(m, n)-rationalizable choices

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Abstract

Rationalizability has been a main topic in individual choice theory since the seminal paper of Samuelson (1938). The rationalization of a multi-valued choice is classically obtained by maximizing the binary relation of revealed preference, which is fully informative of the primitive choice as long as suitable axioms of choice consistency hold. In line with this tradition, we give a purely axiomatic treatment of the topic of choice rationalization. In fact, we introduce a new class of properties of choice coherence, called axioms of replacement consistency, which examine how the addition of an item to a menu may cause a substitution in the selected set. These axioms are used to uniformly characterize rationalizable choices such that their revealed preferences are quasi-transitive, Ferrers, semitransitive, and transitive. Further, regardless of rationalizability, we study the relationship of these new axioms with some classical properties of choice consistency, such as standard contraction, standard expansion, and WARP. To complete our analysis of the transitive structure of rationalizable choices, we examine the case of revealed preferences satisfying weak (m, n)-Ferrers properties in the sense of Giarlotta and Watson (2014). Originally introduced with the purpose of extending the notions of interval orders and semiorders, these Ferrers properties give a descriptive taxonomy of binary relations displaying a transitive strict preference but an intransitive indifference. Here we suggest a possible economic interpretation of weak (m, n)-Ferrers properties, showing that, in a suitable model of transactions, they provide a way of controlling phenomena of money-pump due to the presence of mixed cycles of preference/indifference. Finally, we define (m, n)-rationalizable choices as those having a weakly (m, n)-Ferrers revealed preference, and characterize these choices by means of additional axioms of replacement consistency.

Key words: Individual choice; rational choice; revealed preference; axioms of choice consistency; interval order; semiorder; total preorder; (m, n)-Ferrers; (m, n)-rationalizability; money-pump.

1 Introduction and motivation

The rationalization of an observed choice behavior is a topic in individual choice theory which has been given a lot of attention since the pioneering paper of Samuelson (1938): see – among

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several contributions lying down the fundamentals of the associated theory of revealed preferences – Houthakker (1950), Chernoff (1954), Arrow (1959), Richter (1966), Hansson (1968), and Sen (1971). See also Suzumura (1983), Moulin (1985), and Aleskerov *et al.* (2007) for an introduction to rational choices and revealed preference theory. Informally, a choice correspondence[1] is *rationalizable* whenever it can be recovered by maximizing the induced relation of revealed preference.[2] Thus rationalizability is equivalent to say that the original correspondence collapses to a binary relation, whose maximization fully explains the economic agent's observed behavior.

Rationalizable choices are known to satisfy some structural properties, called *axioms of choice consistency*. These axioms range from properties that guarantee the mere rationalizability of a choice in the sense described above (Chernoff's axiom of contraction consistency, and Sen's axiom of expansion consistency), to properties ensuring that the relation of revealed preference satisfies highly desirable structural properties (e.g., Samuelson's Weak Axiom of Revealed Preferences, WARP). In fact, it is well known that the preference revealed by a rationalizable choice is always acyclic, however it may well fail to satisfy any form of pseudo-transitivity.

The possibility to have a rationalizable choice with an intransitive revealed preference has been regarded as a drawback of the approach by many economists, who argue that the explanation of an observed choice behavior provided by a binary maximization cannot be taken *per se* as indicative of a perfect rationality. In fact, the transitivity of the revealed preference (hence the satisfaction of WARP) has been traditionally seen as an unquestionable rationality tenet, whose violation may give rise to perverse effects, such that the *money-pump* phenomenon observed by Davidson, Mc Kinsey, and Suppes (1955). Indeed, as described by Tversky (1969):

> *Transitivity, however, is one of the basic and the most compelling principles of rational behaviour. For if one violates transitivity, it is a well known conclusion that he is acting, in effect, as a "money-pump". Suppose an individual prefers y to x, z to y, and x to z. It is reasonable to assume that he is willing to pay a sum of money to replace x by y. Similarly, he should be willing to pay some amount of money to replace y by z, and still a third amount to replace z by x. Thus, he ends up with the alternative he started with but with less money.*

As a matter of fact, the mere presence of a strict cycle of preferences indeed puts the economic agent at the risk of losing all her money, since she may get involved in another cycle of money-pump, and continue in this fashion until her financial resources are totally exhausted.

Admittedly, the above money-pump effect requires strict cycles of preferences, which are forbidden in the case of a preference revealed by a choice (which is by definition acyclic, at least whenever the choice domain is finitely complete[3]). However, many contributions to the economic literature show that a money-pump effect may also arise – under suitable assumptions – in the presence of *mixed cycles*, that is, whenever there are cycles involving both strict preferences and indifferences: see, e.g., Restle (1961), who argues that a strict cycle can be easily induced by a mixed cycle using a "small bonus" approach.[4] Furthermore, several additional ways to induce

[1] Recall that given a set X of alternatives, a choice correspondence c on X is a contractive set-function defined on a suitable subset Ω of the powerset of X. The correspondence c associates to each menu S in Ω a nonempty subset $c(S) \subseteq S$, which comprises all elements of S that are deemed choosable.

[2] An alternative x is *revealed to be preferred* to another alternative y whenever there is a menu containing both of them and such that x is selected from it.

[3] A *finitely complete* domain contains all finite subsets of the ground set X.

[4] For some recent examples of this approach, see also Hansson (1993) and Rabinowicz (2008).

a money-pump from mixed cycles of strict preferences/indifferences have been proposed in the literature, e.g., in a multiple-criteria set up (Schumm, 1987), or using the notion of dominance in cases of preferences under uncertainty (Gustafsson, 2010).[5]

Although the classical approach described above is motivated by justifiable (someone would say compelling) economic reasons, a strong opposition to regarding transitivity as an undisputed feature of rationality has naturally emerged since long time ago. Within the field of individual choice, it was Sen (1971) who started describing revealed preference theory as "obsessed with transitivity". More generally, research collecting evidence of the natural intransitivity of indifference and preference has became abundant over time in all economic theory. In this respect, Bleichrodt and Wakker (2015) argue that the year 1982 is a sort of "breaking point" in the economic literature. Indeed, in that very year the transitivity axiom was given up in three seminal papers, related to the so-called *regret theory*: the axiomatic approach of Fishburn (1982), a decision analysis oriented paper by Bell (1982), and the fundamental contribution of Loomes and Sudgen (1982). As a matter of fact, one should regard the probabilistic choice model proposed by Luce (1959) as the pioneering example of intransitive preferences.

In the same stream of research that opposes the traditional approach based on transitivity, one cannot avoid mentioning the extraordinary amount of literature on semiorders, interval orders, and similar preference structures, which describe forms of rational behavior characterized by weaker forms of transitivity. Anticipated by intuitions of authors such as Fechner (1860), Poincaré (1908), Georgescu-Roegen (1936), Armstrong (1939), and Halphen (1955), research on possibly intransitive preference structures had its definitive consecration by the seminal papers of Luce (1956) and Fishburn (1970), who formally introduced the notions of semiorder and interval order, respectively. The semiorder/interval order approach is based on the idea of weakening the axiom of transitivity, rather than abandoning it all together. In fact, as Luce's coffee/sugar example suggests, the transitivity of indifference should be questioned and regulated, whereas that of strict preference may well be retained.

Weak (m, n)*-Ferrers properties*, introduced by Giarlotta and Watson (2014) as a generalization of the approach proposed by Öztürk (2008), go exactly in the direction of considering binary structures with a transitive strict preference but a possibly intransitive indifference. Originally designed to provide a combinatorial extension of both the Ferrers condition and the semitransitivity – which coincide, respectively, with weak $(2, 2)$-Ferrers and weak $(3, 1)$-Ferrers – these new properties display an exhaustive (finite) taxonomy of enhanced forms of quasi-transitivity. In fact, roughly speaking, weak (m, n)-Ferrers properties classify transitive strict preferences by means of the types of forbidden mixed cycles of preference/indifference. It follows that such an approach may be relevant for economic applications insofar as weak (m, n)-Ferrers properties prompt a possible recognition of money-pump effects due to the presence of mixed cycles of a certain length and type.

The analysis conducted in this paper employs interval orders, semiorders, and, in general, weak (m, n)-Ferrers properties to classify rational choice behaviors. In fact, we make a systematic attempt to explicitly separate two issues: the rationalizability of a choice on one hand, and the internal structure of its revealed preference on the other one. To that end, we introduce suitable

[5]However, as an objection to the money-pump argument, let us mention the contributions of Schick (1986) and McClennen (1990), who basically claim that after transactions between indifferent alternatives, an economic agent may well refuse a transaction between strictly preferred alternatives, thus avoiding a money-pump situation. As observed by Piper (2014), both solutions to the money-pump effect are based on the idea that the economic agent remembers the past and plans the future accordingly.

axioms of replacement consistency, which a rationalizable choice may or may not satisfy. These axioms examine how the addition of an item to a menu causes a substitution in the subset of selected elements. We use these axioms to characterize rationalizable choices whose revealed preference satisfies different levels of transitivity.

Specifically, first we examine those cases in which the revealed preference is a classical binary structure, such as a quasi-transitive relation, an interval order, a semiorder, or a total preorder. In this perspective, our contribution can be seen as an axiomatization that is alternative to those given by Jamison and Lau (1973, 1975), Fishburn (1975), Schwartz (1976), Bandyopadhyay and Sengupta (1991, 1993). Successively, in order to complete a taxonomic classification of rationalizable choices, we also characterize choices with a weakly (m, n)-Ferrers revealed preference by means of additional axioms of replacement consistency. In this way, we provide a uniform treatment of the topic by means of properties of choice consistency that belong to a single category.

The paper is organized as follows. In Section 2 we provide all preliminaries about choices and preferences. In particular, we recall the notion of rationalizability, the relation of revealed preference, and some classical axioms of choice consistency. Section 3 introduces four axioms of replacement consistency. We prove characterizations of rationalizable choices having a revealed preference that is, respectively, a quasi-preorder, an interval order, a semiorder, and a total preorder. Further, we extensively examine the relation of these axioms of replacement consistency with some classical properties of individual choice theory, regardless of rationalizability. Section 4 deals with the general case of (m, n)-rationalizable choice correspondences, that is, choices rationalizable by means of a revealed preference satisfying the weak (m, n)-Ferrers property. We study minimal forbidden configurations for weak (m, n)-Ferrers properties, which link the latter to money-pump phenomena arising from mixed cycles. Finally, we provide a full characterization of (m, n)-rationalizable choices by means of additional axioms of replacement consistency. Section 5 summarizes the findings of the paper, and suggests some natural extensions and applications of our approach. In the Appendix, a picture describes all implications among combinations of weak (m, n)-Ferrers properties.

2 Preliminaries on rationalizable choices

In this section we provide the reader with all the basic ingredients of individual choice theory. Furthermore, we recall the fundamentals of the theory of revealed preferences, and its connection to the rationalizability of a choice correspondence satisfying suitable axioms of consistency.

2.1 Basic definitions and terminology

In what follows, X denotes a nonempty set of alternatives. A *choice domain* on X is a family $\Omega \subseteq 2^X \setminus \{\emptyset\}$, containing all the subsets S of X on which an economic agent makes (realistically or potentially) a selection, that is, she selects a nonempty subset of S comprising the "best" elements of S. In some problems, Ω is assumed to be *complete*, i.e., it is equal to $2^X \setminus \{\emptyset\}$: this is the natural setting whenever one deals with a finite set X of alternatives. However, in the case that X is infinite, the hypothesis of the completeness of Ω is often too demanding for most economic settings. Apart from the cases of choices arising from consumer demand theory[6] (which require different hypotheses), usually the choice domain Ω is assumed to satisfy suitable closure properties such as (i) it contains all singletons, and (ii) it is closed with respect to the operation of taking finite unions of its elements. A choice domain satisfying (i) and (ii) is hereby called *normal*; throughout this paper, the normality of Ω is assumed even without explicit mention. Note the postulate of normality implies that Ω is *finitely complete*, i.e., it contains $[X]^{\text{fin}}$ (which henceforth denotes the family of all finite nonempty subsets of X). In an individual choice context, a finitely complete choice domain is easily justifiable by the fact that an economic agent should be able to make a selection within finite sets.

A *choice space* is a pair (X, Ω), where Ω is a (normal) choice domain on X. We refer to a generic element S of Ω as a *menu*, whereas any element x belonging to a menu is called an *item*. A choice space (X, Ω) is *finite* if $\Omega = [X]^{\text{fin}}$, regardless of the cardinality of the ground set X. A *choice correspondence* (also called a *multi-valued choice*) on a choice space (X, Ω) is a map $c \colon \Omega \to 2^X \setminus \{\emptyset\}$ such that $c(S) \subseteq S$ for all $S \in \Omega$. Thus a choice correspondence c selects within each menu S a nonempty set $c(S) \subseteq S$, composed of those elements of S that are deemed choosable by the economic agent: we denote a choice correspondence on (X, Ω) by $c \colon \Omega \rightrightarrows X$ with the aim of emphasizing this multi-valued feature. In the special case that the economic agent always selects a single element of each menu, we say that c is a *choice function*, and denote it by $c \colon \Omega \to X$. A choice correspondence is *finite* if so is its choice space. To simplify some statements, we employ a special notation for subfamilies of Ω containing fixed elements of X: for instance, Ω_x stands for the subfamily of Ω composed of all menus containing the item x, Ω_{yz} is the subfamily of Ω composed of all menus containing both y and z, etc.

Next, we recall a few basic notions concerning preference relations. A reflexive binary relation on X is denoted by $\succsim$, and is generically called a *weak preference*: we read $x \succsim y$ as "x is weakly preferred to y". Following standard practice, $\succ$ denotes the *strict preference* derived from $\succsim$, i.e., the (irreflexive and) asymmetric relation on X defined by $x \succ y$ if $x \succsim y$ and $\neg(y \succsim x)$, where $x, y \in X$. Further, the *indifference* derived from $\succsim$ is the reflexive and symmetric relation $\sim$ on X defined by $x \sim y$ if $x \succsim y$ and $y \succsim x$. It follows that a weak preference $\succsim$ can always be written as the (disjoint) union of its strict preference $\succ$ and its indifference $\sim$. In particular, x is weakly preferred to y if and only if either x is strictly preferred to y, or x and y are indifferent.

A weak preference $\succsim$ is *complete* (or *total*) if $x \succsim y$ or $y \succsim x$ holds for all (distinct) $x, y \in X$, and *acyclic* if it contains no strict cycles (i.e., there is no sequence $x_1, x_2, \ldots, x_n$ of $n \geq 3$ elements of X such that $x_1 \succ x_2 \succ \ldots \succ x_n \succ x_1$). Further, $\succsim$ is *quasi-transitive* if its asymmetric part $\succ$ is transitive, *Ferrers* if ($x \succsim y$ and $z \succsim w$) implies ($x \succsim w$ or $z \succsim y$), and *semitransitive* if ($x \succsim y$ and $y \succsim z$) implies ($x \succsim w$ or $w \succsim z$), where x, y, z, w range over X. Then, $\succsim$ is a *preorder* if it is transitive, a *quasi-preorder* if it is quasi-transitive, an *interval order* if it is Ferrers, a *semiorder* if it is Ferrers and semitransitive,[7] and a *linear order* if it is an antisymmetric total preorder.

As customary, given a weak preference $\succsim$ on X and a set $S \subseteq X$, the $\succsim$-maximal elements of S are those that are strictly non-dominated by other items of the same set, that is,

$$\max(S, \succsim) \;:=\; \big\{x \in S :\; (\nexists s \in S)\,(s \succ x)\big\}.$$

Note that in the special case that $\succsim$ is complete, the set of maximal elements of $S \subseteq X$ can be equivalently written as $\max(S, \succsim) = \{x \in S :\; (\forall s \in S)\,(x \succsim s)\}$. Observe also that $\max(S, \succsim)$ may be empty for some menus S: consider, e.g., the case of an infinite subset of the set of natural numbers, or a menu composed by the elements of a strict cycle in a non-acyclic relation.

[6]On the topic, see the survey by Varian (2006), and references therein.

[7]Note that a reflexive relation that is either Ferrers or semitransitive is automatically complete.

2.2 Revealed preference and choice rationalizability

The link between the two theories of individual choice and individual preference depends on the set up of the problem at hand. In fact, the literature basically distinguishes two approaches, which are identifiable by the primitive information associated with a choice problem: either (i) a preference (modeled by a binary relation), or (ii) a choice (modeled by a possibly multi-valued function). The relational approach (i) and the functional approach (ii) are alternative to each other, both philosophically and methodologically.

In this paper, we employ a functional approach, thus regarding the observed choice behavior of an economic agent as a *prior*. According to this point of view, a binary relation of preference is canonically derived from the given choice correspondence, with the goal to rationalize (i.e., explain) the observed selection. Informally, a primitive choice c is rationalizable whenever the functional approach can be reduced to a relational approach, that is, c canonically induces a binary relation $\succsim_c$, which in turn allows one to univocally retrieve c. The next definition provides the formal notion.

Definition 2.1 Let $c\colon \Omega \rightrightarrows X$ be a choice correspondence. The *preference revealed by* c is the binary relation $\succsim_c$ on X defined as follows for each $x, y \in X$:

$$x \succsim_c y \quad \overset{\text{def}}{\Longleftrightarrow} \quad (\exists S \in \Omega_{xy})\ [x \in c(S)].$$

Then c is *rationalizable* (or *binary*, or *normal*) if it can be retrieved from its revealed preference $\succsim_c$ via the maximality principle, i.e., $c(S) = \max(S, \succsim_c)$ for each menu $S \in \Omega$.

Given a choice correspondence c, then we read $x \succsim_c y$ as "item x is revealed to be weakly preferred to item y", in the sense that there exists a feasible menu containing both items and such that x is selected in it (whereas y may be selected or not). Note that the revealed preference $\succsim_c$ is reflexive, complete, and acyclic, regardless of the rationalizability of c.[8] Using a recently introduced terminology, the revealed preference is in fact an *extended preference*, that is, it contains a linear order.[9]

On the other hand, $\succsim_c$ need not be transitive in general, in fact it can even fail to be quasi-transitive. The literature on the topic studies types of rationalizability linked to the internal structure of the revealed preference, mostly focusing on choices that are

(A) *transitively rationalizable* (rationalizable by a transitive revealed preference), and

(B) *quasi-transitively rationalizable* (rationalizable by a quasi-transitive revealed preference).

The analysis of alternative types of rationalizability has successively been extended to cases in which the revealed preference satisfies intermediate forms of transitivity.[10] Here we refer to the related choice correspondences as

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[8]For reflexivity, since c is contractive and no menu may have an empty image, we have $c(\{x\}) = \{x\}$ for each $x \in X$, hence $x \succsim_c x$ holds. For completeness, if $x, y \in X$ are such that $\neg(x \succsim_c y)$, then $x \notin c(S)$ for each $S \in \Omega_{xy}$; in particular $c(\{x, y\}) = \{y\}$, hence $y \succsim_c x$ holds. Finally, the acyclicity of $\succsim_c$ is an immediate consequence of the finite completeness of a normal choice domain: indeed, if S is the finite menu composed of the elements of a strict cycle, then the very definition of revealed preference implies that $c(S)$ is empty, a contradiction.

[9]See Definition 2.2, Lemma 2.3, and Figure 1 in Giarlotta and Watson (2014).

[10]For a similar classification within the realm of *social* choice theory, see section 3 of Sen (1986).

(C) *interval order rationalizable* (rationalizable by a revealed preference that is Ferrers), and

(D) *semiorder rationalizable* (rationalizable by a revealed preference that is a semiorder).

The main goals of this paper are (1) to obtain a unified treatment of the above cases (A)-(D), and (2) to describe a more general framework that subsumes (A)-(D) as special cases.

2.3 Classical axioms of choice consistency

Several rationalizability results have been obtained over the last few decades. Most of them are expressed in terms of the satisfaction of suitable *axioms of choice consistency.* These axioms are conditions imposed on the choice structure of an economic agent with the aim of ensuring that the individual choice has some desirable (i.e., rational) features.[11] In fact, it turns out that rationalizability is equivalent to the satisfaction of two standard properties of consistency, which are related to contractions/expansions of feasible menus:

(α)**-axiom** (*standard contraction consistency*)

$$\big[x \in S \subseteq T \ \wedge \ x \in c(T)\big] \quad \Longrightarrow \quad x \in c(S)$$

(γ)**-axiom** (*standard expansion consistency*)

$$x \in \textstyle\bigcap_{j \in J} c(S_j) \quad \Longrightarrow \quad x \in c\left(\textstyle\bigcup_{j \in J} S_j\right)$$

where $x \in X$ and $S, T, S_i, \bigcup_{j \in J} S_j \in \Omega$ (for each $i \in J$). Axiom (α) is also called *Chernoff's condition* (Chernoff, 1954), whereas axiom (γ) was introduced by Sen (1971). Chernoff's condition (α) says that if an item is selected from a feasible menu, then it is still selected from any submenu containing it. Sen's property (γ) goes in the direction opposite to (α), since it requires that whenever an item is selected in each menu of a given family, then it is also selected in the expanded menu obtained as the union of all menus in the family. It is worth noting that the above axioms are universally quantified over a single element of the ground set X, thus they are unary.

It is well known that the standard axioms of contraction/expansion consistency characterize rationalizable choices whenever the choice domain is finitely complete. (Recall that our initial assumption about the normality of the choice domain automatically yields the finite completeness of a choice correspondence.) In fact, we have:

Theorem 2.2 (Sen, 1971) *A (finitely complete) choice correspondence is rationalizable if and only if it satisfies axioms* (α) *and* (γ).

Rationalizability does not guarantee *per se* any special structural property of the revealed preference, which may indeed fail to satisfy even weak forms of transitivity. In fact, in order to handle the two special cases (A) and (B) mentioned at the end of Section 2.2, additional conditions of "symmetric expansion consistency" have been analyzed in the literature: see Sen (1969, 1971), as well as Sen (1986) for an account of several axioms of this kind and an analysis of the relationships among them. Two well known properties belonging to this class are the (β)-axiom and the (δ)-axiom. With an eye to some asymmetric consistency axioms to come, we restate them in the following equivalent way, which emphasizes their symmetric nature:

[11] Here we insist on the fact that the choice is "individual" and not "social", since in *Social Choice Theory* the hypothesis that a choice correspondence satisfies suitable axioms imposed *ab externo* has undertaken severe criticisms: see, e.g., Sen (1993) for the notion of "epistemic value" of a menu.

(β)**-axiom** (*symmetric expansion consistency*)

$$\big[x \neq y \ \wedge \ x, y \in c(S) \ \wedge \ S \subseteq T\big] \quad \Longrightarrow \quad \big[x \in c(T) \iff y \in c(T)\big]$$

(δ)**-axiom** (*symmetric weak expansion consistency*)

$$\big[x \neq y \ \wedge \ x, y \in c(S) \ \wedge \ S \subseteq T\big] \quad \Longrightarrow \quad \{x\} \neq c(T) \neq \{y\}$$

where $x, y \in X$ and $S, T \in \Omega$. Axiom (β) says that if two distinct items x and y are selected from a menu S, then they are simultaneously either selected or rejected in any expanded menu T. On the other hand, axiom (δ) says that under the same hypothesis as (β), it cannot happen that one between x and y is the unique item selected from the expanded menu T. Therefore, axiom (β) is definitively more demanding than axiom (δ). Note also that, contrary to what happens for the unary axioms (α) and (γ), axioms (β) and (δ) are binary, in the sense that they are universally quantified over a pair of elements of X.

The last axiom recalled here – which was proposed by Samuelson (1938) – is probably the most frequently mentioned in the literature:

WARP (*Weak Axiom of Revealed Preferences*)

$$\big[x, y \in S \cap T \ \wedge \ x \in c(S) \ \wedge \ y \in c(T)\big] \quad \Longrightarrow \quad \big[x \in c(T) \ \wedge \ y \in c(S)\big]$$

where $x, y \in X$ and $S, T \in \Omega$. WARP summarizes suitable features of both contraction and expansion consistency in a single – and quite strong, despite its name – axiom. In fact, WARP basically regards two alternatives as choice-equivalent under the sole condition that they are selected from two (possibly different) menus containing both of them. Note that WARP is again a kind of symmetric and binary axiom.

The *Fundamental Theorem of Revealed Preference Theory* solves case (A) mentioned before:

Theorem 2.3 (Arrow (1959), Sen (1971)) *A choice correspondence is transitively rationalizable if and only if it satisfies WARP if and only if it satisfies axioms (α) and (β).*

Concerning case (B), let us mention the following sufficient (but not necessary) condition:[12]

Theorem 2.4 (Sen, 1986) *A choice correspondence satisfying axioms (α), (γ), and (δ) is quasi-transitively rationalizable.*

Finally, for cases (C) and (D), there are several important contributions, which are however scattered in the literature: see, e.g., Jamison and Lau (1973, 1975), Fishburn (1975), Schwartz (1976), Bandyopadhyay and Sengupta (1991, 1993). For a comprehensive analysis of the topic, see the book by Aleskerov *et al.* (2007). In the next section we address cases (A)-(D) by means of unified approach, introducing a new type of consistency properties.

3 Axioms of replacement consistency

In a recent paper on individual choices, Eliaz and Ok (2006) examine a weakening of WARP, called WARNI (*Weak Axiom of Revealed Non-Inferiority*). The authors' goal is to provide a

[12]See part (2) of the so-called "sundry choice – functional lemma" in Sen (1986), page 1099. It is worth noting that in a previous paper, the same author proves that a finite version of δ (quantified over finite menus only) characterizes the quasi-transitivity of a rationalizable choice: see "Property δ" and Theorem 10 in Sen (1971).

choice-theoretic foundation of incomplete preferences. In fact, WARNI can be naturally associated with the construction of a new type of (transitive and possibly incomplete) revealed preference, which still enables one to retrieve the primitive choice using the maximality principle.

It turns out that WARNI is also linked to problem (B) mentioned in Section 2.2. In fact, the proof of Eliaz and Ok's Theorem 1 implicitly shows that a choice correspondence satisfying WARNI is quasi-transitively rationalizable. However, the converse does not hold in general, since a technical hypothesis about the choice domain is needed to prove that WARNI holds under quasi-transitive rationalizability.[13]

In order to provide a general characterization of quasi-transitively rationalizable choice correspondences defined on a normal domain, in this section we introduce a new class of consistency properties in an individual choice setting. These axioms examine how the addition of an item to a menu may cause a "substitution" in the selected set. We shall use these properties to characterize rationalizable choice correspondences such that their revealed preferences satisfy suitable forms of transitivity, giving *inter alia* an alternative solution to cases (A), (B), (C), and (D) mentioned at the end of Section 2.2.

3.1 Definitions and semantics

To start, we introduce four new axioms of choice consistency, and discuss their semantics. These properties are the following (as usual, $x, y, z \in X$ and $S, T \in \Omega$ are universally quantified):

(ρ)**-axiom** (*standard replacement consistency*)
$$\big[y \in c(S) \ \wedge \ y \notin c(S \cup \{x\})\big] \quad \Longrightarrow \quad x \in c(S \cup \{x\})\,;$$

(ρ_f)**-axiom** (*Ferrers replacement consistency*)
$$\big[x \in c(S) \ \wedge \ y \in S \ \wedge \ z \in c(T) \ \wedge \ z \notin c(T \cup \{y\})\big] \quad \Longrightarrow \quad x \in c(T \cup \{x\})\,;$$

(ρ_{st})**-axiom** (*semitransitive replacement consistency*)
$$\big[y \in c(S) \ \wedge \ z \in S \ \wedge \ z \in c(T) \ \wedge \ y \notin c(S \cup \{x\})\big] \quad \Longrightarrow \quad x \in c(T \cup \{x\})\,;$$

(ρ_t)**-axiom** (*transitive replacement consistency*)
$$\big[y \in c(S) \ \wedge \ y \notin c(S \cup \{x\})\big] \quad \Longrightarrow \quad c(S \cup \{x\}) = \{x\}\,.$$

The relationship among these four axioms as well as the level of transitivity of the associated relation of revealed preference will be examined in Section 3.3. The results in that section will also explain the employed notation. Here we only provide a rationale for their introduction. Axiom (ρ) says that given a menu S and an item y in it, if y is selected from S but fails to be chosen as soon as a certain item x is added to S, then the new item x must be selected from the larger menu $S \cup \{x\}$. In some sense, the new element x "replaces" y in the selection taste of the economic agent. (Note that, however, the equality $c(S \cup \{x\}) = (c(S) \setminus \{y\}) \cup \{x\}$ does not hold in general.)

Axioms (ρ_f) and (ρ_{st}) are slightly more demanding than (ρ), despite in different (orthogonal) directions. Specifically, axiom (ρ_f) requires that given two menus $S \in \Omega_{xy}$ and $T \in \Omega_z$, if x is selected from S, and z is selected from T but not from $T \cup \{y\}$, then x must be selected from

[13] This technical condition is the closure of the choice domain with respect to countable subsets of the ground set. This implies that in the special case of a *total* choice correspondence defined on a *finite* ground set, WARNI does characterize quasi-transitive rationalizability.

$T \cup \{x\}$. Note that whenever z is chosen from the menu T but not from the larger menu $T \cup \{y\}$, axiom (ρ) allows one to derive that y is selected from $T \cup \{y\}$, but no conclusion can be inferred about the selection of x from the menu $T \cup \{x\}$. Axiom (ρ_f) is stronger than (ρ) insofar as it allows one to derive the conclusion $x \in c(T \cup \{x\})$ from the fact that x is selected from a menu S containing y among its items (i.e., from the fact that x is revealed to be preferred to y).

The semantics of axiom (ρ_{st}) is somehow similar to that of (ρ_f). Specifically, (ρ_{st}) requires that given two menus $S \in \Omega_{yz}$ and $T \in \Omega_z$, if y is selected from S, z is selected from T, and y fails to be selected as soon as x is added to S, then x is selected from $T \cup \{x\}$. Note that the conclusion of axiom (ρ_{st}) is exactly the same as that of axiom (ρ_f), but the antecedent is different (in fact, neither stronger nor weaker than the antecedent of (ρ_f)). Indeed, it might well happen that the item x selected from $T \cup \{x\}$ belongs to neither S nor T.

Axiom (ρ_t) is the strongest of the four, since it says that given a menu $S \in \Omega_y$, if y is selected from S but fails to be selected as soon as a new item x is added to S, then x becomes the unique item that is selected from the larger menu $S \cup \{x\}$. Note that the antecedent of axiom (ρ_t) is exactly the same as that of axiom (ρ), but its conclusion is drastically stronger, since x "fully replaces" anything that was originally chosen from S. Despite its seemingly unreasonable strength, axiom (ρ_t) is easily justifiable in a situation of perfect comparability. To motivate the introduction of (ρ_t), consider the following example.

Assume that G is a consumer who wishes to buy an apartment in the neighborhood where she was raised, provided that the price does not exceed a certain upper bound. G has restricted her possible choice to two apartments y_1 and y_2, but she is displaying quite some indecisiveness in selecting one of the two. While G is in the process of finalizing her decision – maybe after asking references about the current owners, or on the basis of friends' suggestions, or even by a subjective randomization (e.g., flipping a coin) – a new apartment x located in the same building as apartment y_1 becomes available. If x is definitively larger, with a more refined interior, and a much better view than y_1, G will have no doubts in eliminating y_1 from the set of possible final selections. However, it seems reasonable to assume that G might also choose x over y_2, thus making x as her definite selection. Thus axiom (ρ_t) holds in this scenario.

The statements of (ρ) and (ρ_t) only involve two items and a single menu, hence their semantics is quite simple to understand. On the other hand, the rationale of axioms (ρ_f) and (ρ_{st}), despite being of the same nature, is more subtle, since their statements simultaneously involve three items and two menus. To give a better insight into their semantics, in what follows we reformulate all axioms of replacement consistency using a model-theoretic notation.

To start, we associate to an arbitrary choice correspondence $c\colon \Omega \rightrightarrows X$ two new preference relations, which are inspired by the replacement paradigm. Let $\succsim_c^+$ and $\trianglerighteq_c$ be the relations on X defined as follows for each $x, y \in X$:

$$
\begin{array}{lcl}
x \succsim_c^+ y & \stackrel{\text{def}}{\Longleftrightarrow} & (\exists S \in \Omega)\ \big[y \in S \ \wedge\ c(S \cup \{x\}) = \{x\} \big] \\
x \trianglerighteq_c y & \stackrel{\text{def}}{\Longleftrightarrow} & (\exists S \in \Omega)\ \big[y \in c(S) \ \wedge\ y \notin c(S \cup \{x\}) \big].
\end{array}
$$

We call $\succsim_c^+$ and $\trianglerighteq_c$ the relations of *strong revealed preference* and *replacing preference* associated with c, respectively. Further, we shall employ the following notation:

$$
\begin{array}{lll}
S \models x \succsim_c y & \text{stands for} & y \in S \ \wedge\ x \in c(S) \\
S \models x \succsim_c^+ y & \text{stands for} & y \in S \ \wedge\ c(S \cup \{x\}) = \{x\} \\
S \models x \trianglerighteq_c y & \text{stands for} & y \in c(S) \ \wedge\ y \notin c(S \cup \{x\})
\end{array}
$$

where $S \in \Omega$ and $x, y \in X$. Therefore, $S \models x \succsim_c y$ means that menu S witnesses a revealed preference of x over y; the meaning of $S \models x \succsim_c^+ y$ and $S \models x \trianglerighteq_c y$ is similar. Then the four axioms of replacement consistency can be reformulated as follows using the above notation:[14]

$$
\begin{array}{llcl}
(\rho)\text{-axiom} & S \models x \trianglerighteq_c y & \Longrightarrow & S \cup \{x\} \models x \succsim_c y \\
(\rho_f)\text{-axiom} & \big(S \models x \succsim_c y \big) \;\wedge\; \big(T \models y \trianglerighteq_c z \big) \;\wedge\; z \in c(T) & \Longrightarrow & T \cup \{x\} \models x \succsim_c z \\
(\rho_{st})\text{-axiom} & \big(S \models x \trianglerighteq_c y \big) \;\wedge\; \big(S \models y \succsim_c z \big) \;\wedge\; z \in c(T) & \Longrightarrow & T \cup \{x\} \models x \succsim_c z \\
(\rho_t)\text{-axiom} & S \models x \trianglerighteq_c y & \Longrightarrow & S \cup \{x\} \models x \succsim_c^+ y.
\end{array}
$$

Note that this model-theoretic formulation of the four axioms of replacement consistency reveals a complementarity of (ρ_f) and (ρ_{st}), since they both state a type of transitive coherence of the two binary relations $\succsim_c$ and $\trianglerighteq_c$.[15]

3.2 Examples and counterexamples

We discuss four sets of examples, which illustrate the effect of the different axioms of replacement consistency. In the first three sets, the ground set X is finite, whereas in the last one X is infinite; furthermore, the choice domain is complete in all cases. These examples are designed in a way such that Chernoff's standard axiom (α) of contraction consistency *always* holds, whereas Sen's standard axiom (γ) of expansion consistency fails at times (and so the choice correspondence is not rationalizable in these cases). In fact, our goal is to show that the effect of the axioms of replacement consistency is *not* related to the possibility of rationalizing a choice correspondence, at least in the general case.[16] Note that in order to simplify the proof of some claims, we shall make use of a few results presented in the next section.

Example 3.1 Let $X := \{x, y, z\}$ and $\Omega := 2^X \setminus \{\emptyset\}$. Further, let $c_i \colon \Omega \rightrightarrows X$ be the four choice correspondences defined as follows:

$$
\begin{array}{lllll}
(c_1) & \underline{x}\,\underline{y}\,, & x\,\underline{z}\,, & \underline{y}\,\underline{z}\,, & x\,y\,\underline{z} \\
(c_2) & \underline{x}\,y\,, & x\,\underline{z}\,, & \underline{y}\,\underline{z}\,, & x\,y\,\underline{z} \\
(c_3) & \underline{x}\,\underline{y}\,, & \underline{x}\,\underline{z}\,, & \underline{y}\,\underline{z}\,, & x\,\underline{y}\,\underline{z} \\
(c_4) & x\,\underline{y}\,, & x\,\underline{z}\,, & \underline{y}\,\underline{z}\,, & x\,\underline{y}\,\underline{z}
\end{array}
$$

where the underlined elements of each set are the items selected in that menu.[17] One can readily check that axiom (α) holds for all c_i's. Below we determine whether each c_i satisfies axiom (γ) or any axiom of replacement consistency.

[14] The equivalent formulation of axiom (ρ_f) given below is redundant, since the clause "$z \in c(T)$" is subsumed by the clause "$T \models y \trianglerighteq_c z$". However, this formulation allows us to point out a complementarity of the two replacement axioms (ρ_f) and (ρ_{st}), which otherwise would remain unnoticed.

[15] The transitive coherence associated to the model-theoretic formulation of (ρ_f) and (ρ_{st}) is indeed reminiscent of the mixed transitivity properties of a *NaP-preference*: see Section 5 and references therein.

[16] Under axiom (α), there are special cases in which there exists a relationship between axioms of replacement consistency and axiom (γ). For instance, it is well known that single-valued choice functions are (transitively) rationalizable if and only if (α) holds, whence assuming (α) automatically implies assuming (γ) – in fact, even (β) – in these cases: see Houthakker (1950) for the equivalence between (α) and WARP for choice functions. By Theorem 3 in Sen (1971) – which gives a complete characterization of transitive rationalizability – Houthakker's equivalence for choice functions extends to many other axioms of choice consistency.

[17] Thus, for the choice correspondence c_1, the notations $\underline{x}\,\underline{y}$ and $x\,y\,\underline{z}$ stand for, respectively, $c_1(\{x, y\}) = \{x, y\}$ and $c_1(\{x, y, z\}) = \{z\}$. Obviously, the singletons are always fixed points of c_1 (and, in general, of any choice correspondence), i.e., $c_1(\{a\}) = \{a\}$ for any $a \in X$.

(1) Neither (γ) nor (ρ) holds for c_1: indeed, $y \in c_1(\{x,y\}) \cap c_1(\{y,z\})$ but $y \notin c_1(\{x,y,z\})$, and $y \in c_1(\{y,z\}) \setminus c_1(\{x,y,z\})$ but $x \notin c_1(\{x,y,z\})$.

(2) The choice correspondence c_2 is very similar to c_1, the only difference being the image of the doubleton $\{x,y\}$. This fact yields that – contrary to c_1 – the correspondence c_2 is rationalizable. On the other hand, axiom (ρ) still fails to hold for c_2, exactly for the same reason as for c_1. Note that the preference $\succsim_{c_2}$ revealed by c_2 is not quasi-transitive, since $z \succ_{c_2} x \succ_{c_2} y$ and $z \sim_{c_2} y$.

(3) One can check that c_3 satisfies the axioms of replacement consistency (ρ), (ρ_f), and (ρ_{st}). On the other hand, c_3 satisfies neither (ρ_t) nor (γ): indeed, (ρ_t) fails because $x \in c(\{x,y\}) \setminus c(X)$ and $c(X) \neq \{z\}$, whereas (γ) fails since $x \in c_3(\{x,y\}) \cap c_3(\{x,z\})$ and $x \notin c(X)$.

(4) It is not difficult to verify that c_4 satisfies all the axioms (γ), (ρ), (ρ_f), (ρ_{st}), and (ρ_t).

In the next set of examples, the ground set X has size four. In these examples both the standard axiom (α) of contraction consistency and the axiom (ρ) of replacement consistency *always* hold. However, the standard axiom (γ) of expansion consistency and the other axioms of replacement consistency only hold in selected cases. Specifically, the odd numbered choice correspondences do satisfy (γ), hence they are rationalizable by Theorem 2.2. On the other hand, the even numbered choice correspondences violate axiom (γ).

Example 3.2 Let $X := \{x,y,z,w\}$ and $\Omega := 2^X \setminus \{\emptyset\}$. Further, let $c_i \colon \Omega \rightrightarrows X$ be the following five choice correspondences:

$$\begin{array}{lllll}
(c_5) & \underline{x}\,y,\ \underline{x}\,\underline{z},\ x\,\underline{w},\ \underline{y}\,\underline{z},\ y\,\underline{w},\ \underline{z}\,\underline{w}, & \underline{x}\,y\,\underline{z},\ x\,y\,\underline{w},\ x\,\underline{z}\,\underline{w},\ y\,\underline{z}\,\underline{w}, & x\,y\,\underline{z}\,\underline{w} \\
(c_6) & \underline{x}\,\underline{y},\ \underline{x}\,\underline{z},\ x\,\underline{w},\ \underline{y}\,\underline{z},\ y\,\underline{w},\ \underline{z}\,\underline{w}, & \underline{x}\,y\,\underline{z},\ x\,y\,\underline{w},\ x\,\underline{z}\,\underline{w},\ y\,\underline{z}\,\underline{w}, & x\,y\,\underline{z}\,\underline{w} \\
(c_7) & \underline{x}\,\underline{y},\ \underline{x}\,\underline{z},\ x\,\underline{w},\ \underline{y}\,z,\ \underline{y}\,\underline{w},\ \underline{z}\,\underline{w}, & \underline{x}\,\underline{y}\,z,\ x\,\underline{y}\,\underline{w},\ x\,\underline{z}\,\underline{w},\ \underline{y}\,z\,\underline{w}, & x\,\underline{y}\,z\,\underline{w} \\
(c_8) & \underline{x}\,\underline{y},\ \underline{x}\,\underline{z},\ \underline{x}\,\underline{w},\ \underline{y}\,\underline{z},\ \underline{y}\,\underline{w},\ \underline{z}\,\underline{w}, & \underline{x}\,\underline{y}\,z,\ x\,\underline{y}\,\underline{w},\ x\,\underline{z}\,\underline{w},\ \underline{y}\,z\,\underline{w}, & x\,\underline{y}\,z\,\underline{w} \\
(c_9) & \underline{x}\,y,\ \underline{x}\,\underline{z},\ \underline{x}\,\underline{w},\ \underline{y}\,\underline{z},\ \underline{y}\,\underline{w},\ \underline{z}\,\underline{w}, & \underline{x}\,y\,\underline{z},\ \underline{x}\,y\,\underline{w},\ \underline{x}\,\underline{z}\,\underline{w},\ \underline{y}\,\underline{z}\,\underline{w}, & \underline{x}\,y\,\underline{z}\,\underline{w}\,. \\
(c_{10}) & \underline{x}\,\underline{y},\ \underline{x}\,\underline{z},\ x\,\underline{w},\ \underline{y}\,\underline{z},\ \underline{y}\,\underline{w},\ \underline{z}\,\underline{w}, & \underline{x}\,\underline{y}\,\underline{z},\ x\,\underline{y}\,\underline{w},\ x\,\underline{z}\,\underline{w},\ y\,\underline{z}\,\underline{w}, & x\,y\,\underline{z}\,\underline{w}\,.
\end{array}$$

(5) The choice correspondence c_5 is rationalizable, and satisfies the replacement axioms (ρ) and (ρ_f). On the other hand, the axiom (ρ_{st}) of semitransitive replacement does not hold for c_5: indeed, for $S := \{y,z\}$ and $T := \{z,w\}$, we have $y \in c_5(S)$, $z \in S \cap c_5(T)$, $y \notin c_5(S \cup \{x\})$, and yet $x \notin c_5(T \cup \{x\})$. In view of Theorem 3.5 (proved in the next section), axiom (ρ_t) does not hold for c_5 either.

(6) It is easy to check that c_6 satisfies the standard axiom (ρ) of replacement consistency. Furthermore, the axiom (ρ_f) of Ferrers replacement holds for c_6 as well. (The verification of this last fact is a bit long, and is left to the reader.) On the other hand, c_6 satisfies neither (γ), nor (ρ_{st}), not (ρ_t). (For (γ), note that $y \in c(\{x,y\}) \cap c(\{y,z\})$ but $y \notin c(\{x,y,z\})$). For (ρ_{st}), take $S := \{y,z\}$ and $T := \{z,w\}$, and derive a contradiction. For (ρ_t), use Proposition 3.7(ii).)

(7) The choice correspondence c_7 is rationalizable, and satisfies the axioms of replacement (ρ) and (ρ_{st}). On the other hand, (ρ_f) does not hold for c_7: indeed, for $S := \{x,y\}$ and $T := \{z,w\}$, we have $x \in c_7(S)$, $y \in S$, $z \in c_7(T) \setminus c_7(T \cup \{y\})$, and yet $x \notin c_7(T \cup \{x\})$. In view of Theorem 3.5, axiom (ρ_t) does not hold for c_7 either.

(8) The choice correspondence c_8 is very similar to c_7, the only difference consisting of the images of $\{x, w\}$ and $\{y, z\}$. One can readily check that c_8 satisfies (α) but not (γ) (being $\succsim_{c_8} = X^2$). Furthermore, (ρ) holds for c_8, whereas (ρ_f), (ρ_{st}) and (ρ_t) do not.[18]

(9) It is long but straightforward to check that the rationalizable choice correspondence c_9 satisfies the three replacement axioms (ρ), (ρ_f) and (ρ_{st}). On the other hand, the axiom (ρ_t) of transitive replacement does not hold for c_9: indeed, for $S := \{y, z, w\}$, we have $y \in c_9(S)$ and $y \notin c_9(S \cup \{x\})$, however $c_9(S \cup \{x\}) = \{x, z, w\} \neq \{x\}$.

(10) The choice correspondence c_{10} satisfies both (ρ) and (ρ_{st}). (Again, the long verification that (ρ_{st}) holds is left to the reader.) On the other hand, neither (γ) nor (ρ_f) hold for c_{10}. (For (γ) observe that $y \in c(\{y, z\}) \cap c(\{y, w\})$ and $y \notin c(\{y, z, w\})$, whereas for (ρ_f) take $S := \{x, z\}$ and $T := \{y, w\}$.)

In order to obtain an instance of a rationalizable choice correspondence such that (ρ) holds but all of the other axioms of replacement consistency fail, we use a ground set of size five.

Example 3.3 Let $\succsim_{11}$ be the binary relation on $X := \{x_1, x_2, x_3, y_1, y_2\}$ defined by

$$\succsim_{11} \;:=\; X^2 \setminus \{(x_2, x_1), (x_3, x_1), (x_3, x_2), (y_2, y_1)\}.$$

Further, let $c_{11} \colon 2^X \setminus \{\emptyset\} \rightrightarrows X$ be the (total) choice correspondence induced by $\succsim_{11}$, i.e., $c_{11}(S) := \max(S, \succsim_{11})$ for each $\emptyset \neq S \subseteq X$. Using Theorem 3.5, it is immediate to check that c_{11} is a rationalizable choice correspondence satisfying only (ρ) among all axioms of replacement consistency.

The final example of this section exhibits an infinite choice correspondence satisfying (α) and (ρ_t), but such that neither (γ) nor (δ) hold for it.

Example 3.4 Let $X = Y \cup \{x_\infty\}$ be an infinite set of alternatives (where $x_\infty \notin Y$), and $\Omega := 2^X \setminus \{\emptyset\}$. Further, let $c_\infty \colon \Omega \rightrightarrows X$ be the choice correspondence defined as follows:

$$c_\infty(S) \;:=\; \begin{cases} S & \text{if } S \text{ is finite or } x_\infty \notin S \\ \{x_\infty\} & \text{otherwise.} \end{cases}$$

It is simple to check that axioms (α) and (ρ_t) hold for c_∞. By Proposition 3.7, all axioms of replacement consistency hold as well. Further, c_∞ satisfies the following minimal form of (γ):

$(\gamma)^-$**-axiom** (*singleton expansion consistency*)
$$\big[x \in c(S \cup \{y\}) \;\wedge\; x \in c(S \cup \{z\})\big] \quad \Longrightarrow \quad x \in c(S \cup \{y, z\})$$

where $x, y, z \in X$ and $S \in \Omega$. On the other hand, neither (γ) nor (δ) hold for c_∞. In fact, axiom (γ) does not even hold in its finite form (i.e., with two menus): indeed, for any fixed $y \in Y$, if $S := \{x_\infty, y\}$ and $T := Y$, then $y \in c_\infty(S) \cap c_\infty(T)$ and yet $y \notin c_\infty(S \cup T) = c_\infty(X)$. Furthermore, c_∞ does not satisfy axiom (δ), since for $S := \{x_\infty, y\}$ and $T := X$, we obtain $x_\infty, y \in c_\infty(S)$ and $S \subseteq T$, but $c_\infty(T) = \{x_\infty\}$. Finally, note that the hypothesis that X is infinite is necessary: indeed, by Proposition 3.8(ii), for any choice defined on a finite ground set, if (α) and (ρ_t) hold, then so does (β), and therefore (γ) and (δ) must hold as well.

[18] For instance, to show that c_8 does not satisfy (ρ_{st}) (with a change of variables, namely, y and z are exchanged) let $S := \{y, z\}$ e $T := \{y, w\}$. Then $z \in c(S)$, $y \in S$, $y \in c(T)$, $z \notin c(S \cup \{x\})$, and yet $x \notin c(T \cup \{x\})$.

In Table 1 we summarize the findings of Examples 3.1, 3.2, 3.3, and 3.4 in terms of axioms satisfied by the corresponding choices c_i. (Whenever an axiom holds, it is marked by a star.) The standard axiom (α) of contraction consistency is not mentioned, since it holds in all cases.

Choice	(γ)	(ρ)	(ρ_f)	(ρ_{st})	(ρ_t)
c_1					
c_2	*				
c_3		*	*	*	
c_4	*	*	*	*	*
c_5	*	*	*		
c_6		*	*		
c_7	*	*		*	
c_8		*			
c_9	*	*	*	*	
c_{10}		*		*	
c_{11}	*	*			
c_∞		*	*	*	*

Table 1: Axioms of consistency satisfied by the choices defined in Examples 3.1, 3.2, 3.3, and 3.4.

3.3 A unified characterization of classical forms of rationalizability

Now we link the structural properties of a rationalizable choice to those of its revealed preference. Specifically, the structural properties of a choice correspondence are expressed in terms of the satisfaction of suitable axioms of replacement consistency, whereas those of the associated revealed preferences are related to their degree of transitivity. An advantage of this approach to the topic is the fact that it systematically separates the issues related to the rationalizability of a choice (modeled by the satisfaction of axioms (α) and (γ)) from those related to the transitive structure of its revealed preference (modeled by the satisfaction of the axioms of replacement consistency).

In this section we conduct this analysis for some basic cases. In fact, here we only deal with the classical cases in which the relation of revealed preference is a total preorder, a total quasi-preorder, an interval order, and a semiorder (i.e, respectively, cases (A), (B), (C), and (D) mentioned in Section 2.2). We shall generalize this approach in Section 4, introducing the so-called weak (m, n)-Ferrers properties.

There are two reasons why we reserve a separate section to deal with cases (A)-(D). The first is related to the fact that total preorders, total quasi-preorders, interval orders, and semiorders are classical preference structures, which have been extensively studied in the literature, due to the wide range of their applications: see, e.g., Aleskerov *et al.* (2007), Fishburn (1970, 1985), Luce (1956), Pirlot and Vincke (1997).[19] The second is purely didactic, since dealing directly

[19] In a forthcoming paper, Giarlotta and Watson describe universal types of semiorders, called $\mathbb{Z}$-*products*, which are generalized lexicographic products of three total preorders such that the middle factor is the chain $(\mathbb{Z}, \geq)$ of integers equipped with a shift operator. These semiorders are universal in the sense that any semiorder embeds into a $\mathbb{Z}$-product (in fact, into a $\mathbb{Z}$-*line*, which is a $\mathbb{Z}$-product having linear orders as its two extreme factors).

with the general case of choices that are rationalizable by means of (m,n)-Ferrers revealed preferences would make the semantics of the related axioms of replacement consistency rather obscure.

Theorem 3.5 *Let $c\colon \Omega \rightrightarrows X$ be a rationalizable choice correspondence, and $\succsim_c$ its revealed preference. The following equivalences hold:*

(i) *$\succsim_c$ is quasi-transitive $\iff$ c satisfies axiom (ρ);*

(ii) *$\succsim_c$ is Ferrers $\iff$ c satisfies axiom (ρ_f);*

(iii) *$\succsim_c$ is semitransitive $\iff$ c satisfies axiom (ρ_{st});*

(iv) *$\succsim_c$ is transitive $\iff$ c satisfies axiom (ρ_t).*

Proof. Since c is rationalizable by hypothesis, Theorem 2.2 yields that c satisfies the standard consistency axioms (α) and (γ).

(i): For necessity, assume that c has a quasi-transitive revealed preference $\succsim_c$. We show that c satisfies the axiom of replacement (ρ). To that end, let $x, y \in X$ and $S \in \Omega$ be such that $y \in c(S)$ and $y \notin c(S \cup \{x\})$. Axiom (γ) yields $y \notin c(\{x,y\})$, hence $c(\{x,y\}) = \{x\}$. Now axiom (α) entails $y \notin c(T)$ for each $T \in \Omega_{xy}$. From the definition of revealed preference we obtain $x \succsim_c y$ and $\neg(y \succsim_c x)$, i.e., $x \succ_c y$. Further, we have $y \succsim_c z$ for each $z \in S$, because $y \in c(S)$. Since $\succsim_c$ is complete and quasi-transitive, it follows that $x \succsim_c z$ holds for each $z \in S \cup \{x\}$, i.e., $x \in c(S \cup \{x\})$. This proves that c satisfies axiom (ρ), as claimed.

Conversely, assume that c satisfies axiom (ρ). Since $\succsim_c$ is complete, to prove the quasi-transitivity of $\succsim_c$ is equivalent to show that $x \succsim_c y \succ_c z$ implies $x \succsim_c z$ for each $x, y, z \in X$. To that end, first observe that the definition of revealed preference and axiom (α) yield $x \in c(\{x,y\})$ and $c(\{y,z\}) = \{y\}$. Now, if $x \notin c(\{x,y,z\})$, then $z \in c(\{x,y,z\})$ by axiom (ρ), and so $z \in c(\{y,z\})$ by axiom (α), a contradiction. It follows that $x \in c(\{x,y,z\})$, which implies $x \succsim_c z$. Thus the claim holds.

(ii): For sufficiency, assume that (ρ_f) holds, and let $x, y, z, w \in X$ be such that $x \succsim_c y$ and $z \succsim_c w$. Suppose $z \succsim_c y$ fails, hence $y \succ_c z$ by the completeness of $\succsim_c$. The definition of revealed preference and axiom (α) yield $x \in c(\{x,y\})$, $z \in c(\{z,w\})$ and $z \notin c(\{y,z,w\})$. Now the axiom of replacement (ρ_f) entails $x \in c(\{x,z,w\})$, hence $x \succsim_c w$. This proves that $\succsim_c$ is Ferrers.

Conversely, assume that $\succsim_c$ is Ferrers. Let $x, y, z \in X$ and $S, T \in \Omega$ be such that $x \in c(S)$, $y \in S$, $z \in c(T)$ and $z \notin c(T \cup \{y\})$. We wish to show that $x \in c(T \cup \{x\}) = \max(T \cup \{x\}, \succsim_c)$, i.e., $x \succsim_c t$ for all $t \in T$. The hypotheses $x \in c(S)$ and $y \in S$ yields $x \succsim_c y$. Furthermore, $z \in c(T) = \max(T, \succsim_c)$ implies $z \succsim_c t$ for each $t \in T$. If $z \succsim_c y$ were to hold, then we would have $z \in c(\{z,y\})$ by axiom (α), and so since $z \in c(T)$ by hypothesis, we could derive $z \in c(T \cup \{y\})$ by axiom (γ), a contradiction. In summary, we have $x \succsim_c y$, $z \succsim_c t$ for all $t \in T$, and $\neg(z \succsim_c y)$. Now the Ferrers condition implies $x \succsim_c t$ for all $t \in T$. This proves that (ρ_f) holds for c.

(iii): For sufficiency, assume that (ρ_{st}) holds, and let $x, y, z, t \in X$ be such that $y \succsim_c z \succsim_c t$. To prove that $\succsim_c$ is semitransitive, we show that either $y \succsim_c x$ or $x \succsim_c t$ holds. The definition of revealed preference along with axiom (α) yields $y \in c(\{y,z\})$ and $z \in c(\{z,t\})$. If $y \succsim_c x$, then we are immediately done. Otherwise, we have $x \succ_c y$ by the completeness of $\succsim_c$, whence $y \notin c(\{x,y,z\})$. Now axiom (ρ_{st}) entails $x \in c(\{x,z,t\})$, and so $x \succsim_c t$ holds, as required.

Conversely, assume that $\succsim_c$ is semitransitive. Let $x, y, z \in X$ and $S, T \in \Omega$ be such that $y \in c(S)$, $z \in S \cap c(T)$, and $y \notin c(S \cup \{x\})$. We show that $x \in c(T \cup \{x\}) = \max(T \cup \{x\}, \succsim_c)$, that is, $x \succsim_c t$ for all $t \in T$. To start, we prove by contradiction that $y \succsim_c$ fails. Indeed, if $y \succsim_c x$ were to hold, then we would get $y \in c(\{x,y\})$ by axiom (α), and $y \in c(S \cup \{x\})$ by axiom (γ), which is impossible. Summarizing, we have $y \succsim_c z \succsim_c t$ for all $t \in T$, and $\neg(y \succsim_c x)$. Now semitransitivity yields $x \succsim_c t$ for all $t \in T$, thus proving that (ρ_{st}) holds for c.

(iv): To prove necessity, assume that the revealed preference $\succsim_c$ is transitive (hence a total preorder). By the fundamental theorem of revealed preference theory, it follows that c satisfies WARP. Thus, it suffices to show that axioms (α) and (β) imply axiom (ρ_t). Let $x, y \in X$ and $S \in \Omega$ be such that $y \in c(S)$ and $y \notin c(S \cup \{x\})$. To prove the claim, we show that the equality $c(S \cup \{x\}) = \{x\}$ holds. Since c cannot be empty-valued, it suffices to show that no element of S is chosen in $S \cup \{x\}$. Toward a contradiction, assume that there exists $z \in S$ such that $z \in c(S \cup \{x\})$. Axiom (α) yields $z \in c(S)$. Since $y \in c(S)$ and $y \notin c(S \cup \{x\})$, axiom (β) implies $z \notin c(S \cup \{x\})$, a contradiction.

For sufficiency, assume that the axiom (ρ_t) of transitive replacement consistency holds. Let $x, y, z \in X$ be such that $x \succsim_c y \succsim_c z$. Axiom (α) yields $x \in c(\{x,y\})$ and $y \in c(\{y,z\})$. If $x \notin c(\{x,y,z\})$, then axiom (ρ_t) implies $c(\{x,y,z\}) = \{z\}$. Since the last equality contradicts $y \in c(\{y,z\})$, it follows that $x \in c(\{x,y,z\})$, hence $x \succsim_c z$. This completes the proof of (iv). ☐

As an immediate consequence of Theorems 2.2 and 3.5, we obtain a partial taxonomy of rationalizable choices in terms of (classical and new) axioms of consistency.

Corollary 3.6 *The following equivalences hold for an arbitrary choice correspondence c:*

(i) *c is quasi-transitively rationalizable $\iff$ axioms (α), (γ), and (ρ) hold;*

(ii) *c is interval order rationalizable $\iff$ axioms (α), (γ), and (ρ_f) hold;*

(iii) *c is semiorder rationalizable $\iff$ axioms (α), (γ), (ρ_f), and (ρ_{st}) hold;*

(iv) *c is transitively rationalizable $\iff$ axioms (α), (γ), and (ρ_t) hold.*

The next result summarizes the connections among axioms of replacement consistency, both in the general case and under the standard axiom (α) of contraction consistency.

Proposition 3.7 *The following implications hold:*

(i) *$(\rho_f) \Longrightarrow (\rho)$ and $(\rho_{st}) \Longrightarrow (\rho)$;*

(ii) *under axiom (α), $(\rho_t) \Longrightarrow (\rho_f)$ and $(\rho_t) \Longrightarrow (\rho_{st})$.*

All reverse implications fail.

Proof. Let $c\colon \Omega \rightrightarrows X$ be a choice correspondence.

(i): To prove the implication "$(\rho_f) \Rightarrow (\rho)$", rewrite axiom (ρ) with a change of variables as follows: if $z \in c(T)$ and $z \notin c(T \cup \{x\})$, then $x \in c(T \cup \{x\})$. Now apply axiom (ρ_f) for $x = y$ and $S = \{x\}$ to conclude that (ρ) holds. The implication "$(\rho_{st}) \Rightarrow (\rho)$" can be proved similarly, by letting $y = z$ and $S = T$ in the statement of (ρ_{st}).

(ii): Assume that (α) holds. First we prove "$(\rho_t) \Rightarrow (\rho_f)$". Toward a contradiction, assume that (ρ_t) holds whereas (ρ_f) does not. Therefore, there exist $x, y, z \in X$ and $S, T \in \Omega$ such that (1) $x \in c(S)$, (2) $y \in S$, (3) $z \in c(T)$, (4) $z \notin c(T \cup \{y\})$, and (5) $x \notin c(T \cup \{x\})$. By axiom

(α), (1)&(2) yields (6) $x \in c(\{x, y\})$, whereas (3)&(4) imply (7) $z \notin c(T \cup \{x, y\})$. On the other hand, since axiom (ρ_t) holds, (3)&(5) implies (8) $z \in c(T \cup \{x\})$, which in turn, by (4), implies (9) $x \neq y$. Another application of axiom (ρ_t) to (7)&(8) yields $c(T \cup \{x, y\}) = \{y\}$. Now a final application of axiom (α) entails $c(\{x, y\}) = \{y\}$, which, by (9), contradicts (6). This proves the first implication. For the second implication, assume by contradiction that (ρ_t) holds whereas (ρ_{st}) does not. Therefore, there exist $x, y, z \in X$ and $S, T \in \Omega$ such that (1) $y \in c(S)$, (2) $z \in S$, (3) $z \in c(T)$, (4) $y \notin c(S \cup \{x\})$, and (5) $x \notin c(T \cup \{x\})$. Since (ρ_t) holds, (1)&(4) implies (6) $c(S \cup \{x\}) = \{x\}$, whereas (3)&(5) implies (7) $z \in c(T \cup \{x\})$. The latter together with (5) implies (8) $x \neq z$. Now use axiom (α) to deduce that (2)&(6)&(8) implies $z \notin c(\{x, z\})$, whereas (7) implies $z \in c(\{x, z\})$. This contradiction completes the proof of part (ii).

Finally, Example 3.2 shows that all reverse implications do not hold. (In fact, the reverse of the implications in (i) does not even hold under axiom (α).) $\square$

It is worth emphasizing the difference between Theorem 3.5 and Proposition 3.7. In fact, the former characterizes forms of rationalizable choice correspondences on the basis of the internal structure of the revealed preference. On the other hand, the latter simply states implications among axioms, which hold regardless of rationalizability. In the same spirit as Proposition 3.7, the next result examines the relationship between two axioms of replacement consistency (namely, (ρ) and (ρ_{t})) and two axioms of expansion consistency (namely, (β) and (δ)), provided that the standard axiom (α) of contraction consistency holds.

Proposition 3.8 *Under axiom (α), the following implications hold:*

(i) $(\beta) \Longrightarrow (\rho_t)$;

(ii) $(\rho_t) \Longrightarrow (\beta)$ *for finite choices;*

(iii) $(\delta) \,\&\, (\gamma) \Longrightarrow (\rho)$.

Parts (ii) *and* (iii) *do not hold if we drop finiteness and satisfaction of (γ), respectively.*

PROOF. Part (i) is an immediate consequence of Theorems 2.3 and 3.5(iv).

For part (ii), let $c\colon \Omega \rightrightarrows X$ be a finite choice correspondence satisfying axioms (α) and (ρ_t). Toward a contradiction, assume that (β) does not hold. Then there exist $x, y \in X$ and $S, T \in \Omega = [X]^{\mathrm{fin}}$ such that

$$x \neq y, \quad x, y \in c(S), \quad S \subseteq T, \quad x \in c(T), \quad y \notin c(T) \tag{1}$$

hold. By axiom (α) and the finiteness of Ω, we can assume that T is minimal in satisfying (1), in the sense that $x, y \in c(T')$ holds for every T' such that $S \subseteq T' \subsetneqq T$. It follows that $T = T' \cup \{z\}$ for some $T' \supseteq S$ and $z \notin T'$, where $x, y \in c(T')$. Since $y \notin c(T' \cup \{z\})$, axiom (ρ_t) yields $c(T) = c(T' \cup \{z\}) = \{z\}$, and therefore $x = z$, which contradicts (1).

Part (iii) is an immediate consequence of Theorems 2.4 and 3.5(i).

Since (β) implies (δ), Example 3.4 shows that the assumption of finiteness in (ii) is needed. Further, Example 3.10 proves that the satisfaction of (γ) in (iii) cannot be dropped. $\square$

Proposition 3.8(ii) and the fundamental theorem of revealed preference theory readily yield the following characterization of the transitive rationalizability of finite choices.

Corollary 3.9 *A finite choice correspondence is transitively rationalizable if and only if axioms* (α) *and* (ρ_t) *hold.*

Finally, we show that (α) and (δ) do not imply (ρ), even under the assumption that the ground set is finite.

Example 3.10 Let $X := \{x, y, z, w\}$, $\Omega := 2^X \setminus \{\emptyset\}$, and $c \colon \Omega \rightrightarrows X$ be the choice correspondence defined by

$$c(S) := \begin{cases} S \setminus \{x\} & \text{if } S \neq \{x\} \text{ and } S \neq X \\ \{x\} & \text{if } S = \{x\} \\ \{z, w\} & \text{if } S = X\,. \end{cases}$$

One can readily check that axioms (α) and (δ) hold for c. In addition, for $S := \{y, z, w\}$, we have $y \in c(S)$, $y \notin c(S \cup \{x\})$, and $x \notin c(S \cup \{x\})$. Thus (ρ) fails for c.

4 The (m, n)-rationalizability of a choice

In this section we extend the approach of Section 3, introducing additional axioms of choice replacement and characterizing non-standard types of rationalizability. Specifically, we obtain a taxonomy of rationalizable choices c having a transitive revealed strict preference $\succ_c$ and a possibly intransitive revealed indifference $\sim_c$. To this aim, we employ the recently introduced weak (m, n)-Ferrers properties, which generalize the classical Ferrers property as well as the two properties of quasi-transitivity and semitransitivity.

Weak (m, n)-Ferrers properties describe a finite taxonomy of quasi-transitive preferences having a possibly intransitive indifference, and are therefore interesting for economic theory *per se*, insofar as they provide a systematic approximation to the much celebrated property of transitivity by discrete levels of pseudo-transitivity. However, their economic significance is not limited to a pure abstraction of the notion of transitivity. For instance, here we show that each weak (m, n)-Ferrers property can be viewed as the statement that a money-pump of a particular kind does not exist. In view of the above considerations, the notion of (m, n)-rationalizable choice sheds light on suitable features of rationality displayed by an economic agent's behavior.

4.1 Weak (m, n)-Ferrers properties and levels of transitivity

To start, let us recall the notion of (m, n)-Ferrers properties. The original idea is due to Öztürk (2008), who however deals with (m, n)-Ferrers properties for asymmetric binary relations. More recently, Giarlotta and Watson (2014) distinguish two versions – *weak* and *strict* – of (m, n)-Ferrers properties, and provide a complete classification of the weak ones.

Definition 4.1 Let $\succsim$ be a weak preference (i.e., a reflexive binary relation) on X. For each pair (m, n) of integers such that $m \geq n \geq 1$, we say that $\succsim$ is *weakly* (m, n)*-Ferrers* (or, simply, (m, n)*-Ferrers*) if for each $x_1, \ldots, x_m, y_1, \ldots, y_n \in X$, the following implication holds:

$$\big[(x_1 \succsim \ldots \succsim x_m) \wedge (y_1 \succsim \ldots \succsim y_n)\big] \quad \Longrightarrow \quad \big[(x_1 \succsim y_n) \vee (y_1 \succsim x_m)\big]. \tag{2}$$

Similarly, we say that $\succsim$ is *strictly* (m, n)*-Ferrers* if its asymmetric part $\succ$ satisfies the implication (2), with $\succ$ in place of $\succsim$.

Note that the assumption "$m \geq n$" causes no loss of generality in Definition 4.1, since (in its absence) $\succsim$ is weakly/strictly (m, n)-Ferrers if and only if it is weakly/strictly (n, m)-Ferrers. The next result, which is a reformulation of Lemma 2.6 in Giarlotta and Watson (2014), points out an almost dual behavior of weak and strict Ferrers properties.

Lemma 4.2 *Let $\succsim$ be a reflexive binary relation on X. For all positive integers m, n, m', n' such that $m \geq n$, $m' \geq n'$, $m \geq m'$, and $n \geq n'$, we have:*

(i) *if $\succsim$ is (weakly) (m, n)-Ferrers, then it is (weakly) (m', n')-Ferrers;*

(ii) *if $\succsim$ is strictly (m', n')-Ferrers, with $m' + n' \geq 3$, and $\succ$ is transitive, then $\succsim$ is strictly (m, n)-Ferrers.*

Let us emphasize again that in this paper we only deal with weak (m, n)-Ferrers properties.[20] That is why we shall omit the adjective "weak" throughout, and henceforth simply speak of (m, n)-Ferrers properties.

Remark 4.3 As pointed out in Giarlotta and Watson (2014), a weak preference is:
- $(1, 1)$-Ferrers if and only if it is complete;
- $(2, 1)$-Ferrers if and only if it is a total quasi-preorder;
- $(2, 2)$-Ferrers if and only if it is an interval order;
- $(2, 2)$-Ferrers and $(3, 1)$-Ferrers if and only if it is a semiorder;
- $(3, 2)$-Ferrers if and only if (by definition) it is a *strong interval order*;
- $(3, 2)$-Ferrers and $(4, 1)$-Ferrers if and only if (by definition) it is a *strong semiorder*.[21]

The following result summarizes one of the main findings in Giarlotta and Watson (2014):[22]

Theorem 4.4 *The following equivalences/implications between (m, n)-Ferrers properties hold:*
- $(5, 1)$-*Ferrers* $\iff$ $(m, 1)$-*Ferrers for all* $m \geq 5$;
- $(4, 2)$-*Ferrers* $\iff$ $(m, 2)$-*Ferrers for all* $m \geq 4$;
- $(4, 2)$-*Ferrers* $\Longrightarrow$ $(5, 1)$-*Ferrers;*
- $(3, 3)$-*Ferrers* $\iff$ *transitivity* $\iff$ (m, n)-*Ferrers for each* $m \geq n \geq 3$.

In view of Lemma 4.2, Remark 4.3, and Theorem 4.4 (and some other results), it suffices to study (m, n)-Ferrers properties for (m, n) belonging to the following set of pairs:

$$\mathcal{F} \ := \ \{(1,1), (2,1), (3,1), (4,1), (5,1), (2,2), (3,2), (4,2), (3,3)\}.$$

Figure 1 in the Appendix describes all (non-reversible) implications among all relevant combinations of (m, n)-Ferrers properties.

Before introducing the main notion of this paper – that is, (m, n)-rationalizable choices (see Section 4.4) – we shall conduct a further analysis of (m, n)-Ferrers properties for each $(m, n) \in F$. Specifically:

---

[20] Loosely speaking, given a reflexive and complete binary relation $\succsim$, its weak Ferrers properties assume the transitivity of $\succ$, and are connected to levels of transitivity of the associated indifference $\sim$. On the other hand, strict Ferrers properties of $\succsim$ only deal with levels of transitivity of the associated strict preference $\succ$. It follows that the problem of rationalizable choices whose revealed preference is *strictly* (m, n)-Ferrers requires a quite different analysis. Due to its complexity, we do not address this problem here.

[21] The notions of "strong interval order" and "strong semiorder" are introduced in Giarlotta (2014).

[22] This is Theorem 3.1 in the mentioned paper. See also Examples 3.3-3.10 and Figures 1-3 in the same paper.

- we determine the minimal preference relations that witness the failure of each (m,n)-Ferrers property, while ensuring the satisfaction of all weaker properties (Section 4.2);
- we describe how the satisfaction of (m,n)-Ferrers properties allows one to prevent money-pump phenomena of a certain kind (Section 4.3).

4.2 Forbidden characteristic configurations for (m,n)-Ferrers properties

Here we construct a preference relation $\succsim_{m,n}$ for each $(m,n) \in \mathcal{F}$: this relation is a "forbidden characteristic configuration (FCC)" for the respective (m,n)-Ferrers property. These configurations are unique, up to isomorphisms. Although FCC's are not strictly necessary for the proof of the main result of Section 4 (Theorem 4.14), they are quite useful and of independent (mathematical and economic) interest: see Section 4.3.

Definition 4.5 Fix $m \geq n \geq 1$, with $(m,n) \neq (1,1)$ and $m+n \leq 6$. A reflexive and acyclic[23] binary relation $\succsim$ is a *forbidden characteristic configuration (FCC) for the (m,n)-Ferrers property* if the following conditions hold:

- $\succsim$ fails to be (m,n)-Ferrers;
- if $\succsim'$ is a reflexive and acyclic relation that fails to be (m,n)-Ferrers but is (m',n')-Ferrers for each $m' \leq m$ and $n' \leq n$ with $(m',n') \neq (m,n)$, then $\succsim'$ contains an isomorphic copy of $\succsim$.[24]

For the base case $(m,n)=(1,1)$, a reflexive and acyclic relation $\succsim$ is an *FCC for the $(1,1)$-Ferrers property* if it is not complete, and is defined on a set of minimal size.

Said differently, a forbidden characteristic configuration for the (m,n)-Ferrers property is an acyclic weak preference (which is also complete, with the only exception of the base case of the $(1,1)$-Ferrers property) defined on a minimal set, witnessing the failure of the (m,n)-Ferrers property. We now identify all the preference relations that are relevant to our analysis.

Definition 4.6 Let $X_{m,n} := \{x_1,\ldots,x_m,y_1,\ldots,y_n\}$, with $m \geq n \geq 0$ and $|X_{m,n}| = m+n$. For each $(m,n) \in \mathcal{F}$, define the following binary relations $\succsim_{m,n}$:

$$
\begin{array}{lcl}
\succsim_{1,1} & := & X_{1,1} \times X_{1,1} \setminus \{(x_1,y_1),(y_1,x_1)\} \\
\succsim_{2,1} & := & X_{2,1} \times X_{2,1} \setminus \{(x_1,y_1),(y_1,x_2)\} \\
\succsim_{2,2} & := & X_{2,2} \times X_{2,2} \setminus \{(x_1,y_2),(y_1,x_2)\} \\
\succsim_{3,1} & := & X_{3,1} \times X_{3,1} \setminus \{(x_1,x_3),(x_1,y_1),(y_1,x_3)\} \\
\succsim_{3,2} & := & X_{3,2} \times X_{3,2} \setminus \{(x_1,x_3),(x_1,y_2),(y_1,x_3)\} \\
\succsim_{3,3} & := & X_{3,0} \times X_{3,0} \setminus \{(x_1,x_3)\} \\
\succsim_{4,1} & := & X_{4,1} \times X_{4,1} \setminus \{(x_1,x_3),(x_2,x_4),(x_1,x_4),(x_1,y_1),(y_1,x_4)\} \\
\succsim_{4,2} & := & X_{4,0} \times X_{4,0} \setminus \{(x_1,x_3),(x_2,x_4),(x_1,x_4)\} \\
\succsim_{5,1} & := & X_{5,0} \times X_{5,0} \setminus \{(x_1,x_3),(x_2,x_4),(x_3,x_5),(x_1,x_4),(x_2,x_5),(x_1,x_5)\}.
\end{array}
$$

[23] The required condition of acyclicity is by no means restrictive, since we are dealing with preference relations revealed by choice correspondences (which are always acyclic, as long as the choice domain is finitely complete).

[24] Given two preference relations $\succsim$ (on X) and $\succsim'$ (on X'), we say that $\succsim'$ contains an *isomorphic copy* of $\succsim$ if there is an injective map $f\colon X \to X'$ such that for all $x,y \in X$, we have $x \succsim y$ if and only if $f(x) \succsim' f(y)$.

The main result of this section shows that for each relevant pair (m,n) of integers, the relation $\succsim_{m,n}$ is *the* FCC for the (m,n)-Ferrers property. (We use "the" since FCC's are unique up to isomorphisms.) Before proving it, let us preliminarily observe that – contrary to what one might expect – the three relations $\succsim_{3,3}$, $\succsim_{4,2}$, and $\succsim_{5,1}$ are not defined on, respectively, $X_{3,3}$, $X_{4,2}$, and $X_{5,1}$. Indeed, as we shall see in the proof of Lemma 4.7, the minimality condition of the FCC requires one to use some of the x_i's as y_j's.

Lemma 4.7 *For each $(m,n) \in \mathcal{F}$, the relation $\succsim_{m,n}$ is the FCC for the (m,n)-Ferrers property.*

Proof. The result is trivial for the base case $(m,n) := (1,1)$. Next, let $(m,n) \in \mathcal{F} \setminus \{(1,1)\}$. It is not hard to check that $\succsim_{m,n}$ fails to be (m,n)-Ferrers, but is (m',n')-Ferrers for each $m' \leq m$ and $n' \leq n$ such that $(m',n') \neq (m,n)$. Hence, to prove the lemma, it is enough to show that, for each $(m,n) \in \mathcal{F} \setminus \{(1,1)\}$, if $\succsim$ is a minimal preference relation that fails to be (m,n)-Ferrers, but is (m',n')-Ferrers for each $m' \leq m$ and $n' \leq n$ such that $(m',n') \neq (m,n)$, then $\succsim$ is isomorphic to $\succsim_{m,n}$.

We only show that the above property holds for the pair $(4,2) \in \mathcal{F}$, and leave the remaining cases to the reader. To that end, we develop a formal proof technique for the case of $(4,2)$-Ferrers, which can be adapted to the other Ferrers properties without much difficulty. Let $\succsim$ be a minimal preference relation which fails to be $(4,2)$-Ferrers, but is (m,n)-Ferrers for $(m,n) \in \{(4,1),(3,2)\}$ (hence also for $(m,n) \in \{(1,1),(2,1),(3,1),(2,2)\}$). Since $\succsim$ fails to be $(4,2)$-Ferrers, there exist elements p,q,r,s,t,u (not necessarily distinct) such that

$$p \succsim q \succsim r \succsim s\,, \quad t \succsim u\,, \quad u \succ p\,, \quad s \succ t \tag{3}$$

hold. Set $X := \{p,q,r,s,t,u\}$. By exploiting the relationships (3) and the Ferrers properties satisfied by $\succsim$, we shall see that one can infer whether $(a,b) \in \succsim$ or not, for all $a,b \in X$. To enhance readability, we make use of deduction rules of the following two types:

$$\langle w_1,\ldots,w_m \,|\, z_1,\ldots,z_n\rangle \;\to\; a \succsim b \qquad \text{and} \qquad \langle w_1,\ldots,w_m \,|\, z_1,\ldots,z_n\rangle \;\to\; b \succ a.$$

More specifically, by an application of a rule of type

$$\langle w_1,\ldots,w_m \,|\, z_1,\ldots,z_n\rangle \;\to\; a \succsim b,$$

we mean a deduction of the following form:

> "since $w_1 \succsim \ldots \succsim w_m$, $z_1 \succsim \ldots \succsim z_n$, and $a' \succ b'$ hold, then, by the (m,n)-Ferrers property, $a \succsim b$ must hold as well",

where $\{(a,b),(b',a')\} = \{(w_1,z_n),(z_1,w_n)\}$. Likewise, by an application of a rule of type

$$\langle w_1,\ldots,w_m \,|\, z_1,\ldots,z_n\rangle \;\to\; b \succ a,$$

where $(a,b) \in \{(w_i,w_{i+1}) : 1 \leq i < m\} \cup \{(z_j,z_{j+1}) : 1 \leq j < n\}$, we mean a deduction of the following form:

> "if $a \succsim b$ held, then $\langle w_1,\ldots,w_m \,|\, z_1,\ldots,z_n\rangle$ would be a counterexample to the (m,n)-Ferrers property (in the sense that we would have $w_1 \succsim \ldots \succsim w_m$, $z_1 \succsim \ldots \succsim z_n$, $z_n \succ w_1$, and $w_m \succ z_1$); hence, by the $(1,1)$-Ferrers property, we must have $b \succ a$".

By the minimality of $\succsim$, an application of the following deduction rules (in the given order, starting from the first column)

$$
\begin{array}{rclcrcl}
\langle p,q\,|\,u\rangle & \to & u \succsim q & \qquad & \langle p,q,r,s\,|\,u\rangle & \to & u \succsim s \\
\langle r,s\,|\,t\rangle & \to & r \succsim t & & \langle p,q,r,s\,|\,t\rangle & \to & p \succsim t \\
\langle t,u\,|\,p\rangle & \to & t \succsim p & & \langle u,r,s\,|\,t\rangle & \to & u \succsim t \\
\langle t,u\,|\,s\rangle & \to & s \succsim u & & \langle p,s\,|\,t,u\rangle & \to & s \succ p \\
\langle p,q\,|\,t,u\rangle & \to & t \succsim q & & \langle p,r,s\,|\,t,u\rangle & \to & r \succ p \\
\langle r,s\,|\,t,u\rangle & \to & r \succsim u & & \langle p,q,s\,|\,t,u\rangle & \to & s \succ q \\
\langle p,q,r\,|\,u\rangle & \to & u \succsim r & & \langle q,r\,|\,s\rangle & \to & s \succsim r \\
\langle q,r,s\,|\,t\rangle & \to & q \succsim t & & \langle p,q\,|\,r\rangle & \to & r \succsim q \\
\langle p,q,r\,|\,t,u\rangle & \to & t \succsim r & & \langle q,r\,|\,p\rangle & \to & q \succsim p \\
\langle q,r,s\,|\,t,u\rangle & \to & q \succsim u & & & &
\end{array}
$$

yields that $\succsim\, = X^2 \setminus \{(p,r),(q,s),(p,s),(p,u),(t,s)\}$. Define an equivalence relation $\equiv$ on X by

$$a \equiv b \quad \overset{\text{def}}{\Longleftrightarrow} \quad (\forall x \in X)\,(a \succsim x \Longleftrightarrow b \succsim x) \;\wedge\; (\forall x \in X)\,(x \succsim a \Longleftrightarrow x \succsim b). \tag{4}$$

Plainly, if $a = b$, then $a \equiv b$. In addition, it is not hard to see that the quotient relation $\succsim/\!\equiv$, just as $\succsim$, fails to be $(4,2)$-Ferrers, but is $(4,1)$-Ferrers and $(3,2)$-Ferrers.[25] Therefore, by the minimality of $\succsim$, for all $a, b \in X$, we have that $a = b$ if and only if $a \equiv b$, which implies $\succsim\, = \,\succsim/\!\equiv$. Thus, since the $\equiv$-equivalence classes are $\{p\}, \{q,t\}, \{r,u\}, \{s\}$, it follows that p,q,r,s are pairwise distinct, and the equality $\succsim\, = \{p,q,r,s\}^2 \setminus \{(p,r),(q,s),(p,s)\}$ holds. This proves that $\succsim$ is isomorphic to $\succsim_{4,2}$, as claimed. □

4.3 (m,n)-Ferrers properties and money-pump avoidance

In this section we give an economic interpretation of (m,n)-Ferrers properties, which is connected to money-pump phenomena. To that aim, we introduce a simple model of transactions of goods that is well suited to describe the semantics of Ferrers properties. In fact, we show that in this model, whenever the binary relation modeling the agent's preference structure satisfies a fixed (m,n)-Ferrers property, there exists a strategy that prevents the agent from getting involved in mixed indifference/strict preference cycles of a certain type. To start, we describe the model.

Suppose that in order to convince an economic agent to exchange a certain good with a different one that is indifferent to the former, the agent is offered a *base compensation* B. Suppose also that in case of consecutive exchanges of indifferent goods, the compensation may not equal to the base compensation, but is instead equal to the compensation of the previous transaction times an *indifference factor* $r > 0$.[26] Thus, for example, if three consecutive transaction of

[25] This is a special case of the following more general result: for each acyclic weak preference $\succsim$ and each $m \geq n \geq 1$ such that $(m,n) \neq (1,1)$ and $m+n \leq 6$, the relation $\succsim$ is (m,n)-Ferrers if and only if so is $\succsim / \equiv$ (where $\equiv$ is defined as in (4)).

[26] We assume that the agent might only be sensitive to *consecutive* transactions of indifferent goods. This can be explained, for instance, by the possibility that transactions take place over a long period of time, and the economic agent remembers only the last one. Note that the case of a constant compensation is obtained for $r = 1$.

indifferent goods are made, then the compensation is B for the first, rB for the second, and r^2B for the third. It follows that if no other transactions of this kind are made, then the total compensation R received by the economic agent will be $R = (1 + r + r^2)B$.

The indifference factor r of this model has a natural economic interpretation, since it may be viewed as either a "discount" or a "penalty", depending on the agent's attitude. Specifically, a factor $0 < r < 1$ acts as a discount for consecutive transactions of indifferent goods, because the agent is willing to receive less and less money at each successive switch. This circumstance may be due to the consideration that she is getting paid a positive amount of money versus leaving her total utility basically unchanged. On the other hand, the case $r > 1$ is typical of a situation in which the agent, worried about the possibility of getting trapped into a money-pump cycle, gets more and more suspicious at the proposal of consecutive switches of indifferent goods. As a consequence, at each immediately successive transaction of this kind, she shall ask for a compensation that is larger than the previous one. The intermediate case $r = 1$ identifies a *neutral* agent, who is neither overconfident nor suspicious.

Finally, suppose that in order to switch from a good to a different good that is strictly preferred to the former, the economic agent is willing to pay a certain amount of money, which is a percentage s of the base compensation B, that is, sB: we call s the *(strict) preference factor*. Thus, for instance, if two transactions of this kind are made, then the price S paid by the economic agent is $S = 2sB$. Note that in the described model, additional transactions of strictly preferred goods are all done at the same price, regardless of whether they are consecutive or not. The reason for distinguishing the way we treat consecutive types of transactions – namely, indifferent goods vs. strictly preferred goods – is that in the latter case it is reasonable to assume that discount/penalty phenomena do not arise.[27]

We aim at describing how in this model of transactions the economic agent can avoid getting trapped into a money-pump cycle. This is equivalent to determining under which conditions we shall have $R \geq S$: the total compensation R received for the transactions of indifferent goods must be greater or equal than the total price S paid in transacting for better goods. The next example gives an instance of a money-pump phenomenon in such a model, due to the presence of a certain kind of mixed cycle.

Example 4.8 Assume that the economic agent displays the following mixed cycle of (four) indifferences and (two) strict preferences:

$$x_1 \sim x_2 \sim x_3 \sim x_4 \succ y_1 \sim y_2 \succ x_1. \tag{5}$$

As a natural requirement of a rational economic behavior, if the agent is starting with the good x_1 in her hands, then she should at least make sure that exchanging goods according to certain rules (possibly decided in advance by a contract) will not bring her back to x_1 with less money than she had before starting all transactions. For simplicity, let the base cost B equal to 1 (euro, dollar, etc.). Now assume that the agent is so naive/overconfident to agree to be paid

[27] The rationale for such an assumption is that – differently from what happens in the case of transactions of indifferent goods – there is no ambiguity for strict preference transactions. This is due to the fact that quasi-transitivity holds starting from a $(2, 1)$-Ferrers weak preference, and so we never switch to a different type of preference relation in case of transactions of strictly preferred goods. On the contrary, a sequence of consecutive transactions of indifferent goods may hiddenly lead to a strict preference of the last good over the first one (or vice versa), unless the economic agent's preference structure is modeled by a total preorder (a $(3, 3)$-Ferrers binary relation). At any rate, no discount/penalty phenomenon appears in the case of a neutral agent, who treats all consecutive transactions of (either indifferent or strictly preferred) goods with the same logic.

for indifference switches at a discount factor $r = 0.5$, and instead to pay each strict preference switch at a premium factor $s = 1.4$. Then she will loose money in going from x_1 back to x_1 by a sequence of transactions, passing through y_2, y_1, x_4, x_3, and x_2 (in the given order). In fact, she will receive $R = 1 + 1 + 0.5 + 0.25$ as a total compensation for the indifference switches, but pay a total price $S = 1.4 + 1.4$ for the two strict preference switches, thus enduring a net loss $R - S = -0.05$, and, in turn, triggering a money-pump process. However, if the agent knows in advance that her preference structure is $(4,2)$-Ferrers, then she will not run such a risk, since a configuration as (5) is forbidden, and any mixed cycle with two strict preferences and more indifferences[28] is not risky, since it gives a net profit of at least $R - S = 2.875 - 2.8 = 0.075$.

Example 4.8 is a simple instance of how the satisfaction of the $(4,2)$-Ferrers property may allow one to avoid money-pump phenomena of a certain kind. In the remainder of this section, we perform a similar analysis for each (m,n)-Ferrers property, however only considering the special cases of avoidance of money-pump phenomena arising from mixed indifference/strict preference cycles having *exactly two* strict preferences. A complete analysis of how (m,n)-Ferrers properties may allow an economic agent to avoid money-pump phenomena deriving from arbitrary mixed cycles appears to be doable but quite technical, and exceeds the scope of the present paper.[29]

Definition 4.9 (*Money-Pump Avoidance*) Let $\mathsf{MPA}(r,s)$ stand for the following property: "If r is the indifference factor and s is the strict preference factor, then there are no money-pump phenomena deriving from the presence of mixed cycles with exactly two strict preferences."

Within the transaction model under examination, the next result connects each forbidden characteristic configuration $\precsim_{m,n}$ for the (m,n)-Ferrers property (see Definition 4.6) with the avoidance of special money-pump phenomena.

Proposition 4.10 *For each $(m,n) \in \mathcal{F} \setminus \{(1,1)\}$, we have:*

(i) $\precsim_{2,1}$ *satisfies* $\mathsf{MPA}(r,s) \iff s \le \frac{1}{2}$;

(ii) $\precsim_{3,1}$ *satisfies* $\mathsf{MPA}(r,s) \iff s \le \frac{1+r}{2}$;

(iii) $\precsim_{4,1}$ *satisfies* $\mathsf{MPA}(r,s) \iff s \le \frac{1+r+r^2}{2}$;

(iv) $\precsim_{5,1}$ *satisfies* $\mathsf{MPA}(r,s) \iff s \le \frac{1+r+r^2+r^3}{2}$;

(v) $\precsim_{2,2}$ *satisfies* $\mathsf{MPA}(r,s) \iff s \le 1$;

(vi) $\precsim_{3,2}$ *satisfies* $\mathsf{MPA}(r,s) \iff s \le 1 + \frac{r}{2}$;

(vii) $\precsim_{4,2}$ *satisfies* $\mathsf{MPA}(r,s) \iff s \le 1 + \frac{r+r^2}{2}$;

(viii) $\precsim_{3,3}$ *satisfies* $\mathsf{MPA}(r,s) \iff s \le 1 + r$.

[28] Mixed cycles with two strict preferences and fewer indifferences are forbidden by the fact that the $(4,2)$-Ferrers property implies the (m,n)-Ferrers property for each $m \le 4$ and $n \le 2$: see Theorem 4.4.

[29] See, however, Remark 4.11.

Proof. Assume without loss of generality that the base compensation is $B = 1$. For each $(m,n) \in \mathcal{F} \setminus \{(1,1)\}$, denote by $R_{m,n}$ and $S_{m,n}$ the total compensation and the total price, respectively, connected to $\succsim_{m,n}$. Since each FCC contains exactly two strict preferences, we have $S_{m,n} = 2s$ for all $(m,n) \in \mathcal{F} \setminus \{(1,1)\}$. On the other hand, the value of the total compensation $R_{m,n}$ is the following:

$$\begin{array}{rcl} R_{2,1} & = & 1 \\ R_{3,1} & = & 1+r \\ R_{4,1} & = & 1+r+r^2 \\ R_{5,1} & = & 1+r+r^2+r^3 \\ R_{2,2} & = & 2 \\ R_{3,2} & = & 2+r \\ R_{4,2} & = & 2+r+r^2 \\ R_{3,3} & = & 2+2r. \end{array}$$

Since the satisfaction of $\mathsf{MPA}(r,s)$ is equivalent to requiring that $R_{m,n} \geq S_{m,n} = 2s$, a simple computation shows that the claim holds in all cases. □

Proposition 4.10 has an immediate economic interpretation in terms of the relationship between the level of transitivity of an economic agent's preference structure on one hand, and the caution that she has to exercise whenever indulging in certain types of transactions on the other hand.

To make the above statement more precise, assume that the preference structure of the economic agent is complete but rather irregular, in the sense that it even fails to be quasi-transitive. Therefore, we are in presence of a weak preference $\succsim$ that is $(1,1)$-Ferrers but not $(2,1)$-Ferrers, hence Lemma 4.7 yields that $\succsim$ contains an isomorphic copy of the forbidden characteristic configuration $\succsim_{2,1}$. In this circumstance, it turns out that the agent has to be rather cautious in agreeing to pay a certain amount of money for any switch to a strictly preferred good. In fact, Proposition 4.10(i) suggests that she should pay no more than one half of the base compensation for any such transaction, since otherwise she would become exposed to the risk of losing all her money in a money-pump cycle.

On the contrary, a rather regular preference structure allows the agent to be less parsimonious in her transactions. For instance, assume that her preference structure $\succsim$ is a strong semiorder (i.e., $(3,2)$-Ferrers and $(4,1)$-Ferrers, see Figure 1 in the Appendix),[30] which however fails to be $(5,1)$-Ferrers. In this circumstance, Proposition 4.10(iv) says that the agent is allowed to spend at most $(1+r+r^2+r^3)B/2$ for each switch to a better good, where B is the base compensation and r the indifference factor. For example, if the agent is neutral (that is, neither overconfident nor suspicious, which happens for $r = 1$), then she must pay no more than twice the amount of the base compensation for each strict preference transaction, or else risk getting involved in a money-pump cycle.

Remark 4.11 Observe that Theorem 3.1(vi) in the monograph by Pirlot and Vincke (1997) characterizes semiorders as those reflexive and complete binary relations with the property that every mixed cycle contains more indifferences than strict preferences. It follows that in our model of transactions, a neutral agent ($r = 1$) having a semiordered preference structure will be safe from money-pump phenomena induced by mixed cycles of arbitrary (finite) length as long

[30] For an instance of a strong semiorder, see Example 4.6 and Figure 5(iii) in Giarlotta (2014).

as her strict preference factor s is at least 1. It should be possible to conduct a similar analysis for all types of (m,n)-Ferrers properties.

4.4 (m,n)-rationalizability and axioms of (m,n)-replacement consistency

We finally introduce the notion of (m,n)-rationalizable choices, connecting it to the satisfaction of (m,n)-Ferrers properties. The main result of this section is Theorem 4.14, which extends Theorem 3.5 to cases of (m,n)-rationalizable choices by introducing additional properties of (m,n)-replacement consistency. In view of the discussion in Section 4.3 about the relationship between (m,n)-Ferrers properties on one hand, and the avoidance of certain phenomena of money-pump on the other one, an extension of Theorem 3.5 appears to be susceptible of an economic interpretation.

To start, we employ (m,n)-Ferrers properties to classify rationalizable choices according to the transitive structure of their revealed preferences, thus providing a general framework for cases (A), (B), (C), and (D) discussed in Section 2.

Definition 4.12 A choice correspondence is (m,n)-*rationalizable* (where $m \geq n \geq 1$) if it is rationalizable and its revealed preference is (m,n)-Ferrers.

In order to extend Theorem 3.5 to cases of (m,n)-rationalizability, next we define additional properties of replacement consistency. To that end, consider the following formulae:

$$y \in c(S) \;\wedge\; y \notin c(S \cup \{x\}) \;\Longrightarrow\; x \in c(T \cup \{x\}) \tag{6}$$

$$x \in c(S) \;\wedge\; y \in T \;\wedge\; z \in c(S') \;\wedge\; z \notin c(S' \cup \{y\}) \;\Longrightarrow\; x \in c(S' \cup \{x\}) \tag{7}$$

$$x \in c(S) \;\wedge\; y \in T \;\wedge\; z \in c(S') \;\wedge\; z \notin c(S' \cup \{y\}) \;\Longrightarrow\; x \in c(S \cup T'). \tag{8}$$

Then the axioms $(\rho_{m,n})$ of (m,n)-*replacement consistency* are the following for each $(m,n) \in \mathcal{F}$ (as usual, universal quantifications are omitted):

$(\rho_{1,1})$ $S = S$;[31]

$(\rho_{2,1})$ $S = T$ implies (6);

$(\rho_{3,1})$ $S \cap c(T) \neq \emptyset$ implies (6);

$(\rho_{4,1})$ $S \cap c(U) \neq \emptyset \;\wedge\; U \cap c(T) \neq \emptyset$ implies (6);

$(\rho_{5,1})$ $S \cap c(U) \neq \emptyset \;\wedge\; U \cap c(W) \neq \emptyset \;\wedge\; W \cap c(T) \neq \emptyset$ implies (6);

$(\rho_{2,2})$ $S = T$ implies (7);

$(\rho_{3,2})$ $S \cap c(T) \neq \emptyset$ implies (7);

$(\rho_{4,2})$ $S \cap c(U) \neq \emptyset \;\wedge\; U \cap c(T) \neq \emptyset$ implies (7);

$(\rho_{3,3})$ $S \cap c(T) \neq \emptyset \;\wedge\; S' \cap c(T') \neq \emptyset$ implies (8).

[31] This is obviously a tautology, however we include it here for the sake of completeness.

Remark 4.13 It is long but straightforward to check that for rationalizable choices, $(\rho_{2,1})$, $(\rho_{3,1})$, $(\rho_{2,2})$, and $(\rho_{3,3})$ are equivalent reformulations of, respectively, (ρ), (ρ_{st}), (ρ_f), and (ρ_t). Similarly, again under axioms (α) and (γ), the implication

$$(\rho_{m,n}) \implies (\rho_{m',n'}) \tag{9}$$

holds for all $(m, n), (m', n') \in \mathcal{F}$ such that $m \geq m'$ and $n \geq n'$.

As announced, we have:

Theorem 4.14 *For each $(m, n) \in \mathcal{F}$, a choice correspondence is (m, n)-rationalizable if and only if axioms (α), (γ), and $(\rho_{m,n})$ hold for it.*

Proof. Let $c\colon \Omega \rightrightarrows X$ be a (normal) choice correspondence, and $\succsim_c$ its revealed preference. In view of Sen's Theorem, it suffices to show that for each $(m, n) \in \mathcal{F} \setminus \{(1, 1)\}$, if c is rationalizable, then

$$\succsim_c \text{ is } (m, n)\text{-Ferrers} \iff (\rho_{m,n}) \text{ holds for } c. \tag{10}$$

Thus assume that c is rationalizable. We prove (10) inductively, which means, when proving $\succsim_c$ is (m, n)-Ferrers if and only if c satisfies $(\rho_{m,n})$, we suppose this equivalence holds for all (m', n') such that $m' \leq m$, $n' \leq n$, and $(m', n') \neq (m, n)$. Since Theorem 3.5 yields that the claim holds for $(m, n) \in \{(2, 1), (2, 2), (3, 1), (3, 3)\}$, below we only prove the equivalence (10) for $(m, n) \in \{(3, 2), (4, 2), (4, 1), (5, 1)\}$.

Assume that $\succsim_c$ is not $(3, 2)$-Ferrers. By inductive hypothesis and by (9), we may assume that $(X, \succsim_c)$ contains an isomorphic copy of the FCC for $(3, 2)$-Ferrers. Thus, there exist elements $x_1, x_2, x_3, y_1, y_2 \in X$ such that $y_2 \succ_c x_1$, $x_3 \succ_c y_1$, $x_1 \succsim_c x_2 \succsim_c x_3$, and $y_1 \succsim_c y_2$ hold. Letting $x := x_1$, $y := x_3$, $z := y_1$, $S := \{x_1, x_2\}$, $T := \{x_2, x_3\}$, and $S' := \{y_1, y_2\}$, we obtain that $(\rho_{3,2})$ fails for c. Conversely, assume that c does not satisfy $(\rho_{3,2})$. Then there exist $S, S', T \in \Omega$ and $x, y, z, t \in X$ such that

$$t \in S \cap c(T), \quad x \in c(S), \quad y \in T, \quad z \in c(S'), \quad z \notin c(S' \cup \{y\}), \quad x \notin c(S' \cup \{x\})$$

hold. But then we must have $y \succ_c z$ and $v \succ_c x$, for some $v \in S'$. In addition, we must have $x \succsim_c t \succsim_c y$ and $z \succsim_c v$, so $\succsim_c$ is not $(3, 2)$-Ferrers.

Next, assume that $\succsim_c$ is not $(4, 2)$-Ferrers. Again, by inductive hypothesis and by (9), we may assume that $(X, \succsim_c)$ contains an isomorphic copy of the FCC for $(4, 2)$-Ferrers. Thus, there exist elements $x_1, x_2, x_3, x_4, y_1, y_2 \in X$ such that $y_2 \succ_c x_1$, $x_4 \succ_c y_1$, $x_1 \succsim_c x_2 \succsim_c x_3 \succsim_c x_4$, and $y_1 \succsim_c y_2$ hold. Letting $x := x_1$, $y := x_4$, $z := y_1$, $S := \{x_1, x_2\}$, $U := \{x_2, x_3\}$, $T := \{x_3, x_4\}$, and $S' := \{y_1, y_2\}$, we obtain that $(\rho_{4,2})$ fails for c. Conversely, assume that c does not satisfy $(\rho_{4,2})$. Then there exist $S, U, T, S' \in \Omega$ and $x, y, z, t, w \in X$ such that

$$t \in S \cap c(U), \quad w \in U \cap c(T), \quad x \in c(S), \quad y \in T, \quad z \in c(S'), \quad z \notin c(S' \cup \{y\}), \quad x \notin c(S' \cup \{x\})$$

hold. But then we must have $y \succ_c z$ and $v \succ_c x$, for some $v \in S'$. In addition, we must have $x \succsim_c t \succsim_c w \succsim_c y$ and $z \succsim_c v$, so $\succsim_c$ is not $(4, 2)$-Ferrers.

Next, assume that $\succsim_c$ is not $(4, 1)$-Ferrers. As before, by inductive hypothesis and by (9), we may assume that $(X, \succsim_c)$ contains an isomorphic copy of the FCC for $(4, 1)$-Ferrers. Thus, there exist elements $x_1, x_2, x_3, x_4, y_1 \in X$ such that $y_1 \succ_c x_1$, $x_4 \succ_c y_1$, and $x_1 \succsim_c x_2 \succsim_c x_3 \succsim_c x_4$

hold. Letting $x := y_1$, $y := x_1$, $S := \{x_1, x_2\}$, $U := \{x_2, x_3\}$, and $T := \{x_3, x_4\}$, we obtain that $(\rho_{4,1})$ fails for c. Conversely, assume that c does not satisfy $(\rho_{4,1})$. Then there exist $S, U, T \in \Omega$ and $x, y, t, w \in X$ such that

$$t \in S \cap c(U), \quad w \in U \cap c(T), \quad y \in c(S), \quad y \notin c(S \cup \{x\}), \quad x \notin c(T \cup \{x\})$$

hold. But then we must have $x \succ_c y$ and $z \succ_c x$, for some $z \in T$. In addition, we must have $y \succsim_c t \succsim_c w \succsim_c z$, so $\succsim_c$ is not $(4,1)$-Ferrers.

Finally, assume that $\succsim_c$ is not $(5,1)$-Ferrers. As before, by inductive hypothesis and by (9), we may assume that $(X, \succsim_c)$ contains an isomorphic copy of the FCC for $(5,1)$-Ferrers. Thus, there exist elements $x_1, x_2, x_3, x_4, x_5, y_1 \in X$ such that $y_1 \succ_c x_1$, $x_5 \succ_c y_1$, and $x_1 \succsim_c x_2 \succsim_c x_3 \succsim_c x_4 \succsim_c x_5$ hold. Letting $x := y_1$, $y := x_1$, $S := \{x_1, x_2\}$, $U := \{x_2, x_3\}$, $W := \{x_3, x_4\}$, and $T := \{x_4, x_5\}$, we obtain that $(\rho_{5,1})$ fails for c. Conversely, assume that c does not satisfy $(\rho_{5,1})$. Then there exist $S, U, W, T \in \Omega$ and $x, y, t, w, v \in X$ such that

$$t \in S \cap c(U), \quad w \in U \cap c(W), \quad v \in W \cap c(T), \quad y \in c(S), \quad y \notin c(S \cup \{x\}), \quad x \notin c(T \cup \{x\})$$

hold. But then we must have $x \succ_c y$ and $z \succ_c x$, for some $z \in T$. In addition, we must have $y \succsim_c t \succsim_c w \succsim_c v \succsim_c z$, so $\succsim_c$ is not $(5,1)$-Ferrers. $\square$

We conclude this section by exhibiting minimal choice correspondences that witness a different strength of the axiom of (m,n)-replacement consistency.

Example 4.15 For each $(m,n) \in \mathcal{F}$, let $c_{m,n}$ be the (rationalizable) choice correspondence induced by the FCC $\succsim_{m,n}$ described in Definition 4.6 (and Lemma 4.7). Then we have:

- $c_{2,1}$ satisfies $(\rho_{1,1})$ but not $(\rho_{2,1})$;
- $c_{3,1}$ satisfies $(\rho_{2,2})$ but not $(\rho_{3,1})$;
- $c_{4,1}$ satisfies $(\rho_{3,2})$ but not $(\rho_{4,1})$;
- $c_{5,1}$ satisfies $(\rho_{3,2})$ and $(\rho_{4,1})$ but not $(\rho_{5,1})$;
- $c_{2,2}$ satisfies $(\rho_{5,1})$ but not $(\rho_{2,2})$;
- $c_{3,2}$ satisfies $(\rho_{5,1})$ and $(\rho_{2,2})$ but not $(\rho_{3,2})$;
- $c_{4,2}$ satisfies $(\rho_{5,1})$ and $(\rho_{3,2})$ but not $(\rho_{4,2})$;
- $c_{3,3}$ satisfies $(\rho_{4,2})$ but not $(\rho_{3,3})$.

5 Conclusions and further directions of research

In this paper we have classified rationalizable choices according to the level of transitivity of their revealed preferences, which ranges from mere acyclicity to full transitivity. With the aim of obtaining a unified treatment of the topic, we have addressed the case of choices having a revealed preference such that its asymmetric part is transitive but its indifference may fail to be so. Specifically, we have introduced some axioms of replacement consistency, which describe how the addition of an item to a menu may cause a substitution in the set of selected elements. We have used these axioms to characterize rationalizable choices whose revealed preference is quasi-transitive, Ferrers, semitransitive, and transitive. Further, we have extended this approach by introducing axioms of (m, n)-replacement consistency, which characterize choices having a revealed preference that is weakly (m, n)-Ferrers.

Future research on the topic points in three main directions. A first interesting problem is that of the classification of rationalizable choices such that even the strict part of their revealed preference may fail to be transitive. This naturally prompts a systematic study of choices whose revealed preference is *strictly* (m, n)-Ferrers. In fact, the monotonicity property given by Lemma 4.2(ii) partially holds also in absence of strict transitivity, that is, loosely speaking, the larger m and n are, the weaker the strict (m, n)-Ferrers property becomes. The simplest types of strict (m, n)-Ferrers properties, obtained for $n = 1$, already allow one to conduct this type of analysis.[32] In fact, it can be shown that an acyclic preference is strictly $(m, 1)$-Ferrers for some $m \geq 3$, and each strict $(m, 1)$-Ferrers property implies the strict $(p, 1)$-Ferrers property for all $p \geq \mathrm{st}(m)$, where $\mathrm{st}(m) \geq m$ is the *stabilizer* of m. It follows that strict $(m, 1)$-Ferrers properties provide a good estimate of the level of transitivity of the asymmetric part of the revealed preference. More generally, it would be interesting to obtain a full classification of rationalizable choices on the basis of the satisfaction of strict (m, n)-Ferrers properties.

A second stream of research is related to possible applications in economics of (generalizations of) (m, n)-Ferrers properties, to be interpreted as weaker forms of rationality conditions imposed on possibly intransitive preference structures. Specifically, it appears of some interest to designing strategies that allow the economic agent to detect/prevent any type of money-pump phenomena in suitable transaction models. To that end, it is necessary to introduce new types of Ferrers properties that are connected to the existence of mixed cycles of all kinds, thus generalizing both weak and strict (m, n)-Ferrers properties. Such an approach would operatively create a bridge between regret theory on one hand (which basically abandons transitivity as a rationality tenet), and theories that maintain suitable forms of pseudo-transitivity as minimal rationality requirements (postulating, e.g., the transitivity of the strict preference but not of the associated indifference, such as in interval orders, semiorders, strong interval orders, strong semiorders, etc.).

Last but not least, we are currently studying weaker forms of choice rationalizability, which are alternative to the classical revealed (mono-)preference approach. In this respect, pairs of binary relations satisfying suitable properties – called *NaP-preferences* (*necessary and possible preferences*) – provide a natural tool to rationalize a choice.[33] Recall that a NaP-preference is a pair of binary relations such that the first component is a preorder, the second is a quasi-transitive completion of the second, and the two relations jointly satisfy forms of transitive coherence and mixed completeness. It is possible to associate to an arbitrary choice correspondence a relation of *revealed universal preference*, which complements the classical relation of *revealed (existential) preference*. It turns out that under Sen's axiom (α) of contraction consistency and the standard axiom of replacement consistency (ρ) introduced in this paper, the pair formed by the two types of revealed preferences is indeed a NaP-preference. This NaP-preference enables one to "bi-rationalize" a choice in some cases in which it fails to be (mono-)rationalizable.

[32] In this simplified case, the monotonicity claim is to be intended as eventually true: see Lemma 2.6 and, especially, Conjecture 2.16 in Giarlotta and Watson (2014), which has been proved to hold.

[33] See Giarlotta and Greco (2013) for the introductory paper on NaP-preferences, and Giarlotta (2014, 2015) for further developments of the theory.

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Appendix

In Figure 1 we describe all implications among combinations of weak (m, n)-Ferrers properties.[34] All reverse implications do not hold (see Examples 3.3–3.10 in Giarlotta and Watson (2014)).

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[34] This figure is a slight refinement of Figure 3 in Giarlotta and Watson (2014).

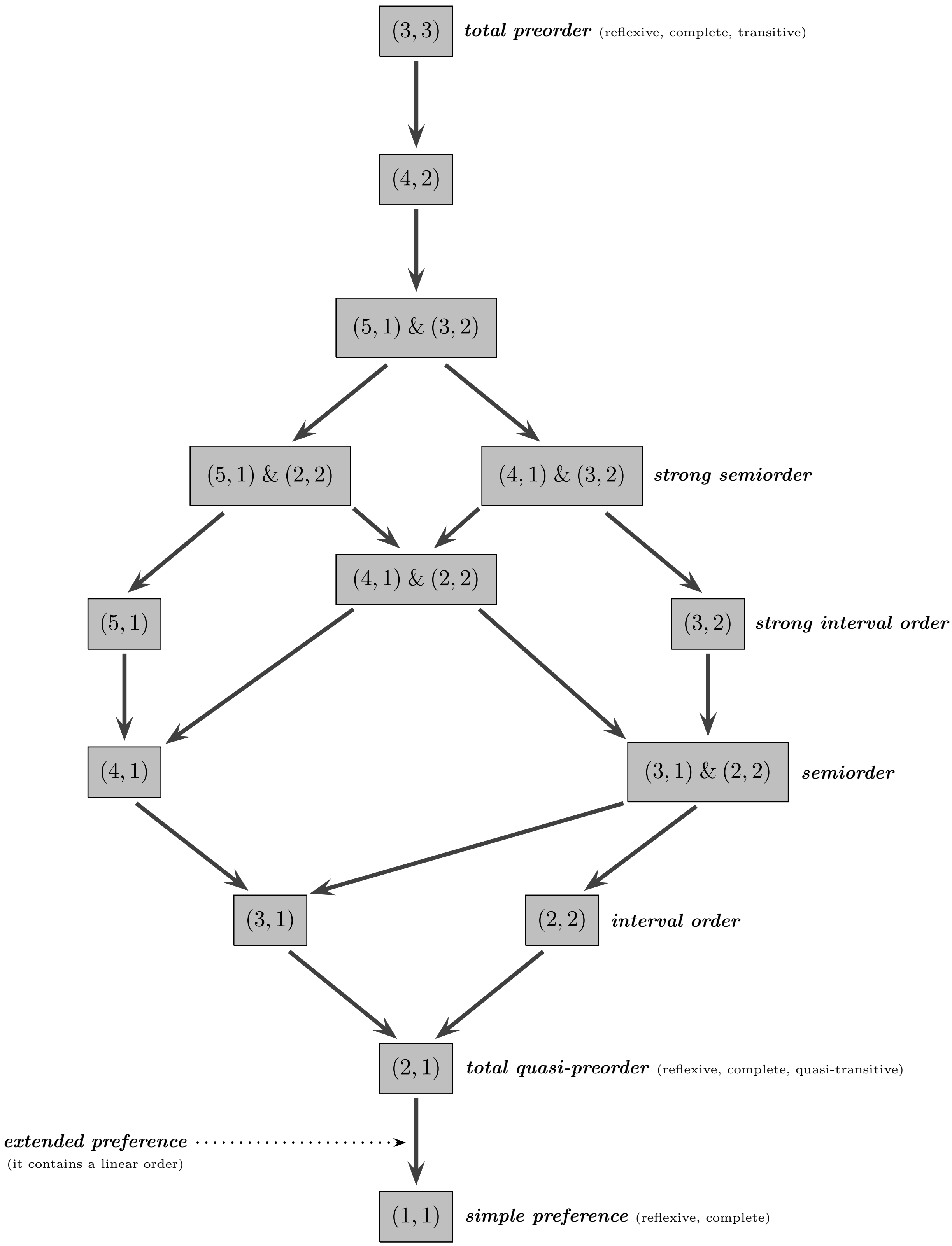

Figure 1: Implications among combinations of weak (m,n)-Ferrers properties

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