CMFT-MS 15060

A cubic Hamiltonian system with meromorphic solutions

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Received: date / Accepted: date

Abstract A Hamiltonian system in two dependent variables is presented with properties analogous to the Hamiltonian systems associated with the six Painlevé equations. Its solutions are meromorphic functions in the complex plane having only simple poles with three possible residues given by the third roots of unity. Like the Painlevé equations $P_{II} - P_{VI}$ the system has families of rational solutions that can be obtained by applying Bäcklund transformations to certain seed solutions.

Keywords Painlevé equations · Hamiltonian systems · meromorphic solutions · Bäcklund transformations

Mathematics Subject Classification (2000) 34M05 · 34M55 · 33E17

1 Introduction

The Painlevé equations $P_I - P_{VI}$ are six non-linear second-order ODEs with the property that all their solutions are single-valued about all their movable singularities in the complex plane, a property known as the Painlevé property. They were discovered in the 1900's by Painlevé [13], Gambier [2] and Fuchs [1] in the classification of all equations with this property of the form $y'' = R(z, y, y')$, R being a rational function in y and y' with coefficients analytic in z. Each of the Painlevé equations can be written as an equivalent Hamiltonian system with z-dependent

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Hamiltonian $H_J(z,p,q),\ J=I,\dots,VI$,

$$\begin{aligned}
H_I &= \frac{1}{2}q^2 - 2p^3 - zp\\
H_{II} &= \frac{1}{2}q^2 - \left(p^2 - \frac{z}{2}\right)q - \kappa p\\
H_{III} &= \frac{1}{z}\left[2q^2p^2 - \left(2\eta_\infty zp^2 + (2\kappa_0+1)p - 2\eta_0 z\right)q + \eta_\infty(\kappa_0+\kappa_\infty)zp\right]\\
H_{IV} &= 2pq^2 - \left(p^2 + 2zp + \kappa_0\right)q + \kappa_\infty p\\
H_V &= \frac{1}{z}\left[p(p-1)^2q^2 - \left(\kappa_0(p-1)^2 + \kappa_t p(p-1) - \eta zp\right)q + \kappa(p-1)\right]\\
H_{VI} &= \frac{1}{z(z-1)}\left[p(p-1)(p-z)q^2 - [\kappa_0(p-1)(p-z) + \kappa_1 p(p-z)\right.\\
&\quad \left.+(\kappa_t-1)p(p-1)]\,q + \kappa(p-t)\right],
\end{aligned}$$

where the various κ's and η's are arbitrary complex parameters, the Hamiltonian equations being given by

$$q' = \frac{\partial H_J}{\partial p}, \quad p' = -\frac{\partial H_J}{\partial q}.$$

These Hamiltonian systems were already found by Malmquist [7] and later have been studied extensively by Okamoto in a series of four papers [9–12]. The Hamiltonian structure of the Painlevé equations can be obtained from isomonodromic deformation problems of associated second-order linear equations [8]. Here H_{VI} arises in the isomonodromic deformation problem of a second-order linear equation with four regular singular points at 0, 1, z and ∞, whereas the other Painlevé Hamiltonians involve associated equations with irregular singular points.

Complete proofs that the first and second Painlevé equations have the Painlevé property appeared in the published literature only in 1999 by Hinkkanen and Laine [5] and for the fourth Painlevé equation in 2000 by Steinmetz [15]. The solutions of these equations are meromorphic functions in $\mathbb{C}$. Proofs of the Painlevé property for all six Painlevé equations were also given by Shimomura in [14], see also the book [3]. The equations *III*, *V* and *VI*, however, have also fixed singularities and their solutions are in general not meromorphic in the whole of $\mathbb{C}$. In this article we present a system of equations with the Painlevé property,

$$q' = p^2 + zq + \alpha, \quad p' = -q^2 - zp - \beta, \quad \alpha, \beta \in \mathbb{C}, \tag{1}$$

which is a Hamiltonian system with Hamiltonian

$$H(z,p,q) = \frac{1}{3}\left(q^3 + p^3\right) + zqp + \alpha p + \beta q.$$

A proof that this system has the Painlevé property is contained in [6] as a special case of Hamiltonian systems with movable algebraic singularities, however we will write down a proof for this specific system below. [1] Unlike the Painlevé equations,

[1] After writing this article it was pointed out to the author by Norbert Steinmetz that the system presented here is in fact closely related to the fourth Painlevé equation. However, since then the article [16] has appeared which further studies the solutions of this system in their own right.

which (apart form P_I) have two different types of leading order behaviour at the simple poles of their solutions, this system exhibits a triplet of leading order behaviours expressed by the third roots of unity $1, \omega, \omega^2$, where $\omega = \frac{-1+i\sqrt{3}}{2}$. Every pole z_* of a solution (p, q) of (1) is a common simple pole of q and p and there exists, for each possible pair of residues, a one-parameter family of Laurent series solutions,

$$\begin{aligned} q(z) =& \frac{-\omega^j}{z - z_*} + \frac{\omega^j z_*}{2} + \left(\omega^j \left(1 + \frac{z_*^2}{4}\right) - \frac{\alpha}{3} + \frac{2}{3}\omega^{2j}\beta\right)(z - z_*) + h(z - z_*)^2 \\ &+ \sum_{n=3}^{\infty} c_n (z - z_*)^n \\ p(z) =& \frac{\omega^{2j}}{z - z_*} + \frac{\omega^{2j} z_*}{2} + \left(\omega^{2j} \left(1 - \frac{z_*^2}{4}\right) - \frac{2}{3}\omega^j \alpha + \frac{\beta}{3}\right)(z - z_*) + k(z - z_*)^2 \\ &+ \sum_{n=3}^{\infty} d_n (z - z_*)^n, \end{aligned} \tag{2}$$

$j \in \{0, 1, 2\}$, where $h = c_2$ and $k = d_2$ are coupled by the linear relation

$$\omega^j h - k = \left(\frac{5}{4}\omega^{2j} - \frac{\omega^j \alpha}{2} + \frac{\beta}{2}\right) z_*,$$

and either h or k can be taken as arbitrary parameter. The remaining coefficients c_n, d_n, for $n \geq 3$, can be determined by the recurrence relation

$$\begin{pmatrix} c_n \\ d_n \end{pmatrix} = \frac{1}{n^2 - 4} \begin{pmatrix} n & 2\omega^{2j} \\ 2\omega^j & n \end{pmatrix} \begin{pmatrix} \sum_{i=1}^n d_{i-1} d_{n-i} + c_{n-2} + z_* c_{n-1} \\ -\sum_{i=1}^n c_{i-1} c_{n-i} - d_{n-2} - z_* d_{n-1} \end{pmatrix}.$$

Thus about every pole z_*, for each leading order behaviour, one can compute the Laurent series solutions $(q(z), p(z))$ and the system is said to *pass the Painlevé test*. This, however, does not show that the system has the Painlevé property which will be done in the next section.

2 Meromorphicity of the solutions

In this section we will prove the following theorem.

Theorem 1 *All solutions (q, p) of the system (1) can be analytically continued to meromorphic functions in the whole complex plane.*

As a preliminary we state the following lemma by Painlevé, a proof of which can be found in [4], for example.

Lemma 1 *Let $F_k(z, f_1, \ldots, f_m)$, $k = 1, \ldots, m$, be analytic functions in a neighbourhood of a point $(z_*, \eta_1, \ldots, \eta_m) \in \mathbb{C}^{m+1}$. Let γ be a curve with end point z_* and suppose that $(f_1, \ldots, f_m)$ are analytic on $\gamma \setminus \{z_*\}$ and satisfy*

$$f_k' = F_k(z, f_1, \ldots, f_m), \quad k = 1, \ldots, m.$$

Suppose there is a sequence $(z_n)_{n \in \mathbb{N}} \subset \gamma$ such that $z_n \to z_*$ and $f_k(z_n) \to \eta_k \in \mathbb{C}$ as $n \to \infty$ for all $k = 1, \ldots, m$.

Then the solution can be analytically continued to include the point z_*.

Proof (of Theorem 1) At any point z where $q(z) \neq 0$ one can apply the following changes of variables,

$$u = \frac{1}{q}, \quad v_j = q\left(1 - \omega^{2j}\alpha + \omega^j\beta - z\omega^{2j}q + \omega^j q^2 + qp\right),$$

where $j \in \{0, 1, 2\}$. In these variables,

$$\begin{aligned}
u' =& -\omega^{2j} + zu + \left(-\omega^j z^2 + 2(-\omega^j + \alpha - \omega^{2j}\beta)\right)u^2 + 2z(\omega^{2j} - \omega^j\alpha + \beta)u^3 \\
&+ 2z\omega^j u^3 v_j - (1 - \omega^{2j}\alpha + \omega^j\beta)^2 u^4 - 2z\omega^{2j}u^4 v_j + 2\left(1 - \omega^{2j}\alpha + \omega^j\beta\right)u^5 v_j \\
&- u^6 v_j^2 \\
v_j' =& -(1 + \omega^j\beta + z^2)(\omega^j - \alpha + \omega^{2j}\beta) - zv_j + 2z(\omega^j - \alpha + \omega^{2j}\beta)^2 u \\
&+ 2(2\omega^j + \omega^j z^2 - \alpha + 2\omega^{2j}\beta)uv_j - (1 - \omega^{2j}\alpha + \omega^j\beta)^3 u^2 \\
&- 6z(\omega^{2j} - \omega^j\alpha + \beta)u^2 v_j - 3\omega^j u^2 v_j^2 + 4(1 - \omega^{2j}\alpha + \omega^j\beta)^2 u^3 v_j \\
&+ 4\omega^{2j} z u^3 v_j^2 - 5(1 - \omega^{2j}\alpha + \omega^j\beta)u^4 v_j^2 + 2u^5 v_j^3,
\end{aligned} \tag{3}$$

which, for any pair of initial values $u(z_0), v_j(z_0) \in \mathbb{C}$, defines a regular initial value problem. We now introduce the following auxiliary function for the system (1),

$$V(z) = H(z, q, p) + \frac{p^2}{q}, \tag{4}$$

which remains bounded at any singularity of a solution $(q(z), p(z))$ as shown below. V satisfies the following first-order linear differential equation,

$$V' + 3\frac{p}{q^2}V = \beta\frac{p}{q} + 2\alpha\left(\frac{p}{q}\right)^2 + 3\left(\frac{p}{q}\right)^3. \tag{5}$$

Let $\Gamma \subset \mathbb{C}$ be a rectifiable curve from some point z_0 to z_* such that (q, p) is analytic on $\Gamma \setminus \{z_*\}$ but cannot be analytically continued to include the point z_*. By [6, Lem. 6] we can deform Γ such that $\frac{1}{q}$ and $\frac{p}{q}$ are bounded on the modified curve $\tilde{\Gamma}$. [2] We can thus integrate equation (5) on $\tilde{\Gamma}$,

$$V(z) = E(z)^{-1}\left(V(z_0) + \int_{\tilde{\Gamma}_z} E(z)\left(\beta\frac{p}{q} + 2\alpha\left(\frac{p}{q}\right)^2 + 3\left(\frac{p}{q}\right)^3\right)\right),$$

where $\tilde{\Gamma}_z$ denotes the part of $\tilde{\Gamma}$ up to the point z and E is the integrating factor

$$E(z) = \exp\left(\int_{\tilde{\Gamma}_z} 3\frac{p}{q^2}\right),$$

showing that V remains bounded as $z \to z_*$ along $\tilde{\Gamma}$. We will now show that $q, p \to \infty$ as $z \to z_*$ along $\tilde{\Gamma}$. For suppose there was a sequence $(z_n)_{n\in\mathbb{N}} \subset \tilde{\Gamma}$ with $z_n \to z_*$ as $n \to \infty$ such that $q(z_n) \to q_*$ for some $q_* \in \mathbb{C} \setminus \{0\}$. Since V and $\frac{1}{q}$ are bounded on $\tilde{\Gamma}$ equation (4) shows that also the sequence $(p(z_n))_{n\in\mathbb{N}}$ is bounded. But in this case, by Lemma 1 the solution $(q(z), p(z))$ would have an analytic

[2] This lemma is quite technical, however it forms an essential part of the proof.

continuation to z_* which is contrary to the assumption. Hence we have $q(z) \to \infty$ as $z \to z_*$ on $\tilde{\Gamma}$ and likewise $p(z) \to \infty$.

We now re-express V in terms of the variables u and v_j,

$$\begin{aligned} V =& \frac{1}{3}u^6 v_j^3 + \left(u^2\left(\alpha\omega^{2j} - \beta\omega^j\right) + uz\omega^{2j} - \omega^j\right)u^3 v_j^2 + v_j\left(\omega^{2j} - uz\right. \\ & -u^2\left(\alpha - 2\beta\omega^{2j} - z^2\omega^j\right) + 2zu^3\left(\alpha\omega^j - \beta\right) - u^4\left(1 + 2\alpha\beta - \alpha^2\omega^j - \beta^2\omega^{2j}\right)\Big) \\ & + \left(\frac{2}{3} - \omega^{2j}\alpha + \omega^j\beta + \frac{1}{3}(\omega^j\alpha - \beta)^3\right)u^3 + \left(\alpha^2 + \beta^2\omega^j - 2\alpha\beta\omega^{2j} - \omega^{2j}\right)zu^2 \\ & + \left(-\alpha - \beta^2 + \alpha\beta\omega^j + \omega^j - \beta z^2\omega^{2j} + \alpha z^2\right)u - z(1 - \omega^j\beta) + \frac{z^3}{3}. \end{aligned}$$

This expression can be viewed as a cubic equation in v_j. We solve it by first eliminating the term quadratic in v_j by letting

$$w = v_j - \omega^j u^{-3} + z\omega^{2j}u^{-2} + \left(\omega^{2j}\alpha - \omega^j\beta\right)u^{-1},$$

obtaining the equation $w^3 + 3fw + 2g = 0$, where

$$f = \left(z + \alpha u - u^3\right)u^{-5}, \quad 2g = \left(1 + 3\beta u^2 - 3(z+V)u^3 - 3\alpha u^4 + 2u^6\right)u^{-9}.$$

Note that these expressions are in fact independent of the choice of $j \in \{0,1,2\}$! By Cardano's formula the three roots of this equation are given by

$$w = -\omega^k\sqrt[3]{g + \sqrt{g^2 + f^3}} + \omega^{2k} f \Big/ \sqrt[3]{g + \sqrt{g^2 + f^3}}, \quad k = 0,1,2,$$

with the same choice of branch for both the square and cubic root in both terms. We will now show that for $u \to 0$, one of the variables v_j, $j \in \{0,1,2\}$, remains bounded. We expand the expressions in w in the following way,

$$\begin{aligned} \sqrt[3]{g + \sqrt{g^2 + f^3}} &= \sqrt[3]{2g\left(1 + \frac{f^3}{4g^2} + \cdots\right)} = \frac{1}{u^3} + \frac{\beta}{u} + O(1), \\ \implies f \Big/ \sqrt[3]{g + \sqrt{g^2 + f^3}} &= \frac{z}{u^2} + \frac{\alpha}{u} + O(1), \\ \implies w &= \frac{-\omega^k}{u^3} + \frac{z\omega^{2k}}{u^2} + \frac{\omega^{2k}\alpha - \omega^k\beta}{u} + O(1). \end{aligned}$$

Although w can take on any of these expressions for $k \in \{0,1,2\}$ we see that, with the correct choice of j, for one of the variables v_j we have

$$v_j = w + \omega^j u^{-3} - z\omega^{2j}u^{-2} - \left(\omega^{2j}\alpha - \omega^j\beta\right)u^{-1} = O(1).$$

Now at any singularity z_* we have $u \to 0$ and v_j is bounded for some $j \in \{0,1,2\}$, hence by Lemma 1 the solution (u, v_j) of the system (3) can be analytically continued to the point z_*. The zero of u corresponds to a pole of q and p in the original system (1) which therefore has meromorphic solutions with Laurent expansions of the form (2) about every pole.

3 Bäcklund transformations and rational solutions

Like the Painlevé Hamiltonian systems $H_{II} - H_{VI}$ the system (1) possesses a set of Bäcklund transformations, i.e. birational canonical transformations leaving the form of the equation invariant up to a change in the parameters (α, β).

Theorem 2 *The form of the system (1) is invariant under the transformations σ_j and ρ_j, $j = 0, 1, 2$, given by*

$$\begin{aligned}\sigma_j: \quad & q \mapsto \tilde{q} = q + \frac{\omega^j - \alpha + \omega^{2j}\beta}{\omega^{2j}q + \omega^j p - z}, \quad p \mapsto \tilde{p} = p - \frac{\omega^{2j} - \omega^j\alpha + \beta}{\omega^{2j}q + \omega^j p - z}, \\ & (\alpha, \beta) \mapsto (\tilde{\alpha}, \tilde{\beta}) = (\omega^{2j}\beta + \omega^j, \omega^j\alpha - \omega^{2j}), \\ \rho_j: \quad & q \mapsto \tilde{q} = \omega^j q, \quad p \mapsto \tilde{p} = \omega^{2j}p, \quad (\alpha, \beta) \mapsto (\tilde{\alpha}, \tilde{\beta}) = (\omega^{2j}\alpha, \omega^j\beta).\end{aligned} \tag{6}$$

The transformations acting on the parameter space $(\alpha, \beta) \in \mathbb{C}^2$ generate a group isomorphic to the extended affine Weyl group of type $A_2^{(1)}$.

Proof That the transformations leave the form of the equation invariant can be checked by direct computation. In the parameter space $(\alpha, \beta) \in \mathbb{C}^2$ the transformations σ_j, $j = 0, 1, 2$, correspond to reflections on the complex lines defined by the equations

$$\omega^{2j}\alpha - \omega^j\beta = 1,$$

in particular we have $\sigma_j^2 = \mathrm{id}$ for all $j \in \{0, 1, 2\}$. Further one can check that the relations $(\sigma_j\sigma_{j+1(\mathrm{mod}\ 3)})^3 = \mathrm{id}$ hold. The group Σ generated by the three elements with these relations is the affine Weyl group of type A_2. The transformations ρ_j, $j \in \{1, 2\}$, correspond to rotations in $\mathbb{C}^2$ by the angle $\frac{4j\pi}{3}$ in the α-plane and independently by $\frac{2j\pi}{3}$ in the β-plane and we denote the cyclic group of rotations by $\Pi = \{\rho_0 = \mathrm{id}, \rho_1, \rho_2\}$. One can also directly verify the relations $\rho_i\sigma_j\rho_i = \sigma_{j+i(\mathrm{mod}\ 3)}$. Thus the group generated by the transformations (6) is the semi-direct product $\Sigma \rtimes \Pi$, called the extended affine Weyl group of type $A_2^{(1)}$.

For particular values of the parameters (α, β) the system (1) has rational solutions which can be obtained by acting on certain seed solutions with the Bäcklund transformations. There are two sets of rational solutions which are disjoint modulo the transformations (6). The simplest rational solution occurs for $\alpha = \beta = 0$, namely $q = p = 0$. Some rational solutions generated from this seed are listed in Table 1. A different solution is given by $q = p = -z$ with $\beta = -\alpha = 1$. Some solutions obtained from this seed are listed in Table 2. The distribution of the rational solutions in the parameter space is shown in Figure 1, projected onto the β plane, where they lie in a triangular grid. The hollow circles represent the solutions generated from $q = p = 0$ whereas the filled circles correspond to the solutions obtained from $q = p = -z$. The elements of the group Σ are represented by the reflections on the dashed lines in the diagram, the generating elements σ_j, $j \in \{0, 1, 2\}$, being given by the reflections on the lines forming the central triangle about the origin.

We conclude this section by proving that the rational solutions thus obtained are the only ones.

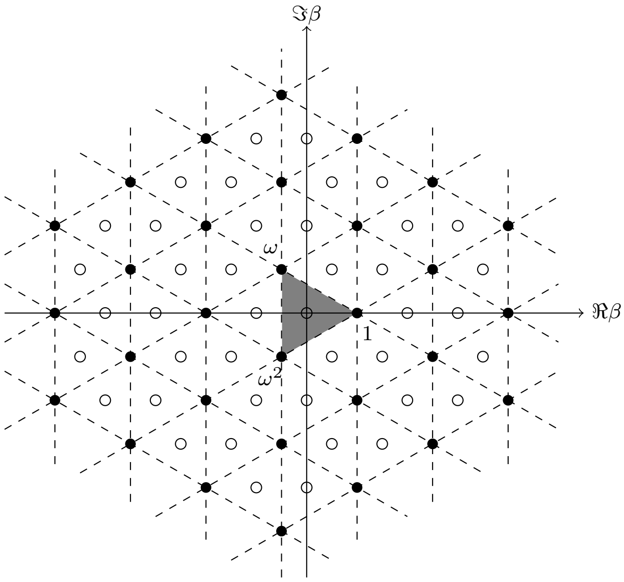
Table 1 Rational solutions generated from $q = p = 0$

α	β	q	p
0	0	0	0
1	-1	$-\frac{1}{z}$	$\frac{1}{z}$
ω	$-\omega^2$	$-\frac{\omega}{z}$	$\frac{\omega^2}{z}$
ω^2	$-\omega$	$-\frac{\omega^2}{z}$	$\frac{\omega}{z}$
-2	2	$\frac{2z^3-6z}{z^4+3}$	$-\frac{2z^3+6z}{z^4+3}$
3	-3	$-\frac{3z^8+6z^6+90z^2+45}{z^9+18z^5+45z}$	$\frac{3z^8-6z^6-90z^2+45}{z^9+18z^5+45z}$
-3	3	$\frac{3z^8-12z^6+18z^4+36z^2-9}{z^9+18z^5+9z}$	$-\frac{3z^8+12z^6+18z^4-36z^2-9}{z^9+18z^5+9z}$

Table 2 Rational solutions generated from $q = p = -z$

α	β	q	p
-1	1	$-z$	$-z$
ω	ω^2	$-\omega z$	$-\omega^2 z$
ω^2	ω	$-\omega^2 z$	$-\omega z$
2	-2	$-\frac{1}{z} - z$	$\frac{1}{z} - z$
-4	4	$\frac{-3z^6+3z^4+5z^2-1}{3z^5+z}$	$\frac{-3z^6-3z^4+5z^2+1}{3z^5+z}$
5	-5	$\frac{3z^9+6z^7-8z^5+10z^3+5z}{-3z^8+2z^4+1}$	$\frac{3z^9-6z^7-8z^5-10z^3+5z}{-3z^8+2z^4+1}$

Fig. 1 Distribution of the rational solutions in the parameter space (β plane)



Theorem 3 *The system (1) admits a rational solution $(q(z), p(z))$ if and only if the parameters (α, β) are such that $\alpha = -\bar{\beta}$ and $\beta \in \mathbb{Z} \oplus \omega\mathbb{Z}$. For any of these parameters, there exists a unique rational solution.*

Proof The sufficiency of these conditions follows from the fact that for any such (α, β) a rational solution can be generated by repeated application of the Bäcklund transformations (6) acting on the trivial solution $q = p = 0$ or $q = p = -z$. For a rational solution $(q(z), p(z))$, by leading-order analysis one can see that the point $z = \infty$ has to be either a simple pole or simple zero of both q and p (unless $q = p = 0$). In case of a simple pole we can express any rational solution by their Laurent series in a neighbourhood of $z = \infty$,

$$q(z) = a_1 z + \sum_{k=1}^{\infty} \frac{a_{-k}}{z^k}, \quad p(z) = b_1 z + \sum_{k=1}^{\infty} \frac{b_{-k}}{z^k}. \tag{7}$$

Inserting these expansions into (1) and comparing coefficients yields the relations

$$\begin{aligned} a_1 + b_1^2 = 0, \quad a_0 + 2b_1 b_0 = 0, \quad a_1 = 2b_{-1}b_1 + a_{-1} + \alpha, \\ b_1 + a_1^2 = 0, \quad b_0 + 2a_1 a_0 = 0, \quad b_1 = -2a_{-1}a_1 - b_{-1} - \beta, \end{aligned}$$

from which we obtain $a_1 \in \{-1, -\omega, -\omega^2\}$, $b_1 = -a_1^2 = \bar{a}_1$, $a_0 = b_0 = 0$ and

$$\alpha = a_1 - 2b_{-1}b_1 - a_{-1}, \quad \beta = -b_1 - 2a_{-1}a_1 - b_{-1}.$$

a_{-1} and b_{-1} being the residua of q and p, respectively, they can be expressed as the sum of residua over all poles of q and p in the finite complex plane,

$$a_{-1} = -l - m\omega - n\omega^2, \quad b_{-1} = l + m\omega^2 + n\omega, \quad l, m, n \in \mathbb{N},$$

i.e. $a_{-1}, b_{-1} \in \mathbb{Z} \oplus \omega\mathbb{Z}$ with $a_{-1} = -\bar{b}_{-1}$. Hence we find $\alpha = -\bar{\beta}$ with $\beta \in \mathbb{Z} \oplus \omega\mathbb{Z}$.

On the other hand, if $(q(z), p(z))$ has a zero at $z = \infty$, by inserting into (1) the expansions

$$q(z) = \sum_{k=1}^{\infty} \frac{a_{-k}}{z^k}, \quad p(z) = \sum_{k=1}^{\infty} \frac{b_{-k}}{z^k}. \tag{8}$$

we find $a_{-1} = -\alpha$ and $b_{-1} = -\beta$, showing that also here α, β are of the form $\alpha = -\bar{\beta}$ with $\beta \in \mathbb{Z} \oplus \omega\mathbb{Z}$.

By a sequence of Bäcklund transformations (6) any solution of (1) with parameters (α, β) can be transformed into a solution of (1) with parameters $(\tilde{\alpha}, \tilde{\beta})$ where $\tilde{\beta}$ is in the central triangle formed by the points 1, ω and ω^2 in the β plane (shaded area in Figure 1). The uniqueness of the rational solutions thus follows from the uniqueness of the trivial solutions as follows. Suppose first that there exists a rational solution other than $q = p = 0$ for the system

$$q' = p^2 + zq, \quad p' = -q^2 - zp. \tag{9}$$

If q and p both have a zero at $z = \infty$ then $q(z) = \frac{f(z)}{g(z)}$, where f and g are polynomials with $\deg f = \deg g - 1$ and $p(z) = \frac{h(z)}{k(z)}$, where h and k are polynomials with $\deg h = \deg k - 1$. Dividing the equations in (9) by q and p, respectively, leads to

$$\frac{q'}{q} = \frac{f'}{f} - \frac{g'}{g} = \frac{p^2}{q} + z, \quad \frac{p'}{p} = \frac{h'}{h} - \frac{k'}{k} = -\frac{q^2}{p} - z,$$

but here the left hand sides of both equations tend to 0 as $z \to \infty$ whereas the right hand sides tend to infinity, so this type of rational solution cannot occur. Therefore we must have $q(z) = \lambda z + \frac{f(z)}{g(z)}$ and $p(z) = \bar{\lambda} z + \frac{h(z)}{k(z)}$, $\lambda \in \{-1, -\omega, -\omega^2\}$. Applying one of the Bäcklund transformations σ_j, $j \in \{0, 1, 2\}$, leads to a rational solution $(\tilde{q}, \tilde{p})$ of the form $\tilde{q} = \mu z + \frac{\tilde{f}}{\tilde{g}}$ with $\deg \tilde{f} = \deg \tilde{g} - 1 = \deg f - 1$ and

$\tilde{p} = \bar{\mu}z + \frac{\tilde{h}}{\tilde{k}}$, with $\deg \tilde{h} = \deg \tilde{k} - 1 = \deg k - 1$, of system (1) with parameters (α, β) corresponding to a hollow circle in Figure 1. So after a finite sequence of Bäcklund transformations one would end up with the solution $q(z) = p(z) = -z$ of system (1) with $\beta = -\alpha = 1$, which, however, corresponds to a filled circle. From this contradiction it follows that $q = p = 0$ is the unique rational solution of (9).

Now let (q, p) be a rational solution of the system (1) with $\beta = -\alpha = 1$,

$$q' = p^2 + zq - 1, \quad p' = -q^2 - zp - 1. \tag{10}$$

Suppose first that q and p have a pole at $z = \infty$. Inserting the series (7) one finds that $a_1 = b_1 = -1$ and all coefficients a_{-k}, b_{-k}, $k = 1, 2, \dots$ are zero. On the other hand let q and p have a zero at $z = \infty$, i.e. they are of the form $q = \frac{f}{g}$ with $\deg f = \deg g - 1$ and $p = \frac{h}{k}$ with $\deg h = \deg k - 1$, corresponding to the expansions (8). Applying one of the Bäcklund transformations σ_j, $j \in \{0, 1, 2\}$, one obtains a rational solution $(\tilde{q}, \tilde{p})$ of degree one less of system (1) with parameters (α, β) corresponding to a filled circle. However, after a finite sequence of Bäcklund transformations one would end up with the solution $q = p = 0$ which corresponds to a hollow circle. So rational solutions of this type cannot exist for (10) and hence $q = p = -z$ is the unique rational solution in this case.

4 Concluding remarks

The Hamiltonian system presented in this article is shown to have meromorphic solutions. In fact, the combinations $w = \omega^j p + \omega^{2j} q - z$, $j \in \{0, 1, 2\}$, satisfy an equation closely related to the fourth Painlevé equation and therefore the solutions can be expressed completely in terms of the fourth Painlevé transcendents. However, the solutions of the system exhibit some interesting properties which are being studied further in the article [16].

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