Hard coloring problems in low degree planar bipartite graphs

Miroslav Chlebík\textsuperscript{a}, Janka Chlebíková\textsuperscript{b}

\textsuperscript{a}Max Planck Institute for Mathematics in the Sciences, Inselstraße 22-26, D-04103 Leipzig, Germany
\textsuperscript{b}Department of Informatics Education, Faculty of Mathematics, Physics and Informatics, Mlynská dolina, 842 48 Bratislava, Slovakia

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Abstract

In this paper we prove that the PRECOLORING EXTENSION problem on graphs of maximum degree 3 is polynomially solvable, but even its restricted version with 3 colors is NP-complete on planar bipartite graphs of maximum degree 4.

The restricted version of LIST COLORING, in which the union of all lists consists of 3 colors, is shown to be NP-complete on planar 3-regular bipartite graphs.

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1. Introduction

Throughout the paper $G = (V, E)$ stands for a finite simple undirected graph, where $V = V(G)$ and $E = E(G)$ denote the set of vertices and edges, respectively. The degree of a vertex $v \in V$ is denoted by $d_G(v)$ and the maximum degree of $G$ by $\Delta(G)$. A coloring of a graph is any mapping that assigns to each vertex a color such that neighboring vertices receive different colors, a $k$-coloring is a coloring using at most $k$ colors. The minimum $k$ for which a $k$-coloring of $G$ exists is called chromatic number of $G$ and denoted by $\chi(G)$. In a partial coloring of a graph only some vertices (called precolored) have been assigned a color.

Many generalizations of the standard VERTEX COLORING problem have been studied, in which the task is to find a coloring of a graph under some additional constraints. In the LIST COLORING problem, which has its origin in works by Vizing [19] and Erdős et al. [7], each vertex of a graph can receive a color only from its prescribed list of admissible colors. In the PRECOLORING EXTENSION problem, a partial coloring of a graph has to be extended to a coloring of the entire graph using only a set of admissible colors. Both problems are well established and provide natural interpretations for various kinds of scheduling problems, issues in VLSI theory, etc. (see, e.g., [2,3]).

For general graphs both problems PRECOLORING EXTENSION and LIST COLORING (strictly speaking, their decision versions) are NP-complete, as they generalize the standard VERTEX COLORING problem. On the other hand, all these problems can be solved in linear time for partial $k$-trees [11]. In many restricted graph classes the PRECOLORING EXTENSION problem is harder than VERTEX COLORING, but easier than LIST COLORING. For example, LIST COLORING is NP-complete for split graphs [18] and for the complements of bipartite graphs [10], although the PRECOLORING EXTENSION problem is polynomially solvable in both cases [9]. For some graph classes, where VERTEX COLORING is
A list assignment $L$ is defined as the size of the union of all lists, i.e., $|L(v)| = \sum_{B \in A(v)} |L_v(B)|$. A list assignment $L$ such that $|L(v)| \geq d_G(v)$ for all $v \in V$ is called supervalent. A block of a graph $G$ is a maximal connected subgraph of $G$ that has no cut-vertex. For a connected graph $G$, a supervalent list assignment $L$ not admitting an $L$-coloring of $G$ exists if and only if $G$ is a Gallai tree (a connected graph in which every block is a complete graph or an odd cycle). Furthermore, a full characterization of all supervalent list assignments not admitting an $L$-coloring is known.

**Theorem 1** (Borodin [4,5], Erdős [7]). Let $L$ be a supervalent list assignment for a connected graph $G$. Then $G$ is not $L$-colorable if and only if the following conditions (1)–(3) hold:

1. $G$ is a Gallai tree,
2. $|L(v)| = d_G(v)$ for every $v \in V$,
3. $L_v(B) = \bigcup_{B \in A(v)} L_B$ for all $v \in V$, where $A(v)$ is the set of blocks of $G$ containing $v$, and for each block $B$, $L_B$ is a set of $\chi(B) - 1$ colors.

A short proof of this theorem appears in [12]. It can be viewed as an extension of the classical Brooks’ theorem [6] which asserts that the chromatic number of a connected graph exceeds its maximum degree only for a complete graph and an odd cycle. Some further extensions of Brooks-type theorem for the (generalized) coloring problems can be found in [8,1]. It is also easy to check that the proof of Theorem 1 can be made algorithmic (see [8], where this aspect is in the focus) and hence the following holds:

**Theorem 2.** There is a polynomial-time algorithm which for a graph $G$ and its supervalent list assignment $L$ decides whether $G$ is $L$-colorable and if so, then the algorithm finds an $L$-coloring of $G$.

**Remark 1.** It is easy to observe that the LIST COLORING problem is polynomially solvable if the size of each list is at most 2 (since the problem can be reduced to 2SAT), or for graphs of maximum degree at most 2 (since component of such graphs are paths and cycles), see, e.g., [19,7,18].

There are straightforward consequences of Theorem 2 to PRECOLORING EXTENSION. In fact, the PRECOLORING EXTENSION problem can be interpreted as a particular case of the LIST COLORING problem as follows: let $G$ be an instance of PRECOLORING EXTENSION. The list $L(v)$ of a vertex $v$, which is not precolored in $G$, will contain all admissible colors except those used on precolored neighbors of $v$. In this way lists of all not precolored vertices of
Theorem 3. There is a polynomial-time algorithm which for a partially colored graph \( G \) and an integer \( k \geq \Delta(G) \) decides whether the partial coloring can be extended to a \( k \)-coloring of the graph \( G \) and if so, then the algorithm finds such a \( k \)-coloring.

Theorem 3 easily implies that the PRECOLORING EXTENSION problem can be solved efficiently for graphs of very small maximum degree.

Theorem 4. The PRECOLORING EXTENSION problem is polynomially solvable for graphs of maximum degree at most 3.

Proof. Given a partially colored graph \( G \) and an integer \( k \) as an input, the task is to extend the partial coloring to a \( k \)-coloring of \( G \). If \( k \geq \Delta(G) \), we can use the algorithm from Theorem 3. Otherwise, \( k \leq \Delta(G) - 1 \leq 2 \) and in this case the problem is clearly at most as difficult as 2-coloring of graphs, that is a polynomially solvable problem. This completes the proof. \( \square \)

3. Hardness results

In [14] Kratochvíl and Tuza proved that LIST COLORING on planar bipartite graphs of maximum degree 3 is NP-complete even if the size of each list is at most 3 (and the number of occurrences of each color in lists is at most 3 as well). However, their proof method does not prove NP-completeness for instances with bounded total number of colors in the same graph class. In this section we provide such hardness result even for more restricted instances of LIST COLORING, namely for 3-regular planar bipartite graphs and lists with the total number of colors equal to 3.

We use a reduction from CUBIC PLANAR MONOTONE 1-IN-3SAT, a restricted version of the MONOTONE 1-IN-3SAT problem. It is well known that the problem and also its restricted version are NP-complete [17] (see also [15]). In the MONOTONE 1-IN-3SAT problem a set \( \varphi \) of variables and a set \( \mathcal{C} \) of clauses of the form \((a \lor b \lor c)\) is given, where \( a, b, \) and \( c \) are distinct variables from \( \varphi \) without negation. The question is to decide if there is an assignment of truth values to the variables such that exactly one of the three variables in each clause is true. The CUBIC PLANAR MONOTONE 1-IN-3SAT problem is restricted to instances whose underlying bipartite \((\varphi, \mathcal{C})\) graph is 3-regular and planar, where \((\varphi, \mathcal{C})\) is the graph with vertex set \( \varphi \cup \mathcal{C} \) and an edge for each pair \((u, C)\) such that \( u \) is a variable appearing in a clause \( C \).

Theorem 5. The LIST COLORING problem with the total number of colors 3 is NP-complete on 3-regular planar bipartite graphs.

Proof. The problem is clearly in NP. To prove its NP-hardness we will reduce the NP-complete problem CUBIC PLANAR MONOTONE 1-IN-3SAT to it.

Consider an instance \( I := (\varphi, \mathcal{C}) \) of MONOTONE 1-IN-3SAT, whose underlying bipartite graph is planar and 3-regular. For such instances it is NP-complete to decide if there is an assignment \( \varphi : \varphi \rightarrow \{0, 1\} \) such that exactly one of three variables in each clause is set to 1 [17,15]. The instance \( I \) will be transformed in polynomial time into an instance \( f(I) := (G, L) \) of the LIST COLORING problem, where \( G = (V, E) \) is a 3-regular planar bipartite graph and \( L : V \rightarrow 2^{[0,1,2]} \) its list assignment, with the property that \( G \) is \( L \)-colorable if and only if \( I \) is a YES instance of 1-IN-3SAT (it means that there is an assignment of truth values to the variables \( \varphi \) in \( I \) such that exactly one of three variables in each clause \( C \) is set to 1).

First, the vertices of \( \varphi \) that represent variables of \( I \) will be the vertices of \( G \) as well and the list assignment is defined \( L(v) = \{0, 1\} \) for each \( v \in \varphi \). Next, any vertex \( C \) representing a clause, say \( u \lor v \lor w \), and its incident edges \((u, C), (v, C), (w, C)\) are replaced by a simple gadget \( G_C \) depicted, along with the list assignment for its vertices,
in Fig. 1. The degree of every vertex is 3 in the constructed graph, but there are parallel edges. To obtain maximum generality, we want to show that the problem is hard for 3-regular graphs even if parallel edges are not allowed. To get rid of a double edge, we replace it by the graph $G_d$ shown in Fig. 2, where the vertices of degree 2 coincide with vertices of $G_C$. It is easy to see that this transformation does not change the solvability of the instances. Having this replacement done, one after another for every $C \in \mathcal{C}$, we obtain a 3-regular instance $G$ of the LIST COLORING problem.

As the gadget $G_C$ is a planar bipartite graph, and the replacement of a vertex $C \in \mathcal{C}$ by the gadget was chosen to preserve the planarity and bipartiteness of the graph, the final graph $G$ is also planar and bipartite.

Now our aim is to show that the corresponding list assignment $L : V \to 2^{\{0,1,2\}}$ is such that $G$ is $L$-colorable if and only if $I$ is a YES instance of 1-IN-3SAT. For this purpose we describe some properties of possible $L$-colorings of the gadget $G_C$ from Fig. 1.

A partial coloring $\varphi : \{u, v, w\} \to \{0, 1\}$ of the gadget $G_C$ is said to be good, if it can be extended to an $L$-coloring of the entire gadget $G_C$.

Claim. A partial coloring $\varphi$ of the gadget $G_C$ is good if and only if exactly one of $\varphi(u)$, $\varphi(v)$, and $\varphi(w)$ is equal to 1.

Proof of Claim. (a) Assume first that $\varphi$ is good. If for some $x \in \{u, v, w\}$ $\varphi(x) = 1$, then in all $L$-colorings extending $\varphi$, colors of vertices on the shortest paths from $x$ to any of the other two vertices of $\{u, v, w\}$ are uniquely determined and, in particular, both vertices $\{u, v, w\}\{x\}$ have color 0. Thus, at most one of $\varphi(u)$, $\varphi(v)$, and $\varphi(w)$ equals 1. Assume that $\varphi(u) = \varphi(v) = \varphi(w) = 0$ and assign greedily colors to all vertices whose color is uniquely determined in all possible $L$-colorings extending $\varphi$. We can see that the central vertex of the gadget whose list is $\{0, 1, 2\}$, has neighbors of all three colors (Fig. 3). Thus, an $L$-coloring extending $\varphi$ does not exist in case $\varphi(u) = \varphi(v) = \varphi(w) = 0$, which contradicts to the fact that $\varphi$ was good. Hence if a partial coloring $\varphi$ is good, exactly one of $\varphi(u)$, $\varphi(v)$, and $\varphi(w)$ equals 1.
(b) For each of three cases of $\varphi$ with exactly one value of 1 we can provide an $L$-coloring of $G_C$ extending $\varphi$ as depicted in Fig. 4, the extending of $G_d$ is uniquely determined. The first coordinate in the figure corresponds to the case of $\varphi$ with $\varphi(u) = 1$, $\varphi(v) = \varphi(w) = 0$, and the other two coordinates to the cases $\varphi(v) = 1$, $\varphi(u) = \varphi(w) = 0$, and $\varphi(w) = 1$, $\varphi(u) = \varphi(v) = 0$, respectively.

Now we use Claim to complete the proof showing that $G$ is $L$-colorable if and only if $I$ is a YES instance of 1-IN-3SAT. Assume first that $G$ is $L$-colorable and let $\psi : V \rightarrow \{0, 1, 2\}$ be a fixed $L$-coloring of $G$. For each clause $C$, say $C := u \lor v \lor w$, the restriction of $\psi$ to $\{u, v, w\}$ is good and, due to Claim, exactly one of $\psi(u)$, $\psi(v)$, and $\psi(w)$ equals 1. Consequently, the restriction $\psi|\mathcal{F} : \mathcal{F} \rightarrow \{0, 1\}$ witnesses that $I$ is a YES instance of 1-IN-3SAT.

Assume now that $I$ is a YES instance of 1-IN-3SAT and let $\psi : \mathcal{F} \rightarrow \{0, 1\}$ be such that for each clause $C \in \mathcal{C}$, say $C := u \lor v \lor w$, exactly one of $\psi(u)$, $\psi(v)$, and $\psi(w)$ equals 1. Using Claim we obtain that $\psi$ can be extended to an $L$-coloring of the corresponding copy $G_C$ of the gadget. Having this extension done, one after another for copies of the gadget corresponding to all clauses $C \in \mathcal{C}$, one obtains an $L$-coloring of $G$.

This completes the proof that our polynomial-time reduction $f$ has the required properties and that the problem under consideration is NP-complete. □
triangle-free graphs of maximum degree 4. In the following, we show that PRECOLORING EXTENSION with 3 colors is NP-complete on planar bipartite graphs of maximum degree 4. Notice that, due to Theorems 3 and 4, the degree bound 4 and the number of colors 3 are the smallest possible values (unless P = NP) for such hardness result.

**Theorem 6.** The PRECOLORING EXTENSION problem with 3 colors is NP-complete on planar bipartite graphs of maximum degree 4.

**Proof.** The problem is clearly in NP. To prove its NP-hardness we will use the hardness result shown in Theorem 5 for the LIST COLORING problem. Recall that each instance provided in the proof of Theorem 5 consists of a 3-regular planar bipartite graph $G$ and a list $L(v) \subseteq \{0, 1, 2\}$ of cardinality 2 or 3 for each vertex $v \in V$. Now we describe how such an instance can be transformed in polynomial time into a partially colored planar bipartite graph $G'$ of maximum degree 4 such that the partial coloring can be extended to a 3-coloring of $G'$ if and only if $G$ is $L$-colorable.

For each vertex $v \in V$ such that $|L(v)| = 2$ we add a new vertex $v'$, precolored by the color $\{0, 1, 2\}\setminus L(v)$, and an edge $(v, v')$. It is easy to see that the partial coloring of the graph $G'$ obtained in this way can be extended to a 3-coloring of $G'$ if and only if $G$ is $L$-colorable. Moreover, $G'$ is a planar bipartite graph of maximum degree 4. Thus, the problem under consideration is NP-complete as well. □

**References**