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# A class of nonlinear second-order ODEs with movable algebraic singularities

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**Abstract.** A class of nonlinear second-order rational ODEs is studied for which it is shown that any movable singularity of a solution that can be reached along a finite length curve is an algebraic branch point. Some conditions need to be imposed on the equations including the existence of certain formal algebraic series solutions. An example is discussed demonstrating the degree of restriction for the parameters of the equation.

**Keywords.** movable singularities, solutions of ODEs.

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## 1. Introduction

The aim of this article is to describe a class of ordinary differential equations (ODEs) the solutions of which have a somewhat simple singularity structure. A simple singularity structure of ODEs has been put forward as a criterion for their integrability [8], however note that the equations considered in this article are in general non-integrable. The simplest possible singularity structure is for the solutions to have only poles as movable singularities, commonly known as the Painlevé property. The next step is to allow also for movable singularities with a finite branching. This property is called the quasi-Painlevé property in [10] or also weak-Painlevé property in [8]. Here, we consider rational second-order ODEs of the form

$$y'' = E(z,y)(y')^2 + F(z,y)y' + G(z,y), \tag{1}$$

where $E, F, G$ are certain types of rational functions in $y$ with coefficients that are analytic in $z$. Under the assumptions listed below it is shown that all movable singularities of solutions of (1) which can be reached along a curve of finite length are at most algebraic branch points. A main ingredient in the proof is to construct a function $W(z, y, y')$ which remains bounded as the singularity is approached along the curve. This method has been applied in many of the proofs

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that the Painlevé equations possess the Painlevé property, e.g. [5], [7], [9], and subsequently for classes of equations with the quasi-Painlevé property, e.g. [10], [1], [2] and [3]. Although the local structure of a solution of (1) is simple in the way that it is finitely branched about the movable singularities described here, its global structure can be complicated. An example of a singularity which is approached along a curve of infinite length is discussed by Smith in [11]. There it is shown that such a singularity is an accumulation point of algebraic branch points.

In [3], $W$ is taken to be strictly quadratic in $y'$ in order for the class of equations considered to include the Painlevé equations $P_{II} - P_{IV}$. In this article we consider a class of equations of the form (1) with the quasi-Painlevé property where it suffices to take $W$ to be linear in $y'$. Although this class of equations does not include any of the Painlevé equations, it generalises the result in [2] for equations of Liénard type.

For the class of equations considered here we make the following assumptions:

(i) There are functions $\alpha_\mu$, $\mu = 1, \ldots, M$, analytic in a neighbourhood of some point $z_\infty \in \mathbb{C}$ such that $\alpha_1(z_\infty), \ldots, \alpha_M(z_\infty)$ are pairwise distinct and $E$, $F$ and $G$ are of the form

$$\begin{aligned} E =& \sum_{\mu=1}^{M} \frac{k_A^\mu}{y - \alpha_\mu(z)}, \\ F =& f(z,y) \prod_{\mu=1}^{M} (y - \alpha_\mu(z))^{-k_F^\mu}, \\ G =& g(z,y) \prod_{\mu=1}^{M} (y - \alpha_\mu(z))^{-k_G^\mu}, \end{aligned}$$

where $k_A^\mu$, $k_F^\mu$, $k_G^\mu$ are integers and $f, g$ are polynomials in $y$ with coefficients analytic in a neighbourhood of $z = z_\infty$ such that $f(z_\infty, \alpha_\mu(z_\infty)) \neq 0$ for all $\mu \in \{1, \ldots, M\}$ and the highest coefficient of $f$ is non-zero at $z_\infty$. For the degree of $f$ we assume that $\deg_y f > \sum_{\mu=1}^{M} k_F^\mu$ and the degree of $g$ is restricted by $\deg_y g \leq 1 + \deg_y f - \sum_{\mu=1}^{M} (k_F^\mu - k_G^\mu)$.

(ii) For all $\mu \in \{1, \ldots, M\}$ for which $\alpha'_\mu \equiv 0$, i.e. $\alpha_\mu = \text{const}$, we have $k_F^\mu > k_G^\mu \geq 0$. For those $\mu$ for which $\alpha'_\mu \neq 0$ we have $k_F^\mu = k_G^\mu > 0$, and the condition

$$G^\mu(z, \alpha_\mu(z)) + \alpha'_\mu(z) F^\mu(z, \alpha_\mu(z)) \equiv 0 \tag{2}$$

is satified identically, where $F^\mu(z,y) = f(z,y) \prod_{\substack{\nu=1 \\ \nu\neq\mu}}^{M} (y - \alpha_\nu(z))^{-k_F^\nu}$ and $G^\mu(z,y) = g(z,y) \prod_{\substack{\nu=1 \\ \nu\neq\mu}}^{M} (y - \alpha_\nu(z))^{-k_G^\nu}$. In the case $k_F^\mu = k_G^\mu = 1$ the following additional condition is satisfied:

$$k_A^\mu \alpha_\mu'(z_\infty) + F^\mu(z_\infty, \alpha_\mu(z_\infty)) \neq 0. \tag{3}$$

(iii) There is a neighbourhood $U$ of $z_\infty$ such that for all $\hat{z} \in U$ there exists a formal series solution

$$y(z) = \alpha_\mu(\hat{z}) + \sum_{k=1}^{\infty} c_k (z - \hat{z})^{k/k_F^\mu}, \quad c_1 \neq 0. \tag{4}$$

(iv) Let $k_F^0 = \deg_y f - \sum_{\mu=1}^{M} k_F^\mu$. There is a neighbourhood $U$ of $z_\infty$ such that for all $\hat{z} \in U$ there exists a formal series solution of the form

$$y(z) = \sum_{k=0}^{\infty} c_k (z - \hat{z})^{(k-1)/k_F^0}, \quad c_0 \neq 0. \tag{5}$$

Under these assumptions we are going to prove the main result of this article:

**Theorem 1.** *Let $\gamma$ be a finite length curve up to, but not including its endpoint $z_\infty$. Let $y$ be a solution of (1) under the assumptions (i) – (iv) which is analytic on $\gamma$ but cannot be analytically continued to $\gamma \cup \{z_\infty\}$. Then $y$ has a convergent Puiseux series expansion in the vicinity of $z_\infty$ of the form*

$$y(z) = \alpha_\mu(z_\infty) + \sum_{k=1}^{\infty} c_k (z - z_\infty)^{k/k_F^\mu}, \tag{6}$$

*or*

$$y(z) = \sum_{k=0}^{\infty} c_k (z - z_\infty)^{(k-1)/k_F^0}. \tag{7}$$

If a solution $y$ has a singularity at $z_\infty$, assumption (i) expresses what we mean by a movable singularity. The condition $f(z_\infty, \alpha(z_\infty)) \neq 0$ is needed to ensure that the leading order behaviour of $F(z, y)$ does not change discontinuously when $z_\infty$ is approached along $\gamma$. In assumption (ii), condition (2) is needed so that in equation (1), when expanded in $y - \alpha_\mu(z)$, the leading orders of the terms $F(z, y)y'$ and $G(z, y)$ cancel. The existence of formal series solutions, assumptions (iii) and (iv), are necessary for a solution to acquire the form (6) or (7). The theorem then states that every solution about any movable singularity, reachable along a finite length curve, is of this algebraic type. The existence of formal series solutions puts some constraints on the parameters of the equation known as resonance conditions. They are differential relations between $f$ and $g$ which can be calculated algorithmically and will be described by means of an example in section 5.

## 2. Preliminary Lemmas

We start with some general lemmas regarding local solutions of systems of ordinary differential equations and their analytic continuation which are standard in the literature, e.g. [4] or [6].

**Lemma 2** (Cauchy)**.** *Let $f_1, \dots, f_m$ be functions in the variables $y_1, \dots, y_m$ and $z$, analytic in the domain $|z - z_0| \leq r$, $|y_k - y_{k,0}| \leq r$. Let $M$ be the maximum modulus of the functions $f_1, \dots, f_m$ in this domain. Then the initial value problem*

$$\frac{dy_k}{dz} = f_k(z, y_1, \dots, y_m), \quad y_k(z_0) = y_{k,0}, \quad k = 1, \dots, m, \tag{8}$$

*has a unique analytic solution in the disc $B(\rho, z_0)$, where*

$$\rho = r\left(1 - e^{-\frac{1}{(m+1)M}}\right). \tag{9}$$

**Lemma 3** (Painlevé)**.** *Let $f_1, \dots, f_m$ be as in Lemma 2. Let $\gamma$ be a curve up to, but not including its endpoint $z_0$. Suppose that $(y_1, \dots, y_m)$ is a solution of the system (8) which is analytic on $\gamma$. Let $(z_n)_{n\in\mathbb{N}} \subset \gamma$ be a sequence of points on the curve such that $z_n \to z_0$ and $y_k(z_n) \to y_{k,0}$ as $n \to \infty$ for all $k = 1, \dots, m$. Then the solution $(y_1, \dots, y_m)$ can be analytically continued to include the point $z_0$ and satisfies $y_k(z_0) = y_{k,0}$, $k = 1, \dots, m$.*

**Proof.** From some $n$ onwards one has $|z_n - z_0| < \frac{r}{2}$ and $|y_k(z_n) - y_{k,0}| < \frac{r}{2}$ for all $k = 1, \dots, m$. Then, by Lemma 2, the solution $(y_1, \dots, y_m)$ is analytic in $B(z_n, \frac{\rho}{2})$ with $\rho$ in (9). Taking $n$ sufficiently large one has $z_0 \in B(z_n, \frac{\rho}{2})$. ∎

**Lemma 4.** *Let $\gamma$ be a finite length curve in the complex plane and let $P(z, w)$, $Q(z, w)$ and $R(z, w)$ be polynomials in $w = w(z)$ with coefficients analytic in $z$ in a neighbourhood of $\gamma$. If $w$ is bounded on $\gamma$ then any solution of the equation*

$$W' = PW + Qw' + R, \tag{10}$$

*is also bounded on $\gamma$.*

**Proof.** The proof is very similar to the corresponding proof in [3], [9] or [10]. The solution of the first-order linear differential equation (10) can be found by the method of variation of the constant. Choosing a point $z_0 \in \gamma$ the solution can be written as

$$W(z) = \hat{Q}(z, w(z)) + \frac{1}{I(z)}\left(W(z_0) - \hat{Q}(z_0, w(z_0)) + \int_{z_0}^{z} (R - \hat{Q}_\zeta + P\hat{Q}) I(\zeta) \mathrm{d}\zeta\right), \tag{11}$$

where $I(z) = \exp\left(-\int_{z_0}^{z} P(\zeta, w(\zeta))\mathrm{d}\zeta\right)$ and $\hat{Q}(z, w)$ is a polynomial in $w$ such that $\hat{Q}_w = Q$. Since $P$, $\hat{Q}$ and $R$ are polynomials in $w$ and $\gamma$ has finite length, the integral in $I(z)$ and the integral in (11) are bounded. It follows that $W(z)$ is bounded on $\gamma$. ∎

The following lemma by Shimomura is essential in our method of proof for Theorem 1. In its original version in [9] it was used for a proof that the Painlevé equations possess the Painlevé property. The first complete proof of this fact not using integrability and isomonodromic methods was published by Hinkkanen and Laine in 1999 [5] although a rigorous proof of this kind was already contained in the lecture notes by Hukuhara from 1960 which, however, were only published in 2001 [7]. The lemma in the form stated here requires only a slight modification in the original proof.

**Lemma 5.** *Let $y$ be a solution of (1) under the assumptions in Theorem 1. Let $c$ be some complex number not equal to $\alpha_1(z_\infty), \ldots, \alpha_M(z_\infty)$. Then $\gamma$ can be continuously deformed, in the region where $y$ is analytic, to a new curve $\tilde{\gamma}$ with same endpoint and of finite length such that there exists $\epsilon > 0$ for which $|y(z) - c| > \epsilon$ for all $z \in \tilde{\gamma}$.*

## 3. Structure of the Proof

Below a summary of the structure of the proof of Theorem 1 is given.

1. Choose a complex number $c \in \mathbb{C} \setminus \{\alpha_1(z_\infty), \ldots, \alpha_M(z_\infty)\}$ and let $w(z) = (y(z) - c)^{-1}$. The structure of the differential equation is invariant under this transformation as shown below.
2. A function $W$, rational in $w$ and linear in $w'$ is constructed which satisfies a first-order linear differential equation of the form (10). Lemma 4 together with Lemma 5 then shows that $W$ is bounded as $z \to z_\infty$ along $\tilde{\gamma}$.
3. Using all the previous lemmas it is now straightforward to show that $w \to \tilde{\alpha}_\mu(z_\infty)$ for some $\mu \in \{0, 1, \ldots, M\}$.
4. The function $W$ is used to construct a regular initial value problem for $z$ and $W$ as functions of $w$ showing that $z$ is analytic in $w$ near $\tilde{\alpha}(z_\infty)$, i.e. one can write a solution for $z$ as a power series expansion in $w - \tilde{\alpha}_\mu(z_\infty)$ for some $\mu \in \{0, 1, \ldots, M\}$. Inverting this power series and transforming back to the original variable $y$ then leads to the form of the solution as described in Theorem 1.

For step 1, the transformation $w(z) = (y(z) - c)^{-1}$ takes equation (1) into

$$w'' = \left(2w^{-1} - w^{-2}E(z, c + w^{-1})\right)(w')^2 + F(z, c + w^{-1})w' - w^2 G(z, c + w^{-1}),$$

which is of the same form as (1),

$$w'' = \tilde{E}(z, w)(w')^2 + \tilde{F}(z, w)w' + \tilde{G}(z, w), \tag{12}$$

with

$$\tilde{E}(z,w) = \sum_{\mu=0}^{M} \frac{k_A^\mu}{w(z)-\tilde{\alpha}_\mu(z)}, \tag{13}$$

$$\tilde{F}(z,w) = \tilde{f}(z,w) \prod_{\mu=0}^{M} (w(z)-\tilde{\alpha}_\mu(z))^{-k_F^\mu}, \tag{14}$$

$$\tilde{G}(z,w) = \tilde{g}(z,w) \prod_{\mu=0}^{M} (w(z)-\tilde{\alpha}_\mu(z))^{-k_G^\mu}, \tag{15}$$

where $\tilde{\alpha}_\mu(z) = (\alpha_\mu(z)-c)^{-1}$ for $\mu = 1,\ldots,M$ and $\tilde{\alpha}_0(z) \equiv 0$, $k_A^0 = 2 - \sum_{\mu=1}^M k_A^\mu$, $k_F^0 = \deg_y f - \sum_{\mu=1}^M k_F^\mu$, $k_G^0 = \deg_y g - \sum_{\mu=1}^M k_G^\mu - 2$ and $\tilde{f}, \tilde{g}$ are polynomials in $w$ with analytic coefficients such that $\tilde{f}(z_\infty, \tilde{\alpha}_\mu(z_\infty)) \neq 0$ for all $\mu \in \{0,1,\cdots,M\}$. Note that the restriction for the degree of $g$ in assumption (i) implies $k_F^0 > k_G^0$.

Proceeding with step 2 we have

**Lemma 6.** *Let $y$ be a solution of (1) under the assumptions in Theorem 1. Then there exist functions $A$ and $B$, rational in $w = (y-c)^{-1}$, with coefficients analytic in $z$, such that*

$$W(z) := A(z,w)w' + B(z,w) \tag{16}$$

*satisfies a first-order linear differential equation of the form*

$$W' = PW + Qw' + R, \tag{17}$$

*where $P(z,w)$, $Q(z,w)$ and $R(z,w)$ are polynomials in $w$ with coefficients analytic in $z$.*

**Proof.** Differentiating (16) with respect to $z$ and using (12) to substitute for $w''$ it follows that $W$ satisfying (17) is equivalent to the equations

$$A\tilde{E} + A_w = 0, \tag{18}$$

$$A\tilde{F} + B_w + A_z = PA + Q, \tag{19}$$

$$A\tilde{G} + B_z = PB + R. \tag{20}$$

In order for $\tilde{E}$ to have the form (13) we choose

$$A(z,w) = \prod_{\mu=0}^{M} (w-\tilde{\alpha}_\mu(z))^{-k_A^\mu},$$

which is rational in $w$ and satisfies (18). To find a suitable rational function $B$ we expand all functions involved in (19) and (20) as Laurent series in $w - \tilde{\alpha}_\mu(z)$ and write down the equations which need to be satisfied by the Laurent coefficients. We will show that for every $\mu = 0,1,\ldots,M$ only finitely many non-zero coefficients of $B$ need to be determined in order to satisfy the two equations.

The Laurent coefficients of every function will be denoted by the corresponding lower case letter, e.g. $\tilde{f}^\mu_k$ denotes the $k$th coefficient of $\tilde{F}$ expanded in $w-\tilde{\alpha}_\mu(z)$,

$$\tilde{F}(z,w)=\sum_{k=-k_F^\mu}^{\infty}\tilde{f}^\mu_k(z)(w-\tilde{\alpha}_\mu(z))^k. \tag{21}$$

Comparing coefficients of power $(w-\tilde{\alpha}_\mu(z))^{k-k_A^\mu-k_F^\mu}$ in equations (19) and (20) we obtain, with $k_B^\mu:=k_A^\mu+k_F^\mu-1$,

$$\sum_{i=0}^{k}a^\mu_{i-k_A^\mu}\tilde{f}^\mu_{k-i-k_F^\mu}+(k-k_B^\mu)(b^\mu_{k-k_B^\mu}-\tilde{\alpha}'_\mu a^\mu_{k-k_B^\mu})+{a^\mu}'_{k-k_B^\mu-1}=\sum_{i=0}^{k-k_F^\mu}p_i^\mu a^\mu_{k-k_B^\mu-1-i}+q^\mu_{k-k_B^\mu-1}, \tag{22}$$

$$\sum_{i=0}^{k-k_F^\mu+k_G^\mu}a^\mu_{i-k_A^\mu}\tilde{g}^\mu_{k-k_F^\mu-i}+(b^\mu_{k-k_B^\mu-1})'-\tilde{\alpha}'_\mu(k-k_B^\mu)b^\mu_{k-k_B^\mu}=\sum_{i=0}^{k-1}p_i^\mu b^\mu_{k-k_B^\mu-1-i}+r^\mu_{k-k_B^\mu-1}. \tag{23}$$

For $k<0$, the only terms remaining in equations (22) and (23) are the terms involving $b_j^\mu$ on the left-hand sides and $q_j^\mu$, $r_j^\mu$, respectively, on the right-hand sides. The equations are therefore consistent with choosing

$$b^\mu_{k-k_B^\mu}=q^\mu_{k-k_B^\mu-1}=r^\mu_{k-k_B^\mu-1}=0 \text{ for all } k<0.$$

We will show in the following that most of the remaining coefficients $q_j^\mu$, $r_j^\mu$, $j<0$, can also be set to zero. If $k_B^\mu<0$ there is nothing to be done.

Distinguish those $\mu\in\{0,1,\ldots,M\}$ for which $\tilde{\alpha}'_\mu\equiv 0$ and those for which $\tilde{\alpha}'_\mu\neq 0$. Consider first the case $\tilde{\alpha}'_\mu\equiv 0$ where $k_F^\mu>k_G^\mu$. Here, equation (23) for $k=0$ reduces to $r^\mu_{-k_B^\mu-1}=0$ and equation (22) becomes $a^\mu_{-k_A^\mu}\tilde{f}^\mu_{-k_F^\mu}-k_B^\mu b^\mu_{-k_B^\mu}=q^\mu_{-k_B^\mu-1}$. If $k_B^\mu=0$ we have $q^\mu_{-1}=a^\mu_{-k_A^\mu}\tilde{f}^\mu_{-k_F^\mu}$. Otherwise, for $k_B^\mu>0$, we can set $q^\mu_{-k_B^\mu-1}=0$ by choosing

$$b^\mu_{-k_B^\mu}=\frac{1}{k_B^\mu}a^\mu_{-k_A^\mu}\tilde{f}^\mu_{-k_F^\mu}. \tag{24}$$

Now consider the case $\tilde{\alpha}'_\mu\neq 0$ where $k_F^\mu=k_G^\mu$. Again, for $k=0$, with the same choice of $b^\mu_{-k_B^\mu}$, we can set $q^\mu_{-k_B^\mu-1}=0$ if $k_B^\mu\neq 0$. However, equation (23) now reduces to $a^\mu_{-k_A^\mu}\tilde{g}^\mu_{-k_F^\mu}+\tilde{\alpha}'_\mu k_B^\mu b^\mu_{-k_B^\mu}=r^\mu_{-k_B^\mu-1}$. By using (24) and assumption (ii) of Theorem 1 one obtains $r^\mu_{-k_B^\mu-1}=a^\mu_{-k_A^\mu}\left(\tilde{g}^\mu_{-k_F^\mu}+\tilde{\alpha}'_\mu\tilde{f}^\mu_{-k_F^\mu}\right)=0$ since, as one can compute, $\tilde{f}^\mu_{-k_F^\mu}=(-1)^{k_F^\mu}(\alpha_\mu(z)-c)^{-2k_F^\mu}F^\mu(z,\alpha_\mu(z))$, $\tilde{g}^\mu_{-k_F^\mu}=(-1)^{k_G^\mu+1}(\alpha_\mu(z)-c)^{-2k_G^\mu-2}G^\mu(z,\alpha_\mu(z))$ and $\tilde{\alpha}'_\mu(z)=-(\alpha_\mu(z)-c)^{-2}\alpha'_\mu(z)$. In case $k_B^\mu=0$ we have $r^\mu_{-1}=a^\mu_{-k_A^\mu}\tilde{g}^\mu_{-k_F^\mu}=-\tilde{\alpha}'_\mu a^\mu_{-k_A^\mu}\tilde{f}^\mu_{-k_F^\mu}=-\tilde{\alpha}'_\mu q^\mu_{-1}$.

If $k_B^\mu \geq 1$ one can now successively determine the coefficients $p_{k-1}^\mu$ and $b_{k-k_B^\mu}^\mu$ for all $k = 1, \dots, k_B^\mu - 1$ by solving the linear system of equations

(25)

$$\delta_{1,k_F^\mu} a_{-k_A^\mu}^\mu p_{k-1}^\mu - (k - k_B^\mu) b_{k-k_B^\mu}^\mu = \sum_{i=0}^{k} a_{i-k_A^\mu}^\mu \tilde{f}_{k-i-k_F^\mu}^\mu + (a_{k-k_B^\mu-1}^\mu)' - \tilde{\alpha}_\mu'(k - k_B^\mu) a_{k-k_B^\mu}^\mu - \sum_{i=0}^{k-k_F^\mu-\delta_{1,k_F^\mu}} p_i^\mu a_{k-k_B^\mu-1-i}^\mu$$

(26)

$$b_{-k_B^\mu}^\mu p_{k-1}^\mu + \tilde{\alpha}_\mu'(k - k_B^\mu) b_{k-k_B^\mu}^\mu = \sum_{i=0}^{k-k_F^\mu+k_G^\mu} a_{i-k_A^\mu}^\mu \tilde{g}_{k-k_F^\mu-i}^\mu + (b_{k-k_B^\mu-1}^\mu)' - \sum_{i=0}^{k-2} p_i^\mu b_{k-k_B^\mu-1-i}^\mu$$

where $\delta_{1,k_F^\mu} = 1$ if $k_F^\mu = 1$ and $\delta_{1,k_F^\mu} = 0$ otherwise.

This system can be solved for each $k = 1, \dots, k_B^\mu - 1$ since

$$\det \begin{pmatrix} \delta_{1,k_F^\mu} a_{-k_A^\mu}^\mu & -(k - k_B^\mu) \\ b_{-k_B^\mu}^\mu & \tilde{\alpha}_\mu'(k - k_B^\mu) \end{pmatrix} = \delta_{1,k_F^\mu} a_{-k_A^\mu}^\mu \tilde{\alpha}_\mu'(k - k_B^\mu) + b_{-k_B^\mu}^\mu (k - k_B^\mu)$$
$$= \left\{ \begin{array}{l} \frac{k-k_B^\mu}{k_B^\mu} a_{-k_A^\mu}^\mu \left(k_B^\mu \tilde{\alpha}_\mu' + \tilde{f}_{-k_F^\mu}^\mu\right) \\ b_{-k_B^\mu}^\mu (k - k_B^\mu) \end{array} \right\} \neq 0,$$

where the first line is valid only for the case $k_F^\mu = k_G^\mu = 1$ and is non-zero in a neighbourhood of $z_\infty$ by the additional condition (3) in assumption (ii), and the second line for all other cases. Having determined $p_{k-1}^\mu$ and $b_{k-k_B^\mu}^\mu$ in this way one obtains $r_{k-k_B^\mu-1}^\mu = q_{k-k_B^\mu-1}^\mu = 0$ for $k = 1, \dots, k_B^\mu - 1$. For $k = k_B^\mu$ one can still determine $p_{k_B^\mu-1}^\mu$ by equation (26) implying that $r_{-1}^\mu = 0$. However, $q_{-1}^\mu$ will then in general be non-zero.

With the coefficients $b_{k-k_B^\mu}^\mu$, $k = 0, 1, \dots, k_B^\mu - 1$, determined above we now consider the principle parts of a Laurent series expansion

$$\frac{b_{-k_B^\mu}^\mu(z)}{(w - \tilde{\alpha}_\mu(z))^{k_B^\mu}} + \frac{b_{1-k_B^\mu}^\mu(z)}{(w - \tilde{\alpha}_\mu(z))^{k_B^\mu-1}} + \cdots + \frac{b_{-1}^\mu(z)}{w - \tilde{\alpha}_\mu(z)}. \tag{27}$$

By Mittag-Leffler's theorem there exists a function, rational in $w$, which has poles at most at $\tilde{\alpha}_\mu(z)$, $\mu = 0, 1, \dots, M$, with prescribed principle parts given by (27) and we take $B(z, w)$ to be such a function. Also, there exists a function $P(z, w)$, polynomial in $w$, such that about $\tilde{\alpha}_\mu(z)$ its Taylor series expansion starts with

$$p_0^\mu(z) + p_1^\mu(z)(w - \tilde{\alpha}_\mu(z)) + \cdots + p_{k_B^\mu-1}^\mu(z)(w - \tilde{\alpha}_\mu(z))^{k_B^\mu-1} + \cdots.$$

Note that the functions $B$ and $P$ are not uniquely determined by these expansions. However, to construct a polynomial $P$ of minimal degree in $w$ one could start with $P(z,w) = P_0(z) + P_1(z)w + \cdots + P_D(z)w^D$ where $D = \sum_{\mu=0}^M k_B^{\mu+} - 1$, ($k_B^{\mu+} = \max\{0, k_B^\mu\}$), and determine the coefficient functions $P_0(z), \ldots, P_D(z)$ by solving the linear system of $D+1$ equations

(28)
$$P(z,\tilde{\alpha}_\mu) = p_0^\mu, \quad \frac{\partial P}{\partial w}(z,\tilde{\alpha}_\mu) = 1! \cdot p_1^\mu, \ldots, \quad \frac{\partial^{k_B^\mu - 1} P}{\partial w^{k_B^\mu - 1}}(z,\tilde{\alpha}_\mu) = (k_B^\mu - 1)! \cdot p_{k_B^\mu - 1}^\mu,$$

$\mu = 0, \ldots, M$. Similarly one could construct a rational function $B$ by choosing a polynomial ansatz for $B(z,w) \cdot \prod_{\mu=0}^M (w - \tilde{\alpha}_\mu(z))^{k_B^{\mu+}}$. $B$ being rational in $w$ and $P$ polynomial, equations (19) and (20) show that $Q$ and $R$ are also rational in $w$, of certain degrees determined by the other terms in (19) and (20), and with at most simple poles at $\tilde{\alpha}_\mu(z)$. However, only the existence of such polynomial $P$ and functions $Q, R$ is important here.

Conditions (iii) and (iv) of Theorem 1 imply the existence of formal series solutions of equation (12) in the transformed variable $w = (y-c)^{-1}$ of the form

$$w(z) = \tilde{\alpha}_\mu(\hat{z}) + \sum_{k=1}^\infty \tilde{c}_k (z - \hat{z})^{k/k_F^\mu}, \quad \tilde{c}_1 \neq 0, \tag{29}$$

for all $\mu \in \{0, 1, \ldots, M\}$.

With the integrating factor $I(z) = \exp\left(-\int_{z_0}^z P(\zeta, w(\zeta)) d\zeta\right)$ equation (17) can be written in the form

$$\frac{d}{dz}(I(z)W(z)) = (Q(z,w)w' + R(z,w))\, I(z). \tag{30}$$

If, in equation (16), we substitute for $w$ the series expansion (29) we see that $W$ has a Laurent series expansion in $(z-\hat{z})^{1/k_F^\mu}$. Also, since $P$ is a polynomial in $w$, $I(z)$ has a power series expansion in $(z-\hat{z})^{1/k_F^\mu}$. Therefore, the product $I(z)W(z)$ has a Laurent series expansion in $(z-\hat{z})^{1/k_F^\mu}$. Similarly, the right-hand side of equation (30) has a Laurent series expansion in $(z-\hat{z})^{1/k_F^\mu}$ of which the coefficient of the term involving $(z-\hat{z})^{-1}$ must be zero since otherwise integration on both sides would imply that $I(z)W(z)$ contained a term $\log(z-\hat{z})$. However, $Q$ has leading order of the form

$$Q(z,w(z)) \sim \frac{q_{-1}^\mu(z)}{w - \tilde{\alpha}_\mu(z)} \sim \frac{q_{-1}^\mu(\hat{z})}{(z-\hat{z})^{1/k_F^\mu}}.$$

Therefore, the right-hand side of equation (30) has leading order

$$\frac{q_{-1}^\mu(\hat{z})}{z - \hat{z}},$$

but by the argument above the existence of the formal series solution implies that $q_{-1}^\mu(\hat{z}) = 0$. Since this condition holds for all $\hat{z}$ in some open neighbourhood of $z_\infty$, we have in fact shown that $q_{-1}^\mu \equiv 0$, which also implies $r_{-1}^\mu \equiv 0$ in the case $k_B^\mu = 0$. This proves that $Q$ and $R$ are in fact polynomials in $w$. ■

## 4. Proof of Theorem 1

By Lemma 5 we can continuously deform $\gamma$ such that $w$ is bounded on the modified curve $\tilde{\gamma}$. Continuing with step 3, the following lemma shows that $w$ has a well-determined behaviour as $z$ approaches $z_\infty$ along $\tilde{\gamma}$.

**Lemma 7.** *Let $y$ be a solution of (1) under the assumptions in Theorem 1 and $w = (y-c)^{-1}$ as before. Then, for some $\mu = 0, 1, \ldots, M$,*

$$\lim_{\tilde{\gamma} \ni z \to z_\infty} w(z) = \tilde{\alpha}_\mu(z_\infty).$$

**Proof.** If this was not the case, there would exist some $\epsilon > 0$ and a sequence $(z_n)_{n\in\mathbb{N}} \subset \tilde{\gamma}$ with $z_n \to z_\infty$ as $n \to \infty$ such that $|w(z_n) - \tilde{\alpha}_\mu(z_\infty)| > \epsilon$ for all $\mu = 0, 1, \ldots, M$. Also, by Lemmas 4 and 6, $W$ in (16) is bounded on $\tilde{\gamma}$. Now $A = \prod_{\mu=0}^{M}(w - \tilde{\alpha}_\mu(z))^{-k_A^\mu}$ is bounded away from zero on the sequence $(z_n)$ and $B$, a rational function in $w$ with only possible poles at $\tilde{\alpha}_\mu(z)$, is bounded on $(z_n)$. Since $W = Aw' + B$ this implies that $w'$ is bounded on the sequence $(z_n)$. However, Lemma 3 applied to the system $w' = w_1, w_1' = \tilde{E}(z,w)(w')^2 + \tilde{F}(z,w)w' + \tilde{G}(z,w)$ now shows that $w$, and therefore $y$, can be analytically continued to $z_\infty$, contradicting the assumption in Theorem 1. ■

*Remark.* In the above lemma, $w(z)$ can in fact only converge to one of those $\tilde{\alpha}_\mu(z_\infty)$ for which $k_B^\mu = k_A^\mu + k_F^\mu - 1 > 0$.

**Proof of Theorem 1.** Solving (16) for $w'$ and taking the reciprocal yields

$$\frac{dz}{dw} = \frac{A}{W - B}. \tag{31}$$

One can then multiply equation (17) by (31) to obtain

$$\frac{dW}{dw} = \frac{dW}{dz}\frac{dz}{dw} = Q + (PW + R)\frac{A}{W - B}. \tag{32}$$

The two equations (31) and (32) form a system of ordinary differential equations for $z$ and $W$ as functions of $w$. By Lemma 7, $w \to \tilde{\alpha}_\mu(z_\infty)$ for some $\mu \in \{0, 1, \ldots, M\}$. Since $W$ is bounded as $z \to z_\infty$, there exists a sequence $(z_n)_{n\in\mathbb{N}} \subset \tilde{\gamma}$ with $z_n \to z_\infty$ as $n \to \infty$ such that $W(z_n) \to W_\infty$ for some $W_\infty \in \mathbb{C}$. $B$ is a rational function with pole of order $k_B^\mu$ at $\tilde{\alpha}_\mu(z)$, i.e. one can write

$$B(z,w) = (w - \tilde{\alpha}_\mu(z))^{-k_B^\mu} b(z,w),$$

where $b(z, w(z))$ is regular on $\tilde{\gamma}$ and $b(z_\infty, \tilde{\alpha}_\mu(z_\infty)) \neq 0$. Therefore, the right-hand side of (31),

$$\frac{A}{W - B} = \frac{\prod_{\substack{\nu=0 \\ \nu\neq\mu}}^{M}(w - \tilde{\alpha}_\nu(z))^{-k_A^\nu}}{W(w - \tilde{\alpha}_\mu(z))^{k_B^\mu} - b(z,w)}(w - \tilde{\alpha}_\mu(z))^{k_B^\mu - k_A^\mu},$$

as well as the right-hand side of (32) are analytic functions in the variables $(w, z, W)$ in some neighbourhood of the point $(\tilde{\alpha}_\mu(z_\infty), z_\infty, W_\infty)$. Applying Lemma 3 to this system of equations shows that $z$ and $W$ are analytic functions of $w$ at $w = \tilde{\alpha}_\mu(z_\infty)$. Since $\frac{dz}{dw} = \frac{A}{W-B}$ has a series expansion with leading term of order $k_B^\mu - k_A^\mu = k_F^\mu - 1$ in $(w - \tilde{\alpha}_\mu(z_\infty))$, $z$ has a series expansion

$$z - z_\infty = \sum_{k=0}^{\infty} \xi_k (w - \tilde{\alpha}_\mu(z_\infty))^{k_F^\mu + k},$$

convergent in a neighbourhood of $\tilde{\alpha}_\mu(z_\infty)$. Taking the $k_F^\mu$-th root on both sides and making a choice of branch shows that

$$(z - z_\infty)^{1/k_F^\mu} = \sum_{k=1}^{\infty} \eta_k (w - \tilde{\alpha}_\mu(z_\infty))^k.$$

By inverting this series one obtains

$$w(z) = \tilde{\alpha}_\mu(z_\infty) + \sum_{k=1}^{\infty} \zeta_k (z - z_\infty)^{k/k_F^\mu},$$

from which follows that $y$ has a series expansion of the form (6) or (7), convergent in a (punctured) neighbourhood of $z_\infty$. ∎

## 5. Resonance Conditions

Assumptions (iii) and (iv) in Theorem 1, the existence of formal series solutions, are equivalent to a number of resonance conditions. These are certain differential relations among the functions $f(z, y)$, $g(z, y)$ and $\alpha_\mu(z)$, $\mu = 1, 2, \ldots, M$, arising from the fact that when inserting the formal series (4) or (5) into equation (1) to obtain a recurrence relation for the coefficients $c_k$, the coefficient to some power $(z - \hat{z})^{k/k_F^\mu}$ must vanish identically. In the proof of Lemma 6 we have used the existence of the formal series to show the vanishing of the coefficients $q_{-1}^\mu(z)$, $\mu = 0, 1, \ldots, M$, which, when written out explicitely, would yield exactly the resonance conditions. However, inserting the formal series expansions (4) or (5) into equation (1) directly is more natural and provides an algorithmic procedure to determine the resonance conditions for any given equation (1) in the specified form. This procedure is demonstrated below for an example. Note, however, that the steps at which the resonances occur depend on the values of the $k_B^\mu = k_A^\mu + k_F^\mu - 1, \mu = 0, \ldots, M$. Although straightforward, the expressions can get rather complex and it is therefore of advantage to use a computer algebra software. Here we have used Mathematica 8.0.1 to carry out the calculations.

**An Example.** Consider the case with only one function $\alpha_1 = \alpha(z) \neq \text{const}$,

$$-y'' + \frac{(y')^2}{y-\alpha} + \frac{f(z,y)}{(y-\alpha)^2}y' + \frac{g(z,y)}{(y-\alpha)^2} = 0, \tag{33}$$

where $f$ and $g$ are polynomials with degrees $\deg_y f = 4$, $\deg_y g = 5$. Since $\alpha' \neq 0$, condition (ii) of Theorem 1 is

$$g(z,\alpha(z)) + \alpha'(z) f(z,\alpha(z)) = 0. \tag{34}$$

By substituting the formal series solution

$$y(z) = \alpha(\hat{z}) + \sum_{k=1}^{\infty} c_k (z-\hat{z})^{k/2}$$

into equation (33) one obtains the following expansion in $(z-\hat{z})^{1/2}$ of the left-hand side:

$$\begin{aligned}
&\frac{f + c_1^2}{2c_1(z-\hat{z})^{3/2}} + \frac{4g + 4f\alpha' + c_1^2\left(3c_2 + \alpha' + 2f_y\right)}{4c_1^2(z-\hat{z})} \\
&+ \frac{1}{4c_1^3(z-\hat{z})^{1/2}}\left[-8g\left(c_2 - \alpha'\right) - 2f\left(c_2^2 - c_1c_3 + 2c_2\alpha' - 3\alpha'^2\right) + \right. \\
&\left. c_1^2\left(c_2^2 + 2c_1c_3 + \alpha'^2 + 4\alpha' f_y + 2c_2\left(\alpha' + f_y\right) + 4g_y + c_1^2 f_{yy} + 2f_z\right)\right] + \mathrm{O}(1),
\end{aligned}$$

where the functions in the numerators are evaluated at $(\hat{z}, \alpha(\hat{z}))$. Equating the coefficient of $(z-\hat{z})^{-3/2}$ to zero gives

$$c_1^2 = -f(\hat{z}, \alpha(\hat{z})),$$

where the choice of branch for $c_1$ can be absorbed into the choice of branch for $(z-\hat{z})^{1/2}$. Equating the coefficient of $(z-\hat{z})^{-1}$ to zero and using (34) one obtains

$$c_2 = -\frac{1}{3}\left(\alpha'(\hat{z}) + 2f_y(\hat{z}, \alpha(\hat{z}))\right).$$

The coefficient of $(z-\hat{z})^{-1/2}$ equated to zero then yields, since valid for all $\hat{z}$ in a neighbourhood of $z_\infty$, the resonance condition

$$4g_y(z,\alpha(z)) + 6\alpha'(z) f_y(z,\alpha(z)) + 2f_z(z,\alpha(z)) + 4\alpha'(z)^2 = f(z,\alpha(z)) f_{yy}(z,\alpha(z)). \tag{35}$$

To express the other resonance condition, corresponding to the existence of the formal series solution

$$y(z) = \sum_{k=0}^{\infty} c_k (z-\hat{z})^{(k-1)/2},$$

write $f(z,y) = f_0(z) + f_1(z)y + \cdots + f_4(z)y^4$, $g(z,y) = g_0(z) + \cdots + g_5(z)y^5$. Inserting the formal series solution into the equation one obtains the following expansion in $(z-\hat{z})^{1/2}$:

$$\frac{-c_0\left(1+c_0^2 f_4\right)}{2(z-\hat{z})^{5/2}}+\frac{\alpha-c_1\left(1+4 c_0^2 f_4\right)-2 c_0^2\left(f_3+2 \alpha f_4\right)}{4(z-\hat{z})^2}$$
$$+\frac{1}{4(z-\hat{z})^{-3/2}}\left[-2 c_0\left(f_2+2 \alpha f_3+c_1^2 f_4+3 \alpha^2 f_4+c_1\left(f_3+2 \alpha f_4\right)\right)\right.$$
$$\left.-2 c_2+c_0^{-1}\left(c_1-\alpha\right)^2-2 c_0^2 c_2 f_4+c_0^3\left(4 g_5-2 f_4^{\prime}\right)\right]+\mathrm{O}((z-\hat{z})^{-1}),$$

where the argument of the functions in the numerators is $\hat{z}$. Equating the coefficient of $(z-\hat{z})^{-5/2}$ to zero gives, since $c_0 \neq 0$ by assumption,

$$c_0^2=-\frac{1}{f_4(\hat{z})},$$

where again the choice of branch for $c_0$ can be absorbed into $(z-\hat{z})^{1/2}$. Equating the coefficient of $(z-\hat{z})^{-2}$ to zero one obtains

$$c_1=\frac{-2 f_3(\hat{z})-5 \alpha(\hat{z}) f_4(\hat{z})}{3 f_4(\hat{z})}.$$

The coefficient of $(z-\hat{z})^{-3/2}$ equated to zero then yields the resonance condition

$$f_2(z) f_4(z)+3 \alpha(z) f_3(z) f_4(z)+6 \alpha(z)^2 f_4(z)^2+2 g_5(z)-f_4^{\prime}(z)=0. \tag{36}$$

For equation (33) we have thus obtained the resonance conditions (35) and (36). If one prescribes the polynomial $f$, condition (36) then determines the coefficient $g_5$ of $g$. Conditions (35) and (34) together determine two further coefficients of $g$, e.g. $g_0$ and $g_1$. There are then three coefficients of $g$ remaining which can be chosen arbitrarily. Given $f$, we have thus specified all possible polynomials $g$ such that assumptions (iii) and (iv) are satisfied for equation (33).

## 6. Conclusion

For a class of rational ODEs of the form (1) Theorem 1 states under the assumptions made that all movable singularities of a solution which are reached along a curve of finite length are of algebraic type. The class of ODEs considered here arises from the fact that the function $W$ which remains bounded as the singularity is approached is taken to be linear in $y'$. This generalises the result for equations of Liénard type in [2].

## References

1. Filipuk G and Halburd R G 2009 Movable algebraic singularities of second-order ordinary differential equations *J. Math. Phys.* **50** 023509
2. Filipuk G and Halburd R G 2009 Movable singularities of equations of Liénard type *Comput. Methods Funct. Theory* **9** 551–63
3. Filipuk G and Halburd R G 2009 Rational ODEs with movable algebraic singularities *Stud. Appl. Math.* **123** 159–80

4. Hille E 1976 Ordinary Differential Equations in the Complex Domain (New York: Wiley-Interscience)
5. Hinkkanen A and Laine I 1999 Solutions of the first and second Painlevé equations are meromorphic *J. Anal. Math.* **79** 345–77
6. Ince E L 1926 Ordinary Differential Equations (London: Longmans)
7. Okamoto K and Takano K 2001 The proof of the Painlevé property by Masuo Hukuhara *Funkcial. Ekvac.* **44** 201–17
8. Rañada A F, Ramani A, Dorizzi B and Grammaticos B 1985 The weak-Painlevé property as a criterion for the integrability of dynamical systems *J. Math. Phys.* **26** 708–10
9. Shimomura S 2003 Proofs of the Painlevé property for all Painlevé equations *Japan. J. Math.* **29** 159–80
10. Shimomura S 2007 A class of differential equations of PI-type with the quasi-Painlevé property *Ann. Mat. Pura Appl.* **186** 267–80
11. Smith R A 1953 On the singularities in the complex plane of the solutions of $y'' + y'f(y) + g(y) = P(x)$ *Proc. London Math. Soc.* **3** 498–512

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