

A characterization of the Copeland solution*

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Abstract

We provide a new characterization of the Copeland solution, based on the number of steps in which candidates beat each other. A *Condorcet winner* is a candidate which beats every other contender in one step. In other words, given m candidates, a Condorcet winner beats all remaining contenders in a total of $m - 1$ steps. When choosing from a tournament, there is universal agreement on the *Condorcet principle* which requires to pick the Condorcet winner, whenever it exists. As a Condorcet winner may fail to exist, the Condorcet principle can be extended to what we call the *minisum principle*: Choose the candidate(s) who beat all remaining contenders in the smallest total number of steps. We show that the minisum principle characterizes the Copeland solution.

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1 Overview

In solving the problem of choosing from a tournament, Copeland (1951) proposes to pick the candidates which beat the highest number of contenders. The proposal of Copeland received attention from a variety of fields, including biology as in Landau (1953); graph theory as in Van den Brink and Gilles (2003); economics as in Paul (1997); computer science as in Singh and Kurose (1991) and social choice theory as in Moulin (1986). As a result, it has been the subject matter of thorough investigations and we know, at present, many of its properties.¹

However, the literature is not very rich in characterizations of the Copeland solution. In fact, the original proposal of Copeland (1951) is not supported by a characterization. Later, Moon (1968) shows the equivalence between the Copeland ranking and the one generated by the *maximum likelihood solution* of Zermelo (1929) which assigns a “strength” to each alternative and derives the social ranking accordingly. The first axiomatic characterization of the Copeland solution is by Rubinstein (1980) who characterizes the “Copeland welfare function” as a method to rank the participants of a tournament. Henriot (1985) extends this characterization to environments which allow ties between candidates. Moreover, he gives three characterizations of the “Copeland choice rule” which chooses among the participants of a tournament. As far as we know, these are the only characterizations of the Copeland solution, all of which use its invariance to the reversal of a cycle’s orientation.

We provide a new characterization based on the number of steps in which candidates beat each other. A *Condorcet winner* is a candidate which beats every other contender in one step. In other words, given m candidates, a Condorcet winner beats all remaining contenders in a total of $m - 1$ steps. When choosing from a tournament, there is universal agreement on the *Condorcet principle* which requires to pick the Condorcet winner, whenever it exists. As a Condorcet winner may fail to exist, the Condorcet principle can be extended to what we call the *minisum principle*: Choose the candidate(s) who beat all remaining contenders in the smallest total of steps. Interestingly, the minisum principle characterizes the Copeland solution.

Section 2 introduces the basic notions. Section 3 states our results.

¹One can see Laslier (1997) for a detailed exposition of tournament solutions, including the one proposed by Copeland.

2 Basic Notions

Let X be a finite set of candidates with $\#X \geq 3$. By a *tournament* over X , we mean a complete and asymmetric binary relation over X . We write Θ for the set of tournaments over X .² A *tournament solution* is a mapping $f : \Theta \rightarrow 2^X \setminus \{\emptyset\}$. For each $T \in \Theta$, let $\delta_T(x) = \#\{z \in X : xTz\}$ be the number of alternatives that x directly beats. The *Copeland rule* is the tournament solution γ defined as $\gamma(T) = \{x \in X : \delta_T(x) \geq \delta_T(y) \forall y \in X\}$.

For any natural number n , we write $I_n = \{0, 1, \dots, n\}$ for the set that consists of zero and the natural numbers from 1 to n . Given any tournament $T \in \Theta$ and any $x, y \in X$, a *path* from x to y in T is a sequence $\{x_i\}_{i \in I_n} \subseteq X$ of alternatives with $x_0 = x$ and $x_n = y$ such that $x_i T x_{i+1} \forall i \in I_n \setminus \{n\}$. We refer to n as the *length of the path*. Let $\lambda_T(x, y)$ be the length of the shortest path from x to y in T , i.e., the length of any path from x to y in T is at least $\lambda_T(x, y)$. We set $\lambda_T(x, y) = \#X$, when T admits no path from x to y and $\lambda_T(x, x) = 0$. If $\lambda_T(x, y) = 1$, then xTy , in which case x directly beats y . A *Condorcet winner* is a candidate which directly beats all other contenders. Given any $T \in \Theta$, let $s_T(x) = \sum_{y \in X} \lambda_T(x, y)$ be the sum of the shortest paths lengths from $x \in X$ to all remaining contenders. We write $\mu(T) = \{x \in X : s_T(x) \leq s_T(y) \forall y \in X\}$ for the set of *minisum candidates*. We characterize the Copeland solution in terms of this minisum principle.

3 The Characterization

Following Miller (1980), given any tournament $T \in \Theta$ and any distinct $x, y \in X$, we say that y *covers* x in T iff $xTz \implies yTz \forall z \in X$. We write $U(T) = \{x \in X : \nexists y \in X \text{ which covers } x \text{ in } T\}$ for the *uncovered set* of T . The transitivity of the covering relation ensures $U(T) \neq \emptyset$. As Miller (1980) shows, when T does not admit a Condorcet winner, we have $U(T) = \{x \in X : \lambda_T(x, y) \in \{1, 2\}\}$, i.e., the uncovered set consists of the candidates which beat every other contender in at most two steps. This is an extension of the Condorcet principle through the requirement of minimizing the maximum

²So given any $T \in \Theta$ and any distinct $x, y \in X$, precisely one of xTy and yTx holds. We understand completeness in the weak sense where xTx holds for no $x \in X$.

number of steps. We call this the *maximin* principle.³ So the uncovered set is equivalent to the maximin principle.

The literature admits various solutions that refine the uncovered set. Proposition 3.1 below quotes a result of Miller (1980) which shows that the Copeland solution is a refinement of the uncovered set.

Proposition 3.1 $\gamma(T) \subseteq U(T)$ at each $T \in \Theta$ while $\gamma \neq U$.

As shown below, the set of minisum candidates also refines the uncovered set.

Proposition 3.2 $\mu(T) \subseteq U(T)$ at each $T \in \Theta$.

Proof. Take any $x \in \mu(T)$. So $s_T(x) \leq s_T(y) \forall y \in X$. Suppose, there exists $c \in X$ that covers x . By the definition of the covering relation and the definition of the shortest path, we have $\lambda_T(c, z) \leq \lambda_T(x, z)$ for all $z \in X \setminus \{x, c\}$. Thus $\sum_{z \in X \setminus \{x, c\}} \lambda_T(c, z) \leq \sum_{z \in X \setminus \{x, c\}} \lambda_T(x, z)$. Moreover, as c covers x , we have $c T x$, implying $\lambda_T(c, x) = 1 < \lambda_T(x, c)$. Hence, $\sum_{z \in X} \lambda_T(c, z) < \sum_{z \in X} \lambda_T(x, z)$ which means $s_T(c) < s_T(x)$, contradicting $s_T(x) \leq s_T(y) \forall y \in X$. Thus, there exists no $c \in X$ that covers x , hence $x \in U(T)$. ■

In fact, the set of minisum candidates coincides with the set of Copeland winners.

Theorem 3.1 $\gamma(T) = \mu(T)$ at each $T \in \Theta$.

Proof. Take any $T \in \Theta$. To show $\gamma(T) \subseteq \mu(T)$, take any $x \in \gamma(T) \subseteq U(T)$. As $x \in U(T)$, for any $y \in X \setminus \{x\}$, we have $\lambda_T(x, y) = 1$ if $x T y$ and $\lambda_T(x, y) = 2$ if $y T x$. So $s_T(x) = \#\{y \in X : x T y\} + 2 \cdot \#\{y \in X : y T x\} = \delta_T(x) + 2(n - 1 - \delta_T(x)) = 2n - 2 - 2\delta_T(x)$. As $x \in \gamma(T)$, hence $\delta_T(x) \geq \delta_T(y) \forall y \in X$, we have $s_T(x) \leq s_T(y) \forall y \in X$, implying $x \in \mu(T)$. To show $\mu(T) \subseteq \gamma(T)$, take any $y \in \mu(T) \subseteq U(T)$. As $\lambda_T(x, z) \in \{1, 2\}$ and $\lambda_T(y, z) \in \{1, 2\}$ for all $z \in X$ and $s_T(x) \leq s_T(y)$, we have $\delta_T(x) \geq \delta_T(y)$, which by definition of γ implies $x \in \gamma(T)$. ■

As the Copeland rule (which is equivalent to the minisum principle) refines the uncovered set (which is equivalent to the maximin principle), tournament solutions exemplify a case where the minisum principle refines the maximin principle - a fact which is not common in the literature.⁴

³Shepsle and Weingast (1984) call this the *two step principle*.

⁴See Brams et al. (2007) for a detailed discussion of these two principles.

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