in \mathcal{J}$ which is more pure than J', a contradiction.

Now it is clear, that if J' is not pure in T_u and the case (i) does not occur, then for some $j \in \{0, 1\}$, $T_u \cap J' = \{u_j, u''_{1-j}\}$. If the special vertex adjacent to u'_j (respectively, to u''_j) does not belong to J', then replacing u''_{1-j} in J' by u'_j (respectively, by u''_j) will result in $J'' \in \mathcal{J}$ which is more pure than J', a contradiction.

Observation 3. $|S \cap J'| \leq 2$.

PROOF. If for $p = 0$ or $p = 1$ we have $|S \cap J'| = 4 - p$, clearly for each $u \in \{x, y, z\}$, $|F_u \cap J'| \leq p$. We can then find $J'' \in \mathcal{J}$ pure in T_x, T_y, and T_z such that $|F_u \cap J''| = 2$ for each $u \in \{x, y, z\}$, and $S \cap J'' = \emptyset$. Clearly $|J''| \geq |J| + 2 - 2p \geq |J|$, and J'' has less special vertices than J', a contradiction.

Now we are ready to complete the proof of the part (b) showing that J' is, in fact, pure in each T_u, $u \in \{x, y, z\}$. Assume, on the contrary, that J' is not pure in at least one of T_x, T_y, T_z. Using Observations 2 and 3, we obtain that $|S \cap J'| = 2$. Let $S \cap J' = \{s_1, s_2\}$. There are 6 theoretical possibilities how this pair $\{s_1, s_2\}$ from S is chosen. But *each* pair $\{s_1, s_2\}$ of vertices from S has the following property that can be easily verified: There is $u \in \{x, y\}$ for which two vertices of F_u adjacent to $\{s_1, s_2\}$ have distinct indices and at least one of them belongs to $\{u'_0, u'_1\}$. This fact (together with $S \cap J' = \{s_1, s_2\}$) easily leads to a contradiction. Hence, J' is pure in each T_u, $u \in \{x, y, z\}$. By Observation 1, it is then even full in each F_u and clearly, $|S \cap J'| \leq 1$ will follow. This completes the proof of the part (b).

(c) We have already seen that an independent set J' satisfying (I), (II), (III), and (IV′) exists. Let for $u \in \{x, y, z\}$, $\psi(u) \in \{0, 1\}$ be such that $F_u \cap J'$ contains exactly all vertices of lower index $\psi(u)$. If $\psi(x) + \psi(y) + \psi(z) = 0$, $J' \cup \{\boxed{\psi(x)\psi(y)\psi(z)}\}$ is an independent set as required.

Otherwise one can add $\{\boxed{\psi(x)\psi(y)(1 - \psi(z))}\}$ to J', remove $z_{\psi(z)}$ from J', if it belongs to it, and modify J' in F_z to obtain J'' such that $F_z \cap J'' = \{z'_{1-\psi(z)}, z''_{1-\psi(z)}\}$. Now J'' is as required.

Hence the theorem is proved for all considered equation gadgets. $\square$

3 Consistency Gadgets

This section is devoted to graphs with certain expanding and mixing properties and therefore it can be also of independent interest. We study parameters of graphs, that are suitable as consistency gadgets for coupling our equation gadgets introduced in Section 2.

Definition 4 *A graph H is called a* consistency $(B, 3k)$-gadget*, if it has the following structure:*

(i) *The degree of each vertex is at most B.*
(ii) *There are $3k$ pairs of* contact vertices $\{(c_0^i, c_1^i) : i = 1, 2, \ldots, 3k\}$.
(iii) *The degree of any contact vertex is at most $B - 1$.*
(iv) *The first $2k$ pairs of contact vertices $\{(c_0^i, c_1^i) : i = 1, 2, \ldots, 2k\}$ are* implicitly linked *in the following sense: whenever J is an independent set in H, there is an independent set J' in H such that $|J'| \geq |J|$, a contact vertex c can belong to J' only if $c \in J$, and for any $i = 1, 2, \ldots, 2k$ at most one vertex of the pair (c_0^i, c_1^i) belongs to J'.*
(v) *The consistency property: Let us denote $C_j := \{c_j^1, c_j^2, \ldots, c_j^{3k}\}$ for $j \in$*

$\{0,1\}$, *and* $M_j := \max\{|J| : J$ *is an independent set in* H *such that* $J \cap C_{1-j} = \emptyset\}$. *Then* $M_1 = M_2$ $(:= M(H))$, *and for every* $\psi : \{1,2,\ldots,3k\} \to \{0,1\}$ *and for every independent set* J *in* $H \setminus \{c^i_{1-\psi(i)} : i = 1,2,\ldots,3k\}$ *we have* $|J| \le M(H) - \min\Big\{|\{i : \psi(i) = 0\}|, |\{i : \psi(i) = 1\}|\Big\}$.

To obtain better inapproximability results we use equation gadgets that require some further restrictions on degrees of contact vertices of a consistency $(B, 3k)$-gadget:

(iii-1) For Max-B-IS, $B \ge 6$, the degree of any contact vertex is at most $B - 2$.
(iii-2) For Max-B-IS, $B \in \{4, 5\}$, the degree of any contact vertex c^i_j with $i \in \{1,\ldots,k\}$ is at most $B-1$, and the degree of c^i_j with $i \in \{k+1,\ldots,3k\}$ is at most $B - 2$, where $j = 0, 1$.

Remark 5 *Let* $j \in \{0,1\}$ *and* J *be any independent set in* $H \setminus C_{1-j}$ *such that* $|J| = M(H)$. *Then necessarily* $J \supseteq C_j$. *To show that, assume that for some* $l \in \{1,2,\ldots,3k\}$, $c^l_j \notin J$. *Keep one such* l *fixed and define* $\psi : \{1,2,\ldots,3k\} \to \{0,1\}$ *by* $\psi(l) = 1-j$, *and* $\psi(i) = j$, *for* $i \ne l$. *Now (v) above says* $|J| < M(H)$, *a contradiction. Hence, in particular,* C_j *is an independent set in* H.

Definition 6 *For integers* $B \ge 3$ *and* $k \ge 1$ *let* $\mathcal{G}_{B,k}$ *stand for the set of corresponding consistency* $(B, 3k)$*-gadgets. Let*

$$\mu_{B,k} := \min\Big\{\frac{M(H)}{k} : H \in \mathcal{G}_{B,k}\Big\},\ \ \lambda_{B,k} := \min\Big\{\frac{|V(H)| - M(H)}{k} : H \in \mathcal{G}_{B,k}\Big\}$$

(if $\mathcal{G}_{B,k} = \emptyset$, *let* $\lambda_{B,k} = \mu_{B,k} = \infty$*),* $\mu_B = \underline{\lim}_{k\to\infty}\mu_{B,k}$, *and* $\lambda_B = \underline{\lim}_{k\to\infty}\lambda_{B,k}$.

The parameters μ_B and λ_B play a role of quantities in which our inapproximability results for Max-B-IS and Min-B-VC can be expressed. Providing upper bounds on those parameters we obtain explicit lower bounds on approximability for both problems.

In what follows we describe some methods for constructing consistency $(B, 3k)$-gadgets. We will confine ourselves to highly regular gadgets. This ensures that our inapproximability results apply also to B-regular graphs. We will look for a bipartite graph with bipartition (D_0, D_1), where $C_0 \subseteq D_0$, $C_1 \subseteq D_1$ and $|D_0| = |D_1|$, as a suitable candidate for a consistency $(B, 3k)$-gadget H. The idea is that if the cardinality of D_j $(j = 0, 1)$ is significantly larger than $3k$ $(= |C_j|)$ then suitable probabilistic model of constructing bipartite graphs with bipartition (D_0, D_1) and prescribed degrees, will produce with high probability a graph H with good "mixing properties" that ensures the consistency property with $M(H) = |D_j|$. We will not develop any probabilistic model here, rather we will rely on what has already been proved (using similar methods) for amplifiers. The starting point to our construction of consistency $(B, 3k)$-gadgets will be amplifiers studied earlier by Berman & Karpinski [3], [4], and by Chlebík & Chlebíková [5].

Definition 7 *A graph $G = (V, E)$ is a $(2,3)$-graph if G contains only vertices of degree 2 (contacts) and 3 (checkers). We denote Contacts* $= \{v \in V : \deg_G(v) = 2\}$, *and Checkers* $= \{v \in V : \deg_G(v) = 3\}$. *Furthermore, a $(2,3)$-graph G is an* amplifier *if for every $A \subseteq V$: $|\mathrm{Cut}\, A| \geq |Contacts \cap A|$, or $|\mathrm{Cut}\, A| \geq |Contacts \setminus A|$, where* $\mathrm{Cut}\, A = \{\{u, v\} \in E$*: exactly one of vertices u and v is in $A\}$. An amplifier G is called a (k, τ)-amplifier if $|Contacts| = k$ and $|V| = \tau k$.*

To simplify proofs, we will use in our constructions only such (k, τ)-amplifiers whose contact vertices are pairwise nonadjacent. Recall, that the infinite families of amplifiers with $\tau = 7$ [3], and even with $\tau \leq 6.9$ constructed in [5], are of this kind.

3.1 Consistency $(3, 3k)$-gadgets

The construction. Let a $(3k, \tau)$-amplifier $G = (V(G), E(G))$ from Definition 7 be fixed, and $x^1, \dots, x^{3k}$ be its contact vertices. We assume, moreover, that there is a matching in G consisting of vertices $V(G) \setminus \{x^{2k+1}, \dots, x^{3k}\}$. Let us point out that both, the wheel-amplifiers with $\tau = 7$ [3], and also their generalization with $\tau \leq 6.9$ given in [5], clearly contain such matchings.

Let one such matching $\mathcal{M} \subseteq E(G)$ be fixed from now on. Each vertex $x \in V(G)$ is replaced by a small gadget A_x. The gadget for $x \in V(G) \setminus \{x^{2k+1}, \dots, x^{3k}\}$ is a path of 4 vertices x_0, X_1, X_0, x_1 (in this order). For $x \in \{x^{2k+1}, \dots, x^{3k}\}$ we take as A_x a pair of vertices x_0, x_1 without an edge. Denote $E_x := \{x_0, x_1\}$ for each $x \in V(G)$, and $F_x := \{X_0, X_1\}$ for $x \in V(G) \setminus \{x^{2k+1}, \dots, x^{3k}\}$. The union of gadgets A_x (over all $x \in V(G)$) contains already all vertices of our consistency $(3, 3k)$-gadget H, and some of its edges. Now we identify the remaining edges of H. For each edge $\{x, y\}$ of G we connect the corresponding gadgets A_x, A_y with a pair of edges in H, as follows: if $\{x, y\} \in \mathcal{M}$, we connect X_0 with Y_1, and X_1 with Y_0; if $\{x, y\} \in E(G) \setminus \mathcal{M}$, we connect x_0 with y_1, and x_1 with y_0. Having this done, one after another for each edge $\{x, y\} \in E(G)$, we obtain the consistency $(3, 3k)$-gadget $H = (V(H), E(H))$ with contact vertices x^i_j determined by contact vertices x^i of G, for $j \in \{0, 1\}$, $i \in \{1, 2, \dots, 3k\}$.

The proofs of all conditions from Definition 4 of a consistency $(3, 3k)$-gadget follow in the series of claims. Clearly, H is a bipartite graph with bipartition (D_0, D_1) where D_j is the set of vertices of H with a lower index j, $j \in \{0, 1\}$. Further, $|D_0| = |D_1| = (6\tau - 1)k =: M(H)$. Moreover, degree of each contact vertex in H is 2, and degree of any other vertex is 3. First we prove that pairs $\{(x_0^i, x_1^i) : i = 1, \ldots, 2k\}$ are implicitly linked. In fact, we will prove the following stronger result:

Claim 8 *Whenever J is an independent set in H, then there is an independent set J' in H such that $|J'| \geq |J|$ and the following holds: if $x \in V(G) \setminus \{x^{2k+1}, \ldots, x^{3k}\}$ with $|E_x \cap J| = 2$, then $|E_x \cap J'| = 1$; in all other cases $E_x \cap J' = E_x \cap J$.*

PROOF. Consider $x \in V(G) \setminus \{x^{2k+1}, \ldots, x^{3k}\}$ with $|E_x \cap J| = 2$ and make the following modification of J. Take $y \in V(G) \setminus \{x^{2k+1}, \ldots, x^{3k}\}$ such that $\{x, y\} \in \mathcal{M}$. As $\{Y_0, Y_1\} \in E(H)$ there is $j \in \{0, 1\}$ such that $Y_j \notin J$. Take one such j and replace x_j in J by X_{1-j}. Having the above modification of J done, one after another for each $x \in V(G) \setminus \{x^{2k+1}, \ldots, x^{3k}\}$, we obtain J' as required. $\square$

Hence J' obtained from J using Claim 8 is an independent set even in the graph $\widetilde{H}$ obtained from H adding an edge $\{x_0, x_1\}$ connecting the pair E_x, for each $x \in V(G) \setminus \{x^{2k+1}, \ldots, x^{3k}\}$. We denote further by $\widetilde{\widetilde{H}}$ the graph obtained from H adding an edge $\{x_0, x_1\}$ for all pairs E_x, $x \in V(G)$.

Now our aim is to prove that H satisfies the consistency property. For this purpose we keep fixed an arbitrary assignment $\psi : \{1, 2, \ldots, 3k\} \to \{0, 1\}$, and denote by $\mathcal{J}$ the set of all independent sets J in H such that $J \cap \{x^i_{1-\psi(i)} : i = 1, 2, \ldots, 3k\} = \emptyset$. If $\psi \equiv 0$ (respectively, $\psi \equiv 1$), then there is $J \in \mathcal{J}$ with $|J| = M(H)$, namely $J := D_0$ (respectively, $J := D_1$). To complete the proof of consistency of H we have to show that

$$|J| \leq M(H) - \min\Big\{|\{i \in \{1, 2, \ldots, 3k\} : \psi(i) = 0\}|, |\{i \in \{1, 2, \ldots, 3k\} : \psi(i) = 1\}|\Big\} \quad (1)$$

for every $J \in \mathcal{J}$. For this purpose we need to introduce some notation: Given an assignment $\sigma : V(G) \to \{0, 1\}$, let $N(\sigma)$ contain for each $x \in V(G)$ exactly those vertices from A_x which have lower index $\sigma(x)$. Clearly, $|N(\sigma)| = M(H)$. In general, $N(\sigma)$ is not an independent set in H. But the structure of violating edges of $N(\sigma)$, i.e., edges of H with both endpoints in $N(\sigma)$, can be described as follows: for each $\{x, y\} \in E(G)$ with $\sigma(x) \neq \sigma(y)$ there is exactly one violating edge in H, namely $\Big\{x_{\sigma(x)}, y_{\sigma(y)}\Big\}$, if $\{x, y\} \in E(G) \setminus \mathcal{M}$; and $\Big\{X_{\sigma(x)}, Y_{\sigma(y)}\Big\}$, if $\{x, y\} \in \mathcal{M}$.

An assignment $\sigma : V(G) \to \{0, 1\}$ is said to be *admissible*, if the set of violating edges of $N(\sigma)$ forms a matching in H. Clearly, σ is admissible if and only if for each $x \in V(G)$ there is at most one $y \in V(G)$ such that $\{x, y\} \in E(G) \setminus \mathcal{M}$ and $\sigma(y) \neq \sigma(x)$.

We will call an independent set J in H (in fact, even in $\widetilde{\widetilde{H}}$) *$\sigma$-regular*, if $J \subseteq N(\sigma)$. To obtain a σ-regular set from $N(\sigma)$ we have to remove at least one endpoint for every violating edge if the set of violating edges forms a matching. The cardinality of the set of violating edges is then the same as of Cut (in G) of the set $\{x \in V(G) : \sigma(x) = 0\}$. As G is an amplifier, this cardinality is at least $\min\Big\{|\{i : \sigma(x^i) = 0\}|, |\{i : \sigma(x^i) = 1\}|\Big\}$. It means, for any admissible assignment $\sigma : V(G) \to \{0, 1\}$ any σ-regular independent set J in H satisfies

$$|J| \leq M(H) - \min\Big\{|\{i \in \{1, 2, \ldots, 3k\} : \sigma(x^i) = 0\}|, |\{i \in \{1, 2, \ldots, 3k\} : \sigma(x^i) = 1\}|\Big\}. \tag{2}$$

Our strategy to prove (1) is to relate it to (2).

Now we are back to our fixed ψ and $\mathcal{J}$ as above. Denote further by $\widetilde{\mathcal{J}}$ the set of $J \in \mathcal{J}$ for which J is also an independent set in $\widetilde{H}$ (in fact, J is then an independent set also in $\widetilde{\widetilde{H}}$). Let $\widetilde{\mathcal{J}}_{\max}$ be the set of all independent sets from $\widetilde{\mathcal{J}}$ of the maximum size, i.e., of size $\max\{|J| : J \in \widetilde{\mathcal{J}}\}$. Using Claim 8 we easily get that this maximum is the same as $\max\{|J| : J \in \mathcal{J}\}$. Hence it is sufficient to prove (1) for an element $J \in \widetilde{\mathcal{J}}_{\max}$.

Clearly, for any $J \in \widetilde{\mathcal{J}}$ all vertices of $A_x \cap J$ have the same index for each $x \in V(G)$. For $J \in \widetilde{\mathcal{J}}_{\max}$ we have, moreover, that $A_x \cap J \neq \emptyset$ for each $x \in V(G) \setminus \{x^{2k+1}, \ldots, x^{3k}\}$. Keep, for a moment, one $J \in \widetilde{\mathcal{J}}_{\max}$ fixed. It determines an assignment σ $(= \sigma_J)$: $V(G) \to \{0, 1\}$ according to the following rules (i) and (ii):

(i) For $x \in V(G) \setminus \{x^{2k+1}, \ldots, x^{3k}\}$, $\sigma(x) \in \{0, 1\}$ is uniquely determined by $(\emptyset \neq)$ $A_x \cap J \subseteq \{x_{\sigma(x)}, X_{\sigma(x)}\}$.

(ii) For $x = x^i$ with $i \in \{2k+1, \ldots, 3k\}$ we take $\sigma(x^i) = \psi(i)$, unless $A_{x^i} \cap J = \emptyset$ and σ assigns (by the rule (i)) $1 - \psi(i)$ to the both neighbors of x in G; in that case we put $\sigma(x^i) = 1 - \psi(i)$.

Clearly, J is σ-regular. We will show that one can take J in such way that σ is, moreover, an admissible assignment. For this purpose we introduce the following notation for elements $J \in \widetilde{\mathcal{J}}_{\max}$:

$$n_1(J) = |\{\{x, y\} \in E(G) : \sigma(x) = \sigma(y)\}|,$$
$$n_2(J) = |\{i \in \{1, 2, \ldots, 2k\} : X^i_{1-\psi(i)} \notin J\}|.$$

For $J_1, J_2 \in \widetilde{\mathcal{J}}_{\max}$ we write $J_1 \prec J_2$ whenever $(n_1(J_1), n_2(J_1)) < (n_1(J_2), n_2(J_2))$ in the lexicographic order.

Let us keep fixed from now on one maximal element J of $(\widetilde{\mathcal{J}}_{\max}, \prec)$. For this choice of J we are able to prove that σ, determined by J as above, is admissible, and to derive (1) from that. We will proceed in several steps.

Claim 9 *Assume that $x \in V(G)$ is a checker vertex, and y, z, $w \in V(G)$ are (distinct) neighbors of x in G, such that $\{x, w\} \in \mathcal{M}$. Suppose $\sigma(x) = j$, and $\sigma(y) = \sigma(z) = 1 - j$. Then $\sigma(w) = j$, and J contains vertices W_j, X_j, and x_j.*

PROOF. Clearly $\sigma(w) = j$, because otherwise one could find larger $J' \in \widetilde{\mathcal{J}}_{\max}$ replacing in J the set $J \cap (A_x \cup \{W_0, W_1\})$ of cardinality at most 2 by $\{x_{1-j}, X_{1-j}, W_{1-j}\}$, a contradiction. It easily follows that J contains vertices W_j and X_j. Assuming $x_j \notin J$ one could obtain contradiction replacing X_j in J by x_{1-j} that leads to $J' \in \widetilde{\mathcal{J}}_{\max}$ with $J \prec J'$. Hence, $x_j \in J$ as well. □

Now we strengthen Claim 9 showing that its assumptions are never satisfied for our extremal J.

Claim 10 *Assume that $x \in V(G)$ is a checker vertex, and y, z are its distinct neighbors such that both edges $\{x, y\}$ and $\{x, z\}$ are from $E(G) \setminus \mathcal{M}$. Then either $\sigma(y) = \sigma(x)$ or $\sigma(z) = \sigma(x)$.*

PROOF. Put $j := \sigma(x)$, and assume for contradiction that $\sigma(y) = \sigma(z) = 1 - j$. Using Claim 9 we conclude that $X_j, x_j \in J$, and consequently $y_{1-j}, z_{1-j} \notin J$. We will discuss several possibilities for the vertex y separately; in all of them we get a contradiction.

(a) Let y be a contact vertex, i.e., $y = x^i$ for some $i \in \{1, 2, \ldots, 3k\}$.

Assume first that $i \in \{1, 2, \ldots, 2k\}$. As $A_{x^i} \cap J \neq \emptyset$ but $x^i_{1-j} \notin J$, clearly $A_{x^i} \cap J = \{X^i_{1-j}\}$. Assuming $\psi(i) = j$ one could replace X^i_{1-j} in J by x^i_j. Otherwise $\psi(i) = 1-j$ and one could replace x_j (resp., X_j) in J by x_{1-j} (resp., y_{1-j}). In both cases it results in $J' \in \widetilde{\mathcal{J}}_{\max}$ with $J \prec J'$, a contradiction.

Assume now that $i \in \{2k + 1, \ldots, 3k\}$. As $\sigma(y) = 1 - j$ but $y_{1-j} \notin J$, it is only possible if $\psi(i) = 1 - j$ and σ assigns $1 - j$ to the second neighbor of y in G. But then replacing x_j and X_j in J by x_{1-j} and y_{1-j} we get $J' \in \widetilde{\mathcal{J}}_{\max}$ with $J \prec J'$, a contradiction.

(b) Let y be a checker. Take $u \in V(G) \setminus \{x\}$ such that $\{y, u\} \in E(G) \setminus \mathcal{M}$. Assuming $\sigma(u) = j$ leads to a contradiction with Claim 9 when applied to the checker y with $1 - j := \sigma(y)$ in the place of x with $j := \sigma(x)$. Namely, by the Claim 9, $y_{1-j} \in J$, a contradiction. Hence, $\sigma(u) = 1 - j$, and in particular, $u_j \notin J$. Consequently, one can replace x_j and X_j in J by x_{1-j} and y_{1-j} to obtain $J' \in \widetilde{\mathcal{J}}_{\max}$ with $J \prec J'$, a contradiction. □

Claim 11 *σ is an admissible assignment.*

PROOF. Assume, for a contradiction, that an assignment σ is not admissible. That means, for some $x \in V(G)$ there are two distinct vertices $y, z \in V(G)$ such that $\{x, y\} \in E(G) \setminus \mathcal{M}$, $\{x, z\} \in E(G) \setminus \mathcal{M}$, and $\sigma(y) = \sigma(z) = 1 - \sigma(x)$. Due to Claim 10, x must be a contact vertex, $x = x^i$. Clearly, $i \in \{2k + 1, \ldots, 3k\}$, because otherwise one of two edges of G adjacent to x belongs to $\mathcal{M}$. Due to our definition of $\sigma(x^i)$ in that case we conclude that necessarily $\sigma(x^i) = \psi(i)$ and $x^i_{\psi(i)} \in J$. Now, y being a checker, take $u \in V(G) \setminus \{x\}$ such that $\{y, u\} \in E(G) \setminus \mathcal{M}$. Assuming $\sigma(u) = \sigma(x^i)$ $(= \psi(i) = 1 - \sigma(y))$, we get a contradiction with Claim 10 (applied to the checker vertex y in the place of x). Hence, $\sigma(u) = 1 - \psi(i)$. But now we can replace $x^i_{\psi(i)}$ in J by $y_{1-\psi(i)}$ to obtain $J' \in \widetilde{\mathcal{J}}_{\max}$ with $J \prec J'$, a contradiction. That completes the proof. $\square$

Claim 12 *Let $x = x^i \in V(G)$ be a contact vertex with $\sigma(x^i) = 1 - \psi(i)$, and y, z be its neighbors in G. Then $\sigma(y) = \sigma(z) = 1 - \psi(i)$.*

PROOF. For $i \in \{2k + 1, \ldots, 3k\}$, it is clear from our definition of σ. Thus assume $i \in \{1, 2, \ldots, 2k\}$. Clearly $X^i_{1-\psi(i)} \in J$. One of neighbors of x, say y, satisfies $\{x, y\} \in \mathcal{M}$. If $\sigma(y) = \psi(i)$, we can replace $X^i_{1-\psi(i)}$ in J by $X^i_{\psi(i)}$; if $\sigma(z) = \psi(i)$ we can replace $X^i_{1-\psi(i)}$ in J by $x^i_{\psi(i)}$. In both cases we would obtain $J' \in \widetilde{\mathcal{J}}$ with $J \prec J'$, a contradiction. Hence, necessarily $\sigma(y) = \sigma(z) = 1 - \psi(i)$. $\square$

Denote $Z := \{x^i_{1-\psi(i)} : \sigma(x^i) = 1 - \psi(i)\}$. From Claims 8–12 it easily follows that σ is an admissible assignment and that even $J \cup Z$ is a σ-regular independent set in H. So we can apply (2) to $J \cup Z$ in place of J to get

$$|J| + |\{i : \sigma(x^i) \neq \psi(i)\}| \leq M(H) - \min\Big\{|\{i : \sigma(x^i) = 0\}|, |\{i : \sigma(x^i) = 1\}|\Big\},$$

from which (1) easily follows verifying that always

$$\begin{aligned} &\min\Big\{|\{i : \psi(i) = 0\}|, |\{i : \psi(i) = 1\}|\Big\} \leq \\ &\qquad \min\Big\{|\{i : \sigma(x^i) = 0\}|, |\{i : \sigma(x^i) = 1\}|\Big\} + |\{i : \sigma(x^i) \neq \psi(i)\}|. \end{aligned}$$

Finally, we have proved that H is a consistency $(3, 3k)$-gadget, as claimed. As $M(H) = (6\tau - 1)k$, $|V(H)| = 2M(H)$, and τ can be taken ≤ 6.9 (see [5]), it easily follows from this construction that $\mu_{3,k} \leq 40.4$ and $\lambda_{3,k} \leq 40.4$ for any k that is sufficiently large.

The construction will be similar as in the case $B = 3$. Given k, we will look for a consistency $(4,3k)$-gadget $H = (V(H), E(H))$ with the following properties:

(A) The first $2k$ pairs $\{(c_0^i, c_1^i), i = 1, 2, \dots, 2k\}$ are connected by edges.
(B) The vertices c_0^i, c_1^i, $i \in \{1, 2, \dots, k\}$ are of degree 3, the vertices c_0^i, c_1^i, $i \in \{k+1, \dots, 3k\}$ are of degree 2. All other vertices of H are of degree 4.
(C) H is a bipartite graph with the bipartition (D_0, D_1), where $C_0 \subseteq D_0$, $C_1 \subseteq D_1$ and $|D_0| = |D_1| = M(H)$. (Here $M(H)$ is as in Definition 4).

The construction. Let a $(3k, \tau)$-amplifier $G = (V(G), E(G))$ be fixed, and $x^1, \dots, x^{3k}$ be its contact vertices. Each vertex $x \in V(G)$ is replaced by a small gadget A_x. The gadget of a checker x is a pair of vertices x_0, x_1 connected by an edge. The same kind of gadget we take for any of the first k contacts, i.e., for each $x \in \{x^1, x^2, \dots, x^k\}$. For $x \in \{x^{2k+1}, x^{2k+2}, \dots, x^{3k}\}$ we take as A_x a pair of nonadjacent vertices x_0, x_1 (i.e., x_0^i and x_1^i, if $x = x^i$ for $i \in \{2k+1, \dots, 3k\}$). For $x \in \{x^{k+1}, x^{2k+2}, \dots, x^{2k}\}$ we take as A_x a 4-cycle (x_0, x_1, X_0, X_1) (with vertices in this order). Denote further $E_x = \{x_0, x_1\}$ for each $x \in V(G)$. The union of gadgets A_x (over all $x \in V(G)$) already contains all vertices of our consistency $(4,3k)$-gadget H, and some of its edges. Now we identify the remaining edges of H. If two vertices $x, y \in V(G)$ are connected by an edge in G, we connect their pairs E_x and E_y with a pair of edges in such way that the vertex of E_x with an index j ($j \in \{0,1\}$) is connected with the vertex of E_y indexed by $1-j$. Having this done, one after another, for each edge $\{x, y\}$ of G, we obtain the graph $H = (V(H), E(H))$. The contact vertices are $c_0^i := X_0^i$, $c_1^i := X_1^i$ for $i \in \{k+1, k+2, \dots, 2k\}$, otherwise $c_0^i := x_0^i$ and $c_1^i := x_1^i$.

Clearly, H is a bipartite graph with the bipartition (D_0, D_1), where D_j is the set of vertices with the lower index j, $j \in \{0,1\}$. Further, $|D_0| = |D_1| = (3\tau + 1)k =: M(H)$. One can easily check that the above requirement (B) on H concerning degrees of vertices is satisfied, as well as (A).

Our aim now is to prove the consistency property. For this purpose, we keep fixed one (arbitrary) assignment $\psi : \{1, 2, \dots, 3k\} \to \{0,1\}$ and denote by $\mathcal{J}$ the set of all independent sets J in H such that $J \cap \{c_{1-\psi(i)}^i : i = 1, 2, \dots, 3k\} = \emptyset$. We have to show that (1) holds for every $J \in \mathcal{J}$. It is clear, that $|J| \le M(H)$, as for each $x \in V(G)$ at most one of x_0 and x_1 can belong to J, and if $x \in \{x^{k+1}, \dots, x^{2k}\}$ at most one of X_0, X_1 as well. Moreover, in the case $\psi \equiv 0$, or $\psi \equiv 1$, one has in fact $\max\{|J| : J \in \mathcal{J}\} = M(H)$, as $|D_0| = |D_1| = M(H)$. Hence the first part of the consistency property is obviously satisfied.

Let us describe our strategy for the proof of (1). We need to introduce some notions: An assignment $\sigma : V(G) \to \{0,1\}$ to the vertices of G is said to be *nice*, if for each $x \in V(G)$ there is at most one neighbor y of x in G such that $\sigma(y) \neq \sigma(x)$. Given an assignment $\sigma : V(G) \to \{0,1\}$, consider the set $N(\sigma) \subseteq V(H)$ which for each $x \in V(G)$ contains exactly the vertices from A_x with the lower index $\sigma(x)$. Clearly, $|N(\sigma)| = M(H)$. In many cases $N(\sigma)$ is not an independent set. But the structure of the set of violating edges of H, i.e., those with both endpoints in $N(\sigma)$, is simple, assuming that σ is nice. In that case they are exactly the edges $\{x_{\sigma(x)}, y_{\sigma(y)}\} \in E(H)$ such that $\{x,y\} \in E(G)$ and $\sigma(x) \neq \sigma(y)$. In particular, they form a matching in H, and the cardinality of this matching is the same as the cardinality of Cut (in G) of the set $\{x \in V(G) : \sigma(x) = 0\}$. As G is an amplifier, this is at least $\min\Big\{|\{i : \sigma(x^i) = 0\}|, |\{i : \sigma(x^i) = 1\}|\Big\}$.

Further, any independent set J in H that is subset of $N(\sigma)$ is said to be σ-regular (in fact, J is an independent set also in a graph obtained from H connecting the pair $E_x = \{x_0, x_1\}$ by an edge, for each contact vertex x). We can now observe that for any *nice* $\sigma : V(G) \to \{0,1\}$, any σ-regular independent set J in H satisfies (2). This is because to obtain an independent set J from $N(\sigma)$ (of cardinality $M(H)$) we have to remove at least one endpoint for every violating edge. So, our strategy to prove (1) is to relate it to (2).

Now we are back to our fixed ψ and $\mathcal{J}$ as above. We want to prove that max $|J|$ over $J \in \mathcal{J}$ is achieved even on "very regular" independent sets from $\mathcal{J}$. Let us introduce the following notation for $J \in \mathcal{J}$:

$$
\begin{aligned}
n_1(J) &= |J|,\\
n_2(J) &= |J \cap (\cup_{x \in V(G)} E_x)|,\\
n_3(J) &= \max\{|J \cap D_0|, |J \cap D_1|\}.
\end{aligned}
$$

For $J_1, J_2 \in \mathcal{J}$, we write $J_1 \prec J_2$ whenever $(n_1(J_1), n_2(J_1), n_3(J_1)) < (n_1(J_2), n_2(J_2), n_3(J_2))$ in the lexicographic order. Let us keep fixed any maximal element J of $(\mathcal{J}, \prec)$. Clearly, $|J| = \max\{|J'| : J' \in \mathcal{J}\}$ for this special J. We are able to relate J to a σ-regular independent set of H for some *nice* assignment $\sigma : V(G) \to \{0,1\}$. In the first stage, let σ be defined only on those $x \in V(G)$ for which $E_x \cap J \neq \emptyset$: let $\sigma(x)$ be the index of a (unique) vertex of $E_x \cap J$ (i.e., $E_x \cap J = \{x_{\sigma(x)}\}$).

Claim 13 *Let $x \in V(G)$ be a checker vertex with $E_x \cap J = \emptyset$. Then for each vertex y such that $\{x,y\} \in E(G)$ the set $E_y \cap J$ is nonempty. In other words, σ is already defined for all three neighbors of x in G. Moreover, σ attains both 0 and 1 as value on neighbors of x.*

PROOF. Let y, z, and w be all three neighbors of x in G. Assume, for example, that $E_y \cap J = \emptyset$. As neither $J \cup \{x_0\}$ nor $J \cup \{x_1\}$ is an independent set in H, necessarily for some $j \in \{0,1\}$ $E_z \cap J = \{z_j\}$ and $E_w \cap J = \{w_{1-j}\}$. But then replacing in J either z_j by x_{1-j}, or w_{1-j} by x_j, will result in $J' \in \mathcal{J}$ with $J \prec J'$ (namely, $n_3(J) < n_3(J')$), a contradiction. Hence σ is already defined for y. The proof for z and w is the same.

Assume now that $\sigma(y) = \sigma(z) = \sigma(w) =: j \in \{0,1\}$. But then adding x_j to J will produce larger $J' \in \mathcal{J}$, a contradiction. $\square$

Claim 14 *Let $x = x^i \in V(G)$ be a contact vertex with $E_x \cap J = \emptyset$, and y, z be its neighbors in G. By our assumption about G, y and z have to be checker vertices with $\sigma(y)$ and $\sigma(z)$ already defined (due to Claim* 13*).*

(a) If $i \in \{k+1, \dots, 2k\}$ then $X^i_{\psi(i)} \in J$ and $\sigma(y) \neq \sigma(z)$.
(b) If $i \in \{1, \dots, k\} \cup \{2k+1, \dots, 3k\}$ then either $\sigma(y) \neq \sigma(z)$, or $\sigma(y) = \sigma(z) = 1 - \psi(i)$. In the latter case $J \cup \{x^i_{1-\psi(i)}\}$ is an independent set in H (and also in a graph obtained from H connecting the pair $E_x = \{x_0, x_1\}$ by an edge, for each contact vertex x), too.

PROOF. (a) In this case clearly $X^i_{\psi(i)} \in J$, due to our choice of maximal J. Further, neither $J' := J \cup \{x^i_{\psi(i)}\}$ nor $J' := J \setminus \{X^i_{\psi(i)}\} \cup \{x^i_{1-\psi(i)}\}$ is an independent set in H (it would imply $J \prec J'$). Hence, for some $j \in \{0,1\}$, $E_y \cap J = \{y_j\}$ and $E_z \cap J = \{z_{1-j}\}$.

(b) In this case $J' := J \cup \{x^i_{\psi(i)}\}$ is not an independent set in H (it would imply $J \prec J'$). Hence, at least for one $u \in \{y, z\}$ we have $\sigma(u) = 1 - \psi(i)$. Moreover, if $\sigma(y) = \sigma(z) = 1 - \psi(i)$, it follows that $y_{\psi(i)} \notin J$ and $z_{\psi(i)} \notin J$, hence $J \cup \{x^i_{1-\psi(i)}\}$ is an independent set as well. $\square$

Now, we are ready to extend σ to a nice assignment for which J is σ-regular.

(i) If $x \in V(G)$ is a checker vertex with $E_x \cap J = \emptyset$, then by Claim 13, σ attains both 0 and 1 on neighbors of x. Necessarily, one $j \in \{0,1\}$ is attained twice there, and we let $\sigma(x) := j$.
(ii) If $x = x^i \in V(G)$ is a contact vertex with $E_x \cap J = \emptyset$, by Claim 14 either both 0 and 1 are attained by σ on neighbors of x, or both neighbors of x have assigned $1 - \psi(i)$ by σ. In the former case, we let $\sigma(x^i) = \psi(i)$, in the latter one $\sigma(x^i) = 1 - \psi(i)$.

Denote further $Z := \{x^i_{1-\psi(i)} : \sigma(x^i) = 1 - \psi(i)\}$. Clearly, J is σ-regular. Using Claim 14(b), even $J \cup Z$ is a σ-regular independent set in H.

Now we want to prove that σ is a nice assignment. Clearly, by our extension of σ based on Claims 13 and 14, for each $x \in V(G)$ with $E_x \cap J = \emptyset$ at most one neighbor u of x in G has $\sigma(u) \neq \sigma(x)$. We have to prove that this is also true for each $x \in V(G)$ with $E_x \cap J \neq \emptyset$.

Claim 15 *Let $x \in V(G)$ be either checker or contact vertex with $E_x \cap J \neq \emptyset$. Then there exists at most one neighbor u of x in G with $\sigma(u) \neq \sigma(x)$.*

PROOF. Consider a neighbor u of x in G with $\sigma(u) \neq \sigma(x)$. Clearly, $E_x \cap J = \{x_{\sigma(x)}\}$ which implies $u_{\sigma(u)} = u_{1-\sigma(x)} \notin J$, hence $E_u \cap J = \emptyset$. If u is a checker vertex, then due to Claim 13 the other two neighbors of u have assigned $1 - \sigma(x)$ already in the first stage, in particular, $(J \setminus \{x_{\sigma(x)}\}) \cup \{u_{1-\sigma(x)}\}$ is an independent set as well. If u is a contact vertex, then due to Claim 14 the other neighbor of u has assigned $1 - \sigma(x)$ already in the first stage, in particular $(J \setminus \{x_{\sigma(x)}\}) \cup \{u_{1-\sigma(x)}\}$ is an independent set in this case as well.

Assume now, that two distinct neighbors y and z of x in G have $\sigma(y) = \sigma(z) = 1 - \sigma(x)$. The analysis above shows, that then $J' := (J \setminus \{x_{\sigma(x)}\}) \cup \{y_{1-\sigma(x)}, z_{1-\sigma(x)}\}$ will be a larger independent set, a contradiction. $\square$

From above Claims 13–15 we know that σ is a nice assignment and that even $J \cup Z$ is a σ-regular independent set in H. So, we can apply (2) to $J \cup Z$,

$$|J| + |\{i : \sigma(x^i) \neq \psi(i)\}| \leq M(H) - \min\{|\{i : \sigma(x^i) = 0\}|, |\{i : \sigma(x^i) = 1\}|\},$$

from which we easily obtain (1) as in case $B = 3$. Hence, H is really a consistency $(4, 3k)$-gadget, as claimed. As $M(H) = (3\tau + 1)k$, $|V(H)| = 2M(H)$, and τ can be taken ≤ 6.9 (see [5]), this gives us estimates $\mu_{4,k} \leq 21.7$ and $\lambda_{4,k} \leq 21.7$ for any k that is sufficiently large.

3.3 Consistency $(B, 3k)$-gadgets for $B \geq 5$

We do not try to optimize our estimates for $B \geq 5$ in this paper, as we are focused on the cases $B = 3$ and $B = 4$. For larger B we provide our inapproximability results based on small degree consistency gadgets constructed above. Of course, one can expect that gadgets with much better parameters can be provided for these cases, by suitable constructions. We can modify the consistency $(4, 3k)$-gadget H to get a slight improvement for the case $B \geq 5$. Namely, also for $x \in \{x^{k+1}, x^{k+2}, \ldots, x^{2k}\}$ we take as A_x a pair of vertices connected by an edge. The corresponding c_0^i, c_1^i vertices of H will have degree 3 in H, and we will have now $M(H) = 3\tau k$. The same proof of consistency for H will work. This consistency gadget H will be clearly simultaneously a consistency $(B, 3k)$-gadget for any $B \geq 5$. In this way we get upper bounds $\mu_{B,k} \leq 20.7$ and $\lambda_{B,k} \leq 20.7$, for any $B \geq 5$ and any k that is sufficiently large.

We can now summarize the results on parameters of consistency $(B, 3k)$-gadgets obtained by constructions above.

Theorem 16 *For any sufficiently large integer k, $\mu_{3,k} \leq 40.4$, $\lambda_{3,k} \leq 40.4$, $\mu_{4,k} \leq 21.7$, $\lambda_{4,k} \leq 21.7$, and $\mu_{B,k} \leq 20.7$, $\lambda_{B,k} \leq 20.7$, for any $B \geq 5$.*

4 Approximation Hardness of Max-IS and Min-VC in Low Degree Graphs

In this section we explore the complexity of the Maximum Independent Set and Minimum Vertex Cover problems in graphs of degree at most B for small value of parameter B.

The following theorem summarizes the results

Theorem 17 *It is* NP*-hard to approximate: the solution of* Max-3-IS *to within any constant smaller than $1 + \frac{1}{2\mu_3+13}$, for $B \in \{4, 5\}$ the solution of* Max-B-IS *to within any constant smaller than $1 + \frac{1}{2\mu_B+3}$, and the solution of* Max-B-IS*, $B \geq 6$, to within any constant smaller than $1 + \frac{1}{2\mu_B+1}$. Similarly, it is* NP*-hard to approximate the solution of* Min-3-VC *to within any constant smaller than $1 + \frac{1}{2\lambda_3+18}$, for $B \in \{4, 5\}$ the solution of* Min-B-VC *to within any constant smaller than $1 + \frac{1}{2\lambda_B+8}$, and the solution of* Min-B-VC*, $B \geq 6$, to within any constant smaller than $1 + \frac{1}{2\lambda_B+6}$.*

PROOF. Let an integer $B \geq 3$ be fixed. For a fixed small $\varepsilon > 0$ consider k large enough such that the conclusion of Theorem 2 for E$[k, k, k]$-Max-E3-Lin-2 is satisfied, and for which there is a consistency $(B, 3k)$-gadget H with $\frac{M(H)}{k} < \mu_B + \varepsilon$ (resp., $\frac{|V(H)|-M(H)}{k} < \lambda_B + \varepsilon$).

Keeping one such gadget H fixed, our reduction f $(= f_H)$ from E$[k, k, k]$-Max-E3-Lin-2 to Max-B-IS (resp., Min-B-VC) is as follows: Let I be an instance of E$[k, k, k]$-Max-E3-Lin-2, $\mathcal{V}(I)$ be the set of variables of I, $m := |\mathcal{V}(I)|$. Hence I has mk equations, each variable $u \in \mathcal{V}(I)$ occurs exactly in $3k$ of them: k times as the first variable, k times as the second one, and k times as the third variable in the equation. Assume, for convenience, that equations are numbered by $1, 2, \ldots, mk$. Given a variable $u \in \mathcal{V}(I)$ and $s \in \{1, 2, 3\}$, let $r_s^1(u) < r_s^2(u) < \cdots < r_s^k(u)$ be the numbers of equations in which variable u occurs as the s-th variable. On the other hand, if for fixed $r \in \{1, 2, \ldots, mk\}$ the r-th equation is $x + y + z = j$ ($j \in \{0, 1\}$), there are uniquely determined numbers $i(x, r)$, $i(y, r)$, $i(z, r) \in \{1, 2, \ldots, k\}$ such that $r_1^{i(x,r)}(x) = r_2^{i(y,r)}(y) = r_3^{i(z,r)}(z) = r$.

Take m disjoint copies of H, one for each variable. Let H_u denote a copy of H that corresponds to a variable $u \in \mathcal{V}(I)$. The corresponding contacts in H_u are denoted by $C_j(u) = \{u_j^i : i = 1, 2, \ldots, 3k\}$, $j \in \{0, 1\}$. Now we take mk disjoint copies of equation gadgets G^r, $r \in \{1, 2, \ldots, mk\}$. More precisely, if the r-th equation reads as $x + y + z = j$ ($j \in \{0, 1\}$), we take as G^r a copy of $G_j[3]$ for Max-3-IS (or $G_j[4]$ for 4(5)-Max-IS or $G_j[6]$ for Max-B-IS, $B \geq 6$). Now the vertices x_0, x_1, y_0, y_1, z_0, and z_1 of G^r are identified with vertices $x_0^{i(x,r)}$, $x_1^{i(x,r)}$ (of H_x), $y_0^{k+i(y,r)}$, $y_1^{k+i(y,r)}$ (of H_y), $z_0^{2k+i(z,r)}$, $z_1^{2k+i(z,r)}$ (of H_z), respectively. It means that in each H_u the first k-tuple of pairs of contacts corresponds to the occurrences of u as the first variable, the second k-tuple corresponds to the occurrences as the second variable, and the third one to occurrences as the third variable in equations. Making the above identification for all equations, one after another, we get a graph of degree at most B, denoted by $f(I)$. Clearly, the above reduction f (using the fixed H as a parameter) to special instances of Max-B-IS, is polynomial. Now we show how the NP-hard gap of E[k, k, k]-Max-E3-Lin-2 is preserved.

We think $f(I)$ as an instance of the Maximum Independent Set problem. An independent set J in $f(I)$ is called *standard*, if for each $u \in \mathcal{V}(I)$ there is (necessarily unique) $\varphi(u) \in \{0, 1\}$ such that $J \cap C_{1-\varphi(u)}(u) = \emptyset$ and $|J \cap V(H_u)| = M(H)$. It implies, in particular, that $J \supseteq C_{\varphi(u)}(u)$ (see Remark 1). Clearly, any standard independent set J in $f(I)$ determines an assignment $\varphi : \mathcal{V}(I) \to \{0, 1\}$. Such independent set J is called, more specifically, φ-standard. Further, it is clear that a φ-standard independent set J can contain one special vertex for each equation satisfied by the assignment φ. More precisely, if r-th equation of I reads as $x + y + z = j$, then J can contain a special vertex from the equation gadget G^r if and only if $\varphi(x) + \varphi(y) + \varphi(z) = j \bmod 2$, namely the special vertex labeled by $\varphi(x)\varphi(y)\varphi(z)$.

Hence, if $\mathrm{sat}(\varphi)$ means the number of equations of I satisfied by φ, one can express easily the maximum cardinality of a φ-standard independent set as $M(H)m + \mathrm{sat}(\varphi)$, for Max-$B$-IS, $B \geq 6$; $M(H)m + mk + \mathrm{sat}(\varphi)$, for Max-4(5)-IS, and $M(H)m + 6mk + \mathrm{sat}(\varphi)$, for Max-3-IS.

Taking φ optimal, i.e., such that $\mathrm{sat}(\varphi) = \mathrm{OPT}(I)|I| = \mathrm{OPT}(I)mk$, allows to express simply $\alpha_{\mathrm{std}}(f(I)) := \max\{|J|\colon J$ is a standard independent set in $f(I)\}$ using $\mathrm{OPT}(I)$. Namely,

- $\alpha_{\mathrm{std}}(f(I)) = mk(\frac{M(H)}{k} + 6 + \mathrm{OPT}(I))$ for Max-3-IS,
- $\alpha_{\mathrm{std}}(f(I)) = mk(\frac{M(H)}{k} + 1 + \mathrm{OPT}(I))$ for Max-4(5)-IS, and

- $\alpha_{\text{std}}(f(I)) = mk(\frac{M(H)}{k} + \text{OPT}(I))$ for B-MIS, $B \geq 6$.

The key point now is that the properties of our consistency gadget H imply, that it is not more advantageous to use an independent set which is not standard, to achieve the maximum cardinality. In other words, $\alpha(f(I))$ is achieved also on some standard independent set, i.e., $\alpha(f(I)) = \alpha_{\text{std}}(f(I))$. For this purpose consider one independent set J of $f(I)$ such that $|J| = \alpha(f(I))$. The aim is to show, that one can modify J to another independent set J' in $f(I)$ such that $|J'| \geq |J|$ and J' is standard.

First, for each $u \in \mathcal{V}(I)$, one after another, modify J inside H_u to obtain another optimal independent set J_0 containing no pair of implicitly linked vertices. In other words, for each $u \in \mathcal{V}(I)$ an independent set $J_0 \cap V(H_u)$ contains at most one vertex from each of the first $2k$ pairs of contact vertices. (This is possible due to property (iv) of a consistency gadget.)

Now, for each equation of I, one after another, modify J_0 inside the corresponding equation gadget G^r according to Theorem 3, to obtain another optimal independent set J_1 with the following properties: For each $u \in \mathcal{V}(I)$ an independent set $J_1 \cap V(H_u)$ contains from each pair of contact vertices at most one vertex, and for each $r = 1, 2, \ldots, mk$ the vertex set $J_1 \cap V(G^r)$ contains exactly one special vertex. If this special vertex for the r-th equation $x + y + z = j$ is labeled by $\psi(x)\psi(y)\psi(z)$, those bits can be viewed as a local satisfying assignment for occurrences of variables x, y, and z in this equation. Moreover, for each $u \in \{x, y, z\}$, the set J_1 in this equation gadget is pure and full in F_u (with vertices of label $\psi(u)$ there), in particular $u_{1-\psi(u)} \notin J_1$. In this way the set J_1 uniquely determines local assignment ψ to all occurrences of each variable. More precisely, as $\psi(u)$ can vary from occurrence to occurrence of u, we should write more precisely $\psi(u^i)$ for particular occurrences of u. For fixed u, we will also write $\psi_u(i) := \psi(u^i)$.

Now for each variable $u \in \mathcal{V}(I)$ we can define $\varphi(u)$ as the prevailing value (0 or 1) of this local assignment to occurrences of u, determined by J_1, as described above. (In the case of the equal number of 0's and 1's, the choice of $\varphi(u) \in \{0, 1\}$ can be arbitrary.)

Keeping $u \in \mathcal{V}(I)$ fixed, denote by $S(u)$ the set of special vertices in J_1 that determine for u the local assignment ψ inconsistent with $\varphi(u)$. Clearly,

$$|S(u)| = |\{i : \psi_u(i) \neq \varphi(u)\}| = \min\{|\{i : \psi_u(i) = 0\}|, |\{i : \psi_u(i) = 1\}|\},$$

hence $|J_1 \cap V(H)| \leq M(H) - |S(u)|$ as follows from the consistency property of H. If there is $u \in \mathcal{V}(I)$ that is inconsistent, i.e., $S(u) \neq \emptyset$, we will further modify J_1 in the following way:

(i) Remove first from J_1 special vertices that caused the inconsistency, i.e.,

$\cup_{u \in \mathcal{V}(I)} S(u)$, of cardinality $|\cup_u S(u)| \leq \sum_u |S(u)|$.

For each inconsistent occurrence of u we further modify J_1 inside the corresponding equation gadget: the vertex $u'_{1-\varphi(u)}$, resp. $u''_{1-\varphi(u)}$, is replaced by $u'_{\varphi(u)}$, resp. $u''_{\varphi(u)}$, if such vertices exist in the equation gadget.

(ii) Then for each u replace $J_1 \cap V(H_u)$ (of cardinality $\leq M(H) - |S(u)|$) by an independent set in $H_u \setminus C_{1-\varphi(u)}(u)$ of cardinality $M(H)$.

The result of (i) and (ii) will be a new independent set J' with $|J'| \geq |J|$. Moreover J' is φ-standard. This completes the proof that maximum independent set is achieved on standard independent set of $f(I)$. Hence we have an affine dependence of $\alpha(f(I))$ on $\mathrm{OPT}(I)$ as described above for $\alpha_{\mathrm{std}}(f(I))$.

Let us now check, how the NP-hard gap of E[k,k,k]-Max-E3-Lin-2 is preserved. If an instance I of E[k,k,k]-Max-E3-Lin-2 has m variables as above, then $f(I)$ has

- for Max-3-IS, $n := m|V(H)| + 16mk$ vertices, and $\alpha(f(I)) = mk(\frac{M(H)}{k} + 6 + \mathrm{OPT}(I))$;
- for Max-4(5)-IS, $n := m|V(H)| + 6mk$ vertices, and $\alpha(f(I)) = mk(\frac{M(H)}{k} + 1 + \mathrm{OPT}(I))$;
- for Max-B-IS, $B \geq 6$, $n := m|V(H)| + 4mk$ vertices, and $\alpha(f(I)) = mk(\frac{M(H)}{k} + \mathrm{OPT}(I))$.

Hence, the NP-hard question of whether $\mathrm{OPT}(I)$ is greater than $(1-\varepsilon)$, or less than $\left(\frac{1}{2}+\varepsilon\right)$, is transformed to the NP-hard partial decision problem of whether

- for Max-3-IS:

$$n\frac{2\frac{M(H)}{k} + 13 + 2\varepsilon}{2\frac{|V(H)|}{k} + 32} > \alpha(f(I)) \quad \text{or} \quad \alpha(f(I)) > n\frac{2\frac{M(H)}{k} + 14 - 2\varepsilon}{2\frac{|V(H)|}{k} + 32};$$

- for Max-4(5)-IS:

$$n\frac{2\frac{M(H)}{k} + 3 + 2\varepsilon}{2\frac{|V(H)|}{k} + 12} > \alpha(f(I)) \quad \text{or} \quad \alpha(f(I)) > n\frac{2\frac{M(H)}{k} + 4 - 2\varepsilon}{2\frac{|V(H)|}{k} + 12};$$

- for Max-B-IS, $B \geq 6$:

$$n\frac{2\frac{M(H)}{k} + 1 + \varepsilon}{2\frac{|V(H)|}{k} + 8} > \alpha(f(I)) \quad \text{or} \quad \alpha(f(I)) > n\frac{2\frac{M(H)}{k} + 2 - 2\varepsilon}{2\frac{|V(H)|}{k} + 8}.$$

Consequently, it is NP-hard to approximate the solution of Max-3-IS within

$1+\frac{1-4\varepsilon}{2M(H)/k+13+2\varepsilon}$; Max-4(5)-IS within $1+\frac{1-4\varepsilon}{2M(H)/k+3+2\varepsilon}$; Max-$B$-IS, $B \geq 6$, within $1+\frac{1-4\varepsilon}{2M(H)/k+1+2\varepsilon}$.

Passing to the complements of graphs, one can state similar results for the Minimum Vertex Cover problem. Clearly, $vc(f(I)) = mk(\frac{|V(H)|-M(H)}{k} + 10 - \mathrm{OPT}(I))$ for Min-3-VC; $vc(f(I)) = mk(\frac{|V(H)|-M(H)}{k} + 5 - \mathrm{OPT}(I))$ for Min-4(5)-VC; $vc(f(I)) = mk(\frac{|V(H)|-M(H)}{k} + 4 - \mathrm{OPT}(I))$ for Min-B-VC, $B \geq 6$. So we get that the partial decision problem

- for Min-3-VC:

$$n\frac{2\frac{|V(H)|-M(H)}{k}+18+2\varepsilon}{2\frac{|V(H)|}{k}+32} > vc \quad \text{or} \quad vc > n\frac{2\frac{|V(H)|-M(H)}{k}+19-2\varepsilon}{2\frac{|V(H)|}{k}+32},$$

- for Min-4(5)-VC:

$$n\frac{2\frac{|V(H)|-M(H)}{k}+8+2\varepsilon}{2\frac{|V(H)|}{k}+12} > vc \quad \text{or} \quad vc > n\frac{2\frac{|V(H)|-M(H)}{k}+9-2\varepsilon}{2\frac{|V(H)|}{k}+12},$$

- Min-B-VC, $B \geq 6$:

$$n\frac{2\frac{|V(H)|-M(H)}{k}+6+2\varepsilon}{2\frac{|V(H)|}{k}+8} > vc \quad \text{or} \quad vc > n\frac{2\frac{|V(H)|-M(H)}{k}+7-2\varepsilon}{2\frac{|V(H)|}{k}+8},$$

is NP-hard. Consequently, it is NP-hard to approximate the solution of Min-3-VC within $1+\frac{1-4\varepsilon}{2(|V(H)|-M(H))/k+18+2\varepsilon}$; Min-4(5)-VC within $1+\frac{1-4\varepsilon}{2(|V(H)|-M(H))/k+8+2\varepsilon}$; Min-$B$-VC, $B \geq 6$, within $1+\frac{1-4\varepsilon}{2(|V(H)|-M(H))/k+6+2\varepsilon}$. □

Using $(B, 3k)$-consistency gadgets studied in Section 3 (with property $|V(H)| = 2M(H)$) and our upper bounds on μ_B and λ_B from Theorem 16 we obtain

Corollary 18 *It is* NP*-hard to approximate the solution of* Max-3-IS *to within* 1.010661 ($> \frac{95}{94}$)*, the solution of* Max-4-IS *to within* 1.0215517 ($> \frac{48}{47}$)*, the solution of* Max-5-IS *to within* 1.0225225 ($> \frac{46}{45}$)*, and the solution of* Max-B-IS*,* $B \geq 6$*, to within* 1.0235849 ($> \frac{44}{43}$)*. Similarly, it is* NP*-hard to approximate the solution of* Min-3-VC *to within* 1.0101215 ($> \frac{100}{99}$)*, the solution of* Min-4-VC *to within* 1.0194553 ($> \frac{53}{52}$)*, the solution of* Min-5-VC *to within* 1.0202429 ($> \frac{51}{50}$)*, and* Min-B-VC*,* $B \geq 6$*, to within* 1.021097 ($> \frac{49}{48}$)*. For each* B*,* $3 \leq B \leq 6$*, the corresponding result applies to* B*-regular graphs as well.*

5 Approximation Hardness of Max-3-DM and Other Problems

Let us explain how inapproximability results for bounded variants of Maximum Independent Set and Minimum Vertex Cover imply the same bounds for some set packing, set covering, and hypergraph matching problems.

Set packing and set cover may be phrased also in hypergraph notation: S is the set of vertices and hyperedges are elements of $\mathcal{C}$. In this notation a set packing is just a matching in the corresponding hypergraph. Given a graph $G = (V, E)$ without isolated vertices, we define its dual hypergraph $\widetilde{G} = (E, \widetilde{V})$ with the set of vertices E, and the set of hyperedges $\widetilde{V} = \{\widetilde{v} : v \in V\}$, where for each $v \in V$ hyperedge $\widetilde{v}$ consists of all $e \in E$ such that $v \in e$ in G. The hypergraph $\widetilde{G}$ defined by this duality is clearly 2-regular, each vertex of $\widetilde{G}$ is contained exactly in two hyperedges. G is of maximum degree B if and only if $\widetilde{G}$ is of dimension B (i.e., the maximum size of a hyperedge in $\widetilde{G}$ is B), in particular, G is B-regular if and only if $\widetilde{G}$ is B-uniform. Independent sets in G are in one-to-one correspondence with matchings in $\widetilde{G}$ (hence with set packings, in the setting of set systems), and vertex covers in G with set covers for $\widetilde{G}$. Hence, any approximation hardness result for Max-B-IS translates via this duality to the one for Max-B-Set Packing (with exactly 2 occurrences), or to Maximum Matching in 2-regular B-dimensional hypergraphs. The relation of results for Min-B-VC to those for Min-B-Set Cover problem is similar.

If G is a B-regular edge B-colored graph, then $\widetilde{G}$ is, moreover, B-partite with balanced B-partition determined by corresponding color classes. It means, that independent sets in such graphs naturally correspond to B-dimensional matchings. Hence, any inapproximability result for the Max-B-IS problem restricted to B-regular edge-B-colored graphs translates directly to the corresponding inapproximability result for Maximum B-Dimensional Matching, even on instances with exactly two occurrences of each element. We prove now that for $B = 3, 4$ our reduction to Max-B-IS problem can be made to produce instances that are edge-B-colored B-regular graphs.

In this way we prove, similarly as for Max-B-IS, the following theorem

Theorem 19 *It is* NP*-hard to approximate the solution of* Max-3-DM *to within* 1.010661 ($> \frac{95}{94}$)*, and the solution of* Max-4-DM *to within* 1.0215517 ($> \frac{48}{47}$)*. Both inapproximability results apply also to instances with each element occurring in exactly two triples, resp. quadruples.*

PROOF. (A) Maximum 3-Dimensional Matching. As it is depicted on Fig. 1, the gadget $G_0[3]$ can be edge-3-colored by colors a, b, c in such way that all edges adjacent to vertices of degree one (contacts) are colored by one fixed color, say a (for $G_1[3]$ we can do the same). As a parameter of our reduction f ($= f_H$) from E$[k,k,k]$-Max-E3-Lin-2 to Max-3-DM we use a consistency $(3,3k)$-gadget $H \in \mathcal{G}_{3,k}$ for Max-3-IS. We will prove and rely on the fact, that in our construction of these gadgets given above we can ensure the following properties of H: degree of any contact vertex is exactly 2, degree of any other vertex is 3, and, moreover, H is edge-3-colorable by colors a, b, c in such way that all edges adjacent to contact vertices are colored by two colors b and c.

We can use the same construction of consistency $(3,3k)$-gadgets as was presented for Max-3-IS, and show that produced graphs H have, additionally, the above property about coloring of edges. Starting from a $(3k,\tau)$-amplifier G and a matching $\mathcal{M} \subseteq E(G)$ of vertices $V(G) \setminus \{x^{2k+1},\dots,x^{3k}\}$, we define such edge coloring of H produced by our construction in two steps: (i) Take preliminary the following edge coloring: for each $\{x,y\} \in \mathcal{M}$ we color the corresponding edges in H as depicted on Fig. 3(i). The remaining edges of H are easily 2-colored by colors b and c, as the rest of the graph is bipartite and of degree at most 2. So, we have a proper edge-3-coloring, but some edges adjacent to contacts are colored by color a. It will happen exactly if $x \in \{x^1,x^2,\dots,x^{2k}\}$, $\{x,y\} \in \mathcal{M}$. (We assume that no two contacts of G are adjacent, hence y is a checker vertex of G.) Clearly, one can ensure that in the above extension of coloring of edges by colors c and b both other edges adjacent to x_0 and x_1 have the same color. (ii) Now we modify our edge coloring in edges violating the required condition as follows. Fix $x \in \{x^1,\dots,x^{2k}\}$, $\{x,y\} \in \mathcal{M}$, and let both other edges adjacent to x_0 and x_1 have assigned color b. Then change coloring according Fig. 3(ii). The case when both edges have assigned color c, can be solved analogously (see Fig. 3(iii)). Recall, that this construction can produce consistency $(3,3k)$-gadgets H with $M(H) \le 40.4k$, for any sufficiently large k.

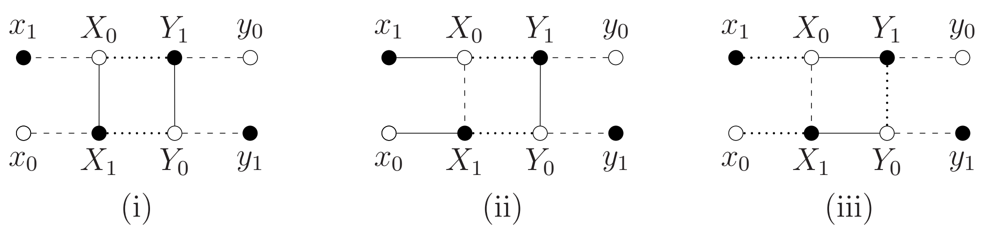

Fig. 3. a color: dashed line, b color: dotted line, c color: solid line

Keeping one such consistency $(3,3k)$-gadget H fixed, our reduction f ($= f_H$) from E$[k,k,k]$-Max-E3-Lin-2 is exactly the same as for Max-3-IS described in Section 4. Let us fix an instance I of E$[k,k,k]$-Max-E3-Lin-2 and consider an instance $f(I)$ of Max-3-IS. As $f(I)$ is an edge 3-colored 3-regular graph, it is at the same time an instance of 3-DM with the same objective function. We can show, that the NP-hard gap of E$[k,k,k]$-Max-E3-Lin-2 is preserved exactly in the same way as for Max-3-IS. Consequently, it is NP-hard to approximate the solution of Max-3-DM to within $1 + \frac{1-4\varepsilon}{2M(H)/k+13+2\varepsilon}$, even on instances with each element occurring in exactly two triples.

(B) Maximum 4-Dimensional Matching. We will use the following edge-4-coloring of our gadget $G_0[4]$ in Fig. 2(i) (analogously for $G_1[4]$): a-colored edges $\{x'_0, \boxed{101}\}$, $\{x'_1, \boxed{011}\}$, $\{y_1, \boxed{000}\}$, $\{y_0, \boxed{110}\}$; b-colored edges $\{x'_0, \boxed{110}\}$, $\{x'_1, \boxed{000}\}$, $\{y_1, \boxed{101}\}$, $\{y_0, \boxed{011}\}$; c-colored edges $\{x_1, x'_0\}$, $\{x_0, x'_1\}$, $\{\boxed{101}, \boxed{110}\}$, $\{z_0, \boxed{011}\}$, $\{z_1, \boxed{000}\}$; d-colored edges $\{x'_0, x'_1\}$, $\{\boxed{000}, \boxed{011}\}$, $\{z_0, \boxed{101}\}$, $\{z_1, \boxed{110}\}$. Now we will show that an edge-4-coloring of a consistency $(4, 3k)$-gadget H exists that fit well with the above coloring of equation gadgets. We suppose that the $(3k, \tau)$-amplifier G from which H was constructed has a matching $\mathcal{M}$ of all checkers. (This is true for amplifiers of [3] and [5]). Color d will be used for edges $\{x_0, x_1\}$, for each $x \in V(G) \setminus \{x^{2k+1}, \dots, x^{3k}\}$. Also, for each $x \in \{x^{k+1}, \dots, x^{2k}\}$, the corresponding $\{X_0, X_1\}$ edge will have color d, too. Color c will be reserved for coloring edges of H "along the matching $\mathcal{M}$", i.e., if $\{x, y\} \in \mathcal{M}$, edges $\{x_0, y_1\}$ and $\{x_1, y_0\}$ have color c. Furthermore, for $x \in \{x^{k+1}, \dots, x^{2k}\}$ the corresponding edges $\{x_0, X_1\}$ and $\{x_1, X_0\}$ will be of color c, too. The edges that are not colored by c or d, form a 2-regular bipartite graph, hence they can be edge 2-colored by colors a and b. The above edge 4-coloring of H and $G_j[4]$ ($j \in \{0, 1\}$) ensures that instances produced in our reduction to MAX-4-IS are edge-4-colored 4-regular graphs. Hence the same approximation hardness result as we obtained for MAX-4-IS applies to these instances of MAX-4-DM as well. □

It is known that MIN-3-SET COVER, resp. MAX-3-SET PACKING, are APX-complete even if the number of occurrences of any element in $\mathcal{C}$ is bounded by a constant $K \geq 2$ ([2], [16]). The MAXIMUM TRIANGLE PACKING problem is APX-complete even for graphs with maximum degree 4 [13]. Some explicit lower bounds on their polynomial time approximability can be obtained from L-reductions used in the proofs of their Max-SNP completeness ([13], [16]). Similarly as in [5], applying the hardness results obtained above for MAX-B-IS and MIN-B-VC to such packing and covering problems, we can improve lower bounds for them as well.

Theorem 20 *It is* NP-*hard to approximate*

(i) MAXIMUM TRIANGLE PACKING *(even on 4-regular line graphs) to within an approximation factor* 1.010661 ($> \frac{95}{94}$),

(ii) MIN-3-SET COVER *with exactly two occurrences of each element to within any constant smaller than* $1 + \frac{1}{2\lambda_3+13} > 1.0101215$ ($> \frac{100}{99}$)*; and* MIN-4-SET COVER *with exactly two occurrences of each element to within any constant smaller than* $1 + \frac{1}{2\lambda_4+8} > 1.0194553$ ($> \frac{53}{52}$),

(iii) MIN-3-SET PACKING *with exactly two occurrences of each element to within any constant smaller than* $1 + \frac{1}{2\mu_3+13} > 1.010661$ ($> \frac{95}{94}$)*; and* MIN-4-SET PACKING *with exactly two occurrences of each element to within any con-*

stant smaller than $1 + \frac{1}{2\mu_4+3} > 1.0215517$ $(> \frac{48}{47})$.

PROOF. A lower bound for Min-B-Set Cover follows from that of Min-B-VC, and a lower bound for Min-B-Set Packing follows from that of Min-B-IS, as was explained in the beginning of this section.

Let us explain briefly, how the result follows for Maximum Triangle packing problem. Consider a 3-regular triangle-free graph G as an instance of Max-3-IS from Theorem 17. (Notice, that instances produced in our approximation hardness result for Max-3-IS were of this form.) The vertices of G are transformed to triangles in the line graph $L(G)$ of G and this is one-to-one correspondence, as G was triangle-free. Clearly, independent sets of vertices in G are in one-to-one correspondence with triangle packings in $L(G)$, so the conclusion easily follows from Theorem 17.

6 Asymptotic Approximability Bounds

This paper is focused mainly on graphs of low degree. But in this section we discuss also the asymptotic relation between hardness of approximation and degree for Maximum Independent Set and Minimum Vertex Cover problem in bounded degree graphs.

For the Maximum Independent Set problem restricted to graphs of degree $B \geq 3$ the problem is known to be approximable with performance ratio arbitrarily close to $\frac{B+3}{5}$ ([2]) for even B and $\frac{B+3}{5} - \frac{4(5\sqrt{13}-18)}{5} \frac{(B-2)!!}{(B+1)!!}$ for odd B ([7]). But asymptotically better ratios can be achieved by polynomial algorithms, currently the best one approximates to within a factor of $O(B\frac{\ln \ln B}{\ln B})$, as follows from [1], [14]. On the other hand, Trevisan [17] has proved NP-hardness to approximate the solution to within $\frac{B}{2^{O(\sqrt{\ln B})}}$.

For the Minimum Vertex Cover problem the situation is more challenging, even in general graphs. The recent result of Dinur and Safra [10] shows that for any $\delta > 0$ the Minimum Vertex Cover problem is NP-hard to approximate to within $10\sqrt{5} - 21 - \delta$. One can observe that their proof can give hardness result also for graphs with (very large) bounded degree $B(\delta)$. This follows from the fact that after their use of Raz's parallel repetition theorem (where each variable appears in only a constant number of tests), the degree of produced instances is bounded by a function of δ. But the dependence of $B(\delta)$ on δ in their proof is quite complicated. The earlier $\frac{7}{6} - \delta$ lower bound proved by Håstad [11] was extended by Clementi & Trevisan [9] to graphs with bounded degree $B(\delta)$.

Our next result improves on theirs: it has better trade-off between non-approximability and degree bound. There are no hidden constants in our asymptotic formula, and it provides good explicit inapproximability results for degree bound B starting from few hundreds. First we need to introduce some notation.

Notation. Denote $F(x) := -x \ln x - (1-x)\ln(1-x)$, $x \in (0,1)$, where ln means the natural logarithm. Further, $G(c,t) := (F(t)+F(ct))/(F(t)-ctF(\frac{1}{c}))$ for $0 < t < \frac{1}{c} < 1$, $g(t) := G(\frac{1-t}{t}, t)$ for $t \in (0, \frac{1}{2})$. More explicitly, $g(t) = 2[-t\ln t - (1-t)\ln(1-t)]/[-2(1-t)\ln(1-t) + (1-2t)\ln(1-2t)]$. Using Taylor series of the logarithm near 1 we see that the denominator here is $t^2 \cdot \sum_{k=0}^{\infty} \frac{2^{k+2}-2}{(k+1)(k+2)} t^k > t^2$, and $-(1-t)\ln(1-t) = t - t^2 \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)} t^k < t$, consequently $g(t) < \frac{2}{t}(1 + \ln \frac{1}{t})$.

For large enough B, we look for $\delta \in (0, \frac{1}{6})$ such that $3\lfloor g(\frac{\delta}{2})\rfloor + 3 \le B$. As $g(\frac{1}{12}) \approx 75.62$ and g is decreasing in $(0, \frac{1}{12}\rangle$, we can see that for $B \ge 228$ any $\delta > \delta_B := 2g^{-1}(\lfloor \frac{B}{3} \rfloor)$ will do. Trivial estimates on δ_B (using $g(t) < \frac{2}{t}(1+\ln\frac{1}{t})$) are $\delta_B < \frac{12}{B-3}(\ln(B-3) + 1 - \ln 6) < \frac{12 \ln B}{B}$.

We will need the following lemma about regular bipartite expanders to prove the Theorem 22.

Lemma 21 *Let $t \in (0, \frac{1}{2})$ and d be an integer for which $d > g(t)$. For every sufficiently large positive integer n there is a d-regular n by n bipartite graph H with bipartition (V_0, V_1), such that for each independent set J in H either $|J \cap V_0| \le tn$, or $|J \cap V_1| \le tn$.*

PROOF. In the standard model of random d-regular bipartite graphs it is well known (and easy to prove) that the conditions $0 < t < \frac{1}{c} < 1$ and $d > G(c,t)$ are sufficient for the existence, for every sufficiently large n, of a d-regular bipartite graph with n by n bipartition (V_0, V_1), which is a (c,t,d)-expander (i.e., $U \subseteq V_0$ or $U \subseteq V_1$, and $|U| \le tn$ imply $|\Gamma(U)| \ge c|U|$; here $\Gamma(U) := \{y\colon y$ is a vertex adjacent to some $x \in U\}$) (see, e.g., Theorem 6.6 in [8] for this result). If $d > g(t)$ $(= G(\frac{1-t}{t}, t))$, by the continuity of G also $d > G(c,t)$ for some $c > \frac{1-t}{t}$. So with these parameters (c,t,d)-expanders exist for n sufficiently large, and they clearly have the required property. □

Theorem 22 *For every $\delta \in (0, \frac{1}{6})$, it is* NP*-hard to approximate* Minimum Vertex Cover *to within $\frac{7}{6} - \delta$ even in graphs of maximum degree $\le 3\lfloor g(\frac{\delta}{2})\rfloor + 3 \le 3\lceil \frac{4}{\delta}(1 + \ln \frac{2}{\delta})\rceil$. Consequently, for any $B \ge 228$, it is* NP*-hard to approximate* Min-B-VC *to within any constant smaller than $\frac{7}{6} - \delta_B$, where $\delta_B := 2g^{-1}(\lfloor \frac{B}{3} \rfloor) < \frac{12}{B-3}(\ln(B-3) + 1 - \ln 6) < 12\frac{\ln B}{B}$.*

PROOF. Let $\delta \in (0, \frac{1}{6})$ be given, put $d := \lfloor g(\frac{\delta}{2}) \rfloor + 1$. Then we choose $t \in (0, \frac{\delta}{2})$ so close to $\frac{\delta}{2}$ that $d > g(t)$. Further, we choose $\varepsilon \in (0, \frac{1}{4})$ such that $(\frac{7}{2} - \varepsilon - 6t)/(3 + \varepsilon) > \frac{7}{6} - \delta$. Now a positive integer k is chosen so large that

(i) NP-hard gap $\langle \frac{1}{2}+\varepsilon, 1-\varepsilon \rangle$ of Theorem 2 applies to the problem Ek-MAX-E3-LIN-2, and
(ii) there is a d-regular $2k$ by $2k$ bipartite graph H with bipartition (V_0, V_1), such that for each independent set J in H either $|J \cap V_0| \leq 2kt$, or $|J \cap V_1| \leq 2kt$ (see Lemma 21). Keep one such H fixed from now on.

We will describe reduction f from Ek-MAX-E3-LIN-2 to the MIN-VC problem in graphs and we will check how the NP-hard gap of (i) is preserved.

Let I be an instance of Ek-MAX-E3-LIN-2, $\mathcal{V}(I)$ be the set of variables of I, and $m := |\mathcal{V}(I)|$. Clearly, the system I has $\frac{mk}{3}$ equations. For each equation of I we take a quadruple of labeled vertices. More precisely, if the equation reads as $x+y+z = j$ $(j \in \{0,1\})$ we take 4 vertices with labels $\boxed{xyz = 00j}$, $\boxed{xyz = 01(1-j)}$, $\boxed{xyz = 10(1-j)}$ and $\boxed{xyz = 11j}$. Notice, that these vertices correspond to all partial assignments to variables making the equation satisfied. Denote by G_I the graph whose vertex set consists of the union of vertices of those $\frac{mk}{3}$ quadruples, with an edge added for each pair of inconsistently labeled vertices. The pair of vertices is inconsistent, if a variable $u \in \mathcal{V}(I)$ exists that is assigned differently in their labels. It is clear, that independent sets in G_I correspond to subsets of I satisfied by an assignment to variables. Consequently, $\alpha(G_I) = \frac{mk}{3}\mathrm{OPT}(I)$. Clearly, the hard gap of (i) is preserved for the MAX-IS problem and translates to another one for the problem MIN-VC for graphs G_I.

Using our fixed expander H we can enforce similar preserving of that NP-hard gap even in graphs of maximum degree $\leq 3d$.

Consider a variable $u \in \mathcal{V}(I)$. Let $V_j(u)$ $(j \in \{0,1\})$ be the set of all $2k$ vertices in which u has assigned bit j. Choose any bijection between $V_0(u)$ and V_0 (of H), and between $V_1(u)$ and V_1 (of H). Now take edges between $V_0(u)$ and $V_1(u)$ exactly as prescribed by our expander H. Having this done, one after another, for each $u \in \mathcal{V}(I)$, we get the graph $G_I^H =: f(I)$. Clearly, the transformation f is polynomial, and the maximum degree of G_I^H is at most $3d$.

Any independent set in G_I is an independent set also in G_I^H, hence $\alpha(G_I^H) \geq \alpha(G_I) = \frac{mk}{3}\mathrm{OPT}(I)$ and $vc(G_I^H) \leq vc(G_I) = \frac{mk}{3}(4 - \mathrm{OPT}(I))$.

On the other hand, one can show that $\alpha(G_I^H) \leq \alpha(G_I) + 2kmt$ as follows: Consider an independent set J of G_I^H with $|J| = \alpha(G_I^H)$. For each $u \in \mathcal{V}(I)$, one after another, remove exactly one of sets $J \cap V_0(u)$, $J \cap V_1(u)$ from J, namely the one with cardinality $\leq 2kt$. (The existence is ensured by properties of our expander H, and the way how G_I^H was created.) Having this done for all $u \in \mathcal{V}(I)$, we get an independent set of G_I (hence of size $\leq \alpha(G_I)$), removing no more than $2kmt$ vertices. Hence $\alpha(G_I^H) \leq \alpha(G_I) + 2kmt = \frac{mk}{3}(\mathrm{OPT}(I) + 6t)$, and $vc(G_I^H) \geq \frac{mk}{3}(4 - \mathrm{OPT}(I) - 6t)$. Hence, the NP-hard question of whether $\mathrm{OPT}(I)$ is greater than $(1 - \varepsilon)$, or less than $(\frac{1}{2} + \varepsilon)$, is transformed to the one of whether $vc(G_I^H)$ is less than $\frac{mk}{3}(3 + \varepsilon)$, or greater than $\frac{mk}{3}(\frac{7}{2} - \varepsilon - 6t)$. Consequently, it is NP-hard to approximate MIN-VC to within $(\frac{7}{2} - \varepsilon - 6t)/(3 + \varepsilon) > \frac{7}{6} - \delta$ on instances G_I^H of maximum degree $\leq 3d$.

The consequence about inapproximability of MIN-B-VC is straightforward. □

Conclusion remarks

One possible way how to improve further our inapproximability results is to give better upper bounds on parameters λ_B and μ_B. We think that there is still a potential for improvement here, using a suitable probabilistic model for the construction of amplifiers and gadgets.

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