Testing gravity and dark energy with gravitational lensing

by

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Abstract

Forthcoming wide field weak lensing surveys, such as DES and Euclid, present the possibility of using lensing as a tool for precision cosmology. This means exciting times are ahead for cosmological constraints for different gravity and dark energy models, but also presents possible new challenges in modelling, both non-standard physics and the lensing itself.

In this thesis I look at how well DES and Euclid will be able to discriminate between different cosmological models and utilise lensing’s combination of geometry and growth information to break degeneracies between models that fit geometrical probes, but may fail to fit the observed growth. I have focussed mainly on the non-linear structure growth regime, as these scales present the greatest lensing signal, and therefore greatest discriminatory power.

I present the predicted discriminatory power for modified gravities models, DGP and $f(R)$, including non-linear scales for DES and Euclid. Using the requirement that modified gravities must tend to general relativity on small scales, we use the fitting formula proposed by Hu & Sawicki to calculate the non-linear power spectrum for our lensing predictions. I demonstrate the improved discriminatory power of weak lensing for these models when non-linear scales are included, and show that not allowing for the GR asymptote at small scales can lead to an overestimation in the strength of the constraints. I then parameterise the non-linear power spectrum to include the growth factor, and demonstrate that even including these extra parameters there is still more power in a full non-linear analysis than just using linear scales.

I then present non-linear weak lensing predictions for coupled dark energy models using the CoDECS simulations. I obtain predictions for the discriminatory power of DES and Euclid in distinguishing between $\Lambda$CDM and coupled dark energy models; I show that using the non-linear lensing signal we could discriminate between $\Lambda$CDM and exponential constant coupling models with $\beta_0 \geq 0.1$ at 99.994% confidence level with a DES-like survey, and $\beta_0 \geq 0.05$ at 99.99994% confidence level with Euclid. I also demonstrate that estimating the coupled dark energy models’ non-linear power spectrum,
using the $\Lambda$CDM Halofit fitting formula, results in biases in the shear correlation function that exceed the survey errors.

I then present weak lensing predictions for DES and Euclid, and the CMB temperature power spectrum expected for Planck for fast transition adiabatic unified dark matter models. I demonstrate that in order to constrain the parameters in this model a high and low redshift observational probe is required. I show that for a $\Lambda$CDM fiducial, Planck could constrain $z_t > 5$ at a 95% confidence level, and DES and Euclid could constrain the maximum time the transition can take to $< 5 \times 10^{-6} / H_0$ at a 95% confidence level.

Finally I look at a full general relativistic model of lensing. I adopt the use of a Lemaitre-Tolman-Bondi model, with and without pressure, to model an overdensity in an expanding background in a continuous spacetime. I use this to examine how the modelling of intermediate scales affects lensing quantities, and whether, as has been suggested recently, the cosmological constant has a direct effect on the lensing observables.
Preface

The work of this thesis was carried out at the Institute of Cosmology and Gravitation, University of Portsmouth, United Kingdom

Whilst registered as a candidate for the above degree, I have not been registered for any other research award. The results and conclusions embodied in this thesis are the work of the named candidate and have not been submitted for any other academic award.


The work in Chapter 4 was carried out in collaboration with Marco Baldi, David Bacon, Kazuya Koyama and Cristiano Sabiu in ‘Weak lensing predictions for coupled dark energy cosmologies at non-linear scales’, Beynon E., Baldi M., Bacon D.J., Koyama, K., Sabiu C., 2012, Mon. Not. Roy. Astron. Soc., 422, 3546

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The work in Chapter 6 was carried out in collaboration with David Bacon and Kazuya Koyama in Beynon et al. (2013, in prep.).
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Chapter 1

Introduction

Current observations (such as Komatsu et al., 2011a; Conley et al., 2011; Anderson et al., 2012; Reid et al., 2012) show that our Universe is currently undergoing a period of accelerated expansion, and that the baryonic matter, such as protons and neutrons which make up stars and galaxies in the Universe, only makes up a fifth of the total matter content that we observe. To explain these observations the standard model of cosmology introduces a dark matter component, a component that only interacts gravitationally with baryons and photons so it can only be detected from its gravitational potential, and a dark energy component with negative pressure which causes the accelerated expansion we see, to a universe where the gravitational physics are described by general relativity. This model fits all current data very well; however it has many problems which are discussed in section 1.9.

Forthcoming weak lensing surveys such as DES\(^1\), Pan-STARRS\(^2\) and LSST\(^3\), and future space surveys such as Euclid\(^4\), will allow a combination of growth of structure and expansion history to be probed to considerably higher precision, which will allow many cosmological models to be excluded. The addition of weak lensing observations can help break degeneracies between cosmological parameters found in observations that only probe the expansion history.

In this chapter I will look at the current standard model of cosmology and the observations that seem to give evidence for this being the correct model of the Universe. I will then look at alternatives to the standard model.

I will use \(c = 1\) and \(8\pi G = 1\) throughout this work.

\(^1\)http://www.darkenergysurvey.org
\(^2\)http://pan-starrs.ifa.hawaii.edu
\(^3\)http://www.lsst.org
\(^4\)http://www.ias.u-psud.fr/imEuclid
1.1 The Einstein field equations

The standard model of cosmology is built upon Einstein’s theory of general relativity (GR). In this section I will look at the principles and equations of this theory.

The Einstein equivalence principle states that acceleration due to a uniform gravitational field cannot be distinguished from an accelerated frame of reference. The laws of special relativity hold in these circumstances (e.g. Carroll, 1997). However gravitational fields are generally not uniform, so these are local inertial frames not global ones. In order to construct a fully relativistic model we must move from the flat Euclidean geometry of Newton’s theory to the curved Riemannian geometry of GR.

Since the geometry is no longer necessarily Euclidean, the length of a path cannot be described by $ds^2 = dx^2$; instead we write $ds^2 = g_{ij}dx^idx^j$ where the metric, $g_{\mu\nu}$, depends on the spacetime curvature. We can then use the metric to calculate the curvature of a Riemannian manifold from the Riemann tensor

$$R^\rho_{\lambda\mu\nu} = \Gamma^\rho_{\nu\lambda,\mu} - \Gamma^\rho_{\mu\lambda,\nu} + \Gamma^\rho_{\mu\alpha}\Gamma^\alpha_{\nu\lambda} - \Gamma^\rho_{\nu\alpha}\Gamma^\alpha_{\mu\lambda} \quad (1.1)$$

This describes how the distance between the path of two freely falling particles deviates from the Euclidean result (Baez, 2005; Loveridge, 2004), which I will discuss further in Chapter 2. The Christoffel symbols, $\Gamma^\lambda_{\lambda\mu\nu}$, are related to the metric in the following way

$$\Gamma^\lambda_{\lambda\mu\nu} = \frac{1}{2}(g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda}) \quad (1.2)$$

Taking the trace of the Riemann tensor gives the Ricci tensor, which describes how a ball of freely falling particles changes in volume:

$$R_{\mu\nu} = g^{\rho\lambda}R_{\rho\lambda\mu\nu} = \Gamma^\rho_{\nu\lambda,\mu} - \Gamma^\rho_{\mu\lambda,\nu} + \Gamma^\rho_{\mu\alpha}\Gamma^\alpha_{\nu\lambda} - \Gamma^\rho_{\nu\alpha}\Gamma^\alpha_{\mu\lambda} \quad (1.3)$$

and taking the trace of the Riemann tensor gives the Ricci Scalar, which describes how the area of a ball of freely falling particles changes:

$$R = g^{\mu\nu}R_{\mu\nu} = g^{\mu\nu}(\Gamma^\rho_{\mu\nu,\rho} - \Gamma^\rho_{\mu\rho,\nu} + \Gamma^\lambda_{\mu\nu}\Gamma^\rho_{\rho\lambda} - \Gamma^\lambda_{\mu\rho}\Gamma^\rho_{\nu\lambda}) \quad (1.4)$$

The traceless part of the Riemann tensor gives the Weyl tensor, which describes how a ball of freely falling particles changes in shape:

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - \frac{1}{2}(g_{\alpha\gamma}R_{\delta\beta} + g_{\beta\delta}R_{\gamma\alpha} - g_{\alpha\delta}R_{\gamma\beta} - g_{\beta\gamma}R_{\alpha\delta} + \frac{1}{6}(g_{\alpha\gamma}g_{\delta\beta} - g_{\alpha\delta}g_{\beta\gamma})R) \quad (1.5)$$
CHAPTER 1. INTRODUCTION

To derive the Einstein field equations (EFEs) we would like to find a form of the Poisson equation that is described by the curvature of spacetime. We first use a tensor generalisation of the density which is the energy momentum tensor $T_{\mu\nu}$, so we need to find a tensor where $A_{\mu\nu} = T_{\mu\nu}$ and in the Newtonian limit reduces to $\nabla^2\Phi = \frac{1}{2}\rho$, where $T_{\mu\nu} = \rho$ for zero pressure. Using the geodesic equation as discussed later in Chapter 2 we obtain (Carroll, 1997)

$$\ddot{x}^\mu = -\Gamma^\mu_{\nu\rho}\dot{x}^\nu \dot{x}^\rho$$

(1.6)

where dots represent the derivative w.r.t. an affine parameter $\sigma$. This is an acceleration equation so the quantity on the right hand side is the force due to the gravitational field. Assuming that the gravitational field is weak and static implies $|\dot{x}^i| \ll \dot{t}$ so we can write $g_{\mu\nu}\ddot{x}^\mu \ddot{x}^\nu \approx \dot{t}^2$, $\Gamma^\mu_{\nu\rho}$ reduces to $\Gamma^\mu_{00} = -\frac{1}{2}g_{\mu\rho}g_{00,\nu}$ and the metric can be written as the Minkowski metric plus a small perturbation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. Making these approximations and dividing by $\dot{t}$ gives

$$\frac{dx^i}{dt^2} = \frac{1}{2}h_{00,i}$$

(1.7)

For a Newtonian potential this gives $h_{00} = -2\Phi$ since we want $A_{00} = \nabla^2\Phi$. We can now relate the geometrical quantities of Riemannian geometry with those in Newton’s theory, so in the Newtonian limit where $g_{\mu\nu} \propto \Phi$, the Christoffel symbols $\Gamma_{\lambda\mu\nu} \propto \nabla \Phi$ which is proportional to the gravitational acceleration according to Newton’s second law, and the Ricci tensor $R_{\mu\nu} \propto \nabla^2\Phi$ which is proportional to the density, $\rho$, according to Poisson’s equation (Plebanski and Krasinski, 2006). Using these relations one can show that $A_{ij}$ must contain second derivatives of the metric, so we postulate that $A_{ij} = R_{ij}$. In addition we require the conservation of the energy momentum tensor

$$T^{\mu\nu};_{\mu} = 0$$

(1.8)

which gives $A_{\mu\nu} = 0$; but if we assume $A_{\mu\nu} = R_{\mu\nu}$ we obtain $R^{\mu\nu};_{\mu} = \frac{1}{2}R_{\nu}$, so therefore $A_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R\eta_{\mu\nu}$ and this gives the form of Einstein’s equation (Einstein, 1916)

$$G_{\mu\nu} = T_{\mu\nu} + \Lambda g_{\mu\nu}$$

(1.9)

where the Einstein tensor, $G_{\mu\nu}$, is defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$$

(1.10)
where $g_{\mu\nu}$ is the metric tensor, $T_{\mu\nu}$ is the stress-energy-momentum tensor, $\Lambda$ is the cosmological constant, which I will discuss later.

The EFEs can also be rewritten in the form of the Einstein-Hilbert action

$$S = \frac{1}{2} \int R \sqrt{-g} d^4x$$

(1.11)

where $g = \det(g_{\mu\nu})$. This yields the EFEs when the action is varied with respect to the metric and $\delta S = 0$. This form can be very useful as it only requires that one defines an action in order to find the form of the field equations for any theory of gravity, and will be used in Chapter 3.

Throughout this thesis I will restrict myself to looking at perfect fluids which are completely characterised by only three quantities: a 4-velocity $u^\mu$, a rest frame density, $\rho$ and rest frame pressure $p$, giving a stress-energy-momentum tensor of the form (Misner et al., 1974)

$$T_{\mu\nu} = (\rho + p)u^\mu u^\nu - pg^{\mu\nu}$$

(1.12)

### 1.2 The cosmological principle

The cosmological principle claims that viewed on sufficiently large scales properties of the Universe are the same for all observers, so our location is not special and the laws of physics do not vary from point to point.

Two consequences of assuming the cosmological principle are that the Universe must be homogeneous, the Universe looks the same at each point at a given cosmic time; and isotropic, the Universe looks the same in all directions. While we cannot directly test the homogeneity of the Universe, as we cannot observe about every point, we have seen that the Universe is isotropic from observing that the CMB is the same in all directions to $10^5$ part in $10^5$ and also by observing the large scale distribution of galaxies, which again are isotropic. Combining our observations of isotropy with the Copernican principle allows us to claim that the Universe is homogeneous too.

### 1.3 FRW models

Assuming the Universe is homogeneous and isotropic means that the metric can change over time but not by a change of position. The metric that satisfies these constraints is called the Friedmann Robertson Walker (FRW) metric (Friedman, 1922; Lemaître, 1933;
Robertson, 1935; Walker, 1937) and has the following form in spherical polar coordinates \((r, \theta, \phi)\)

\[
\text{ds}^2 = dt^2 - a(t)^2 \left[ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right]
\]

(1.13)

where \(t\) is proper time, \(a\) is the scale factor and \(k\) is the spatial curvature which can take the values -1, 0 and 1 which correspond to open (negative curvature), flat (zero curvature) and closed (positive curvature) spacetimes respectively. We can see from this metric that \(r\) corresponds to a comoving distance, which means if a particle only moves with the Hubble flow then \(r\) remains constant.

This spacetime metric is the most widely used to construct cosmological models and I will use this metric to derive all quantities in this chapter. Later in Chapter 6 I will look at how using a generalised form of this metric, the Lemaitre-Tolman-Bondi (LTB) metric, where the scale factor depends on position as well as time, affects lensing predictions.

### 1.4 Expansion history

The expansion history for the FRW metric can be found by solving the Einstein field equations (EFEs) from Equation 1.1 with Equations 1.2-1.4 and Equation 1.10 using the metric components given in Equation 1.13

\[
G_{tt} = 3 \frac{\dot{a}^2}{a^2} + 3 \frac{k}{a^2} = \rho + \Lambda
\]

(1.14a)

\[
G_{rr} = G_{\theta\theta} = G_{\phi\phi} = 2 \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \Lambda - p
\]

(1.14b)

and the Ricci scalar gives

\[
R = -\frac{6}{a^2} \left( a\ddot{a}^2 + \dot{a}^2 + k \right)
\]

(1.15)

The conservation of the energy momentum tensor (Equation 1.8) gives

\[
\rho_t + 3H(\rho + p) = 0
\]

(1.16)

Using this equation we can write the densities in terms of their value today and how they evolve with density. Defining \(w = p/\rho\) gives for constant \(w\)

\[
\rho \propto a^{-3(1+w)}
\]

(1.17)
This means for pressureless matter \( w_m = 0 \) so matter energy density is inversely proportional to volume, for radiation \( w_r = 1/3 \) since the radiation energy density is inversely proportional to volume but also loses energy due to the expansion of the Universe, and for the cosmological constant \( w_\Lambda = -1 \) often just denoted \( w \) so \( \rho_\Lambda \) is constant. Figure 1.1 shows how these densities evolve with time resulting in a radiation dominated period before the matter radiation equality at \( z \approx 3000 \), followed by a matter domination period until very recently when \( \Lambda \) has started to dominate. We can rewrite the densities as \( \Omega = \rho/\rho_{\text{crit}0} \) where \( \rho_{\text{crit}0} = 3H_0^2/8\pi G \) and \( \Omega_\Lambda = \Lambda/3H_0^2 \). We can also rewrite the pressures as \( P = p/(\rho_{\text{crit}0}c^2) \). Throughout this work I will use \( \Omega = 8\pi G \rho/3H_0^2 \).

![Figure 1.1: Evolution of densities, \( \Omega = 8\pi G \rho/3H_0^2 \), over time.](image)

Then Equation 1.14a gives the evolution of the scale factor as

\[
H(a)^2 = \frac{\Omega_r}{a^4} + \frac{\Omega_m}{a^3} + \frac{\Omega_k}{a^2} + \Omega_\Lambda
\]

(1.18)

where \( H(a) = \frac{a}{a}/H_0 \), which is the definition I will use throughout this work, and \( H_0 \) is the present day Hubble constant, which is often written in the form \( H_0 = 100h \text{ km/s Mpc} \). This means that the distance between two galaxies moving only with the Hubble flow is given by \( r = ax \) and the recession velocity is given by \( u = \dot{a}x = Hr \).
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Substituting Equation 1.14a into Equation 1.14b gives the acceleration of the expansion of the Universe as follows

$$\frac{a_{,tt}}{a} = -\frac{1}{6}(\rho + 3p) + \frac{\Lambda}{3} \quad (1.19)$$

Equations 1.14a and 1.19 will be used throughout this thesis to explain how the background evolves in a homogeneous and isotropic spacetime.

1.5 Redshift

The expansion of the Universe causes light to be redshifted; here I will show the origin of this effect. The cosmological redshift of an object is defined as

$$1 + z = \frac{\lambda_o}{\lambda_e} = \frac{\nu_e}{\nu_o} \quad (1.20)$$

where $\lambda_e$ is the wavelength of the light emitted by an object, $\lambda_o$ is the wavelength observed elsewhere at some later time, and $\nu$ is the frequency.

This can be related to the scale factor in the FRW model by integrating the null geodesic equation for the FRW metric (Equation 1.13) giving

$$\int_{t_e}^{t_o} \frac{dt}{a} = \int_0^\chi \frac{dr}{\sqrt{1 - kr^2}} \quad (1.21)$$

To relate this to a wavelength we need to detect the time between subsequent crests of the light wave, $\delta t$, where $\nu = 1/\delta t$. Since $r$ is comoving, and therefore does not change with time, we can relate the emitted and observed time differences by

$$\frac{\delta t_e}{a(t_e)} = \frac{\delta t_o}{a(t_o)} \Rightarrow \frac{\nu_o}{\nu_e} = \frac{a_e}{a_o} \quad (1.22)$$

So the redshift, $z$ is related to the scale factor by

$$1 + z = \frac{a(t_o)}{a(t_e)} \quad (1.23)$$

We can observe these frequency shifts in the emission and absorption lines from galaxy spectra and from this we can find galaxy redshifts using Equation 1.22.
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1.6 Distance measures

Many cosmological observations are associated with distance measurements, which can be derived from the spacetime metric. Again in this section the FRW metric is used to derive all quantities.

Integrating the definition for the Hubble parameter, \( H(a) = \frac{\dot{a}}{a} / H_0 \), allows us to calculate the time between when a photon was emitted from a source and when it is observed, which is named the lookback time, \( t_L \),

\[
t_L = \frac{1}{H_0} \int_0^z \frac{dz'}{(1 + z')H(z')} \tag{1.24}
\]

So the distance the light has travelled in this time is \( t_L \) multiplied the speed of light. We can write this in terms of comoving distances by dividing \( cdt_L \) by the scale factor giving the comoving distance along the line of sight as

\[
D_{\text{C,LoS}} = \frac{1}{H_0} \int_0^z \frac{dz'}{H(z')} \tag{1.25}
\]

The transverse comoving distance, the distance between two objects separated by the angle \( \delta \theta \) is given by \( D_{\text{C,trans}} \delta \theta \) where \( D_{\text{C,trans}} \) is given by (Hogg, 1999)

\[
D_{\text{C,trans}} = \begin{cases} 
\frac{1}{D_0 \sqrt{|\Omega_k|}} \sinh(\sqrt{\Omega_k} D_{\text{C,LoS}} / D_H) & \text{for } \Omega_k > 0 \\
D_{\text{C,LoS}} & \text{for } \Omega_k = 0 \\
\frac{1}{H_0 \sqrt{|\Omega_k|}} \sin(\sqrt{|\Omega_k|} D_{\text{C,LoS}} / D_H) & \text{for } \Omega_k < 0
\end{cases} \tag{1.26}
\]

These relations come from integrating the distance interval \( \sqrt{a^2/(1 + \Omega_k r^2)} dr \) from the FRW metric, so in curved space distances scale as if they were on a sphere for \( \Omega_k < 0 \), or on a hyperbolic surface for \( \Omega_k > 0 \).

While these distance measures are the easiest to derive from the spacetime metric, they cannot be measured directly, so we need to define some observed distance measures. The most commonly used are the luminosity distance and the angular diameter distance.

Luminosity distance

The apparent luminosity \( \ell \) of an object is related to the absolute luminosity \( L \) by \( \ell = L / 4\pi d^2 \) where \( d \) is the proper distance to the object. For cosmological distances this relation needs to be modified to include the fact that the Universe is expanding so \( d \rightarrow ar \).

We also need to include the fact that the photons have been redshifted so their rate of arrival is lowered by \( a_e / a_o \) and their energy is reduced by this factor too, giving
\[ \ell = \frac{L}{4\pi a^2 r^2 (1 + z)^2} \]. The luminosity distance is then defined as this modified \( d \) (e.g. Weinberg, 2008)

\[ D_L = r(1 + z) \]  

(1.27)

A similar distance measurement is the distance modulus, which is the magnitude difference between an object’s observed flux and what it would be if it were at 10pc (e.g. Hogg, 1999).

\[ \mu = 5 \log \left( \frac{D_L}{10\text{pc}} \right) \]  

(1.28)

**Angular diameter distance**

A method which uses geometrical arguments as opposed to luminosity ones is the angular diameter distance, which is defined as the ratio of the proper distance between two points and the angle between them. The angular part of the FRW metric shows that the proper distance between two points separated by an angle \( \delta \theta \) is \( aD_{\text{trans}} \delta \theta \), so the angular diameter distance is given by

\[ D_A = aD_{\text{trans}} = \frac{D_{\text{trans}}}{1 + z} = \frac{D_L}{(1 + z)^2} \]  

(1.29)

This is the distance measurement we will use most frequently in lensing.

### 1.7 Structure formation

Thus far this chapter has mainly looked at how background density quantities affect the cosmology, however for structure formation we need to have inhomogeneities to form gradients in the gravitational potential. This section will show how we can model these inhomogeneities as perturbations of the background density, using perturbation theory, simulations and exact GR models.

It is useful to introduce the correlation function of the density field \( \xi(r) = \langle \delta(x) \delta(x + r) \rangle \). Equally we can describe the density fluctuations in terms of the Fourier modes, \( k = 2\pi / r \), as the power spectrum \( P(k) = \langle |\delta_k|^2 \rangle \). The power spectrum can also be written in dimensionless form \( \Delta^2(k) = \frac{V}{(2\pi)^3} 4\pi k^3 P(k) \), where \( V \) is a normalisation volume, and related to the correlation function by (Peacock, 1999)

\[ \xi(r) = \int \Delta^2 \frac{\sin kr}{kr} \frac{dk}{k} \]  

(1.30)
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The simplest single scalar field inflation theories predict perturbations that are adiabatic, so $\delta \rho_\alpha / (\bar{\rho}_\alpha)_{,t}$ is equal for all the individual constituents $\alpha$ of the Universe, Gaussian, so perturbations obey Gaussian statistics, and are scale invariant (e.g. Liddle and Lyth, 2000). These scale invariant perturbations give an initial power spectrum that is a pure power law $P(k) \propto k^n$. As the Universe evolves linear adiabatic perturbations scale as $\delta \propto a^2$ during radiation domination and $\delta \propto a$ during matter domination, so only the amplitude of the primordial power spectrum is changing with time and the overall shape remains the same at linear scales, as we will show below.

1.7.1 Linear structure growth

The evolution of cosmological perturbations can be calculated using GR, however for the linear regime where we only look at small perturbations ($\delta \ll 1$) a Newtonian approach can be used (Peacock, 1999). This means that the linear growth rate can be found using the following fundamental equations that govern fluid motion. These are the continuity equation which represents mass conservation

$$ (\rho_{,t})_r + \nabla_r \cdot (\rho u) = 0 \quad (1.31) $$

the Euler equation which represents momentum conservation

$$ (u_{,t})_r + (u \cdot \nabla_r) u + \nabla p = -\nabla_r \Phi \quad (1.32) $$

and the Poisson equation which shows how the gravitational potential relates to the density

$$ \nabla^2 \Phi = \frac{1}{2} \rho \quad (1.33) $$

We can recast these equations in terms of comoving coordinates, $x = r/a$, giving $\nabla_r = \nabla / a$, $(\rho_{,t})_r = (\bar{\rho}_{,t})_x - \frac{4}{3} a \cdot \nabla$ and $u = (ax)$.

We can also write the quantities in these equations in terms of the background value and the perturbation of the background. We write $\rho = \bar{\rho}(t)[1 + \delta(x, t)]$ where $\bar{\rho}$ is the background density, $u = a_{,t} x + v(x, t)$ where $v$ is the peculiar velocity and $\Phi = \bar{\Phi} + \phi$.

Substituting these values into Equations 1.31-1.33, subtracting the background equation and removing products of first order terms gives

$$ \delta_{,t} + \frac{1}{a} \nabla \cdot v = 0 \quad (1.34) $$
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\[ \mathbf{v}_t + \frac{a_t}{a} \mathbf{v} + \frac{1}{a} \nabla \phi + \frac{1}{a\tilde{\rho}} \nabla \delta p = 0 \]  
\( (1.35) \)

\[ \nabla^2 \phi = \frac{1}{2} \tilde{\rho} a^2 \delta \]  
\( (1.36) \)

We can eliminate the peculiar velocity term by taking the time derivative of the continuity equation (Equation 1.31)

\[ \delta_{tt} - \frac{a_t}{a^2} \nabla \cdot \mathbf{v} + \frac{1}{a} \mathbf{v}_t = 0 \]  
\( (1.37) \)

and substituting in the form of \( \dot{\mathbf{v}} \) from the Euler equation (Equation 1.35)

\[ \delta_{tt} - \frac{a_t}{a^2} \nabla \cdot \mathbf{v} - \frac{1}{a} \nabla \cdot \left( \frac{a_t}{a} \mathbf{v} + \frac{1}{a} \nabla \phi + \frac{1}{a\tilde{\rho}} \nabla \delta p \right) = 0 \]  
\( (1.38) \)

Using the Poisson equation (Equation 1.36) for \( \nabla^2 \phi \) and the continuity equation (Equation 1.34) for \( \nabla \cdot \mathbf{v} \) gives

\[ \delta_{tt} + 2 \frac{a_t}{a} \delta_t = \frac{1}{2} \tilde{\rho} \delta + \frac{1}{a^2 \tilde{\rho}} \nabla^2 \delta p \]  
\( (1.39) \)

This equation allows us to calculate the linear growth of the power spectrum from early times to today. It demonstrates the competition between gravitational infall and the pressure support. However since this is only valid on linear scales, we need to invoke some method of finding the growth due to non-linear physics, which is most interesting to observations such as lensing. I will discuss methods used to model the non-linear physics in Section 1.7.2.

For scales smaller than the horizon during radiation domination, the shape of the power spectrum as well as the amplitude changes, creating the peak seen in the power spectrum, as shown in Figure 1.2.

One effect that alters the shape of the power spectrum is the Meszaros effect (Meszaros, 1974) shown in Figure 1.3(a), which describes the phenomenon that matter perturbations become frozen when they enter the horizon until matter-radiation equality, while modes outside the horizon continue to grow. This is caused by the rapid expansion during the radiation dominated epoch stopping the growth of CDM perturbations, and can be shown analytically (Coles and Lucchin, 2002) for \( \lambda \gg \lambda_J \) using Equation 1.39 and changing variable to \( y = \rho_{\text{rel}}/\rho_{\text{rel}} = a/a_{\text{eq}} \), where \( \rho_{\text{rel}} \) and \( \rho_{\text{rel}} \) is the density of the non-relativistic
Figure 1.2: Matter power spectrum for WMAP7 best fit cosmology at $z = 0$ calculated using CAMB (Lewis et al., 2000a).
and relativistic components respectively and $a_{eq}$ is the scale factor at matter radiation equality:

$$\frac{\partial^2 \delta}{\partial y^2} + \frac{2 + 3y}{2y(1+y)} \frac{d\delta}{dy} - \frac{3\delta}{2y(1+y)} = 0 \tag{1.40}$$

The growing solution gives

$$\delta_+ \propto 1 + \frac{3}{2} \frac{1}{y} \tag{1.41}$$

For $y < 1$ which corresponds to $a < a_{eq}$, the total growth is only $\delta_+|_{y=0}/\delta_+|_{y=1} = 5/2$, whereas after $a_{eq}$ we obtain the growth in a matter-dominated EdS universe $\delta_+|_{y\gg 0} \propto y \propto a \propto t^{2/3}$. A similar effect where the rapid expansion prevents structures forming also occurs at late times during $\Lambda$ domination.

![Figure 1.3: Meszaros Effect](image)

(a) Evolution of perturbations  
(b) Evolution of PS

Another effect, which alters the baryon power spectrum, is due to radiation and baryons being tightly coupled by Thomson scattering forming a hot plasma (Liddle and Lyth, 2000) during radiation domination. Overdensities in this hot plasma cause gravitational infall of matter. If we assume that the pressure only depends on the density, $P(\rho)$, then we can write

$$\delta p = c_s^2 \bar{\rho} \delta \Rightarrow \delta_{,tt} + \frac{a}{a} \delta_{,t} = 4\pi G \bar{\rho} \delta - \frac{c_s^2 k^2}{a^2} \delta \tag{1.42}$$

where the sound speed $c_s^2 = \frac{\partial P}{\partial \rho}$.
Equation 1.42 looks like a damped oscillator when \( 4\pi G \bar{\rho} \delta < \frac{c^2 k^2}{a^2} \) causing acoustic oscillations in the plasma. When \( 4\pi G \bar{\rho} \delta > \frac{c^2 k^2}{a^2} \) then gravitational infall dominates and the overdensity collapses. The Jeans’ length corresponds to the smallest scale structure that can collapse. The proper wavelength is given by \( \lambda = 2\pi a/k \) so solving \( k \) for \( 4\pi G \bar{\rho} \delta = \frac{c^2 k^2}{a^2} \) gives the Jeans’ wavelength as

\[
\lambda_J = \sqrt{\frac{\pi c^2}{G \rho}}
\]  

So perturbations on scales larger than the Jeans’ length grow and those on smaller scales don’t grow, leading to a suppression in the growth of the power spectrum on small scales. This along with the suppression of CDM perturbations, due to the rapid radiation driven expansion, leads to the peak in the power spectrum at the matter-radiation equality horizon size, as shown in Figure 1.3(b).

The acoustic oscillations formed in the plasma are also damped by Silk Damping (Peebles, 1994). This damping is due to photon diffusion out of perturbations, taking the coupled baryons with them erasing perturbations in the plasma (Silk, 1968; Bychkov, 1975).

Another effect that causes small scale perturbation damping is free streaming (Peacock, 1999), where random velocities of DM particles cause structures to disperse. At early times the DM particles travel at the speed of light erasing all perturbations that enter the horizon. This process stops when the DM particles become non-relativistic leading to all perturbations being erased up to the horizon size at this time. The size of structures erased depends on the mass of the DM particles.

These effects explain the shape of the power spectrum on linear scales; however there is another bump in the power spectrum around \( k \sim 1 \ h/\text{Mpc} \) (see Figure 1.2) due to non-linear physics increasing the number of small scale structures.

### 1.7.2 Non-linear structure growth

There is no analytical way to model the non-linear regime, as perturbation theory is no longer valid, so simulations must be used. However we can obtain exact solutions for simpler models, such as the spherical collapse model, which while not entirely realistic can help us understand what happens during collapse of an overdensity.

**Spherical Collapse**

We can model a spherical overdensity as a closed universe inside our own expanding Universe. This allows us to use the positive curvature form of the FRW equations (see
Equation 1.18) to model the collapse of the overdensity (e.g. Peacock (1999)). In this case
\[ H = \sqrt{\frac{\Omega_m}{a^3} + \frac{(1 - \Omega_m)}{a^2}} \] (1.44)
This has a cycloid solution as follows
\[ t(\theta) = \frac{\Omega_m}{2(\Omega_m - 1)^{3/2}}(\theta - \sin \theta) \] (1.45a)
\[ a(\theta) = \frac{\Omega_m}{2(\Omega_m - 1)}(1 - \cos \theta) \] (1.45b)

From these solutions we can see how the perturbation evolves over time. First the overdensity is growing with the Hubble expansion, then the sphere breaks away from the Hubble expansion reaching a maximum radius when \( \theta_{\text{max}} = \pi \) and \( t_{\text{max}} = \frac{\pi \Omega_m}{2(\Omega_m - 1)^{3/2}} \) and collapses completely when \( \theta_{\text{coll}} = 2\pi \) and \( t_{\text{coll}} = \frac{2\pi \Omega_m}{(\Omega_m - 1)^{3/2}} = 2t_{\text{max}}. \)

To relate this to linear theory we need to investigate what happens for small values of \( \theta \). This allows us to do a Taylor expansion of Equations 1.45 and rearrange them to get a form for \( a \) in terms of \( t \) giving
\[ a(t) \approx \frac{\Omega_m}{4} \left( \frac{12t}{\Omega_m} \right)^{2/3} \left[ 1 - \frac{\Omega_m - 1}{20} \left( \frac{12t}{\Omega_m} \right)^{2/3} \right] \] (1.46)

Then the linear evolution of \( \delta \), which we shall denote \( \delta_l \), can be calculated using \( \rho = \Omega_m/a^3 \), where the first term above is the unperturbed value of \( a_{\text{unperturb}} = \frac{\Omega_m}{4} \left( \frac{12t}{\Omega_m} \right)^{2/3} \) giving the expected \( t^{2/3} \) behaviour for an \( \Omega_m = 1 \) universe and therefore \( \bar{\rho} = \Omega_m \left( \frac{2}{\Omega_m} \right)^2 \), so we can write to first order
\[ \delta_l = \frac{\rho - \bar{\rho}}{\bar{\rho}} = \frac{a_{\text{unperturb}}}{a_{\text{perturb}}} - 1 \approx \left[ 1 - \frac{\Omega_m - 1}{20} \left( \frac{12t}{\Omega_m} \right)^{2/3} \right]^{-3} - 1 \approx \frac{3(\Omega_m - 1)}{20} \left( \frac{12t}{\Omega_m} \right)^{2/3} \] (1.47)
If we extrapolate this behaviour to large \( \theta \) then when at maximum radius \( \delta_l \approx 1.06 \) and at collapse \( \delta_l \approx 1.69 \).

The spherical collapse model predicts that the density of the collapsed object goes to infinity at the point of collapse. This is not realistic, since velocity dispersions generated during collapse will balance gravity leading to a virialised halo. Therefore to calculate the density of the collapsed halo we need to consider a gravitationally bound system of \( N \) particles with mass \( m_n \) with positions \( \mathbf{x}_n \) relative to the centre of mass. For each of
the particles the equations of motion are

\[ m_n \ddot{x}_{n,t} = -\frac{\partial U}{\partial x_n} \]  

(1.48)

where \( U \) is the total gravitational potential \( U = \frac{1}{2} \sum_{n \neq l} \frac{G m_n m_l}{x_n - x_l} \). Multiplying this by \( x_n \) and summing over all the particles in the virialised structure gives

\[ \sum_n m_n \dot{x}_{n,t} \cdot x_n = -\sum_n x_n \frac{\partial U}{\partial x_n} \]  

(1.49)

which can rewritten as

\[ \frac{1}{2} \frac{d^2}{dt^2} \sum_n m_n x_n^2 - 2K = U \]  

(1.50)

where \( K \) is the total kinetic energy \( K = \frac{1}{2} \sum_n m_n (x_n)^2 \). Since we are considering a collapsed virialised structure we can assume that although \( \frac{d^2}{dt^2} m_n x_n^2 \neq 0 \) for each particle, on average these motions are not aligned so we can write \( \frac{d^2}{dt^2} \sum_n m_n x_n^2 = 0 \), which allows us to write

\[ 2K_{\text{coll}} + U_{\text{coll}} = 0 \Rightarrow E_{\text{coll}} = K_{\text{coll}} + U_{\text{coll}} = U_{\text{coll}}/2 \]  

(1.51)

where \( r_{\text{max}} \) is the maximum radius of the virialised object and \( r_{\text{coll}} \) is the radius of the virialised object at collapse. Since energy is conserved and \( K_{\text{max}} = 0 \)

\[ E_{\text{max}} = U_{\text{max}} \Rightarrow U_{\text{max}} = U_{\text{coll}}/2 \Rightarrow \frac{1}{r_{\text{max}}} = \frac{1}{2r_{\text{coll}}} \Rightarrow \rho(t_{\text{coll}}) = 8\rho(t_{\text{max}}) = \frac{8\Omega_m}{a_{\text{max}}^3} \]  

(1.52)

So the value of \( \delta \) required to create a virialised structure is given by

\[ \delta_{\text{vir}} = \frac{8\Omega_m}{a_{\text{max}}^3} \left( \frac{3t_{\text{coll}}}{2} \right)^2 = 18\pi^2 \approx 178 \]  

(1.53)

Normally this is approximated to 200 and the size of a virialised structure is defined as \( r_{200} \), the radius at which the density is 200 times the critical density.

The spherical collapse model and the linear structure growth in Section 1.7.1 demonstrate how structure grows in the extreme non-linear and linear regimes, but they do not model the scales in between. Two of the most popular methods to map these linear and non-linear regimes are the HKLM (Hamilton et al., 1991) procedure, refined by Peacock
and Dodds (1996) which rely on the stable clustering hypothesis, and HALOFIT (Smith et al., 2003) which uses the halo model. We will now look at these models.

**Stable clustering**

The stable clustering hypothesis states that a non-linear collapsed object decouples from the global expansion of the Universe to form a virialised system. On small scales the shape of the correlation function \( \xi(r) = \langle \delta(r')\delta(r'+r) \rangle \) can be directly related to the density profile of the cluster \( \rho(r) \propto r^{-\gamma} = (ax)^{-\gamma} \) and the amplitude should be related to the mean density, which scales as \( r^3 = (ax)^3 \), so on very non-linear scales we can write (Peebles, 1994)

\[
\xi(r, z) \propto r^{3-\gamma} = (ax)^{3-\gamma}
\]

(1.54)

On linear scales we know if the initial power spectrum is a power law, \( P(k) \propto k^n \), taking the Fourier transform of this relation gives \( \xi(r) \propto x^{-3-n} \), where I have used \( k \propto 1/x \). Since we know that in the linear regime the amplitude evolves as \( a^2 \), as \( \delta \propto a \), we get

\[
\xi(r, z) \propto a^2 x^{-3-n}
\]

(1.55)

To connect these two regimes we can match them at the scale of quasilinearity, \( \xi(r, z) \approx 1 \), where clusters form, giving the characteristic clustering length \( x_0(z) \). Substituting this into Equations 1.54 and 1.55 gives the following relation between the slope of the non-linear correlation function and the spectral index \( n \)

\[
\xi(r, t) \propto (ax)^{-\gamma} = r^{-\gamma}; \quad \gamma = \frac{3(3+n)}{5+n}
\]

(1.56)

Assuming the stable clustering hypothesis the HKLM procedure proposes that non-linear correlations are a universal function of linear correlations appropriately scaled, so we can write

\[
\xi_{nl}(r_{nl}, z) = f(\bar{\xi}(r_1, z))
\]

(1.57)

where

\[
\bar{\xi}(r) = \frac{3}{r^3} \int_0^r \xi(r) r^2 dr
\]

(1.58)
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The reasoning behind this comes from a mass conservation argument, since if we assume there is no shell crossing we can see that

\[ M_1 = M_{nl} \Rightarrow \rho(< r_1) r_1^3 = \rho(< r_{nl}) r_{nl}^3 \]  

(1.59)

where \( \rho(< r_1) / \rho(< r_{nl}) = (1 + \xi_{nl}) \) giving the \( r \) scaling as

\[ r_{nl}(r_{nl}, z) = (1 + \xi_{nl})^{1/3} r_1 \]  

(1.60)

The functional form of \( f \) is already known in the linear and non-linear extremes and the correlation function for the scales in between is found by empirically fitting to N-body simulations.

The HKLM approach assumes \( \Omega_m = 1 \) and was generalised by Peacock and Dodds (1996) (PD96) to include \( \Omega \neq 1 \) models and calibrated by N-body simulations to improve its accuracy. However this approach is inconsistent with hierarchical models, where objects continually grow through accretion or mergers since it assumes stable clustering.

It has also been shown by Smith et al. (2003) that the HKLM approach generalised and calibrated by PD96 underpredicts the amount of power on quasilinear scales and overpredicts the power on non-linear scales (see Figure 1.4). Therefore Smith et al. (2003) propose a new model HALOFIT that is based on the halo model, which does not assume stable clustering.

The halo model

In the halo model the power spectrum can be written as the sum of two terms:

\[ P(k) = P^{1h}(k) + P^{2h}(k) \]  

(1.61)

On large scales the halos are correlated with each other, which is represented by the two halo term \( P^{2h}(k) \), and on small scales there are also correlations between dark matter particles within the same halos, which is represented by the one halo term \( P^{1h}(k) \).

The one halo term has the form (Peacock and Smith, 2000; Seljak, 2000)

\[ P^{1h}(k) = \frac{1}{\rho^2 (2\pi)^3} \int n(m) |\rho(k|m)|^2 dm \]  

(1.62)

and the two halo term is given by

\[ P^{2h}(k) = P_{hh}(k|m_1, m_2) \int n(m_1) \frac{1}{\rho} \rho(k|m_1) dm_1 \int n(m_2) \frac{1}{\rho} \rho(k|m_2) dm_2 \]  

(1.63)
where \( n(m_i) dm_i \) is the number density of halos in the mass range \( dm_i \), \( \rho(k, m_i) \) is the Fourier transform of the density profile and \( P_{hh}(k|m_1, m_2) = b(m_1)b(m_2)P_l(k) \) where \( b(m_i) \) is the halo bias. The two halo term must reduce to \( P_l \) on linear scales so \( \bar{\rho} = \int dm_i b(m_i) n(m_i) \rho(k|m_i) \).

Another approach taken by Peacock and Smith (2000) is to approximate \( P_{2h}(k) = P_l(k) \) for all scales. However neither of these approaches result in the correct power spectrum so Smith et al. (2003) propose a two halo term that uses a scaling of the linear power spectrum with a cut off at high \( k \), since the halo model says \( P_{2h}(k) \) should be negligible on small scales and the one halo term should dominate.

Figure 1.4 shows how the one and two halo terms proposed in Peacock and Smith (2000) contribute to the overall HALOFIT function.

![Figure 1.4: Comparison of HALOFIT (solid line) and PD96 (dashed line) fitting formula to the Virgo simulations (triangles) at \( z = 0, 0.5, 1, 2, 3 \) from Smith et al. (2003). The dotted lines show the contributions from the one halo and two halo terms in the halo model.](image)

This analytical model parameterises the power spectrum and calibrates these parameters with a set of simulations with \( \Omega_{\text{tot}} = 1 \), \( \Omega_{\text{tot}} < 1 \) and \( \Omega_m = 1 \) with \(-2 \leq n \leq 0\), to
produce one of the most widely used non-linear fitting formulae for $\Lambda$CDM, which I will use to calculate the non-linear power spectrum in Chapter 3.

Simulations

The main method used to calculate the non-linear power spectrum is N-body simulations, such as the N-body codes GADGET (Springel et al., 2001; Springel, 2005a) and RAMSES (Teyssier, 2002). Generally these set up a distribution of particles at a suitably large redshift and evolve their positions by $dx = v dt$ where their velocities are found by $dv = v_t dt$ with $v_t$ given by the Euler equation (Equation 1.35) and background is evolved according to the Friedmann equations (Equations 1.18 to 1.19).

The $\Phi$ term in the Euler equation, found by solving the Poisson equation (Equation 1.36), is time consuming to calculate if the potential between each particle in the simulation is to be calculated, as for a simulation with $N$ particles this results in $N^2$ computations, however there many methods used that do not require this such as tree codes and particle mesh codes. Tree codes split up the spacetime into cubic cells, so that only particles from nearby cells are treated individually and particles in cells further away are treated as a single mass at the centre of the cell, where the size of the cells is smaller in high density regions to reduce the number of particle-particle calculations. Particle mesh (PM) codes work in Fourier space. This involves putting the density field onto a mesh and taking the Fourier transform simplifying $\nabla^2 \Phi = \frac{1}{2} \rho a^2 \Rightarrow \Phi = \frac{1}{2} \rho a^2 k^2$, so $\Phi$ can be found by just multiplying the density by $k^2$. This method is highly dependent on the mesh size, as too small and the calculations are time consuming, or too large and the mesh is not a good approximation to the real density field, so again adaptive tree methods are used where the mesh size depends on the local density. There are different methods to assigning the density field positions on the mesh (Martinez, 2008) which include using the nearest grid point to the particle and Cloud-in-Cell mass assignment, which makes the point particle a cubic cloud with edges equal to the mesh size, it then weights the mass of the particle over the cells it overlaps proportional to the amount it intersects.

In Chapter 4 I will look use simulation data from Baldi (2011b), which uses a modified version of GADGET, which is a tree-PM N-body code.

While these simulations allow very large regions of space to be simulated, they do not use fully general relativistic physics, so there is still much interest in exact GR models to model structure formation to test these approximations.
1.8 Cosmological Probes

There is much observational evidence that shows that the Universe is currently undergoing a period of accelerated expansion. This section will look at three of the main observations, the cosmic microwave background (CMB), large scale structure measurements (LSS) and supernovae (Sne). I will look at weak lensing results in chapter 2.

1.8.1 CMB

As the Universe expanded and cooled, photons from the hot plasma, formed in the early Universe, decoupled from the baryons and travelled unimpeded for the remainder of the Universe’s evolution. These photons have been redshifted during this time and have therefore cooled to $\sim 3$K today from $\sim 3000$K at decoupling. The isotropy of these photons, isotropic to $1/10^5$, hints that they all must have been in causal contact when they were first emitted at decoupling.

![Figure 1.5: WMAP7 TT power spectrum from Larson et al. (2011)](image)

The small temperature anisotropies observed are correlated due to several effects producing peaks as shown in Figure 1.5 from Larson et al. (2011). The positions of the peaks correspond to the sound horizon size at decoupling and are mainly dependent on the curvature of the Universe, where decreasing curvature results in a peak at smaller scales, and the presence of dark energy, where increasing the amount of dark energy moves the peak to larger scales, since the distance to the CMB increases. The amplitude of the odd
numbered peaks are associated with how much the early Universe plasma compresses due to gravity and the amplitude of the even numbered peaks are associated with how the plasma rarefies due to the radiation pressure (Hu and Dodelson, 2002). These effects depend on the baryon density since more baryons result in more compression before the pressure pushes the plasma out again, increasing the amplitude of the odd peaks. This increase in baryon density also results in the slowing of the oscillations which moves the peaks to smaller scales. The damping at small scales is due to Silk damping as discussed in Section 1.7.1. I will use predictions for the temperature correlation function from Planck in Chapter 5.1 to constrain the high redshift physics of the UDM model.

1.8.2 LSS

The acoustic oscillations in Section 1.7.1 provide a standard ruler for cosmology (Eisenstein et al., 2005). The oscillations in the plasma, caused by pressure, cease once the baryons and photons have decoupled and the baryons are left in a shell surrounding the point of gravitational infall at the sound horizon. This leads to a preferred length scale the size of the sound horizon creating a bump in the real space correlation function at this scale, which can be used as a standard ruler to constrain cosmology. This is the cause of the oscillations seen in the baryon power spectrum around $k \approx 0.1 \text{ h/Mpc}$, as this is the Fourier transform of the correlation function, as shown in Figure 1.6 (Anderson et al., 2012).

Figure 1.6: BOSS BAO results from Anderson et al. (2012)
et al., 2012), where the results from BOSS (Baryon Oscillation Spectroscopic Survey), a 10,000 square degree spectroscopic ground-based survey, are shown. These oscillations in the baryon power spectrum cause oscillations in the dark matter power spectrum (see Figure 1.2) due to the gravitational potential of the baryons.

Another probe of the underlying cosmology is redshift space distortions (Kaiser, 1987). These are due to measuring the redshift to an object along the line of sight and inferring its physical distance by the Hubble relation. However peculiar velocities due to gravitational infall mean that the observed redshift will not be only measuring the Hubble flow, but also the motions of the galaxies as shown in Figure 1.7 (Reid et al., 2012), which results in a squashed distribution of galaxies along the line of sight with elongations at small scales. These distortions can be used to infer the peculiar velocities of the galaxies and therefore measure $\Psi$, using Equation 4.7b where $\Psi$ is defined in Equation 4.1. Combining these observations with observations measuring $\Phi + \Psi$, such as lensing (see Chapter 2), gives a measurement of the anisotropic stress $\Phi - \Psi$.

![Figure 1.7: Left: Diagram showing how real space differs from the observed redshift space due to peculiar velocities. Right: Redshift space distortions from Reid et al. (2012). The top figure shows that there is squashing on large scales and the bottom figure shows there are “fingers of God” on small scales.](image-url)
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1.8.3 Type Ia Sne

A white dwarf that gains enough mass, either through accretion or mergers, to reach the Chandrasekhar limit (\( \sim 1.4 M_\odot \)) is no longer able to support itself, by electron degeneracy pressure, against gravitational collapse and explodes (Mazzali et al., 2007). The temporal evolution of the luminosity of the supernova follows a characteristic light curve, as shown in Figure 1.8 (Guy et al., 2007), which is generated by the decay of the Ni\(^{56} \) produced in the explosion. Using these light curves the supernovae can be used as standardisable candles, correcting for their colour and stretch (amplitude). From this the luminosity distance to each supernova can be calculated and plotted against their redshift in a Hubble diagram as shown in Figure 1.9 (Conley et al., 2011), measuring the expansion history.

1.9 Dark Energy and Alternatives

The current best fit model for the observed accelerated expansion and growth of large scale structure is \( \Lambda \)CDM. This model uses GR, but requires two additional components in the form of cold dark matter, CDM, to fit the observed structure growth, and a cosmological constant, \( \Lambda \), to fit the accelerated expansion. Although for CDM there are many proposed candidates in particle physics, there are no good candidates for \( \Lambda \) due to its small size. This along with other problems with \( \Lambda \) (Weinberg, 1989), such as the coincidence problem, have caused many models for alternatives to \( \Lambda \) to be proposed. Here I will look at a few of the proposed alternatives to a cosmological constant.

1.9.1 Modified Gravity

Lovelock (1971) showed that by seeking all tensors with the following properties

(a) \( A^{ij} \) is a function of the metric tensor \( g_{ab} \) and its first two derivatives

\[
A^{ij} = A^{ij}(g_{ab}; g_{ab,c}; g_{ab,cd})
\]

(b) \( A^{ij} \) is divergence free

\[
A^{ij}_{;j} = 0
\]

(c) \( A^{ij} \) is symmetric (This condition is not required in the case of \( n = 4 \) since (a) and (b) imply \( A^{ij} \) is symmetric)

\[
A^{ij} = A^{ji}
\]
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(d) $A^{ij}$ is linear in the second derivatives of $g_{ab}$

The field equations in vacuo are then assumed to take the form

$$A^{ij} = 0$$

The only $A^{ij}$ these constraints allow is (Cartan, 1922; Weyl, 1952)

$$A^{ij} = aG^{ij} + bg^{ij}$$

where $a$ and $b$ are constants. These are just the EFEs with a cosmological constant.

This means that in order to modify GR while keeping a metric theory of gravity we must break away from these restrictions and do one (or more) of the following as listed in Clifton et al. (2012):

(a) Consider other fields, beyond (or rather than) the metric tensor.

(b) Accept higher derivatives of the metric in the field equations.

(c) Have more dimensions.

(d) Give up on symmetric, divergence free field equations.

(e) Give up locality.

In Chapter 3 I will consider two modifications in the form of (b) for $f(R)$ gravity and (c) for the DGP model.

1.9.2 Dark Energy

Quintessence

In quintessence models, the equation of state of the dark energy becomes a function of the scale factor so $w \rightarrow w(a)$. This can avoid the problem of the small value of the cosmological constant, since it is not required that the vacuum energy was always so small. The action of a scalar field, $\Phi$, is given by (Peebles and Ratra, 2003)

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} g^{\mu\nu} \Phi,_{\mu} \Phi,_{\nu} - V(\Phi) \right]$$

where $V(\phi)$ is the scalar field self-interaction potential and $g = \text{det}(g_{\mu\nu})$. To derive the energy momentum tensor for this action we need to vary the action with respect to the
where the variation of the metric is $\delta g_{\mu\nu}$ and then the variation of the determinant is given by (Hawking and Ellis, 1973)

$$\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}$$

Substituting this into Equation 1.70 gives

$$\delta S = \frac{1}{2} \int d^4x \sqrt{-g} g^{\mu\nu} \left( \Phi,_{\mu} \Phi,_{\nu} + g_{\mu\nu} \left[ \frac{1}{2} g^{\rho\lambda} \Phi,_{\rho} \Phi,_{\lambda} - V(\Phi) \right] \right)$$

Since

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \delta L_m$$

we obtain

$$T^{\mu\nu} = \Phi,_{\mu} \Phi,_{\nu} - g_{\mu\nu} \left[ \frac{1}{2} \Phi,_{\rho} \Phi,_{\rho} - V(\Phi) \right]$$

This has the same form as the energy-momentum tensor for a perfect fluid (Equation 1.12) with a density and pressure given by

$$\rho_\Phi = \frac{1}{2} \Phi^2,_{t} + V(\Phi)$$

$$p_\Phi = \frac{1}{2} \Phi^2,_{t} - V(\Phi)$$

Substituting these forms for the density and pressure into the energy-momentum tensor conservation equation, Equation 1.16, gives

$$\Phi,_{\mu} + 3H \Phi,_{t} + \frac{dV}{d\Phi} = 0$$

Since the Hubble equation is just the sum of the background densities and $\rho \propto a^{-3(1+w)}$ for $w=\text{constant}$ then the introduction of a time varying scalar field alters the Hubble equation so that

$$H^2 = \frac{\Omega_r}{a^4} + \frac{\Omega_m}{a^3} + \frac{\Omega_k}{a^2} + \frac{\Omega_{DE}}{a^{3(1+w)}}$$
where \( w = p_\Phi / \rho_\Phi \). The growth equation remains the same as 1.39 where the difference in growth is only seen through the change in the Hubble evolution.

I will use a quintessence model to compare to DGP in Chapter 3 and these equations will be used also in Chapter 4 where they will be modified to include a coupling between the components of the model.

### Coupled Dark Energy

This is a class of models where the dark sector (i.e. dark matter and dark energy) is coupled, which could alleviate the coincidence problem. In these models the conservation of the total energy momentum tensor is not violated so

\[
T_{\nu\mu} = \sum_\alpha T_{(\alpha)\nu\mu} = 0 \tag{1.78}
\]

but the individual stress energy tensor for each component is not conserved giving

\[
T_{(\alpha)\nu\mu} = Q_{(\alpha)\nu} \tag{1.79}
\]

which allows us to show that the conservation of the total energy momentum tensor is not violated provided we constrain ourselves to couplings that abide by the following (Amendola, 2000)

\[
\sum_\alpha Q_{(\alpha)\nu} = 0 \tag{1.80}
\]

where \( Q_{(\alpha)\nu} \) accounts for the coupling between each species. Since we do not observe any coupling between baryons and the dark sector, the coupling between these components must be close to zero; however we can still look at coupling dark matter and dark energy.

This class of models will be discussed in more detail in Section 4.2.

### Unified Dark Matter

Unified dark matter models have a single component that acts as both the dark matter and dark energy, providing the observed accelerated expansion and growth. There are many proposed UDM models (e.g. Bertacca et al., 2010, 2011a,b; De-Santiago and Cervantes-Cota, 2011; Piattella and Bertacca, 2011; Pujolas et al., 2011; Camera et al., 2012) including:
• Chaplygin gas (Kamenshchik et al., 2001; Bilic et al., 2002; Bento et al., 2002) is a perfect fluid with an EoS of the form

\[ p = -\frac{A}{\rho} \Rightarrow \rho = \sqrt{A + \frac{B}{a^6}} \]  

(1.81)

where \( A \) and \( B \) are constants and the conservation of the energy momentum tensor has been used to get the second expression. This gives \( \rho \approx \sqrt{B/a^3} \) for small \( a \), and \( \rho \approx \sqrt{A} \) and \( p \approx -\sqrt{A} \) for large \( a \).

• Scherrer model (Scherrer, 2004; Bertacca et al., 2007) where the equation of state is purely kinetic so \( p = F(X) \) and \( \rho = 2X \frac{dF}{dX} - F(X) \) where \( X = \frac{1}{2} \nabla_\mu \Phi \nabla^\mu \Phi \). Suppose \( F(X) \) is a function with a maximum or minimum at \( \hat{X} \), which indicates where a transition in the EoS occurs. This means \( F(X) \) can be approximated as a parabola around \( \hat{X} \) so we can expand \( F(X) = F_0 + F_2(X - \hat{X})^2 \). Substituting this form for \( F(X) \) into \( \rho \) and \( p \) shows the UDM behaves like radiation in the early Universe \( (X \gg \hat{X}) \) and at late times it acts like the sum of a dark matter term and dark energy term.

• Single dark perfect fluid with an affine barotropic EoS (Quercellini et al., 2007; Pietrobon et al., 2008). In this model we parameterise a barotropic (i.e. pressure only depends on density \( p(\rho) \)) EoS giving \( p_X = p_0 + \alpha \rho_X \) and substituting this into \( T_{\mu\nu} = 0 \) gives \( \rho_X(a) = \rho_\Lambda + (\rho_X|_{z=0} - \rho_\Lambda)a^{-3(1+\alpha)} \), and therefore \( w_X = -(1+\alpha)\frac{\Delta a}{\rho_X} \). so at \( z = 0 \) the EoS tends towards a cosmological constant as desired.

There are many other models that have been proposed; however, many have problems with giving the observed structure formation or with looking like \( \Lambda \)CDM at all times. In Chapter 5.1 I will look at a model with a barotropic EoS with a fast transition, which can give the observed structure formation and does not look like \( \Lambda \)CDM at all times.

### 1.9.3 Inhomogeneous models

Models which do not alter the components or the model of gravity, but instead they alter the spacetime geometry, are inhomogeneous models. There are many variations of these such as

• Void models (e.g. February et al., 2010), where we are located in a large void modelled by the Lemaitre-Tolman-Bondi metric. This breaks the Copernican principle as it requires we have a special location in the Universe resulting in us observing an accelerated expansion.
• Swiss cheese models (e.g. Kantowski, 1969), which are created using a FRW modified by the introduction of mass-compensating homogeneities modelled by the LTB metric.

• Backreaction (e.g. Schwarz, 2010) which occurs because in GR averaging does not commute, since $G_{\mu\nu}$ depends non-linearly on the metric so $\langle G(g) \rangle \neq G(\langle g \rangle)$. To find the form of the backreaction term we need to take the average of the Raychaudhuri equation, which tells us about how nearby particles move with respect to each other

$$\theta_{,t} = -2\sigma^2 - \frac{1}{3}\theta^2 - \frac{1}{2}\rho$$

where $\theta$ is the expansion, and $\sigma$ is the shear. This looks like Equation 1.19 with $\theta = 3H$ without the $\sigma$ term (since there is no shear in a FRW spacetime), so the difference between Equation 1.19 and the averaged form of the Raychaudhuri equation is the backreaction term $Q_D(t) = 2/3(\langle \theta^2 \rangle_D - \langle \theta \rangle_D^2) - 2\langle \sigma^2 \rangle_D$. If the backreaction term is large enough then the average acceleration of the expansion $(a_D)_{,tt} > 0$.

Using constraints from current observations (e.g Zibin et al., 2008; Garcia-Bellido and Haugboelle, 2008; Moss et al., 2011; Zumalacarregui et al., 2012; de Putter et al., 2012) shows that these models cannot explain the recent accelerated expansion; however they are still of interest on scales where homogeneity cannot be assumed.

1.10 Outline of thesis

The observed expansion history can be explained using some form of dark energy or a modification of gravity; however there can be a distinct change in the growth of structures depending on the model adopted. I am going to investigate the effect of these models on the observations made by lensing, since lensing probes both the expansion history and the growth of structure, and can break degeneracies in parameters from observations that measure the expansion history alone.

In Chapter 2 I will look at gravitational lensing in more detail, along with the equations that I will use throughout this thesis. I will also look at previous results and future lensing surveys.

In Chapter 3 I will look at whether upcoming weak lensing surveys will be able to distinguish between different models of gravity, specifically in the non-linear regime since it provides much of the power for lensing and can be most easily probed by current and upcoming lensing surveys. The effect of including the non-linear regime in modified
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Gravity lensing predictions is examined, including the small-scale GR limit, to see how useful weak lensing will be overall when trying to determine the correct model of gravity. First I look at DGP and f(R) gravity models as examples, and investigate weak lensing’s ability to differentiate between these models and dark energy models. I then will take a more phenomenological point of view, by parameterising the shape of the matter power spectrum and examining the sensitivity of weak lensing observables to changes to the matter distribution when the expansion history is the same for each model considered. Using these parameters I will show how strongly DES and Euclid will be able to discriminate between different growth histories with identical expansion histories.

In Chapter 4 I present non-linear weak lensing predictions for coupled dark energy models using the CoDECS simulations. I calculate the shear correlation function and error covariance expected for these models, for forthcoming ground-based (such as DES) and space-based (Euclid) weak lensing surveys. I obtain predictions for the discriminatory power of a ground-based survey similar to DES and a space-based survey such as Euclid in distinguishing between $\Lambda$CDM and coupled dark energy models; I show that using the non-linear lensing signal we could discriminate between $\Lambda$CDM and exponential constant coupling models with $\beta_0 \geq 0.1$ at 99.994% confidence level with a DES-like survey, and $\beta_0 \geq 0.05$ at 99.99994% confidence level with Euclid. I also demonstrate that estimating the coupled dark energy models’ non-linear power spectrum, using the $\Lambda$CDM Halofit fitting formula, results in biases in the shear correlation function that exceed the survey errors.

In Chapter 5 I present constraints on linear scales on the time and speed of equation of state transitions for adiabatic fast transition unified dark matter models. I calculate the shear correlation function for DES and Euclid, and combine this with the CMB temperature correlation predicted for Planck. The combination of these high and low redshift probes show that the transition redshift $z_t > 5$ at a 95% confidence level and the maximum time the transition can take $\tau < 5 \times 10^{-6}/H_0$ at a 95% confidence level.

In Chapter 6 I will examine whether the approximations we make in lensing are good enough for forthcoming surveys, where the errors will no longer be statistically dominated. Here I look at how lensing quantities are affected by the density profile, and the presence of pressure, additional curvature and the cosmological constant in a full GR model. The lack of analytical models for length scales between the very linear and very non-linear means that we have some freedom about the model at those scales. I will investigate how much the model at these length scales alters the expected lensing result. In my approach I will use an LTB model to model a gravitational lens and solve the null geodesics for the light rays numerically. From the paths of these rays I will then calculate the lensing quantities and compare with the usual result.
Finally in Chapter 7 I will draw my conclusions and discuss how this work will be extended in the future.
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Figure 1.8: SNe light curve from Guy et al. (2007)

Figure 1.9: Combined SNe results from several surveys from Conley et al. (2011)
Chapter 2

Gravitational lensing

The theory of General Relativity shows that light rays are deflected by the presence of a large gravitational potential. This means that light rays from distant sources are bent and their image as seen by observers is distorted. In cases where the gravitational potential is large enough, strong lensing takes place and the source is multiply imaged to form an Einstein ring or arcs. However in the majority of the Universe, where there are not strong enough potentials, the source is not multiply imaged, it is instead just distorted. This weak lensing of many images has to be measured and averaged in order to get a detectable signal, since the distortions are very small. In this chapter I will discuss how light rays are bent by gravity, how these light rays deviate from each other and how we can use these observed distortions to constrain the underlying theory of gravity and components of the Universe.

2.1 Geodesic Equation

In GR we regard gravity not as a force but as a consequence of spacetime geometry. Particles with no external forces upon them will move along geodesics where their acceleration in the ray direction is zero, so the magnitude of their velocity, their tangent vector $\dot{x}^\mu$, is constant allowing us to write (Eddington, 1923)

$$D_\sigma \dot{x}^\mu = 0$$  \hspace{1cm} (2.1)

where dots represent $\frac{d}{d\sigma}$ and $D_\sigma$ represents the covariant derivative w.r.t. an affine parameter $\sigma$, which is chosen such that the condition set above is true and is unique up to the transformation $\sigma \rightarrow a\sigma + b$. 
CHAPTER 2. GRAVITATIONAL LENSING

Writing out the covariant derivative explicitly gives

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\gamma} \dot{x}^\nu \dot{x}^\gamma = 0$$

(2.2)

Specifically light follows null geodesics so the interval $ds^2 = 0$. Using these equations we can calculate the trajectory of light rays in any spacetime.

### 2.2 Lensing geometry

Here I define the typical geometry we have in mind when describing how a light ray is bent by the presence of a gravitational lens, which is shown in Figure 2.1.

![Figure 2.1: Geometry of a gravitational lens system](image)

From this figure it can be seen that the positions of the source and image can be related by the lens equation

$$\tilde{\theta} = \beta + \tilde{\alpha} = \beta + \frac{D_{ls}}{D_s} \tilde{\alpha}$$

(2.3)

where I have used the geometrical relation between $\alpha$ and the bend angle, $\tilde{\alpha}$, by $\tilde{\alpha} = \frac{D_{ls}}{D_s} \alpha$, which can be found from Figure 2.1. These $D_s$ are angular diameter distances as we will see in the Section 2.4.2. Using this relation it is clear that we can map the initial angular position onto the final observed angular position if we know the physics of what causes the bend angle.
2.3 Lensing in the Schwarzschild metric

As a first example of the bending of light by gravity I will use the Schwarzschild metric, since the mass $M$ is a point mass and has no spatial or temporal dependence, which simplifies the geodesic equations.

The Schwarzschild metric is given by solving the EFEs in a static isotropic metric, $ds^2 = A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$, where the spacetime is empty $T_{\mu\nu} = 0$ around a point mass $M$ giving, (Schwarzschild, 1916)

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.4}$$

By comparing this with the Minkowski metric we can see that the proper time interval $d\tau$ and $dt$ are related by $d\tau = \left(1 - \frac{2M}{r}\right)^{1/2}dt$, so at large $r$ where $\frac{2M}{r} \to 0$, $t \to \tau$ and at small $r$ clocks run more slowly. The proper radial distance interval $d\tilde{r}$ is related to $dr$ by $d\tilde{r} = \left(1 - \frac{2M}{r}\right)^{-1/2}dr$, so at large distances $r \to \tilde{r}$ and at small $r$ the geometry is stretched in the radial direction. However the angular part of the metric is the same as for the Minkowski metric so we can see that $r$ is the angular diameter distance.

Looking in just the plane where $\theta = \pi/2$ the geodesic equations from Equation 2.2 give

$$\left(1 - \frac{2M}{r}\right)\dot{t} = k \tag{2.5a}$$

$$\ddot{r} + \left(1 - \frac{2M}{r}\right)\frac{M}{r^2}\dot{r}^2 - \left(1 - \frac{2M}{r}\right)^{-1}\frac{M}{r^2}\dot{r}^2 - \left(1 - \frac{2M}{r}\right)r^2\dot{\phi}^2 = 0 \tag{2.5b}$$

$$\dot{\phi} = \frac{J}{r^2} \tag{2.5c}$$

where $J$ and $k$ are constants of integration.

Substituting Equations 2.5a and 2.5c into the null geodesic condition, $\left(1 - \frac{2M}{r}\right)\dot{t}^2 - \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = 0$, and rearranging to find $\dot{r}$ gives

$$\dot{r}^2 = k^2 - \left(1 - \frac{2M}{r}\right)\frac{J^2}{r^2} \tag{2.6}$$
In order to find the bend angle of a photon around a mass $M$ we need to solve for the \( \phi \) coordinate, which we can do as follows

\[
\frac{d\phi}{dr} = \frac{\phi}{r^2} = \left( \frac{1}{\xi^2} - \frac{1}{r^2} \left( 1 - \frac{2M}{r} \right) \right)^{-1/2} \Rightarrow \phi = \int_{-\infty}^{\infty} \frac{1}{r^2} \left( \frac{1}{\xi^2} - \frac{1}{r^2} \left( 1 - \frac{2M}{r} \right) \right)^{-1/2} \, dr \tag{2.7}
\]

where the new constant \( \xi = J/k \) is the impact parameter. This can be used to find the exact \( \phi(r) \) for any ray, however for an analytical result we need to restrict ourselves to \( M/\xi \ll 1 \). We can see that when \( M = 0 \) this gives the expected result of a straight line, and expanding in terms of \( \frac{2M}{r} \) to first order and integrating gives the bend angle

\[
\hat{\alpha} = \frac{4M}{\xi} \tag{2.8}
\]

From Figure 2.1 we can see \( \xi = \theta D_i \), and substituting this and 2.8 into Equation 2.3 we obtain

\[
\hat{\alpha} = \frac{4M}{\theta D_i} \Rightarrow \theta - \beta = \frac{4M}{\theta} \frac{D_{ls}}{D_l D_s} \tag{2.9}
\]

If \( \eta = 0 \) (see Figure 2.1) then \( \beta = 0 \) and then this gives a characteristic radius, denoted the Einstein radius \( \theta_E \) given by substituting \( \beta = 0 \) in the equation above

\[
\theta_E = \sqrt{4M} \frac{D_{ls}}{D_l D_s} \tag{2.10}
\]

If the lens is circularly symmetric then this results in an Einstein ring otherwise the ring is broken into multiple images and arcs typically separated by a distance \(~ 2\theta_E\).

I will return to the Schwarzschild metric in Chapter 6, but for the rest of this chapter and chapters 3 to 5 I will be looking at weak lensing since most gravitational potentials in the Universe are not strong enough to create the arcs and rings seen in strong lensing.

### 2.4 Weak lensing

I will now, and for the rest of this chapter look at weak lensing. We define weak lensing for small potentials \(|\Phi| \ll 1\) and therefore in a cosmological context we will be able to use a perturbed FRW to derive the weak lensing quantities

\[
ds^2 = (1 + 2\Psi) \, dt^2 - (1 - 2\Phi) \, a^2 \, dx^i dx^i \tag{2.11}
\]
where $\Phi$ and $\Psi$ are potentials. Comparing to the Minkowski metric we can see $d\tau = \sqrt{1 + 2\Psi} \, dt$ and $d\tilde{x} = \sqrt{1 - 2\Phi} \, dx$, so for small potentials $t \approx \tau$ and $x \approx \tilde{x}/a$ so is the comoving distance. In GR, without modifying gravity, $\Psi = \Phi$ in the absence of anisotropic stress, which is the form I will use for the rest of this chapter; however in Chapter 3 I will look at the effect of modified gravity on weak lensing, and hence allow $\Psi$ and $\Phi$ to have distinct values.

I will use this metric throughout the rest of this chapter since it is the most widely used to calculate weak lensing quantities.

### 2.4.1 Geodesic deviation

![Figure 2.2: Two geodesics separated by a distance $\xi$](image)

In this section and throughout this thesis I will look at how the path of light rays deviate due to gravity. I will then characterise these deviations into observable quantities that can be used to discriminate between cosmological models.

If we consider two geodesics $\bar{x}^\mu$ and $x^\mu$ separated by a vector $\xi^\mu$ as shown in Figure 2.2 so that (Hobson et al., 2006)

$$\bar{x}^\mu = x^\mu + \xi^\mu$$  \hspace{1cm} (2.12)

We can use the geodesic equation (Equation 2.2) to find the path of the geodesics

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha \beta} \dot{x}^\nu \dot{x}^\sigma = 0$$  \hspace{1cm} (2.13a)

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha \beta} \dot{x}^\alpha \dot{x}^\beta = 0$$  \hspace{1cm} (2.13b)
We can relate the two Christoffel symbols using a Taylor expansion in $\xi$, so at first order we obtain

$$\bar{\Gamma}^\mu_{\alpha\beta} = \Gamma^\mu_{\alpha\beta} + \Gamma^\mu_{\alpha\beta,\gamma} \xi^\gamma$$

(2.14)

where Substituting 2.12 and 2.14 into 2.13 gives to first order in $\xi$

$$\ddot{\xi}^\mu + \Gamma^\mu_{\alpha\beta,\gamma} \dot{x}^\alpha \dot{x}^\beta \dot{\xi}^\gamma + \Gamma^\mu_{\alpha\beta} \dot{x}^\alpha \ddot{\xi}^\beta + \Gamma^\mu_{\alpha\beta,\gamma} \dot{\xi}^\alpha = 0$$

(2.15)

We can substitute in the covariant derivative of $\xi$ and use $\frac{d}{dx^\mu} = \frac{d}{d\sigma}/\dot{x}^\mu$ to obtain

$$\xi^\mu_{;\alpha} = \xi^\mu_{,\alpha} + \Gamma^\mu_{\alpha\beta} \xi^\beta \Rightarrow D\xi^\mu_{\sigma} = \frac{d}{d\sigma} = \dot{\xi}^\mu + \Gamma^\mu_{\alpha\beta} \dot{x}^\alpha \xi^\beta$$

(2.16)

Taking this to the second derivative gives

$$D^2_{\sigma} \xi^\mu = \frac{d}{d\sigma} \left( \dot{\xi}^\mu + \Gamma^\mu_{\alpha\beta,\nu} \dot{x}^\nu \xi^\alpha \right) + \Gamma^\mu_{\gamma\delta} \left( \dot{\xi}^\gamma + \Gamma^\gamma_{\alpha\beta} \dot{x}^\beta \xi^\alpha \right) \dot{x}^\delta$$

(2.17)

$$= \ddot{\xi}^\mu + \Gamma^\mu_{\alpha\beta,\nu} \dot{x}^\nu \xi^\alpha + \Gamma^\mu_{\alpha\beta} \dot{x}^\beta \ddot{\xi}^\alpha + \Gamma^\mu_{\alpha\beta} \dot{\xi}^\alpha \dot{x}^\beta + \Gamma^\mu_{\gamma\delta} \left( \ddot{\xi}^\gamma + \Gamma^\gamma_{\alpha\beta} \dot{x}^\beta \xi^\alpha \right) \dot{x}^\delta$$

Substituting Equation 2.13a into Equation 2.17 gives

$$D^2_{\sigma} \xi^\mu = \ddot{\xi}^\mu + \Gamma^\mu_{\alpha\beta,\nu} \dot{x}^\nu \xi^\alpha + \Gamma^\mu_{\alpha\beta} \dot{x}^\beta \ddot{\xi}^\alpha + \Gamma^\mu_{\alpha\beta} \dot{\xi}^\alpha \dot{x}^\beta + (\Gamma^\mu_{\beta\delta} \Gamma^\beta_{\alpha\gamma} - \Gamma^\mu_{\alpha\beta} \Gamma^\beta_{\gamma\delta}) \dot{x}^\gamma \dot{x}^\delta \xi^\alpha$$

(2.18)

Substituting in Equation 2.15 and Equation 1.1 reduces this to the form

$$D^2_{\sigma} \xi^\mu + R^\mu_{\alpha\beta\gamma} \dot{x}^\alpha \dot{x}^\beta \xi^\gamma = 0$$

(2.19)

which can also be written as (using $\frac{d}{dx^\mu} = \frac{d}{d\sigma}/\dot{x}^\mu$)

$$\xi^\mu_{;\alpha\beta} - \xi^\mu_{;\beta\alpha} + R^\mu_{\alpha\beta\gamma} \xi^\gamma = 0$$

(2.20)

This is the equation of geodesic deviation, where we can see that the deviation of light rays only depends on the curvature of spacetime characterised by the Riemann tensor.
2.4.2 Shear, convergence and rotation

The Riemann tensor can be decomposed into its trace (Ricci tensor $R_{\mu\nu}$) and traceless (Weyl tensor $C_{\mu\alpha\beta\gamma}$) part (Wald, 1984)

$$R_{\mu\alpha\beta\gamma} = C_{\mu\alpha\beta\gamma} - \frac{1}{2} \left( g_{\mu\beta} S_{\alpha\gamma} + g_{\alpha\gamma} S_{\mu\beta} - g_{\mu\gamma} S_{\alpha\beta} - g_{\alpha\beta} S_{\mu\gamma} \right) - \frac{R}{12} \left( g_{\mu\beta} g_{\alpha\gamma} - g_{\mu\gamma} g_{\alpha\beta} \right)$$

(2.21)

where the $S_{\mu\nu}$ component appears because we have further decomposed the Ricci tensor into its trace ($R$) and traceless parts ($S_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R$). This decomposition allows us to see how different parts of the spatial curvature affect the light rays. The first component of this gives the source of shear, since it is traceless, so changes the shape but not the volume; the second component is antisymmetric so is the rotation, and the third component is the source of the convergence since it changes the volume isotropically.

We can also decompose the geodesic deviation vector $\xi^\mu$ into components orthogonal to the direction of the light ray $\dot{x}^\mu$, by introducing a basis which has two vectors $E_1^\mu$ and $E_2^\mu$ along the light ray that are orthonormal and parallel transported along the ray, and orthogonal to $\dot{x}^\mu$, giving the conditions (Perlick, 2010)

$$E_i^\mu E_\mu^\alpha = \delta_{ij} \quad E_i^\mu \dot{x}_\mu = 0$$

(2.22)

$E_1^\mu$ and $E_2^\mu$ are unique up to rotations in the plane orthogonal to $\dot{x}^\mu$, so using the conditions in Equation 2.22 and that the vectors are parallelly transported along the ray, a useful choice of frame provides

$$\dot{x}^\mu = \left[ \frac{1}{\sqrt{g_{11}}}, 0, 0, \frac{1}{\sqrt{g_{33}}} \right]$$

$$E_1 = \left[ 0, \frac{1}{\sqrt{g_{11}}}, 0, 0 \right]$$

(2.23)

$$E_2 = \left[ 0, 0, \frac{1}{\sqrt{g_{22}}}, 0 \right]$$

so any derivative in $x_1$ or $x_2$ are orthogonal to the null geodesic and each other, and any derivative in $x_3$ is along the geodesic.

Using this basis we can decompose the deviation vector as $\xi^\mu = \xi_1 E_1^\mu + \xi_2 E_2^\mu + \xi_0 \dot{x}^\mu$, where $\xi_0$ is the magnitude of the vector in the $\dot{x}^\mu$ direction and $\xi_1$ and $\xi_2$ are the magnitudes of the vector in the $E_1^\mu$ and $E_2^\mu$ directions respectively. Using this decomposition with the geodesic deviation equation (Equation 2.19), taking the dot product with $\xi_i^\mu$, so $\xi_1^\mu = \xi_1 E_1^\mu$, $\xi_2^\mu = \xi_2 E_2^\mu$ and $\xi_0^\mu = \xi_0 \dot{x}^\mu$, and summing over all directions gives (Peebles,
where for the second step I have used the orthogonality conditions in Equation 2.22 and defined $e^\alpha = E_1^\alpha + i E_2^\alpha$. Then using the Ricci decomposition in Equation 2.21 we can write the source of convergence and rotation $\mathcal{R}(\sigma)$ as (Seitz, 1993)

$$\mathcal{R} = \frac{1}{2} R_{\mu\nu} \ddot{x}^\mu \dot{x}^\nu$$

(2.25)

and the source of shear $\mathcal{F}(\sigma)$ as

$$\mathcal{F} = -\frac{1}{2} C_{\alpha\beta\gamma\delta} \dot{x}^\alpha \dot{x}^\beta \dot{x}^\gamma \dot{x}^\delta$$

(2.26)

These quantities are then integrated back along the line of sight to calculate the lensing quantities seen by the observer. This means that the observer is at $\sigma = 0$. We can scale $\sigma$ appropriately so that close to the observer $\sigma$ equals the proper length, so for small angles $\xi|_{\sigma=0} = \tilde{\theta}$, which can be more generally written as

$$\tilde{\xi}(\sigma) = D(\sigma) \tilde{\theta}$$

(2.27)

where the $2 \times 2$ Jacobian matrix $D(\sigma)$ linearly maps $\tilde{\theta}|_{\sigma}$ to $\tilde{\xi}(\sigma)$. The geodesic deviation, Equation 2.19, can then be rewritten as

$$\ddot{D}(\sigma) \tilde{\theta} + \mathcal{T}(\sigma) D(\sigma) \tilde{\theta} = 0$$

(2.28)

which gives

$$\ddot{D}(\sigma) + \mathcal{T}(\sigma) D(\sigma) = 0$$

(2.29)

where $R^\mu_{\alpha\beta\gamma} \dot{x}^\alpha \dot{x}^\beta$ has also been replaced with the optical tidal matrix given by

$$\mathcal{T}(\sigma) = \begin{pmatrix}
\text{Re}(\mathcal{R}(\sigma)) + \text{Re}(\mathcal{F}(\sigma)) & \text{Im}(\mathcal{F}(\sigma)) + \text{Im}(\mathcal{R}(\sigma)) \\
\text{Im}(\mathcal{F}(\sigma)) - \text{Im}(\mathcal{R}(\sigma)) & \text{Re}(\mathcal{R}(\sigma)) - \text{Re}(\mathcal{F}(\sigma))
\end{pmatrix}$$

(2.30)

The initial conditions for Equation 2.29 are $D(0) = 0$ and $\dot{D}(0) = I$, where $I$ is the identity matrix, since $D(\sigma) = \sigma I$ close to the observer. This means the Jacobian can be
\[ D(\sigma) = \sigma I + \int_0^\sigma d\sigma'(\sigma - \sigma') T(\sigma') D(\sigma') \]  

(2.31)

Using the form of \( E_1 \) and \( E_2 \) given in Equation 2.23 and substituting in the metric from Equation 1.13 and Equations 1.1 and 1.5 into 2.25 and 2.26 we find the sources of convergence and shear for an FRW to be

\[ R_{\text{FRW}} = -\frac{H_t}{a^2} = \frac{1}{a^2} \left( \frac{1}{2}(\rho + p) - \frac{\Lambda}{3} \right) \]  

(2.32a)

\[ F_{\text{FRW}} = 0 \]  

(2.32b)

where I have used \( \dot{a}/\dot{a}_0 = \frac{1}{a} \). These equations show that while there is convergence in a homogeneous and isotropic spacetime with no perturbations, there is no shear, giving \( T_{\text{FRW}} = R_{\text{FRW}} I \), so \( D(\sigma) = D_A(\sigma) I \), where \( D_A(\sigma) \) is the angular diameter distance, which can be seen by substituting the form for \( R_{\text{FRW}} \) from Equation 2.32a into Equation 2.29 and solving for \( D(\sigma) \).

For a perturbed FRW \( R \) and \( F \) are modified to give in the Newtonian limit to first order in perturbations with the background subtracted (which can be found using the same method as above but with the metric from Equation 4.1)

\[ R_{\text{perturbed}} = -\frac{1}{a^2} \left( \nabla^2 \Phi + i(\Phi_{,12} - \Phi_{,21}) \right) \]  

(2.33a)

\[ F_{\text{perturbed}} = -\frac{1}{a^2} \left( \Phi_{,11} - \Phi_{,22} + i(\Phi_{,12} + \Phi_{,21}) \right) \]  

(2.33b)

This gives an optical tidal matrix of the form

\[ T_{\text{perturbed}} = -\frac{1}{a^2} \begin{pmatrix} \nabla^2 \Phi + (\Phi_{,11} - \Phi_{,22}) & 2\Phi_{,21} \\ 2\Phi_{,21} & \nabla^2 \Phi - (\Phi_{,11} - \Phi_{,22}) \end{pmatrix} \]  

(2.34)

Therefore the form of the optical tidal matrix shows that the geodesic deviation is purely due to the gradient of the potential, so lensing is a direct probe of the gravitational potentials. It should be noted that in the more general form for the perturbed FRW where \( \Phi \neq \Psi \), \( \Phi \to \frac{1}{2}(\Phi + \Psi) \), so lensing measures the combination of \( \Phi \) and \( \Psi \), which I will use in Chapter 3.

To calculate the amount of lensing along the line of sight we need to solve the geodesic deviation equation. This equation can be simplified if we treat inhomogeneities as being geometrically thin, so they have no width in the line of sight direction, and
project the optical tidal matrix onto a plane at $\sigma'$

$$\mathcal{T}_{\text{projected}} = \delta(\sigma - \sigma') \int_{-\infty}^{\infty} \mathcal{T}_{\text{perturbed}} d\sigma$$  \hspace{1cm} (2.35)$$

Projecting the potentials in the perturbed optical tidal matrix, Equation 2.34 gives

$$\mathcal{T}_{\text{projected}} = -\frac{2}{a^2} \begin{pmatrix} \psi_{,11} & \psi_{,21} \\ \psi_{,21} & \psi_{,22} \end{pmatrix}$$  \hspace{1cm} (2.36)$$

and substituting this into Equation 2.31 gives the usual Jacobian for mapping between an unlensed image to the lensed image due to a projected potential $\psi$. We can define a dimensionless Jacobian $\mathcal{A}$ by dividing $D$ by the angular diameter distance $D_A$ giving

$$\mathcal{A} = \mathcal{I} + \int_0^{\sigma} d\sigma' \frac{(\sigma - \sigma')\sigma'}{\sigma} \mathcal{T}(\sigma')$$  \hspace{1cm} (2.37)$$

where here I have used the Born approximation, which is the approximation that as long as the deflections are small compared to the scale on which the mass distribution changes significantly, then we can integrate along a straight line, which is true in most astrophysical applications. This allows us to write the total deflection along the line of sight as the sum of the deflections due to the potential along the undeflected trajectory.

The dimensionless Jacobian $\mathcal{A}$ is given by

$$\mathcal{A} = \begin{pmatrix} 1 - \kappa - \gamma_1 & -\omega - \gamma_2 \\ \omega - \gamma_2 & 1 - \kappa + \gamma_1 \end{pmatrix}$$  \hspace{1cm} (2.38)$$

where we have decomposed $\mathcal{T}$ into its trace $\kappa$, an isotropic dilation that we denote the convergence

$$\kappa = \frac{1}{2} (\psi_{,11} + \psi_{,22}) = \frac{1}{2} \nabla^2 \psi$$  \hspace{1cm} (2.39)$$

$\omega$ is the antisymmetric part so is a rotation

$$\omega = \frac{1}{2} (\psi_{,12} - \psi_{,21})$$  \hspace{1cm} (2.40)$$

and $\gamma$ is the symmetric traceless part as it changes the shape but not the area at linear order and therefore is a shear, where $\gamma = \gamma_1 + i\gamma_2 = |\gamma|e^{2i\varphi}$.

$$\gamma_1 = \frac{1}{2} (\psi_{,11} - \psi_{,22})$$  \hspace{1cm} (2.41)$$
The subscripts 1 and 2 here correspond to the $E_1$ and $E_2$ directions respectively. It can be easily seen that if $\psi_{12} = \psi_{21}$ then there is no rotation and $\gamma_2 = \psi_{12}$, which I will assume for the rest of this chapter and chapters 3, 4 and 5. This is true here as $\Phi$, and therefore $\psi$, is a scalar. The effect of $\kappa$, $\omega$ and $\gamma$ are shown in Figure 2.3.

Writing $\vec{\xi}_{\text{FRW}} = D_A \vec{\beta}$ and using Equation 2.27 gives $\vec{\beta} = A \vec{\theta}$ showing the direct mapping from the source size and shape to the observed image size and shape.

The actual distortion we measure is the reduced shear $g_i = \frac{\gamma_i}{1 - \kappa}$ giving the distortion matrix as

$$
A = \begin{pmatrix}
1 - g_1 & -g_2 \\
-g_2 & 1 + g_1
\end{pmatrix},
$$

where we have split the matrix from Equation 2.38 into the quantity we seek to measure $g_{1,2}$ and unknown part $\kappa$. It is often assumed that $g \approx \gamma$ which is true to first order since in weak lensing $\kappa$ and $\gamma$ are $\sim 1\%$; however it has been shown in Shapiro (2009) that this
approximation can lead to biases in cosmological constraints and may be an important effect for forthcoming surveys.

Substituting the form for the potential from the Poisson equation, \( \nabla^2 \Phi = \frac{3}{2} H_0^2 \Omega \delta / a^3 \) into the perturbed optical tidal matrix (Equation 2.34), then substituting this into Equation 2.37 with \( A \) given by Equation 2.38 gives the form for the effective convergence

\[
\kappa_{\text{eff}}(\vec{\theta}, \chi) = \frac{3 H_0^2 \Omega_m}{2a} \int_0^\chi d\chi' \frac{\chi'(\chi - \chi') \delta(\chi' \vec{\theta}, \chi')}{\chi} 
\]

(2.44)

where \( \chi \) is the comoving distance and I have used \( \chi = \sigma / a \). This gives the effective convergence for a given \( \chi \), however to find the total \( \kappa \) along the ray we need to integrate \( \kappa_{\text{eff}}(\vec{\theta}, \chi) \) over the source redshift distribution \( G(\chi) \)

\[
\kappa_{\text{eff}}(\vec{\theta}) = \int_0^{\chi_{\text{H}}} d\chi G(\chi) \kappa_{\text{eff}}(\vec{\theta}, \chi) = \frac{3 H_0^2 \Omega_m}{2} \int_0^{\chi_{\text{H}}} d\chi' W(\chi) \frac{\delta(\chi' \vec{\theta}, \chi')}{a} 
\]

(2.45)

where

\[
W(\chi) = \int_\chi^{\chi_{\text{H}}} d\chi' G(\chi') \left( 1 - \frac{\chi}{\chi'} \right), 
\]

(2.46)

This formula shows the strength of weak lensing as a probe of cosmology, as it includes the growth in \( \delta \) and the expansion in \( \chi \), so lensing observations constrain both the growth and the expansion. Now I will look at how we can correlate these distortions to produce a measurable signal.

### 2.4.3 Shear and convergence correlation function

Since the distortions from weak lensing are small, they need to be correlated in order to get an observable signal, so I will now look at how we can convert the projected lensing quantities above into a projected correlation function. First we can write \( \kappa_{\text{eff}}(\vec{\theta}) = \int d\chi q(\chi) \delta(\chi \vec{\theta}, \chi) \) where \( q(\chi) = \frac{3 H_0^2 \Omega_m W(\chi) \chi}{a} \) then (Bartelmann and Schneider, 2001)

\[
C_\kappa(\theta) = \left\langle \kappa_1(\vec{\theta}) \kappa_2(\vec{\theta}') \right\rangle = \int d\chi q(\chi) \int d\chi' q(\chi') \left\langle \delta(\chi \vec{\theta}, \chi) \delta(\chi' \vec{\theta}', \chi') \right\rangle 
\]

(2.47)

taking the Fourier transform of the \( \delta \)s gives

\[
C_\kappa(\theta) = \int d\chi q(\chi) \int d\chi' q(\chi') \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \left\langle \delta(\vec{k}, \chi) \delta^*(\vec{k}', \chi') \right\rangle e^{-i\chi(\vec{k} \cdot \vec{\theta} - k_3)} e^{i\chi'(\vec{k}' \cdot \vec{\theta}' + k'_3)}
\]

(2.48)
CHAPTER 2. GRAVITATIONAL LENSING

and using the definition for the matter power spectrum given in Section 1.7 gives

\[ C_\kappa(\theta) = \int d\chi q^2(\chi) \int \frac{d^3k}{(2\pi)^3} P_\delta(|\vec{k}|, \chi) e^{-i\chi\vec{k}_\perp(\vec{\theta} - \vec{\theta}')} e^{-ik_3\chi} \int d\chi' e^{ik_3\chi'} \] (2.49)

where the spatial variation of \( \chi \) and \( q(\chi) \) is assumed to be small, using Limber’s approximation, so \( \chi \approx \chi' \) and therefore \( q(\chi) \approx q(\chi') \). Integrating this w.r.t. \( k_3 \) gives

\[ C_\kappa(\theta) = \int d\chi q^2(\chi) \int \frac{k dk}{2\pi} P_\delta(k, \chi) J_0(\chi\theta k) \] (2.50)

This can be written as

\[ C_\kappa(\theta) = \int_0^\infty dl \frac{l}{2\pi} P_\kappa(l) J_0(l\theta) \] (2.51)

where \( l = k\chi \) and substituting in the form for \( q(\chi) \) gives

\[ P_\kappa(l) = \frac{9}{4} H_0^4 \Omega_m^2 \int_0^{\chi_H} d\chi \frac{W(\chi)^2}{a^2} P_\delta \left( \frac{l}{\chi}, \chi \right) \] (2.52)

This shows that the convergence power spectrum is directly related to the form of the matter power spectrum, therefore changes in \( P_\delta \) are reflected in \( P_\kappa \).

In the limit of weak lensing the two point statistical properties of \( \kappa_{\text{eff}}(l) \) and \( \gamma(l) \) are identical since

\[ \left\langle \kappa_{\text{eff}}(l) \kappa_{\text{eff}}(l') \right\rangle \propto (\psi_{11} + \psi_{22})^2 \propto (l_1^2 + l_2^2)^2 = |l|^4 \] (2.53a)

\[ \left\langle \gamma(l) \gamma^*(l') \right\rangle \propto (\psi_{11} - \psi_{22} + 2i\psi_{12})(\psi_{11} - \psi_{22} - 2i\psi_{12}) \propto (l_1^2 - l_2^2 + 2il_1l_2)(l_1^2 - l_2^2 - 2il_1l_2) = l_1^4 + 2l_1^2l_2^2 + l_2^4 = |l|^4 \] (2.53b)

and therefore we can use the convergence correlation function given in Equation 2.51 to estimate the observed shear correlation function. The correlation function derived here is sometimes denoted \( \xi_\pm \), where

\[ \xi_\pm(\theta) = \left\langle \gamma_1 \gamma_1 \right\rangle \pm \left\langle \gamma_2 \gamma_2 \right\rangle = \int_0^\infty dl \frac{l}{2\pi} P_\kappa(l) J_{0,4}(l\theta) \] (2.54)

Another way we can decompose the shear correlation is into \( E \) and \( B \) modes (Crittenden et al., 2002), where the \( E \) mode is the divergence component of the signal and the \( B \) mode
is the curl component of the signal. Since for scalar perturbations there is no rotation, and therefore no curl, the $B$ mode should be zero in these models.

This is the main equation I will use throughout the next three chapters to calculate the predicted weak lensing signal for forthcoming surveys, such as DES and Euclid.

**Tomography**

Lensing can also constrain the redshift evolution of the underlying cosmology by binning the sources in the redshift direction and measuring the lensing signal from the sources in each redshift bin. This allows the evolution of the growth of structures over time to be measured.

In this work I will investigate the angular and redshift dependence of the correlation function and therefore the redshift dependence of the underlying cosmology by using the form of $W(\chi)$ to bin the correlation function in $\theta$ and $z$.

**2.4.4 Systematics**

**Intrinsic Alignments**

Since each galaxy has some unknown intrinsic shape, so the ellipticity we actually measure, $\epsilon_i$, is a combination of the shear, $\gamma_i$, and the intrinsic ellipticity, $\epsilon_i^{\text{int}}$, giving

$$\epsilon_i = \gamma_i + \epsilon_i^{\text{int}} \quad (2.55)$$

Therefore the amount of lensing cannot be estimated by a single source; however one can correlate the shear estimators of many sources, in which case randomly oriented intrinsic ellipticities will average out, leaving the gravitational shear signal. This will not succeed if galaxy ellipticities are physically aligned since

$$\langle \epsilon_i \epsilon_j \rangle = \langle \gamma_i \gamma_j \rangle + \langle \epsilon_i^{\text{int}} \epsilon_j^{\text{int}} \rangle + \langle \gamma_i \epsilon_j^{\text{int}} \rangle + \langle \epsilon_i^{\text{int}} \gamma_j \rangle \quad (2.56)$$

If the intrinsic ellipticities are randomly distributed on the sky then only the first term (the 'GG' term) contributes, however galaxy ellipticities are aligned to some degree due to intrinsic alignments (observed by e.g. Okumura et al., 2009; Joachimi et al., 2011). The second term (the 'II' term) can be non-zero if the galaxies were subject to the same tidal forces which causes them to align adding to the lensing signal. The third and fourth terms (the 'GI' terms) which can have an effect if a galaxy is aligned by tidal forces due to a lens that contributes to the lensing signal of another galaxy at a higher redshift. These galaxies will point in orthogonal directions so will partially cancel out the lensing signal.
The redshift dependence of the GG, GI and II components are shown in Figure 2.4. This shows that the II component is only important for sources at low redshifts, since this term mainly affects close pairs and sources at low redshifts will be in closer proximity to a larger fraction of the lenses, whereas the GI component can be of similar amplitude to the GG component throughout.

It should be noted that Kirk et al. (2011) and Laszlo et al. (2011) have recently shown that the effects of modified gravity and alternative dark energy models can be degenerate with systematics due to intrinsic alignments. In this work we do not include these effects, as we are seeking to present the pure shear signal predictions. This is important as we can’t see how the signal will degrade without good predictions of the signal. Throughout this work however it will be assumed that the resulting physical correlation signal can be removed, leaving only the lensing signal. To the degree that this cannot be achieved our results should be considered best-case scenarios.

The effect of baryons

It should be noted that in this work we have not included the effects of baryons, which are known to have an effect on the matter power spectrum for \( k \geq 0.1 h Mpc^{-1} \) as shown in White (2004); Zhan and Knox (2004); Jing et al. (2006); Rudd et al. (2008); Hearin and Zentner (2009); Casarini et al. (2012) and possibly for scales as large as \( k=0.3 h/Mpc \) (van Daalen et al., 2011; Semboloni et al., 2011b). These changes are due to baryon physics, such as AGN outflows, causing changes in the gravitational potentials and therefore the distribution of dark matter. For non-radiative gas simulations this changes the amplitude of the matter power spectrum by a few percent, however if gas cooling and star formation are included this effect could be considerably larger.

The effect on the lensing correlation function is shown in Figure 2.6 from Semboloni et al. (2011a), which shows that including baryons can alter the correlation function by up to \( \sim 15\% \) for \( \xi_+ \) high redshifts with AGN feedback. Therefore for a full lensing analysis in the non-linear regime these effects must be included as well.

Photometric redshifts

Most current and forthcoming optical lensing surveys rely on photometric redshifts, which can also result in errors in the interpretation of the measured lensing signal. This is because these redshifts are measured using strong broad features of the spectrum, such as the 4000Å break, instead of the narrow lines used for spectroscopic redshifts, so the errors are larger than for spectroscopic redshifts. There also exist degeneracies between the optical spectral energy distributions of galaxies with \( z < 0.2 \) and \( z > 1.5 \) (Fu et al., 2008), resulting in galaxies that are at low redshifts being mistaken for high redshift
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galaxies and vice versa. These redshift errors can be calibrated using a spectroscopic sample of the photometric redshifts.

Other systematics

The majority of the other systematics that lensing has to overcome are observational. These include the PSF (point spread function) since an anisotropic PSF, which is affected by wind shake, the atmosphere and the telescope optics, could result in false shear signal, and charge and transfer effects in the CCD can also result in false shear signal. Other problems that are generally of a more random nature are the distortion of images due to the atmosphere and noise on the CCD. These all lead to distortions in the images we observe, and these images are pixelated making their shape more difficult to measure when only over a small number of pixels. These effects are shown in Figure 2.5 and they have been the subject of much work, including the GREAT08 (Bridle et al., 2009) and GREAT10 challenges (Kitching et al., 2012), to find the best way to measure the shape of a galaxy when all these effects are included.

2.5 Observational results

Although the theoretical framework for weak lensing has been around since the 1990s (e.g. Bla; Miralda-Escude (1991); Kaiser (1992) the signal from weak lensing is small and therefore hard to detect, and so far weak lensing has given modest constraints on cosmology. However large wide field surveys on the horizon mean lensing constraints will soon become comparable to other probes, if the systematics can be dealt with.

There are two main approaches to measuring weak lensing: measuring the shear or magnification. Measuring shear requires telescopes with very good imaging (i.e. resolution \( \simeq 0.5" \) with \( \epsilon_{PSF} \lesssim 10\% \)), and there are several forthcoming wide field surveys with imaging good enough for this purpose.

The first statistically significant detection of cosmic shear was found in 2000 by four independent groups Maoli et al. (2000); Van Waerbeke et al. (2000); Kaiser et al. (2000); Bacon et al. (2000); Wittman et al. (2000) each using a different survey and giving remarkably consistent results as shown in Figure 2.7 (Mellier and Van Waerbeke, 2001).

More recently the COSMOS field from the Hubble Space Telescope was used to measure the shear correlation. COSMOS is a 2 square degree deep field survey using photometric redshifts from ground based surveys. The shear correlation function for several redshift and angular bins was observed as shown in Figure 2.8 (Massey et al., 2007). This clearly shows the redshift evolution and angular shape of the correlation.
function, both of which can be used to put constraints on cosmological models. Using the 3D correlation function they constrain $\sigma_8 (\Omega_m/0.3)^{0.44} = 0.866^{+0.089}_{-0.068}$ with $\Omega_m \geq 0.3$.

The latest lensing survey is CFHTLS, which is a ground based 170 square degree survey with a median redshift of $\sim 0.8$, where the results for 57 square degrees by Fu et al. (2008) are shown in Figure 2.9 (Fu et al., 2008) for the shear correlation function with no binning in redshift. The constraints for the full tomographic analysis are shown in Figure 2.10 (Kilbinger et al., 2008). Moving from small area surveys to this large area has caused much of the way we analyse lensing data to be revised including PSF modelling (Heymans et al., 2011) and measuring photometric redshifts (Hildebrandt et al., 2011), so it should be noted that these results shown are not the final results with all these revisions in place.

These detections have all shown the potential weak lensing has to constrain cosmology. Several wide field lensing surveys are on the horizon, such as Pan-STARRS and LSST, which are 30,000 square degree surveys with $z_m = 0.6$ and $z_m = 1$ respectively, DES, a ground based 5000 square degree survey with a median redshift of $\sim 0.8$ due to start taking data in December 2012, and Euclid a space based 15,000 square degree survey with a median redshift similar to that of DES, but the imaging from space should be far superior without having to correct for effects such as atmospheric effects (I will focus on DES and Euclid in this thesis and describe the survey parameters chosen for these surveys in forthcoming chapters). This demonstrates the exciting times that are ahead of us in constraining cosmology with gravitational lensing.
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Figure 2.4: This demonstrates how the GG, GI and II components of the measured ellipticity vary with redshift for $l = 200$ for $\Lambda$CDM using a Euclid galaxy redshift distribution from Joachimi and Schneider (2010).

Figure 2.5: Diagram of how the atmosphere, PSF and pixel noise affect the shape of an image of a galaxy (top) and a star (bottom) from the GREAT08 handbook (Bridle et al., 2008)
Figure 2.6: This demonstrates how the lensing correlation function can be affected by the presence of baryons from Semboloni et al. (2011a). The green curves show the difference between the lensing signal obtained in a simulation where radiative cooling, star formation, supernovae driven winds, and stellar evolution and mass loss were included, and a dark matter only simulation. The pink curve shows the same, but with a modified stellar initial mass function to produce more massive stars when in high pressure gas environments, i.e. close to galactic centres. The blue curve is the same as the green curve, but includes AGN feedback.
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Figure 2.7: Shear correlation function for Maoli et al. (2000) (MvWM), Van Waerbeke et al. (2000) (vWME+), Kaiser et al. (2000) (KWL), Bacon et al. (2000) (BRE) and Wittman et al. (2000) (WTK). This shows a remarkable agreement between different groups and surveys from Mellier and Van Waerbeke (2001).

Figure 2.8: Evolution of the shear correlation function for COSMOS where the points are the observed data binned at $z = 0.1 - 1$ (blue), $z = 1 - 1.4$ (green) and $z = 1.4 - 3$ (red) with the black lines showing the predictions for a flat $\Lambda$CDM model with $\Omega_m = 0.3$ and $\sigma_8 = 0.85$ for the same bins from Massey et al. (2007).
Figure 2.9: Shear correlation function for CFHTLS from Fu et al. (2008) where the red points show the E-mode and the black points the B mode using 57 square degrees of the full survey.

Figure 2.10: $\Omega_m$ and $\sigma_8$ constraints from Kilbinger et al. (2008) using the same dataset as above. The red curves are the constraints from weak lensing only and the blue curves are from CMB only.
Chapter 3

Weak lensing in modified gravities at non-linear scales

There are many different ways that gravity and/or the equation of state of the dark energy can be modified to allow for the expansion history observed, which makes it impossible to differentiate between the effects of modified gravity and dark energy by measuring the background expansion history alone, as we will see below when looking at the Hubble evolution of quintessence. However, modifying gravity also produces a distinct growth rate of structure; thus the expansion history and growth history together can be used to distinguish between various models of gravity. This consistency relation to test GR has been proposed and explored by many papers e.g. Uzan and Bernardeau (2001); Lue et al. (2004a,b); Ishak et al. (2006); Kunz and Sapone (2007); Chiba and Takahashi (2007); Wang et al. (2007); Bertschinger and Zukin (2008); Jain and Zhang (2008); Daniel et al. (2008); Song and Koyama (2009).

There has been a great deal of work showing how to use weak lensing to discriminate between different gravity models; however this has been restricted to probing the linear regime of the matter power spectrum (e.g. Afshordi et al., 2008; Schmidt, 2008; Song and Dore, 2008; Thomas et al., 2008; Tsujikawa and Tatekawa, 2008; Zhao et al., 2009b,a), or uses methods that do not obtain GR at small scales (Knox et al., 2006; Yamamoto et al., 2007; Amendola et al., 2008b). The non-linear regime provides much of the power for lensing and can be most easily probed by current and upcoming lensing surveys. This chapter examines the effect of including the non-linear regime in modified gravity lensing predictions, including the small-scale GR limit, to see how useful weak lensing will be overall when trying to determine the correct model of gravity. First we look at DGP and $f(R)$ gravity models as examples, and investigate weak lensing’s ability to differentiate between these models and dark energy models. We then take a more phenomenological point of view, by parameterising the shape of the matter power spectrum and examining
the sensitivity of weak lensing observables to changes to the matter distribution when the expansion history is the same for each model considered. Using these parameters we show how strongly a ground-based survey similar to DES and a space-based survey such as Euclid will be able to discriminate between different growth histories with identical expansion histories.

This chapter is organised as follows. In section 3.2 we briefly describe the DGP and $f(R)$ models of gravity and how they compare with dark energy models. We describe how we calculate matter power spectra for these models, including the GR small-scale limit. We also describe how we proceed to calculate weak lensing observables from these power spectra. In section 3.3 we present the resulting lensing correlation functions, including realistic errors for future surveys taking into account shape measurement noise and cosmic covariance. In section 3.4 we take the alternative approach of parameterising the non-linear power spectrum, and we investigate how sensitive weak lensing is to these parameters which go beyond the usual growth parameter. We present our conclusions in section 3.5.

Throughout this chapter we will use a flat cosmology with the WMAP5+SNe+BAO best fit cosmological parameters, which are determined by the background evolution of the Universe. We use a $\Lambda$CDM background for both $\Lambda$CDM and $f(R)$, in which case we take $n_s = 0.96$, $h = 0.71$, $\Omega_m = 0.27 \pm 0.02$ and $\sigma_8 = 0.81 \pm 0.03$ Komatsu et al. (2009). When we use a DGP background, we have $n_s = 0.998$, $h = 0.66$, $\Omega_m = 0.26 \pm 0.02$ Fang et al. (2008) giving a $\sigma_8 = 0.66 \pm 0.03$ for an equivalent $\Lambda$CDM model.

## 3.1 Modified Gravity

In this chapter, we consider DGP (Dvali et al., 2000) and $f(R)$ as examples of modified gravity models, as the non-linear power spectra have been studied in great detail in these two models using perturbation theory and N-body simulations. For some reviews of modified gravity models see Nojiri and Odintsov (2006); Durrer and Maartens (2008); Koyama (2008). These models alter gravity on large scales to create the observed accelerated expansion, while keeping small scale gravity intact.

### 3.1.1 DGP

In DGP (Dvali et al., 2000), spacetime has five dimensions, while we live on a 4D Minkowski brane in the 5D Minkowski bulk. Deffayet (2001) generalised the model
to a 4D Friedmann brane in a 5D Minkowski bulk. The extra dimension contributes a further term to the action

\[
S = \frac{1}{2} \left[ \frac{1}{r_c} \int_{\text{bulk}} R^{(5)} \sqrt{-g^{(5)}} dx^5 + \int_{\text{brane}} R \sqrt{\text{det} g} dx^4 \right]
\]  

(3.1)

where Standard Model particles are bound on the 4D brane, as is gravity on small scales where the second term of the action dominates, since the first is divided by the crossover scale \( r_c \); however on large scales gravity leaks off the brane causing late time acceleration, as the first term in the action dominates. Therefore the scale of the transition from 4D to 5D gravity is governed by the crossover scale, which results in the following behaviour for the weak field gravitational potential:

\[
\Psi \propto \begin{cases} 
    r^{-1} & r \ll r_c \\
    r^{-2} & r \gg r_c 
\end{cases}
\]  

(3.2)

To calculate the Friedmann equation we begin with a metric which is homogeneous and isotropic on the brane, but varies in the fifth dimension \( y \), where the brane is the hypersurface defined by \( y = 0 \) (Binetruy et al., 2000a,b; Deffayet, 2001)

\[
ds^2 = n^2(t, y) dt^2 - a^2(t, y) \delta_{ij} dx^i dx^j - b^2(t, y) dy^2
\]  

(3.3)

This is a normal homogeneous and isotropic metric, but with a dependence on the fifth dimension.

The energy momentum tensor can be split into contributions from the brane

\[
T_{\mu \nu} |_{\text{brane}} = \delta(y) \text{diag}(\rho_b, -p_b, -p_b, -p_b, 0)
\]  

(3.4)

where \( \delta(y) \) is the width of the brane, and contributions from the bulk

\[
T_{\mu \nu} |_{\text{bulk}} = \text{diag}(\rho_B, -p_B, -p_B, -p_B, -p_T)
\]  

(3.5)

Calculating \( G_{\mu \nu} \) for the metric given in Equation 3.3 and equating with the \( T_{\mu \nu} \) components gives the five dimensional field equations \( G_{\mu \nu} = G^{(5)}_{\mu \nu} \),

\[
G_{tt} = 3 \left( \frac{a_t^2}{a^2} - \frac{n^2 a_y^2}{a^2 b^2} + \frac{n^2 a_y b_y}{ab^3} - \frac{n^2 a_{yy}}{a b^2} + \frac{a_t b_t}{ab} \right) = G^{(5)}(\delta(y) b \rho_b + \rho_B)
\]  

(3.6a)

\[
G_{ty} = 3 \left( \frac{a_t y}{a} + \frac{n_y a_t}{an} + \frac{b_y a_y}{ab} \right) = 0
\]  

(3.6b)
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\[ G_{x'x} = \delta_{ij} \left( \frac{2 a_{a} t n_{,i}}{n^3} - \frac{2 a_{a} t}{n^2} + \frac{2 a_{y} n_{,y}}{b^2} - \frac{a_{x}^2}{n^2} + \frac{a_{y}^2}{b^2} - 2 a_{a} b_{,y} + 2 a a_{y y} - \right. \]
\[ \left. \frac{2 a a_{b} t}{n^2 b} - \frac{n_{,y} b_{,y} a^2}{n b^3} + \frac{a^2 n_{,y}}{n b^2} - \frac{a^2 b_{,t}}{n b^2} + \frac{a^2 n_{,t} b_{,t}}{n^3 b} \right) = -G_{(5)} (\delta(y) p_b + p_B) \]  

(3.6c)

\[ G_{y y} = 3 \left( -b^2 a_{,t} a_{n} \frac{1}{n^2} + \frac{b^2 a_{,n} a_{t}}{n^3} + \frac{a_{y} n_{,y}}{a n} - \frac{b^2 a^2_{,t}}{a^2 n^2} + \frac{a^2_{,y}}{a^2} \right) = -G_{(5)} \rho_T \]  

(3.6d)

where I have used the assumption that the brane is infinitely thin in the fifth dimension, so there is no flow of matter along the fifth dimension \( T_{y} = 0 \).

Looking at the \( tt \) component and \( yy \) component this can be written as (Binetruy et al., 2000a)

\[ F_{,y} = G_{(5)} \frac{2}{3} a_{,y} a^3 \left( \frac{\delta(y)}{b} \rho_b + \rho_B \right) \]  

(3.7)

\[ F_{,t} = G_{(5)} \frac{2}{3} a_{,t} a^3 \rho_B \]  

(3.8)

where

\[ F(t, y) \equiv \left( \frac{a_{,y} a}{b^2} \right)^2 - \left( \frac{a_{,t} a}{n} \right)^2 - ka^2 \Rightarrow \left( \frac{a_{,y} a}{b} \right)^2 - \left( \frac{a_{,t} a}{n} \right)^2 - ka^2 + G_{(5)} \frac{\rho_B}{6} a^4 + C = 0 \]  

(3.9)

where I have integrated Equation 3.7 w.r.t. \( y \), assuming \( \rho_B \) is a constant, to obtain the second expression and \( C \) is the constant of integration. Using the approach of Binetruy et al. (2000a) we require the metric to be continuous across the brane, (where \( y = 0 \)), but its derivatives with respect to \( y \) can be discontinuous across \( y = 0 \), so we can define the junction, \( [a_{,y}] \), as the change in derivatives across the brane

\[ [a_{,y}] = a_{,y}(+) - a_{,y}(-) \]  

(3.10)

Using \( G_{\mu \nu} = G_{(5)} T_{\mu \nu} \) gives the junction in \( a_{,y} \) as

\[ \left[ a_{,y} \right]_{a|y=0}^{b|y=0} = -\frac{G_{(5)}}{3} \rho_b \]  

(3.11)

and applying the same method to \( n \) gives

\[ \left[ n_{,y} \right]_{n|y=0}^{b|y=0} = \frac{G_{(5)}}{3} (2\rho_b + 3p_b) \]  

(3.12)
CHAPTER 3. WEAK LENSING IN MODIFIED GRAVITIES

Assuming that the solutions are left invariant under the transformation \( y \rightarrow -y \) then Equations 3.11 and 3.12 can be used to find \( a_y \) and \( n_y \) on both sides of the brane as \( y \rightarrow 0 \). Substituting Equations 3.11 and 3.12 and \( n|_{y=0} = 1 \) into Equation 3.9 gives

\[
\left( \frac{a_x}{a} \right)^2 \bigg|_{y=0} = \frac{G_5}{6} \rho_B + \left( \frac{G_5}{6} \rho_b \right)^2 + \frac{C}{a^4} \bigg|_{y=0} - \frac{k}{a^2} \bigg|_{y=0} \quad (3.13)
\]

where \( C \) is a constant of integration. Taking \( C = 0 \) and \( k = 0 \), using \( \Omega_\alpha = \frac{\rho_\alpha}{G_4} \) and defining \( H = \frac{a_x}{a} \bigg|_{y=0} \) gives

\[
H^2 = \frac{G_5}{2G_4} \Omega_B + \left( \frac{G_5}{2G_4} \Omega_b \right)^2 \quad (3.14)
\]

Splitting \( \Omega_b \) into a term which is like the usual density in a 4D model that I will denote \( \Omega \) and the energy density due to the constant intrinsic tension of the brane \( \Omega_\Lambda = -H^2 - k/a^2 \) and setting \( \Omega_B = 0 \) allows us to write the effective brane cosmology (where \( y = 0 \)) where the extra dimension contributes a further term to the Friedmann equation whose amplitude is governed by \( r_c = 2G_5/G_4 \):

\[
H^2 - \frac{H}{r_c} = \Omega(a) \quad (3.15)
\]

where \( H = a_x \bigg|_{y=0} / (a_{y=0} n_{y=0}) \). From this form of the Friedmann equation we can see \( r_c = (1 - \Omega_m)^{-1} \) for a matter dominated universe at late times.

From the conservation of the energy momentum tensor, Equation 1.8, we can show that the equation of matter conservation takes the usual form for an unperturbed metric as shown in Equation 1.16.

Perturbing the metric on the brane gives a modified Poisson equation, which with the energy momentum conservation equations shows that the growth history is altered in the following way (Lue et al., 2004b; Koyama and Maartens, 2006):

\[
\delta'' + \left( \frac{3}{a} + \frac{H'}{H} \right) \delta' = \frac{1}{H^2 a^2} \frac{3 H r_c \left( 1 + \frac{aH}{3H} \right)}{2 H r_c \left( 1 + \frac{aH}{3H} \right) - 1} \Omega_m \delta, \quad (3.16)
\]

I will use the modified Friedmann equation, Equation 3.15, and the linear growth equation, Equation 3.16 in this chapter to calculate the linear power spectrum and distances required to obtain weak lensing predictions for DGP.
3.1.2 \( f(R) \)

In \( f(R) \) gravity the Einstein-Hilbert action is modified to include an arbitrary function of the Ricci scalar, \( R \), giving (Hu and Sawicki, 2007a)

\[
S = \int \left[ \left( \frac{R}{2} - \Lambda \right) \sqrt{-g} + \mathcal{L}_m \right] d^4x \rightarrow \int \left[ \left( \frac{f(R)}{2} - \Lambda \right) \sqrt{-g} + \mathcal{L}_m \right] d^4x \quad (3.17)
\]

To derive the field equations for this metric we need to vary the action with respect to the metric.

I denote the variation of the metric as \( \delta g_{\mu\nu} \) and the variation of the determinant as defined in Equation 1.71.

The variation of the Ricci scalar, whose form is given in Equation 1.4, is

\[
\delta R = \delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu} \nonumber
\]

\[
= \delta g^{\mu\nu} R_{\mu\nu} + (g^{\mu\nu} \delta \Gamma^\sigma_{\mu\nu} - g^{\mu\sigma} \delta \Gamma^\lambda_{\mu\lambda})_{\sigma} \nonumber
\]

\[
= \delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu} \delta g^{\mu\nu}_{\sigma;\sigma} - \delta g^{\lambda\nu}_{;\sigma\lambda} \nonumber
\]

\[
= \delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu} \Box \delta g^{\mu\nu} - \delta g^{\mu\nu}_{;\mu\nu} \quad (3.18)
\]

where \( \Box \) is the D’Alembert operator and I have used

\[
\delta \Gamma^\rho_{\mu\nu} = \frac{1}{2} g^\rho^\lambda (\delta g_{\lambda\nu;\mu} + \delta g_{\lambda\mu;\nu} - \delta g_{\mu\nu;\lambda}) \quad (3.19)
\]

Substituting Equations 1.71 and 3.18 and \( \delta f = f_R \delta R \) into Equation 3.17 giving the variation of the action as (Guarnizo et al., 2010)

\[
\delta S = \int \frac{1}{2} \left[ \delta f \sqrt{-g} + f \delta \sqrt{-g} + 2 \delta \mathcal{L}_m - 2 \delta \sqrt{-g} \Lambda \right] d^4x \nonumber
\]

\[
= \int \frac{1}{2} \sqrt{-g} \left[ f_R \delta R - \frac{1}{2} f g_{\mu\nu} \delta g^{\mu\nu} + 2 \delta \mathcal{L}_m \sqrt{-g} + g_{\mu\nu} \delta g^{\mu\nu} \Lambda \right] d^4x \nonumber
\]

\[
= \int \frac{1}{2} \sqrt{-g} \left[ f_R (\delta g^{\mu\nu} R_{\mu\nu} + g_{\mu\nu} \Box \delta g^{\mu\nu} - \delta g^{\mu\nu}_{;\mu\nu}) - \frac{1}{2} f g_{\mu\nu} \delta g^{\mu\nu} + 2 \delta \mathcal{L}_m \sqrt{-g} + g_{\mu\nu} \delta g^{\mu\nu} \Lambda \right] d^4x \nonumber
\]

\[
= \int \frac{1}{2} \sqrt{-g} \delta g^{\mu\nu} \left[ f_R R_{\mu\nu} - \left( \frac{1}{2} - \Box f_R \right) g_{\mu\nu} - f_{R;\mu\nu} + T_{\mu\nu} + \Lambda g_{\mu\nu} \right] d^4x \quad (3.20)
\]

where \( f = f(R), f_R = df/dR \). The third step uses Equation 1.73 so \( T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_m}{\delta g^{\mu\nu}} \) and the last step uses integration by parts on the last two terms.
By imposing that the action remains invariant, $\delta S = 0$, with respect to $\delta g_{\mu\nu}$ we obtain the field equations

$$f_R R_{\mu\nu} - \left( \frac{f}{2} - \Box f_R \right) g_{\mu\nu} - f_{R;\mu\nu} = T_{\mu\nu} + \Lambda g_{\mu\nu}$$  \hspace{1cm} (3.21)

Using the FRW metric to solve the field equations results in an altered Hubble equation (De Felice and Tsujikawa, 2010)

$$H^2 - f_R (H H' + H^2) + \frac{1}{6} f + H^2 f_{RR} R' = \Omega$$  \hspace{1cm} (3.22)

and using a perturbed FRW metric we obtain a modified Poisson equation from the field equations

$$\Phi = -\frac{1}{2} \frac{a^2}{f_R} \frac{1 + 4 \frac{k^2 f_{RR}}{a^2 f_R}}{1 + 3 \frac{k^2 f_{RR}}{a^2 f_R}} \rho$$  \hspace{1cm} (3.23)

which combined with the conservation equations from the conservation of the energy momentum tensor gives the evolution of density perturbations as

$$\delta'' + \left( \frac{3}{a} + \frac{H'}{H} \right) \delta' = \frac{3}{2} \frac{\dot{a}}{H^2 a^2} \frac{1 + 4 \frac{k^2 f_{RR}}{a^2 f_R}}{1 + 3 \frac{k^2 f_{RR}}{a^2 f_R}} \Omega_m \delta$$  \hspace{1cm} (3.24)

In this study we use an $f(R)$ function of the form Hu and Sawicki (2007a)

$$f = -\frac{R_0^2}{R} f_{R_0},$$  \hspace{1cm} (3.25)

where $R$ is the Ricci scalar, $R_0$ is the present day Ricci scalar and $f_{R_0} = \frac{df}{dR}|_{R=R_0}$. We use $|f_{R_0}| = 10^{-4}$, which has been found to fit with cluster constraints Schmidt et al. (2009), and gives a background evolution which is approximately $\Lambda$CDM to sub-percent level. This allows us to use the $\Lambda$CDM Friedmann equation and only alter the density evolution equation (Lue et al., 2004b; Zhang, 2006; Koyama et al., 2009).

We can generalise the density perturbation evolution equation (Equation 1.39) to include an effective gravitational constant as follows:

$$\delta'' + \left( \frac{3}{a} + \frac{H'}{H} \right) \delta' = \frac{3 \tilde{G}_{eff}}{2 H^2 a^2} \frac{\Omega_m \delta}{a^3},$$  \hspace{1cm} (3.26)

where primes denote differentiation with respect to $a$. This equation is valid for both dark energy and modified gravity models, where $\tilde{G}_{eff}$ is the effective gravitational constant.
normalised by the gravitational constant $G$; hence $\tilde{G}_{\text{eff}} = 1$ for dark energy models, while for modified gravity models

$$\tilde{G}_{\text{eff}} = 1 + \frac{1}{3\beta},$$  \hspace{1cm} (3.27)

where $\beta$ is determined by the model.

For DGP using Equation 3.16 we obtain

$$\beta = 1 - 2Hr_c \left(1 + \frac{aH'}{3H}\right).$$  \hspace{1cm} (3.28)

and for the chosen form of $f(R)$ in this chapter using 3.24 we obtain

$$\beta = 1 + \frac{1}{3c^2 \frac{df}{dR}} \left(\frac{a}{k}\right)^2,$$  \hspace{1cm} (3.29)

where $\bar{k}$ is the dimensionless wavenumber defined as $k(c/H_0)$, $k$ is the wavenumber and $c$ is the speed of light.

### 3.1.3 Screening mechanisms

While these theories can create the large scale accelerated expansion we desire they do not reconcile with small scale observations without considering screening mechanisms, such as the Vainshtein mechanism for DGP and the Chameleon mechanism for $f(R)$.

When DGP reduces to its four dimensional form (i.e. when $r < r_c$) it does not reduce to four dimensional Einstein gravity, but instead includes an extra gravitational scalar, which does not decouple in the massless limit, shown by van Dam and Veltman (1970). This alone would make DGP not a viable theory of gravitation. However Vainshtein (1972) proposed that this coupling is only present at the linear level. If non-linear self interaction effects are taken into account, then in high density regions where the scalars become strongly coupled, the force is suppressed and GR is recovered (Lue, 2006; Clifton et al., 2012; de Rham et al., 2012). The scale at which this mechanism comes into play is denoted the Vainshtein radius $r_V \sim (r_s r_c^2)^{1/3}$, where $r_s$ is the Schwarzschild radius.

In $f(R)$ gravity the chameleon mechanism (e.g. De Felice and Tsujikawa (2010)) is used to recover GR on small scales. This is achieved since the light scalar fields present in the theory have a local matter density dependent mass. This density dependent mass results from the combination of two terms in its effective potential, which are a self-interaction term of the form of a monotonically-decreasing potential, and a term due to the coupling to matter of the form $e^{\beta_i \phi}$, so $V_{\text{eff}}(\phi) = V(\phi) + \rho_i e^{\beta_i \phi}$. Figure 3.1 shows
Figure 3.1: Effective chameleon potential $V_{\text{eff}}$ (solid curve) as the sum of the actual chameleon potential $V(\phi)$ (dashed curve) and the potential from its coupling to the matter density $\rho_m$ (dotted curve). Figure from Khoury and Weltman (2004).
the shape of the effective potential in high and low density regions and shows how as $\rho$ increases, the minimum of the potential shifts to smaller values of $\phi$ and therefore tends towards the expected GR result.

Here I will not use these screening mechanisms explicitly, but will instead try to mimic their behaviour using an interpolation between the expected modified gravity results and the GR results on very small scales, which I will discuss next.

### 3.2 Lensing in DGP and $f(R)$ models

#### 3.2.1 Modified gravity non-linear power spectra

As we have already mentioned, there are two key phenomena to model in any gravity in order to calculate the matter power spectrum: the expansion history, quantified by the evolution of the Hubble parameter, and the growth history, quantified by the evolution of density perturbations $\delta$ in the Universe.

For $\Lambda$CDM, the expansion history is given by the Friedmann equation, Equation 1.18, and the growth history is described by the density perturbation evolution equation, Equation 1.39. At this point we will limit ourselves to the regime where density perturbations evolve linearly. The equivalent Hubble and growth equations for DGP and $f(R)$ are given by Equations 3.15, 3.22 and 3.27-3.29.

For modified gravity to agree with solar system observations it must approach a GR solution on small scales. This means that we can attempt to model the non-linear power spectrum by interpolating the modified gravity non-linear power spectrum with no mechanism to obtain the GR result on small scales, $P_{\text{non-GR}}(k, z)$, and the GR non-linear power spectrum with the same expansion history as the modified gravity model, $P_{\text{GR}}(k, z)$. A fitting formula for this interpolation was proposed by Hu and Sawicki (2007b):

$$P(k, z) = \frac{P_{\text{non-GR}}(k, z) + c_{\text{nl}}(z)\Sigma^2(k, z)P_{\text{GR}}(k, z)}{1 + c_{\text{nl}}(z)\Sigma^2(k, z)},$$

where $\Sigma^2(k, z)$ picks out non-linear scales and $c_{\text{nl}}(z)$ determines the redshift at which the power spectrum approaches the GR result.

In this chapter, we use the fitting formulae for $\Sigma^2(k, z)$ and $c_{\text{nl}}(z)$ obtained by perturbation theory Koyama et al. (2009) and confirmed by N-body simulations Oyaizu et al. (2008); Schmidt (2009),

$$\Sigma^2(k, z) = \left(\frac{k^3}{2\pi^2}P_{\text{lin}}(k, z)\right)^{\alpha_1}, \quad c_{\text{nl}}(z) = A(1 + z)^{\alpha_2}. \tag{3.31}$$
where $P_{\text{lin}}(k, z)$ is the modified gravity linear power spectrum. The non-linear power spectrum for both the $P_{\text{non-GR}}$ and $P_{\text{GR}}$ is found using the Smith et al. (2003) fitting formula from the linear power spectrum. For DGP, $A = 0.3$, $\alpha_1 = 1$ and $\alpha_2 = 0.16$ and for $f(R)$ with $f_{R_0} = 10^{-4}$ we use $A = 0.08$, $\alpha_1 = 1/3$ and $\alpha_2 = 1.05$ for $0 \leq z \leq 1$. It should be noted these values are not valid for all $\Omega_m$ and $\sigma_8$. However, in DGP, testing with simulations showed these values depend on $\Omega_m$ and $\sigma_8$ very weakly (Oyaizu et al., 2008; Schmidt, 2009), so within our priors for $\Omega_m$ and $\sigma_8$ we can assume the values are constant.

We should also emphasise that these fits are confirmed only up to $k = 1h$/Mpc due to the lack of resolution in N-body simulations, so we are extrapolating the fits beyond this regime. Clearly it is necessary to check the validity of this extrapolation using N-body simulations with higher resolution (see Schmidt et al., 2008, for a different approach using the halo model). However, since the modified gravity power spectrum should approach the GR non-linear power spectrum with the same expansion history, and since the fitting formula (Equation 3.30) ensures this, our extrapolation is justifiable.

In applying this formalism, we found that although $f(R)$ fits the N-body results at small $k$, it failed to converge with $\Lambda$CDM at larger $k$ if $\alpha_1 = 1/3$. This is due to the strong scale dependence of the linear power spectrum, since the effective "gravitational constant" in Equation 3.29 depends on $k$, such that $P_{\text{non-GR}}$ deviates from $P_{\text{GR}}$ strongly on small scales and Equation 3.30 with $\alpha_1 = 1/3$, which fits N-body results well up to $k = 1h$/Mpc, fails to converge with $P_{\text{GR}}$, as seen in Figure 3.2. On the other hand, power spectra with $\alpha_1 = 1$ and $\alpha_1 = 2$ show clear convergence; this is shown more explicitly in Figure 3.3. Thus, we also consider $\alpha_1 = 1$ and $\alpha_1 = 2$ cases for $f(R)$ which have more physical behaviour at high $k$ to investigate how this high $k$ convergence affects our constraints.

Since we are interested in how sensitive weak lensing is to different growth histories with the same expansion history, we will also consider a quintessence cold dark matter (QCDM) model (See Section 1.9.2). In this case, the equation of state of the dark energy is altered to match the expansion history of DGP, while the density perturbation evolution equations are the same as $\Lambda$CDM. We show a comparison between DGP and QCDM power in Figure 3.4, including our non-linear prescription. In the linear regime the DGP power spectrum receives scale independent suppressions, since the effective "gravitational constant" in Equation 3.28 does not depend on $k$ and reduces $G_{\text{eff}}$ on all scales, but it converges to the QCDM power spectrum on non-linear scales due to our inclusion of the GR asymptote.
Figure 3.2: Matter power spectrum for ΛCDM, DGP and $f(R)$ at $z=0$.

Figure 3.3: Relative difference between matter power spectra for ΛCDM and $f(R)$ at $z=0$ for different $\alpha_1$. 
3.2.2 Weak Lensing

We will now calculate results for realistic notional surveys: a ground-based survey similar to that of the Dark Energy Survey (DES), and a space-based survey such as that of Euclid, using redshift distributions shown in Figure 3.5; the redshift distribution for our ground-based survey was chosen to be the same as for CFHTLS given by Fu et al. (2008) giving a median redshift, $z_m$, of 0.825 and for Euclid we used the distribution given by Hawken and Bridle (2009) giving $z_m = 0.9$.

The shear correlation function for each of these models will be calculated using the previously derived form of $C_\kappa$ (Equation 2.51) using a modified $P_\kappa$, which includes cross-correlations of sources at different redshifts (e.g. Bacon et al. (2005); Massey et al. (2007))

$$P_\kappa(l) = \frac{9}{4} \left( \frac{H_0}{c} \right)^4 \Omega_m^2 \int_0^{\chi_m} d\chi W_1(\chi) W_2(\chi) \frac{P_\delta \left( \frac{1}{\chi}, \chi \right)}{a^2},$$

where $W_i$ include the galaxy distributions $G_i$ appropriate for the $i$th redshift bin. This equation together with equation (2.51) relates the matter power spectra from our gravity models to the predicted lensing signal; we will now use these tools to calculate lensing predictions for our models.
CHAPTER 3. WEAK LENSING IN MODIFIED GRAVITIES

3.3 Results

We calculate the convergence (combined shear) correlation function (Equation 2.51) for all of our models, and estimate measurement errors due to intrinsic ellipticity, for the notional ground-based and Euclid surveys using bins with error

\[ \sigma_{\text{shape}} = \sqrt{2} \frac{\sigma_\gamma}{\sqrt{N_{\text{pairs}}(\theta, \Delta \theta)}} \]  

(3.33)

where \( \sigma_\gamma = 0.3 \). The errors were estimated using 13.3 galaxies arc min\(^{-2}\) and a survey area of 5000 square degrees for our ground-based survey (as is appropriate for DES), while for Euclid we use 35 galaxies arc min\(^{-2}\) and 20000 square degrees. The covariance matrix for the intrinsic ellipticity noise is diagonal for bins in redshift and angular separation (c.f. Bacon et al., 2003).

We also include the covariance due to sample variance due to the cosmic matter distribution, \( C_{\text{cos}} \), which is estimated using the Horizon simulation Teyssier et al. (2009). 3-D convergence maps were calculated from the 3-D overdensity field, using the relation between \( \kappa \) and \( \delta \) given by Equation 2.39, for 75 patches of area 2 square degrees;

Figure 3.5: Redshift distributions used for survey predictions: for ground-based survey with \( z_m = 0.825 \), and for Euclid with \( z_m = 0.91 \).
convergence correlation functions were then measured in each patch. The covariance between the resulting patch correlation functions was measured as an estimator of the true covariance, in 8 angular separation bins logarithmically spaced from 1′ to 90′ and in 3 redshift bins (leading to 6 redshift pair bins). The diagonal elements of the covariance matrix for mean correlation functions are measured to be approximately $10^{-11} - 10^{-9}$ per square degree, making the sample covariance the dominant source of error for larger angles and higher redshifts for both our ground-based survey and Euclid, with diagonal element values of $10^{-15} - 10^{-13}$. These should be compared with shape noise covariance contributions of $10^{-15} - 10^{-11}$ for ground-based and $10^{-16} - 10^{-12}$ for Euclid. The covariances were included in our $\chi^2$ estimations using the unbiased inverse covariance matrix proposed by Hartlap et al. (2007) which demonstrated that the inverse of the covariance matrix is biased, and this bias depends on the number of bins in the covariance matrix and the number of realisations used to predict the covariance matrix giving the log likelihood

$$-\log L = \frac{1}{2} \sum_{i,j} \left( d_i - t_i \right) \left( \frac{n_o - n_b - 2}{n_o - 1} C_{\cos}^{-1} + \sigma_{\text{shape}}^{-2} \right)_{ij} \left( d_j - t_j \right), \quad (3.34)$$

where $d$ is the ‘data’, here the fiducial $\Lambda$CDM correlation function in redshift and angular separation bins; $t$ is the alternative gravity model correlation function in those bins, $n_o = 75$ is the number of realisations of correlation functions used in the calculation of $C_{\cos}$ and $n_b = 48$ is the total number of bins in angular separation and redshift. Note that we assume the errors are Gaussian, so $-\log L = \frac{1}{2}\chi^2$, and use the sample covariance estimate from Horizon (which follows $\Lambda$CDM) for both $\Lambda$CDM and QCDM cases; the QCDM error bars should therefore only be considered as approximate.

We calculate for each of our models the difference in $\chi^2$ between the fiducial modified gravity model and a dark energy model with the same $H(z)$ (either $\Lambda$CDM or QCDM), applying WMAP+SNe+BAO priors. Note that for $\Lambda$CDM and $f(R)$, we used the $\Lambda$CDM background Komatsu et al. (2009) and for DGP and QCDM, the DGP background Fang et al. (2008) was used (see §1). We compare the same $H(z)$ to investigate whether a modified gravity model, which fits the expansion history perfectly, would give a difference in the lensing signal.

Figure 3.6 shows example results for our ground-based survey and Euclid using the central cosmological parameter values for WMAP+SNe+BAO described in §1; this is for the 2-D projection case where we have not divided the catalogue tomographically. We see from figures (a) and (b) that the difference between models is substantially greater in the nonlinear regime ($\theta \leq 30′$) than in the linear regime ($\theta \geq 30′$), as is the amplitude of
(a) Including non-linear effects for sources with \( z_m = 0.825 \), with ground-based survey errors.

(b) Including non-linear effects for sources with \( z_m = 0.9 \) with Euclid errors.

(c) Not including non-linear effects for sources with \( z_m = 0.825 \) with ground-based survey errors.

(d) Not including non-linear effects for sources with \( z_m = 0.9 \) with Euclid errors.

Figure 3.6: Correlation function predicted for ΛCDM, DGP and \( f(R) \) with error estimates for ground-based survey and Euclid. Models are for the central cosmological parameter values fitting WMAP+BAO+SNe described in §1, using the ΛCDM background (for ΛCDM and \( f(R) \)) and the DGP background (for DGP).
(a) Including non-linear effects for sources with $z_m = 0.825$ with ground-based errors

(b) Including non-linear effects for sources with $z_m = 0.9$ with Euclid errors

Figure 3.7: Correlation function predicted for the QCDM model with the expansion history as DGP and DGP with error estimates for ground-based survey and Euclid. The solid lines show the correlation function for the QCDM model for the central cosmological parameter values fitting WMAP+BAO+SNe, using the DGP background. The dashed line shows the best fit QCDM model to the DGP model obtained by varying $\Omega_m$ and $\sigma_8$. 
Figure 3.8: Correlation function predicted for $\Lambda CDM$, DGP and $f(R)$ with error estimates for ground-based survey at different $z$ using redshift bins with width $\Delta z = 0.4$. 
The signal. As (c) and (d) show, it is also the case that the linear correlation function is small in the low-θ regime, if nonlinear corrections are not included.

We present the $\chi^2$ differences between the modified gravities and fiducial dark energy models in Table 3.1, for the 2-D (non-tomographic) cases including non-linear power. We see that there is indeed strong discriminatory power between modified gravity models and $\Lambda$CDM with the notional ground-based survey; the precision of Euclid is even more impressive.

We also compare the constraints on DGP and a QCDM model of the same expansion history (i.e. a DGP background). The correlation functions for these models are shown in Figure 3.7. One can either consider a QCDM model with cosmological parameters equal to their central values in a fit to WMAP+BAO+SNe, or more realistically the best fit QCDM model to the DGP model obtained by varying $\Omega_m$ and $\sigma_8$. We see that there is a choice of $\Omega_m$ and $\sigma_8$ that make the QCDM and DGP models virtually indistinguishable. This is confirmed by the bottom row of Table 3.1, which shows that the difference in $\chi^2$ for DGP and this QCDM is insignificant. This is clearly partly due to the existence of a QCDM model with rather similar growth to the DGP, but also because of the low amplitude of the DGP correlation function, with the result that the error bars are larger in proportion to the signal than for other models.

The power of future surveys to discriminate between gravity models is borne out by the tomographic results. Examples of these are shown in Figure 3.8, where we see the different redshift evolutions and amplitudes of the signal in the different gravities. Table 3.2 confirms that using the redshift information affords us better discrimination between dark energy and modified gravity models in every case, by a factor of 50 to 100% in the $\Delta \chi^2$ value. Because of this, we will only consider tomographic results from now on in the chapter.

Table 3.3 shows the impact of including non-linear power on our ability to discriminate between modified gravities. Comparing these results with Table 3.2 we can see the improvement that measurements from the non-linear regime of the correlation functions

<table>
<thead>
<tr>
<th>Fiducial Model</th>
<th>Modified gravity</th>
<th>Ground-based $\Delta \chi^2$</th>
<th>Euclid $\Delta \chi^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda$CDM</td>
<td>$f(R)$, $\alpha_1 = 1/3$</td>
<td>500</td>
<td>$6 \times 10^3$</td>
</tr>
<tr>
<td></td>
<td>$f(R)$, $\alpha_1 = 1$</td>
<td>200</td>
<td>$2 \times 10^3$</td>
</tr>
<tr>
<td></td>
<td>$f(R)$, $\alpha_1 = 2$</td>
<td>40</td>
<td>500</td>
</tr>
<tr>
<td>QCDM</td>
<td>DGP</td>
<td>0.5</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 3.1: $\Delta \chi^2$ for DGP and $f(R)$ using errors from our ground-based survey and Euclid, with no redshift information, and using priors from WMAP+SNe+BAO. The top section shows results compared to $\Lambda$CDM, while the bottom row is compared to QCDM.
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Table 3.2: Same as Table 3.1, but using tomographic information. In each case we have redshift bins of width $\Delta z = 0.4$ between $z = 0.3$ and 1.5.

<table>
<thead>
<tr>
<th>Fiducial Model</th>
<th>Modified gravity</th>
<th>Ground-based $\Delta \chi^2$</th>
<th>Euclid $\Delta \chi^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda$CDM</td>
<td>$f(R)$, $\alpha_1 = 1/3$</td>
<td>600</td>
<td>$8 \times 10^3$</td>
</tr>
<tr>
<td></td>
<td>$f(R)$, $\alpha_1 = 1$</td>
<td>300</td>
<td>$3 \times 10^3$</td>
</tr>
<tr>
<td></td>
<td>$f(R)$, $\alpha_1 = 2$</td>
<td>60</td>
<td>$1 \times 10^3$</td>
</tr>
<tr>
<td>QCDM DGP</td>
<td>DGP</td>
<td>0.5</td>
<td>5</td>
</tr>
</tbody>
</table>

Table 3.3: $\Delta \chi^2$ if only linear power is included for $\theta = 30' - 90'$, for 0.4 redshift bins between 0.3 and 1.5 using priors from WMAP+SNe+BAO.

<table>
<thead>
<tr>
<th>Fiducial Model</th>
<th>Modified gravity</th>
<th>Ground-based $\Delta \chi^2$</th>
<th>Euclid $\Delta \chi^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda$CDM</td>
<td>$f(R)$</td>
<td>500</td>
<td>3000</td>
</tr>
<tr>
<td>QCDM DGP</td>
<td>DGP</td>
<td>0.2</td>
<td>2</td>
</tr>
</tbody>
</table>

The improvement is very substantial, amounting to an order of magnitude in $\chi^2$ difference.

It is important to note that using only the Smith et al. (2003) formula, without the GR asymptote, causes an overestimation in our ability to discriminate between modified gravity and dark energy models as shown in Table 3.4. This can amount to up to a 90% difference in $\Delta \chi^2$ for some models, due to the difference in power at small scales that is present when there is no attempt to recover GR. This shows the importance of careful modelling of the nonlinear regime, including the appropriate small-scale GR limit. We are aware that more sophisticated statistical analysis is possible and desirable (e.g. Bayesian evidence) but this initial level of detail indicates the importance of the effect.

Table 3.4: Difference in $\Delta \chi^2$ if the Smith et al. (2003) formula is used with no attempt to fit GR at small scales, compared to using the Hu & Sawicki fitting formula. All results are tomographic with WMAP+SNe+BAO priors as before.

<table>
<thead>
<tr>
<th>Fiducial Model</th>
<th>Modified gravity</th>
<th>Ground-based $\Delta \chi^2$ difference</th>
<th>Euclid $\Delta \chi^2$ difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda$CDM</td>
<td>$f(R)$, $\alpha_1 = 1/3$</td>
<td>-180</td>
<td>-2100</td>
</tr>
<tr>
<td></td>
<td>$f(R)$, $\alpha_1 = 1$</td>
<td>-240</td>
<td>-3200</td>
</tr>
<tr>
<td></td>
<td>$f(R)$, $\alpha_1 = 2$</td>
<td>-210</td>
<td>-2400</td>
</tr>
<tr>
<td>QCDM DGP</td>
<td>DGP</td>
<td>-0.35</td>
<td>-4</td>
</tr>
</tbody>
</table>
3.4 Parameterisation of the Power Spectrum

The sensitivity of lensing to changes in the matter power spectrum will be very important in determining the correct theory of gravity or dark energy in the near future. In this section we will therefore parameterise the non-linear power spectrum, in order to more fully understand what aspect of the power spectrum it is which lensing surveys will be sensitive to.

We use the growth factor $\gamma$ Linder (2005) as is used in (Amendola et al., 2008b), but we also include the parameters used in the Hu and Sawicki fitting formula $A$, $\alpha_1$ and $\alpha_2$ (Equation 3.30 and 3.31), where increasing $A$ makes the modified power spectrum tend towards the GR power spectrum more quickly linearly in redshift, while increasing $\alpha_2$ makes the modified power spectrum tend towards the GR power spectrum more quickly in redshift as a power law, and increasing $\alpha_1$ makes the modified power spectrum tend towards the GR power spectrum more quickly in redshift and $k$ according to a power of the linear power spectrum. This parameterisation allows us to probe many models with an altered growth while allowing for the GR asymptote required to fit GR at very small scales. In the formalism of Linder (2005) the growth history, $g(a) = \delta(a)/\delta(1)$, is given by

$$g(a) = \exp \left( \int_a^1 \left[ 1 - \left( \frac{\Omega_m}{a^3 H^2} \right)^\gamma \right] \frac{da'}{a'^2} \right),$$

(3.35)

where $\gamma$ is set by the model and a larger $\gamma$ implies less growth.

This parameterisation cannot model all theories of gravity, since it does not allow for growth histories which have $k$ dependency, such as $f(R)$. It is also only valid for gravity models where the combination of $\Phi + \Psi$ is the same as in GR, which is true for DGP (Koyama, 2006) and $f(R)$ for $f_{R0} \ll 1$ (Oyaizu et al., 2008).

Figures 3.9(a) and 3.9(b) demonstrate the dependence of the parameters on one another when fitting weak lensing predictions for varying $\gamma$, $A$, $\alpha_1$ and $\alpha_2$ to a $\Lambda$CDM fiducial model when $\Omega_m$ and $\sigma_8$ are fixed at the central values fitting WMAP+BAO+SNe, using a simple grid based method to sample the parameter space. The slight widening in the $\gamma$ constraint as $A$, $\alpha_1$ and $\alpha_2$ increase is due to being able to recover $\Lambda$CDM at non-linear scales by increasing $A$ and $\alpha_1$ as $\gamma$ varies. This means that the constraint on $\gamma$ degrades slightly by including the parameters in the Hu and Sawicki fitting formula ($A$, $\alpha_1$ and $\alpha_2$). The constraint obtained by marginalising over all $\Omega_m$ and $\sigma_8$ shown in Figures 3.10(a) and 3.10(b) shows that the constraint for $\gamma$ for a $\Lambda$CDM fiducial model is very good, as shown in Table 3.5, measuring $\gamma$ at the 68% confidence limit within 20% of its value for the ground-based survey and within 5% for Euclid, while the other parameters are difficult to constrain.
Figure 3.9: Constraints on $\gamma$, $\alpha_1$, $\alpha_2$ and $A$ from our ground-based survey and Euclid, using 0.4 redshift bins between 0.3 and 1.5 for the central cosmological parameter values fitting WMAP+BAO+SNe described in §1. The light grey contours show the 68% confidence limits and the dark grey show the 95% confidence limits.
Figure 3.10: Constraints on $\gamma$, $\alpha_1$, $\alpha_2$ and $A$ from our ground-based survey and Euclid, using 0.4 redshift bins between 0.3 and 1.5, where we have marginalised over all $\Omega_m$ and $\sigma_8$. The light grey contours show the 68% confidence limits and the dark grey show the 95% confidence limits.
Table 3.5: The 68% and 95% confidence limits for the growth factor $\gamma$ obtained for our parameterisation with $\Lambda \text{CDM}$ as the fiducial model compared to those obtained using only linear scales and compared to the constraint from using Smith et al. 2003 to model the non-linear. These are marginalised over $\Omega_m$, $\sigma_8$, $A$, $\alpha_1$ and $\alpha_2$.

<table>
<thead>
<tr>
<th>Survey</th>
<th>Our parameterisation</th>
<th>Linear</th>
<th>Smith et al. (2003)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ground-based</td>
<td>68% 0.10 0.23 0.091</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>95% 0.24 0.42 0.18</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Euclid</td>
<td>68% 0.030 0.12 0.026</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>95% 0.069 0.23 0.051</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3.6: The 68% and 95% confidence limits for the growth factor $\gamma$ obtained for our parameterisation with DGP as the fiducial model compared to those obtained using only linear scales and compared to the constraint from using Smith et al. 2003 to model the non-linear. These are marginalised over $\Omega_m$, $\sigma_8$, $A$, $\alpha_1$ and $\alpha_2$.

<table>
<thead>
<tr>
<th>Survey</th>
<th>Our parameterisation</th>
<th>Linear</th>
<th>Smith et al. (2003)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ground-based</td>
<td>68% 0.22 0.38 0.25</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>95% 0.59 0.68 0.48</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Euclid</td>
<td>68% 0.082 0.20 0.052</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>95% 0.12 0.39 0.10</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

A better constraint on the parameters can be found for a growth history that is not $\Lambda \text{CDM}$, such as DGP, as shown in Figures 3.10(c) and 3.10(d). This provides a better constraint on $A$, $\alpha_1$ and $\alpha_2$, but the constraint on $\gamma$ is not as tight, as shown in Table 3.6, measuring $\gamma$ at the 68% confidence limit within 30% of its value for the ground-based survey and within 12% for Euclid. This is due to the degeneracy between $\gamma$ and the other parameters in this instance. These degeneracies can be seen more clearly before the results are marginalised over $\Omega_m$ and $\sigma_8$ as shown in Figures 3.9(c) and 3.9(d). The large dependence on the other fitting parameters demonstrates that care should be taken when predicting $\gamma$ constraints using this parameterisation.

One might think then that it is better not to include non-linear scales and constrain only $\gamma$ on linear scales. However, there is substantial extra signal coming from the non-linear regime. In fact with our parameterisations, Tables 3.5 and 3.6 show the percentage difference between the 68% confidence limit obtained for $\gamma$ if only a linear analysis is used compared to the full non-linear analysis with the fitting formula is 100% for the ground-based survey and 300% for Euclid with a $\Lambda \text{CDM}$ fiducial model, and 70% for the ground-based survey and 140% for Euclid with a DGP fiducial model.

The percentage overestimation, shown in Tables 3.5 and 3.6, at the 68% level, in the ability of the ground-based survey and Euclid to constrain $\gamma$ if only the Smith et. al. fitting formula is used is 10% for the ground-based survey and 40% for Euclid with a
ΛCDM fiducial model, and 10% for ground-based survey and 60% for Euclid with a DGP fiducial model. This demonstrates that if a full non-linear analysis is to be used then it is necessary to ensure that GR is obtained at small scales, and the extra parameters from the Hu and Sawicki fitting formula must also be measured.

3.5 Conclusions

In this chapter we have presented weak lensing predictions for modified gravity models, including the non-linear regime of the power spectrum.

We have shown how the power spectrum is calculated for DGP, \( f(R) \) and QCDM models, using the fitting function of Hu and Sawicki (2007b) to explore deep into the non-linear regime, while including the fact that gravities should tend towards GR on small scales.

We have calculated the total shear power spectrum given the modified gravity power spectrum, and have shown that this will be measured with high signal-to-noise with future lensing surveys such as Euclid and DES. We have taken into account the cosmic covariance in addition to the noise due to the intrinsic shapes of galaxies.

We have shown that there is substantial additional discriminatory power between modified gravity models which is now afforded to us by the inclusion of the non-linear power regime. We have also shown that using only the Smith et al. (2003) formula without any attempt to obtain the GR non-linear power spectrum on small scales leads to an overestimation in the ability of future surveys to differentiate between different growth histories.

We have parameterised the dark matter power spectrum using the growth factor \( \gamma \) and the parameters in the non-linear fitting function to see how well a ground-based survey similar to DES, and a space-based survey such as Euclid, will be able to put constraints on these. We have compared the results from this parameterisation with results obtained from using only linear scales, and have shown the constraint on \( \gamma \) to be much tighter in the former case.
Chapter 4

Weak lensing in coupled dark energy models

In order to overcome the problems of a cosmological constant, alternative models based on the dynamic evolution of a classical scalar field have been proposed (Wetterich, 1988; Ratra and Peebles, 1988; Armendariz-Picon et al., 2000) as shown in Section 1.9.2. Abandoning the simple picture of a cosmological constant, however, necessarily requires us to consider and to include in our models of the Universe the presence of spatial fluctuations and of possible interactions of the new physical degree of freedom represented by the DE scalar field in order to obtain the observed accelerated expansion.

It is in this context that models of interacting DE have been proposed as a natural extension of the minimally coupled dynamic scalar field scenario (Wetterich, 1995; Amendola, 2000; Farrar and Rosen, 2007). Although an interaction of the DE scalar field with baryonic particles is tightly constrained by observations (Hagiwara et al., 2002), the same bounds do not apply to the case of a selective interaction between DE and CDM, as first speculated by Damour et al. (1990), which has therefore received substantial attention as a realistic competitor to the standard ΛCDM model.

Various different forms of interactions between DE and CDM particles (including massive neutrinos) have been proposed and investigated in the literature (as e.g. by Amendola, 2004; Caldera-Cabral et al., 2009b; Pettorino and Baccigalupi, 2008; Amendola et al., 2008a; Boehmer et al., 2010; Koyama et al., 2009; Honorez et al., 2010), and their impact on the linear growth of density perturbations (see e.g. Di Porto and Amendola, 2008; Caldera-Cabral et al., 2009a; Valiviita et al., 2008; Majerotto et al., 2010; Valiviita et al., 2010; Clemson et al., 2011) and on the nonlinear regime of structure formation (Macciò et al., 2004; Baldi et al., 2010; Baldi, 2011a; Li and Barrow, 2011b; Baldi and Pettorino, 2010; Li, 2010; Li and Barrow, 2011a) has been extensively studied in recent
years. For many such models, robust and realistic observational constraints on the interaction strength have been derived based on CMB and LSS data (Bean et al., 2008; La Vacca et al., 2009; Xia, 2009), local dynamical tests using the motions of satellite galaxies (Kesden and Kamionkowski, 2006; Keselman et al., 2009), and Lyman-α observables (Baldi and Viel, 2010). Although these observational bounds have strongly restricted the allowed parameter space for interacting DE cosmologies, none of them has yet been able to rule out the model, or to unambiguously detect the presence of a DE-CDM interaction with compelling statistical significance.

In this respect, exciting times are ahead of us, with the realistic possibility of exploiting the joint power of forthcoming high-precision cosmological observations to break many of the existing degeneracies between competing cosmological models and finally disentangle the distinctive features of alternative scenarios. Dark energy interactions will be one of the issues that can be tested, and so the next generation of cosmological data will possibly provide a real indication of the nature of the DE phenomenon.

This chapter examines the usefulness of weak gravitational lensing for discriminating between interacting dark energy models. I wish to show how the lensing signal depends on the dark energy interaction, and whether this dependence is sufficiently strong that it could be detected with forthcoming lensing surveys. In particular, I will provide forecasts for the capability of future large Weak Lensing (WL) surveys –both a ground-based survey similar to the DES and a space-based survey, i.e. Euclid– to detect a DE-CDM interaction. The particular focus in this work is the non-linear regime, as this regime provides much of the power for lensing. To this end, we exploit the full non-linear matter power spectrum as predicted by the CoDECS simulations (Baldi, 2011b), the largest suite of self-consistent and high-resolution N-body simulations for interacting DE cosmologies to date.

The chapter is organised as follows: Section 4.2 describes the main features of the interacting DE models under investigation; Section 4.2.2 discusses gravitational lensing in the context of interacting DE models, and Section 4.2.1 describes the methods used to compute the necessary nonlinear power spectra. The results of our analysis are presented in Section 4.2.3, giving forecasts for forthcoming lensing surveys; with conclusions in Section 4.2.4.

### 4.1 Perturbed FRW

In order to model how the introduction of a scalar field affects the spacetime we look at the perturbed FRW metric, which is often used to model the regime where $\Phi$ is small.
The perturbed FRW metric has the form

\[ ds^2 = (1 + 2\Psi)dt^2 - a^2(1 - 2\Phi)\delta_{ij}dx^idx^j \]  
(4.1)

In order to calculate the evolution in the components of the model I will look at the field equations where \( G_{\mu\nu} \) is given by

\[
G^t_t = 3H^2 + 2\left[ \frac{1}{a^2} \nabla^2 \Phi - 3H(\Phi_t + H\Phi) \right] 
\]
(4.2a)

\[
G^i_t = -\frac{2}{a^2}(\Phi_t + H\Phi)_i 
\]
(4.2b)

\[
G^i_j = \delta^i_j(2H_t + 3H^2) + 2\delta^i_j[\Phi_{tt} + \frac{1}{2a^2} \nabla^2(\Psi - \Phi) + H(\Phi_t + 2\Phi, t) \\
+ (2H_t + 3H^2)(\Phi + \Psi)] + \frac{1}{a^2}(\Phi - \Psi),_{ij} 
\]
(4.2c)

where \( i, j = 1, 2, 3 \) and represent the spatial dimensions. The first terms in Equations 4.2a and 4.2c are the terms from the background.

Equation 1.12 gives the form for \( T^{\mu\nu} \) where the form of the perturbed \( u_\mu \) is found by solving \( g_{\mu\nu}u^\mu u^\nu = 1 \), which gives (Liddle and Lyth, 2000; Rahvar, 2003)

\[
1 = i^2(g_{tt} + g_{ij}x^i_t x^j_t) 
\]
(4.3)

where \( x^i_t = v^i \) which is the peculiar velocity. Assuming \( v^i \) is small gives

\[
1 = i^2g_{tt} = i^2(1 + 2\Psi) \Rightarrow i = 1 - \Psi 
\]
(4.4)

so \( u_\mu = (1 - \Psi, v_i) \). Using this form of \( u_\mu \) we can calculate \( T^{\mu}_{\nu} \), which to first order in the perturbed quantities is

\[
T^t_t = \rho + \delta\rho \quad (4.5a)
\]

\[
T^t_i = (\bar{\rho} + \bar{\rho})v^i \quad (4.5b)
\]

\[
T^i_t = -a^2(\bar{\rho} + \bar{\rho})v^i 
\]
(4.5c)

\[
T^i_j = -(\bar{\rho} + \delta\rho)\delta^i_j + \Sigma^i_j 
\]
(4.5d)
where $\Sigma_{ij}$ is the anisotropic stress, which can be rewritten in its dimensionless form $\Pi_{ij}$

$$\Pi_{ij} \equiv \frac{\Sigma_{ij}}{\rho} \Rightarrow \Psi = \Phi + a^2 k^2 \rho \Pi_{ij} \quad (4.6)$$

where I have used $G^\mu_\nu = T^\mu_\nu$ with $i \neq j$ from Equations 4.2c and 4.5d to give $\frac{1}{a^2}(\Phi - \Psi),_{ij} = \Sigma^i_j$ and taken the Fourier transform.

I will also look at the energy momentum conservation equations using Equation 1.8, which subtracting the background, given by Equation 1.16, give

$$\delta \rho,_{t} + (\bar{\rho} + \bar{p})v^i,_{i} - 3(\bar{\rho} + \bar{p})\Phi,_{t} + 3H(\delta \rho + \delta p) = 0 \quad (4.7a)$$

$$v_{i,_{t}} + \frac{\delta p,_{i}}{a(\bar{\rho} + \bar{p})} - \frac{\Pi^j_{i,j}}{\bar{\rho} + \bar{p}}v_i + Hv_i + \Psi,_{i} = 0 \quad (4.7b)$$

For a perfect fluid the off-diagonal space-space components vanish, so there is no anisotropic stress and from Equation 4.6 $\Phi = \Psi$. Here I will look at the field equations for CDM so $p = 0$. Equating 4.2 with 4.5, subtracting the background, setting $\Phi = \Psi$ and $p = 0$ gives a modified Poisson equation

$$\nabla^2 \Phi = \frac{1}{2}\bar{\rho}a^2(\delta - 3a^2 Hv) \quad (4.8)$$

where $\delta = \delta \rho/\rho$ and I have used the $ti$ component of the EFEs $\Phi,_{t} + H\Phi = -\frac{1}{2}a^2 \bar{\rho}v$. We can also find the temporal evolution of $\Phi$ using the $ij$ component of the field equation giving

$$\Phi,_{tt} + 4H\Phi,_{t} + (2H,_{t} + 3H^2)\Phi = 0 \quad (4.9)$$

We can also find the continuity equation using the $t$ component of Equation 4.7

$$\delta,_{t} + \nabla \bar{v} - 3\Phi,_{t} = 0 \quad (4.10)$$

and the Euler equation using the $i$ component of Equation 4.7

$$\bar{v},_{i} + 2H\bar{v} + \frac{1}{a^2} \nabla \Phi = 0 \quad (4.11)$$

Performing the same steps as in Section 1.7.1 and taking the Newtonian subhorizon limit ($v \ll c$) gives the same growth equation as previously derived in Equation 1.39.

These conservation equations are altered when a coupling is included as shown in the next section.
4.2 Coupled dark energy models

Coupled DE (cDE) models have been widely investigated in the literature concerning their cosmological background evolution as well as the behaviour of linear and nonlinear density perturbations in these models (see e.g. Amendola, 2000, 2004; Pettorino and Baccigalupi, 2008; Di Porto and Amendola, 2008; Baldi et al., 2010; Li and Barrow, 2011b; Baldi, 2011a, and references therein).

The evolution equations used here are the same as those in quintessence (See Section 1.9.2) combined with the coupling conditions set out in Section 1.9.2. So using the zero-component of Equation 1.79, and the restriction placed on the size of the couplings in Equation 1.80, alters the rhs of Equation 1.76 for dark energy and the rhs of Equation 1.16 for CDM giving a new set of evolution equations

\[
\begin{align*}
\phi_{,tt} + 3H\phi_{,t} + \frac{dV}{d\phi} &= Q_{(\phi)0} \\
\rho_c_{,t} + 3H\rho_c &= -Q_{(\phi)0} \\
\rho_b_{,t} + 3H\rho_b &= 0 \\
\rho_r_{,t} + 4H\rho_r &= 0 \\
H^2 &= (\Omega_r + \Omega_c + \Omega_b + \Omega_\phi)
\end{align*}
\]

where the subscripts \(b, c, r\), indicate baryons, CDM, and radiation, respectively.

This work considers cDE models with couplings of the form following the notation of Amendola (2000)

\[
Q_{(\phi)0} = \sqrt{6}\beta_c(\phi)\Omega_c\phi_{,t}
\]

The function \(\beta_c(\phi)\) sets the direction and the strength of the energy-momentum flow between the DE scalar field \(\phi\) and the CDM fluid, while the function \(V(\phi)\) determines the dynamical evolution of the DE density. Here two possible choices for each of these two functions are considered, namely an exponential, which is a tracking solution where the scalar field density remains close to that of the dominant background matter during most of cosmological evolution (Lucchin and Matarrese, 1985; Wetterich, 1988; Sahni, 2002) and a SUGRA potential, which is chosen because supergravity is one possible explanation for quintessence and the form is the simplest scalar potential that can be
deduced from supergravity (Brax and Martin, 1999):

\[
\text{EXP : } V(\phi) = A e^{-\alpha \phi} \tag{4.18}
\]

\[
\text{SUGRA : } V(\phi) = A \phi^{-\alpha} e^{\phi^2/2} \tag{4.19}
\]

where \(\alpha\) is a positive constant and where for simplicity the field \(\phi\) has been expressed in units of the reduced Planck mass \(M_{\text{Pl}}\).

In addition we consider both a constant and an exponentially growing coupling function \(\beta_c(\phi)\):

\[
\beta_c(\phi) = \beta_0 e^{\beta_1 \phi}, \tag{4.20}
\]

characterised by \(\beta_1 = 0\) and \(\beta_1 > 0\), respectively. The most relevant difference between the exponential potential and the SUGRA potential relies on the fact that the latter features a global minimum at finite scalar field values; this allows for a change of direction of the scalar field motion, which is the main feature of the recently proposed “Bouncing cDE” scenario (Baldi, 2011a). One should also notice that the notation introduced in Eqs. (4.12-4.16) corresponds to the original convention proposed by Amendola (2000) and has been adopted by several other studies, including the CoDECS project considered in the present work, but it differs by a constant factor \(\sqrt{2/3}\) from what is used in another part of the related literature (as e.g. Pettorino and Baccigalupi, 2008; Baldi et al., 2010). The specific models considered in the present work have been described in full detail by Baldi (2011a) and Baldi (2011b); summarised in Table 4.1, where the features and the specific parameters of each model are outlined.

The evolution equations for linear density perturbations in the context of a cDE cosmology can be found by using Equations 4.8, 4.9 with \(T_{\nu\mu}^\alpha = Q(\alpha)\nu\) instead of \(T_{\nu\mu}^\alpha = 0\) for Equation 4.7, where \(Q(\alpha)\nu\) is given by 4.17 and \(H\) is given by 4.16 giving

\[
\delta_{c,tt} = -2H \left[1 - \beta_c \frac{\dot{\phi}_c}{H\sqrt{6}}\right] \delta_{c,t} + \frac{3}{2} \left[\Omega_b \delta_b + \Omega_c \delta_c \Gamma_c\right] \tag{4.21}
\]

\[
\delta_{b,tt} = -2H \delta_{b,t} + \frac{3}{2} \left[\Omega_b \delta_b + \Omega_c \delta_c\right] \tag{4.22}
\]

where \(\delta_{c,b}\) are the relative density perturbations of the coupled CDM and uncoupled baryonic fluids, respectively, and where the scalar field dependence of the coupling function \(\beta_c(\phi)\) has been omitted for simplicity. In Eq. (4.21), the factor \(\Gamma_c \equiv 1 + 4\beta_c^2(\phi)/3\) represents an additional fifth-force mediated by the DE scalar field \(\phi\) for CDM perturbations, while the second term in the first square bracket at the right-hand-side of Eq. (4.21) is
Table 4.1: Interacting dark energy models considered in this chapter. In addition to the concordance $\Lambda$CDM model, we consider the exponential potential with three interaction strengths; the exponential potential with a time-varying strength; and the SUGRA potential. The scalar field is normalised to be zero at the present time for all the models except the SUGRA model, for which the normalisation is set in the very early universe by placing the field at rest in its potential minimum following Baldi (2012). All the models have the same amplitude of scalar perturbations at $z_{\text{CMB}} = 1100$, as shown by the common value of the amplitude $A_s$, but have different values of $\sigma_8$ at $z = 0$, again with the sole exception of SUGRA003.

an extra friction term on CDM fluctuations arising as a consequence of momentum conservation (see e.g. Amendola, 2004; Pettorino and Baccigalupi, 2008; Baldi et al., 2010; Baldi, 2011b, for a derivation of Eqs. (4.12-4.16,4.21,4.22) and for a detailed discussion of the extra friction and fifth force corrections to the evolution of linear perturbations). As a consequence of these two additional terms in the perturbed dynamic equations, CDM fluctuations will grow faster in cDE models with respect to a standard $\Lambda$CDM cosmology, thereby reaching a higher $\sigma_8$ normalisation at $z = 0$ if starting from the same amplitude at the last scattering surface $z_{\text{CMB}} \approx 1100$, as shown in the last column of Table 4.1. However, in the nonlinear regime the interplay between the friction term and the fifth force is not so straightforward as for the case of linear perturbations, due to the fact that as a consequence of virialisation processes, the local velocity field will not necessarily be aligned to the local gradient of the gravitational potential, as one can see from the three-dimensional generalisation of Eq. (4.21) to a system of point-like massive particles, which can be found taking the $t$ component of $T_{\nu\mu}^{\mu} = Q_{(\alpha)\nu}$ where $T_{\nu\mu}^{\mu}$ is given by 4.10 and $Q_{(\alpha)\nu}$ is given by Equation 4.17 so the evolution of $\vec{v}_c$ is

$$\vec{v}_{c,t} = \beta_c(\phi)\frac{\phi_{,t}}{\sqrt{6}}\vec{v}_c - \vec{\nabla} \left[ \sum_c \frac{GM_c(\phi)}{r_c} \Gamma_c + \sum_b \frac{GM_b}{r_b} \right]$$ (4.23)

where $r_{c,b}$ is the physical distance of the target coupled particle from the other CDM and baryonic particles, respectively, the first term in this equation is the friction term and the second term is the fifth force.
The effect of the friction term in the nonlinear regime has been shown to induce a suppression of small-scale power in the cDE models with respect to the nonlinear power that would be inferred based on the large-scale $\sigma_8$ normalisation in the context of a $\Lambda$CDM universe (Baldi, 2011b). Such suppression will have important consequences on the weak lensing constraints on cDE models that we want to address here. Therefore, although it is possible to estimate the full matter power in cDE scenarios by applying nonlinear corrections (calibrated on $\Lambda$CDM simulations) to the re-normalised linear power spectrum (as recently done e.g. by Amendola et al., 2011), in order to reach high accuracy at scales relevant for present and future large lensing surveys it is necessary to rely on a fully nonlinear treatment of cDE scenarios via specific N-body simulations. A discussion on the comparison between these two approaches is presented in Section 4.2.3.

Figure 4.1 shows the full non-linear power spectra as calculated using the CoDECS simulations (see next section) for each of the constant coupling ($\beta_1 = 0$) models normalised by WMAP7. The values of these couplings were chosen since cDE models with $\beta_0 \leq 0.15$ can fit the angular diameter distance to decoupling measured by WMAP7, so these are of particular interest as they are consistent with current observations of the background, but may on the other hand affect the growth of structures. It can be seen that there is a 2-7% difference in the $z = 0$ power spectrum between $\Lambda$CDM and EXP001, the lowest of the couplings investigated here, and a 25-65% difference between $\Lambda$CDM and the highest of the couplings, EXP003. Figure 4.2 shows the difference in the Hubble evolution between the constant coupling models and $\Lambda$CDM. It can be seen there is a maximum of a $\sim 6\%$ difference in the cDE model with the largest coupling, EXP003, and only around a 1% difference in the smallest coupling, EXP001.

### 4.2.1 Simulations

For our analysis we will rely on the public nonlinear power spectrum data computed from the CoDECS simulations (Baldi, 2011b), the largest suite of cosmological N-body simulations for cDE models to date, carried out with the modified version by Baldi et al. (2010) of the widely used Tree-PM parallel N-body code GADGET (Springel, 2005b). In particular we will consider the H-CoDECS suite that includes hydrodynamical simulations of all the cDE models summarised in Table 4.1 on relatively small scales. More specifically, the H-CoDECS runs follow the evolution of $512^3$ CDM and $512^3$ gas particles in a cosmological comoving box of 80 Mpc/$h$ a side, with a mass resolution at $z = 0$ of $m_c = 2.39 \times 10^8$ $M_\odot/h$ for CDM and $m_b = 4.78 \times 10^7$ $M_\odot/h$ for baryons, and a force resolution set by the gravitational softening $\epsilon_g = 3.5$ kpc/$h$. Gravitational softening is implemented to prevent accelerations that are not physical due to the particles being point masses, so the gravitational force is set to a constant below a chosen separation $\epsilon_g$. 
CHAPTER 4. WEAK LENSING IN COUPLED DARK ENERGY MODELS

Figure 4.1: Power spectrum for $\Lambda$CDM and cDE models with constant coupling at $z=0$.

Figure 4.2: Difference in Hubble evolution between cDE models with constant coupling and $\Lambda$CDM.
Parameter & Value \\ 
--- & --- \\ 
$H_0$ & 70.3 km s$^{-1}$ Mpc$^{-1}$ \\ 
$\Omega_{\text{CDM}}$ & 0.226 \\ 
$\Omega_{\text{DE}}$ & 0.729 \\ 
$A_s$ & $2.42 \times 10^{-9}$ \\ 
$\Omega_b$ & 0.0451 \\ 
$n_s$ & 0.966 \\

Table 4.2: The set of cosmological parameters at $z = 0$ assumed for all the models included in the CoDECS project, consistent with the latest results of the WMAP collaboration for CMB data alone (Komatsu et al., 2011b).

Adiabatic hydrodynamical forces on the gas particles are computed by means of the entropy conserving formulation of Smoothed Particle Hydrodynamics (SPH, Springel and Hernquist, 2002) and other radiative processes such as gas cooling, star formation, or feedback mechanisms are not included in the simulations.

Initial conditions are generated at $z_i = 99$, which is chosen since the physics should be sufficiently linear, but it isn’t too time consuming to evolve the simulation from here. These initial conditions are generated by rescaling, with the appropriate growth factor for each specific model, the displacements obtained for a particular random field realisation of the linear power spectrum $P_{\text{lin}}(k)$ at $z_{\text{CMB}}$. This power spectrum is computed by the publicly available Boltzmann code CAMB (Lewis et al., 2000b) for a $\Lambda$CDM cosmology with parameters consistent with the latest “CMB only Maximum Likelihood” constraints from WMAP7 (Komatsu et al., 2011b), which are summarised in Table 4.2. This means that all the different simulations have exactly the same initial conditions at $z_{\text{CMB}}$, and their different features at low redshifts depend uniquely on the different cosmology in place between last scattering and the present time.

The H-CoDECS matter power spectra have been computed by evaluating the density of the different matter components on a grid with the same size of the PM grid ($i.e.$ 512$^3$ grid nodes) through a Cloud-in-Cell mass assignment (See Section 1.7.2) of the different matter species and of the total matter distribution. This procedure allows us to compute the power spectrum up to scales corresponding to the Nyquist frequency of the grid, $i.e.$ $k_{\text{Ny}} = \pi N/L \approx 20.0 \ h/\text{Mpc}$. Beyond this limiting frequency, the power spectrum has been computed with the folding method of Jenkins et al. (1998); Colombi et al. (2008), and the two estimations have been smoothly interpolated around $k_{\text{Ny}}$. Finally, the combined power spectrum has been truncated at scales where the shot-noise reaches 10% of the measured power.
CHAPTER 4. WEAK LENSING IN COUPLED DARK ENERGY MODELS

With the power spectra computed with the procedure just described, we have investigated how future weak lensing probes could perform in constraining cDE cosmologies, as discussed in the next Section.

4.2.2 Lensing in coupled dark energy cosmologies

This analysis uses the form for \( C_\kappa \) derived in Equation 2.51, but the form for Equation 2.52 usually assumes \( \Omega_m(a) = \Omega_m / a^3 \); however in this work the following form is used which does not make such an assumption

\[
P_\kappa(l) = \frac{9}{4} \left( \frac{H_0}{c} \right)^4 \int_0^{\chi_H} d\chi W_1(\chi) W_2(\chi) a^4 \Omega_m(a)^2 P_\delta \left( \frac{l}{\chi}, \chi \right) \tag{4.24}
\]

as coupling CDM and DE means that \( \Omega_m \) has a different dependence on time, as shown in Eqs. (4.12-4.15). We then use this form of \( P_\kappa \) in Equation 2.51 to find the shear correlation function.

I have modified the COSMOS CosmoMC code (Lesgourgues et al., 2007; Lewis and Bridle, 2002; Massey et al., 2007), which calculates the combined shear correlation function from the theoretical power spectrum prediction given by CosmoMC, to include cross-correlation of redshift bins and to calculate the predicted weak lensing signal directly from the cDE model power spectra, according to Eqs. (2.51-2.46). We will now use these results to estimate the discriminatory power from lensing to distinguish between different coupled DE models.

CosmoMC (Lewis and Bridle, 2002) is a Markov-Chain Monte-Carlo (MCMC) code used to explore large parameter spaces, where using a grid method (where every point is sampled, no matter how likely it is to be the best fit to the data), would be too time consuming. MCMC codes take a random walk throughout the parameter space with the step size and direction of the steps determined by an algorithm, such as the Metropolis-Hastings algorithm and the the slice sampling method.

The Metropolis-Hastings (Metropolis et al., 1953; Hastings, 1970) algorithm works by taking an initial value \( x(t) \) and drawing a sample \( x'(t) \) from a proposal distribution \( q(x'|x) \). The probability that \( x'(t) \) is accepted as the next step in the chain is given by

\[
p(x, x') = 1 - \min \left\{ \frac{f(x') q(x|x')}{f(x) q(x'|x)}, 1 \right\} \tag{4.25}
\]

where \( f(x) \) is the probability density function. This is continued until the parameter space has been sampled adequately. This method is fast when a good estimate for the proposal distribution is used, however a poor estimate can result in the chain converging.
very slowly for a distribution that is too narrow, or even worse not sampling the space properly, possibly missing out on important features in the probability density, for a proposal distribution that is too wide.

When the estimate for the proposal distribution is poor it is more efficient to use the slice sampling method (Neal, 2000), which takes longer to take a single step, but doesn’t depend on the proposal distribution. This works by drawing a real value $y_0$ uniformly from $(0, f(x_0))$ to fix the vertical position of the horizontal slice and using a fixed width to sample from $x_0$ to find where the edges of the probability density are. These are then used as the slice boundaries, as shown in Figure 4.3, and $x_1$ is sampled uniformly from the horizontal slice where $x_1$ defines the new sample point.

Here I use the slice sampling method initially to get a good estimate for the covariance matrix, which gives information about the parameter correlations, and width of the proposal distribution, and then use the Metropolis-Hastings algorithm for faster sampling of the parameter space.

The length of the chains used is determined by when the result has converged by computing the Gelman and Rubin $R$ statistic (Gelman and Rubin, 1992; Brooks and Gelman, 1998) which is given by the variance of the chain means divided by the mean of the chain variances. This statistic should be as close to 1 as possible, where $\delta R$ is the difference between $R$ and 1 and is chosen to give the convergence required. Here I choose $\delta R = 0.01$.

4.2.3 Results

We calculated the combined shear correlation function for each of our models using equations 2.51-2.46. We consider two types of survey: a ground-based survey modelled on
<table>
<thead>
<tr>
<th>Survey</th>
<th>$n/g$ arcmin$^{-2}$</th>
<th>Area/degree$^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>DES</td>
<td>13</td>
<td>5000</td>
</tr>
<tr>
<td>Euclid</td>
<td>30</td>
<td>15000</td>
</tr>
</tbody>
</table>

Table 4.3: Galaxy density, $n$, and area assumed for our fiducial DES and Euclid surveys.

DES, and a space-based survey, Euclid; the adopted galaxy density and survey area are shown in Table 4.3. In calculating the shear correlation function for these surveys we therefore use a DES-like redshift distribution given by Fu et al. (2008)

\[ n(z) = \frac{(z^a + z^{ab})}{(z^b + c)} \quad (4.26) \]

where $a = 0.612$, $b = 8.125$, $c = 0.62$, and a space survey redshift distribution for Euclid given by Hawken and Bridle (2009)

\[ n(z) = \alpha \Sigma_0 \frac{z^2}{z_0^2} \exp\left(-\frac{(z/z_0)^\beta}{2}\right) \quad (4.27) \]

where $\alpha = 2$, $\beta = 3/2$, $z_0 = 0.63$ and $\Sigma_0 = 27$ as used in Chapter 3. We also calculated simulated covariance matrices including sample variance and shape noise in the same way to that calculated in 3 using the Horizon simulation (Teyssier et al., 2009); here we used 81 patches of 3.4 square degrees to estimate cosmological sample variance, and assumed an intrinsic shape noise of $\sigma_\gamma = 0.2$ on each component of the shear.

In order to examine whether interacting dark energy models can be detected by forthcoming space and ground-based missions, we can assess the difference in $\chi^2$ between the best-fit $\Lambda$CDM and best-fit interacting DE model for a given dataset. One could choose a fiducial $\Lambda$CDM shear correlation function with realistic error-bars, and find the best-fit interacting DE model for this; but it is more convenient computationally to choose a fiducial interacting DE model and vary parameters of the easily obtained $\Lambda$CDM models to find the best standard cosmology fit. The difference in $\chi^2$ between the two best-fit models is the same whichever way round we choose, and is a measure of our ability to distinguish between the two types of model.

We ran CosmoMC to find the best fit $\Lambda$CDM models for each of the cDE models with different CDM couplings. We used the following parameter space: $0 \leq \Omega_m \leq 0.5$, $0.5 \leq \sigma_8 \leq 1$, $0.4 \leq h \leq 1$, $-2 \leq w \leq 0$ and $0.01 \leq \Omega_b \leq 0.15$. The tomographic lensing results were studied for 3 cross-correlated redshift bins of equal size between $z = 0.3$ and $z = 1.5$ and $1' \leq \theta \leq 90'$. 

CHAPTER 4. WEAK LENSING IN COUPLED DARK ENERGY MODELS

<table>
<thead>
<tr>
<th>Model</th>
<th>$\beta_0$</th>
<th>$\Delta \chi^2_{\text{DES}}$</th>
<th>$\Delta \chi^2_{\text{Euclid}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>EXP001</td>
<td>0.05</td>
<td>3</td>
<td>30</td>
</tr>
<tr>
<td>EXP002</td>
<td>0.1</td>
<td>48</td>
<td>480</td>
</tr>
<tr>
<td>EXP003</td>
<td>0.15</td>
<td>340</td>
<td>3300</td>
</tr>
</tbody>
</table>

Table 4.4: Best fit $\Delta \chi^2$ for different couplings, using errors calculated for DES and Euclid surveys.

**Constant coupling models with an exponential potential**

In this section we look at how introducing a constant coupling between DM and DE (models EXP001-3 in Table 4.1) affects the weak lensing signal. The shear correlation functions, with WMAP7 initial conditions, are shown in Figures 4.4 and 4.5. Note that $\beta_0$ primarily changes the amplitude of the correlation function, with an additional slight alteration in slope. The difference in $\chi^2$ for each of the different constant couplings is shown in Table 4.4, and we see that lensing with Euclid should be able to discriminate between $\beta_0 \geq 0.05$ and $\Lambda$CDM at a confidence level of 99.99994%, while DES should be able to discriminate between $\beta_0 \geq 0.1$ and $\Lambda$CDM at a confidence level of 99.994%.

Figures 4.6, 4.7, 4.8, and 4.9 show that the best fit $\Lambda$CDM models for each of the couplings occupy quite different areas of the parameter space, especially for Euclid. The discrepancies between DES and Euclid predictions in these plots are found to be due to the off-diagonal covariance matrix terms; this can be seen by examining the best fit models for DES and Euclid along with the cDE model we are trying to fit. The best fit for our DES survey appears to be a worse fit at small $\theta$ and a better fit at large $\theta$ than the Euclid best fit. This is due to the covariance being largest for large angles and high redshifts. So while DES has a larger contribution from shape noise at small $\theta$ allowing a worse fit on small scales, conversely Euclid is more sensitive to the covariance on large scales. This discrepancy between the DES and Euclid best fit $\Lambda$CDM increases as $\beta_0$ increases.

These results show that if dark energy and dark matter truly do interact in the way described by our class of models, and we attempt to fit a $\Lambda$CDM cosmology to the observations, then we will infer increased values of $H_0$ and $\sigma_8$, and a decrease in $w$ and $\Omega_m$ as $\beta_0$ increases.

**Other potentials and coupling**

Although for the previous section we restricted ourselves to looking at constant coupling models with an exponential potential, the cDE model has the freedom to examine different potentials and an evolving coupling. Two of the CoDECS simulations explore this
Figure 4.4: Correlation function predicted for cDE models with error estimates for DES using WMAP7 best fit parameters.

Figure 4.5: Correlation function predicted for cDE models with error estimates for Euclid.
Figure 4.6: Constraints on $\Omega_m$, $\sigma_8$, $n_s$, $w$ and $H_0$. The light grey contours show the 68% and 95% confidence limits for DES, while the dark grey contours show the 68% and 95% confidence limits for Euclid for $\Lambda$CDM fiducial.

Figure 4.7: Same as above using an EXP001 ($\beta_0 = 0.05$) fiducial.
Figure 4.8: Constraints on $\Omega_m$, $\sigma_8$, $n_s$, $w$ and $H_0$. The light grey contours show the 68% and 95% confidence limits for DES, while the dark grey contours show the 68% and 95% confidence limits for Euclid for EXP002 ($\beta_0 = 0.1$) fiducial.

Figure 4.9: Same as above using an EXP003 ($\beta_0 = 0.15$) fiducial.
## Table 4.5: Marginalised parameters for $\Lambda$CDM fit to models for DES and Euclid surveys with 1σ errors.

<table>
<thead>
<tr>
<th>Survey</th>
<th>Model</th>
<th>$w$</th>
<th>$H_0$</th>
<th>$\sigma_8$</th>
<th>$\Omega_m$</th>
<th>$n_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>DES</td>
<td>EXP001</td>
<td>$-0.974 \pm 0.020$</td>
<td>$69.2 \pm 3.5$</td>
<td>$0.834 \pm 0.005$</td>
<td>$0.264 \pm 0.003$</td>
<td>$0.952 \pm 0.013$</td>
</tr>
<tr>
<td></td>
<td>EXP002</td>
<td>$-1.012 \pm 0.047$</td>
<td>$82.7 \pm 9.9$</td>
<td>$0.881 \pm 0.010$</td>
<td>$0.259 \pm 0.005$</td>
<td>$0.973 \pm 0.012$</td>
</tr>
<tr>
<td></td>
<td>EXP003</td>
<td>$-1.110 \pm 0.045$</td>
<td>$95.1 \pm 2.8$</td>
<td>$0.946 \pm 0.008$</td>
<td>$0.258 \pm 0.004$</td>
<td>$0.947 \pm 0.009$</td>
</tr>
<tr>
<td></td>
<td>EXP008e3</td>
<td>$-0.981 \pm 0.048$</td>
<td>$77.3 \pm 10.0$</td>
<td>$0.889 \pm 0.010$</td>
<td>$0.262 \pm 0.005$</td>
<td>$0.954 \pm 0.014$</td>
</tr>
<tr>
<td></td>
<td>SUGRA003</td>
<td>$-0.755 \pm 0.044$</td>
<td>$81.1 \pm 6.1$</td>
<td>$0.760 \pm 0.013$</td>
<td>$0.305 \pm 0.008$</td>
<td>$0.760 \pm 0.013$</td>
</tr>
<tr>
<td>Euclid</td>
<td>EXP001</td>
<td>$-0.974 \pm 0.020$</td>
<td>$69.2 \pm 3.5$</td>
<td>$0.834 \pm 0.005$</td>
<td>$0.264 \pm 0.003$</td>
<td>$0.952 \pm 0.013$</td>
</tr>
<tr>
<td></td>
<td>EXP002</td>
<td>$-0.888 \pm 0.020$</td>
<td>$66.1 \pm 1.5$</td>
<td>$0.918 \pm 0.004$</td>
<td>$0.251 \pm 0.002$</td>
<td>$0.956 \pm 0.018$</td>
</tr>
<tr>
<td></td>
<td>EXP003</td>
<td>$-1.004 \pm 0.020$</td>
<td>$73.3 \pm 1.3$</td>
<td>$1.060 \pm 0.002$</td>
<td>$0.218 \pm 0.001$</td>
<td>$1.009 \pm 0.007$</td>
</tr>
<tr>
<td></td>
<td>EXP008e3</td>
<td>$-0.881 \pm 0.020$</td>
<td>$65.6 \pm 0.5$</td>
<td>$0.935 \pm 0.004$</td>
<td>$0.247 \pm 0.002$</td>
<td>$0.922 \pm 0.016$</td>
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<tr>
<td></td>
<td>SUGRA003</td>
<td>$-0.804 \pm 0.020$</td>
<td>$85.4 \pm 2.2$</td>
<td>$0.745 \pm 0.004$</td>
<td>$0.314 \pm 0.004$</td>
<td>$1.092 \pm 0.007$</td>
</tr>
</tbody>
</table>
freedom: EXP008e3, which has the same potential as the models in the previous section but with an evolving coupling, and SUGRA003, which has a SUGRA potential with a constant coupling. Since there is not yet a suite of these types of simulations exploring the full range of parameter space, we have included them as lone examples simply to demonstrate the range of the cDE model. The power spectrum for these models is shown in Figure 4.10, where we can see that for the EXP008e3 model we get similar differences between the cDE model and $\Lambda$CDM to those shown in the larger constant coupling models (EXP002/3). On the other hand, the SUGRA003 model has smaller differences to this at large scales and much larger differences at small scales (almost 100% at $k = 10h$/Mpc) demonstrating how important it is to carry out full simulations of these models in order to obtain small scale predictions. Figure 4.11 shows the difference between the Hubble evolution for $\Lambda$CDM and the cDE models presented in this section. The SUGRA model differences are up to 7% and again the differences show variations, and with the evolving coupling model only has differences up to 4%, which are fairly consistent across all redshifts.

We again attempted to find a best fit $\Lambda$CDM model using CosmoMC and the $\chi^2$ for the best fit result, shown in Table 4.6, demonstrates that for these particular models we would be able to exclude both models at $> (100 - 3 \times 10^{-10})\%$ for both DES and Euclid if the true cosmological model is $\Lambda$CDM. Further investigation of these types of model would allow constraints to be made on the parameters characterising the coupling and the potential.

### Comparison of simulations and Halofit

In section 4.2 we discussed the importance of using N-body simulations over using $\Lambda$CDM non-linear fitting formulae such as Halofit (Smith et al., 2003) to estimate the non-linear power spectrum for cDE models. In Figure 4.12 we show that the use of Halofit, on the linear power spectrum for each of the models, to estimate the non-linear power spectrum results in differences in the shear correlation function that exceed the statistical errors, for each of the surveys and for all of the models considered. This demonstrates that the error due to using Halofit far exceeds the statistical errors and

<table>
<thead>
<tr>
<th>Model</th>
<th>DES $\Delta \chi^2$</th>
<th>Euclid $\Delta \chi^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>EXP008e3</td>
<td>64</td>
<td>570</td>
</tr>
<tr>
<td>SUGRA003</td>
<td>16</td>
<td>100</td>
</tr>
</tbody>
</table>

Table 4.6: Best fit $\Delta \chi^2$ for EXP008e3 and SUGRA003 using errors calculated for DES and Euclid.
CHAPTER 4. WEAK LENSING IN COUPLED DARK ENERGY MODELS

Figure 4.10: Power spectrum for an evolving coupling model with an exponential potential (EXP008e3), and a constant coupling model with a SUGRA potential.

Figure 4.11: Difference in Hubble evolution between \( \Lambda \)CDM and an evolving coupling model with an exponential potential (EXP008e3) and a constant coupling model with a SUGRA potential.
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demonstrates the importance of using N-body simulations to predict the non-linear matter power spectrum for cDE models, and that further simulations for a variety of cDE models are required to make accurate weak lensing forecasts using non-linear scales.

4.2.4 Conclusions

In this chapter we have presented weak lensing predictions for cDE models using the non-linear power spectrum calculated by the CoDECS simulations.

We have calculated the total shear power spectrum for each of the models, and used CosmoMC to find the best fit $\Lambda$CDM model; we have demonstrated the discriminatory power of future lensing surveys such as DES and Euclid, where it should be possible to tightly constrain constant coupling models with exponential potentials to $\beta_0 < 0.05$ with Euclid, or $\beta_0 < 0.1$ with DES. However, this should be considered a best-case scenario, since the inclusion of intrinsic alignments and baryonic physics may impact the constraining power; this will be the subject of future work.

We have shown that for cDE models with larger coupling there is a clear difference between the best fit $\Lambda$CDM for the same model but different surveys. This difference is due to the dominance of the off-diagonal covariance matrix terms over the diagonal for larger surveys, and shows the importance of including these off-diagonal terms in weak lensing predictions.

We have also calculated the expected signal for a non-constant coupling model and a non-exponential potential model. These models could be excluded by $\geq 95.4\%$ for a DES-like survey and $> (100 - 3 \times 10^{-10})\%$ for Euclid. However we have not obtained constraints on the parameters of these types of model, since currently N-body simulations for these models have only been run with one parameter set. A substantial set of simulations would be required in order to properly sample the parameter space of these more complex scenarios. This will be a worthwhile task, as the effects of these cosmologies appear to be more difficult to detect in the background and in the linear regime with respect to standard interacting dark energy models, making non-linear N-body simulations vital for realistic lensing predictions.

We have also shown the size of the error on weak lensing predictions if a $\Lambda$CDM non-linear fitting formula, such as Halofit, is used to estimate the matter power spectrum, instead of using simulations. We find that this Halofit error is larger than the statistical error for the DES and Euclid surveys, and for all the models considered here. This demonstrates the importance of using a full N-body code to estimate the non-linear power spectrum.
Figure 4.12: Difference between shear correlation function calculated using simulations and shear correlation function calculated using Halofit. Also shown is the measurement error in the solid line (from sample variance and shape noise) for DES (top) and Euclid (bottom) using WMAP7 best fit parameters.
Chapter 5

Unified dark matter models

In this chapter I will look at how unifying dark matter and dark energy into one component can affect observations. I examine a phenomenological model where one fluid acts like dark matter at early times and transitions into a dark energy like state at late times. The evolution of the single dark component is governed by three parameters, which are the time of transition, the speed of transition and the amount of the dark component with dark energy like properties at present day.

The particular class of UDM phenomenological models considered here was proposed by Piattella et al. (2010) on the basis of some general considerations to construct viable UDM models, such as being able to make the dark fluid cluster, and model the observed structure formation and producing the correct Integrated Sachs Wolfe (ISW) effect. It is described by a barotropic \( p_X = p_X(\rho_X) \) equation of state that admits an effective cosmological constant. We maintain the same equation of state in the inhomogeneous universe, which gives adiabatic perturbations. These models are characterised by a transition between an early matter dominated era (with an Einstein de Sitter (EdS) evolution) and a more recent epoch whose dynamics, both background and perturbative, are very close to that of a standard \( \Lambda \)CDM model. The equation of state depends on two parameters which tune the time (or redshift) and the speed of the transition, and peculiar features of the models depend on this redshift and transition speed. It follows that submitting these models to minimal requirements to fit observations naturally split them into two regions in parameter space (Piattella et al., 2010): if the transition is slow then these models become indistinguishable from \( \Lambda \)CDM and are therefore uninteresting; if the transition is fast enough then these models retain interesting (potentially observable) features while satisfying current observational constraints. To achieve this, both the evolution of the background and of the perturbations are important. The fast, sudden transition of the background suppresses the otherwise very large difference between the integrated Sachs-Wolfe (ISW) of these models and the ISW in \( \Lambda \)CDM. At the same time,
the fast transition allows the Jeans length to remain small, even if the adiabatic speed of sound becomes very large for a very short time, so that the matter power spectrum is acceptable (see Equation 5.9). As we will illustrate in Section 5.1, other subtle effects (partly depending on the redshift of the transition) also play a role in constraining the model.

Adiabatic UDM models with a fast transition are therefore the focus of this chapter. In particular, we are interested in predicting the constraints that weak lensing will be able to put on the UDM transition parameters with forthcoming surveys such as DES\(^1\) and Euclid\(^2\) together with Planck. We are particularly interested in weak lensing constraints since lensing probes both the expansion history and the growth of structure. This allows us to test the UDM model’s ability to create structure and cause the accelerated expansion using a single fluid. Here we will only investigate linear scales because, for non-linear scales, one needs to go beyond the perturbative regime and increase the sophistication of the UDM model in order to properly take into account the greater complexity of small scale non-linear physics.

The chapter is organised as follows. In Section 5.1 we briefly describe the fast transition UDM model. In Section 5.2 we present the resulting lensing correlation functions, including realistic errors for future surveys taking into account shape measurement noise and cosmic covariance, and the CMB constraints. We present our conclusions in Section 5.3.

5.1 Unified dark matter models with fast transition

I will now derive the constraint equations for the fast transition model proposed by Piattezza et al. (2010) using the perturbed FRW EFEs and conservation equations.

Assuming a spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology the metric is 
\[ ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j, \]
where \( t \) is the cosmic time, \( a(t) \) is the scale factor and \( \delta_{ij} \) is the Kronecker delta. The Friedmann equations (Equations 1.18 and 1.19) are

\[
H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{1}{3}(\rho_r + \rho_b + \rho_X) \quad (5.1)
\]

\[
\frac{a_{,tt}}{a} = -\frac{1}{6} \left( 2\rho_r + \rho_b + \rho_X + 3p_X \right) \quad (5.2)
\]

\(^1\)http://www.darkenergysurvey.org
\(^2\)http://www.ias.u-psud.fr/imEuclid
where I have used $p_r = \rho/3$, $H = \dot{a}/a$ is the Hubble expansion scalar and the dot denotes derivative with respect to the cosmic time. Here $\rho_r$ and $\rho_b$ are the radiation and baryon densities and $\rho_X$ and $p_X$ represent the energy density and the pressure of the UDM component. Each component is separately conserved since there is no coupling between the baryons and photons and the dark fluid. In particular, conservation of energy for the UDM gives the same relation as in Equation 1.16

$$\rho_{X,t} = -3H(\rho_X + p_X) \quad (5.3)$$

and the evolution equations for the UDM component are the same as those detailed in Section 4.1 with $\rho_X$ and $p_X$. Therefore the Poisson equation is given by Equation 4.8

$$\nabla^2 \Phi = \frac{1}{2} \bar{\rho} a^2 (\delta - 3a^2 Hv(1 + w)) \quad (5.4)$$

The temporal evolution of $\Phi$ is given by Equation 4.9 with a non-zero pressure

$$\Phi_{,tt} + 4H \Phi_{,t} + (2H_{,t} + 3H^2) \Phi = -\frac{c_s^2 \bar{\rho} \delta}{2} \quad (5.5)$$

the perturbed continuity equation is given by Equation 4.10 with a non-zero pressure

$$\delta_{,t} + (1 + w)\nabla \vec{v} + 3H \delta \left(c_s^2 - w\right) - 3(1 + w) \Phi_{,t} = 0 \quad (5.6)$$

and the Euler equation is given by Equation 4.11 with a non-zero pressure

$$\vec{v}_{,t} + 2H \vec{v} + \frac{1}{a^2} \nabla \Phi + \frac{\nabla (c_s^2 \delta)}{a^2 (1 + w)} - 3H \frac{\partial \bar{p}}{\partial \bar{\rho}} \vec{v} = 0 \quad (5.7)$$

where I have used $c_s^2 = \frac{\delta p}{\delta \bar{\rho}}$ and $w = \bar{p}/\bar{\rho}$.

Taking the the spatial derivative of Equation 5.7 and substituting it into the $t$ derivative of Equation 5.6 gives the density evolution as

$$\delta_{,tt} + 2H(1 - 3w) \delta_{,t} + \bar{p} \left[ \frac{3}{2} (1 + w) \left( w - \frac{\partial \bar{p}}{\partial \bar{\rho}} \right) \left( \frac{3a^2 H^2}{k^2} - 2 \right) + 9w^2 \left( \frac{1}{2} - \frac{a^2 H^2}{k^2} \right) \right] - \frac{3H^2 a^2}{k^2} (1 + 4w) - \frac{1}{2} - w \right] \delta + \left[ \frac{3}{H} \frac{\partial \bar{p},t}{\partial \bar{\rho}} - \frac{k^2}{3a^2 H^2} + 2 - 3w \right] \delta \rho = 0 \quad (5.8)$$

In the same way as in Section 1.7.1 we can see that when $\delta \rho$ dominates the potential oscillates, and when $\delta \rho$ dominates we get collapse; so we can define a Jeans’ scale, where
the sum of the \( \delta p \) and \( \delta \rho \) terms is zero (Bertacca et al., 2010)

\[
 k^2 = \frac{2}{3} \frac{\bar{\rho}(1 + w) a^2}{c_s^2} \left[ \frac{1}{2} (c_s^2 - w) - \rho \frac{dc_s^2}{d\rho} + \frac{3(c_s^2 - w)^2 - 2(c_s^2 - w)}{6(1 + w)} + \frac{1}{3} \right] \tag{5.9}
\]

From this we can see that the Jeans scale is small for small \( c_s^2 \) but can also be small for large \( c_s^2 \) provided \( \frac{dc_s^2}{d\rho} \) is large. This defines the transition as “fast” from the point of view of perturbations, as this term determines the rapidity of change of the speed of sound.

A barotropic equation of state (EoS) \( p_X = p_X(\rho_X) \) that satisfies our fast condition, along with acting like CDM at early times and like \( \Lambda \) at late times, is (Piattella et al., 2010)

\[
p_X = -\rho_\Lambda \left[ \frac{1 - \tanh \left( \frac{\rho_X - \rho_\Lambda}{\rho_\Lambda} \right)}{1 - \tanh \left( \frac{\rho_\Lambda - \rho_t}{\rho_s} \right)} \right] \tag{5.10}
\]

which is illustrated in Figure 5.1, showing \( p = 0 \) at early times (\( \rho \) large) and \( p = -1 \) at late times (\( \rho \) small) as required. This depends on the three parameters \( \rho_\Lambda, \rho_s \) and \( \rho_t \). We assume \( \rho_X < \rho_\Lambda \), otherwise we would have a phantom model.

Substituting Equation 5.10 into Equation 5.3 and taking the early and late time limits shows that the EoS (Equation 5.10) describes a transition between two phases joined by the flex in the EoS curve in Figure 5.1 at \( \rho_X = \rho_t \). At early times the pressure \( p_X \) is negligible, for \( \rho_X \gg \rho_t \), and in this regime the UDM behaves as CDM, \( \rho_X \sim a^{-3} \). At late times, assuming \( \rho_t/\rho_\Lambda \) is large enough,\(^1\) \( p_X \simeq -\rho_\Lambda \) and the UDM evolves like the total density in \( \Lambda \)CDM (Balbi et al., 2007; Pietrobon et al., 2008; Piattella et al., 2010)

\[
\rho_X \simeq \rho_\Lambda + (\rho_{X0} - \rho_\Lambda)a^{-3} \tag{5.11}
\]

where \( \rho_{X0} \) is the UDM density today. The challenge for this UDM model is therefore to have a transition between the early CDM phase, evolving like an EdS model, and the late \( \Lambda \)CDM phase, satisfying observational constraints while retaining distinguishable features.

It is clear from the EoS (Equation 5.10) that \( p_X = -\rho_\Lambda \) for \( \rho_X = \rho_\Lambda \), and then Equation 5.3 consistently implies that \( \rho_\Lambda \) plays the role of an effective cosmological constant even if there are no \( \Lambda \) terms in Equations 5.1 and 5.2. Equation 5.3 with the EoS (Equation 5.10) implies that at late times \( \rho_X \to \rho_\Lambda \) and the Universe inevitably evolves toward an asymptotically \( \Lambda \)-dominated (de Sitter) phase.

\(^1\)Given that today \( \Omega_\Lambda < 1 \), it is reasonable to assume that \( \rho_t/\rho_\Lambda \) has to be large, so that the transition is in the past. Then it follows from the EoS (Equation 5.10) that the UDM has now been \( \Lambda \)-dominated for a while, \( p_X \simeq -\rho_\Lambda \).
Figure 5.1: The EoS $p_X = p_X(\rho_X)$ of our UDM model illustrated here (solid line) for parameters values $\rho_t/\rho_\Lambda = 3.5$ and $\rho_s/\rho_\Lambda = 0.5$. The thin horizontal solid lines respectively represent $\Lambda$CDM ($p/\rho_\Lambda = -1$) and pure CDM (an EdS model with $p = 0$). The other thin solid line represents the $p = -\rho$ line. The energy density of the barotropic UDM decreases with time, asymptotically approaching the effective cosmological constant, i.e. the point $\rho/\rho_\Lambda = 1$, $p/\rho_\Lambda = -1$. The dashed line represents the $p = -\rho/3$ line, the boundary between the decelerated expansion phase of the Universe and the accelerated one; the dotted line $p = -\rho/10$ represents a notional boundary, above which pressure is negligible and the CDM-like behaviour of the UDM fluid dominates.
Taking derivatives of Figure 5.1 allows us to investigate how $c_s^2$ changes with $\rho$ for our chosen EoS with $\rho_t/\rho_\Lambda = 5$ and $\rho_s/\rho_\Lambda = 1$.

The effective cosmological constant $\rho_\Lambda$ fixes an energy density scale, like in $\Lambda$CDM, then the model depends on the two extra dimensionless parameters $\rho_s/\rho_\Lambda$ and $\rho_t/\rho_\Lambda$. Equivalently, we find it useful to use the corresponding density parameters $\Omega_\Lambda$, $\Omega_s$ and $\Omega_t$, where $\Omega_\alpha = \rho_\alpha/3H_0^2$. The values of $\rho_s/\rho_\Lambda$ and $\rho_t/\rho_\Lambda$ (or $\Omega_s$ and $\Omega_t$) respectively regulate how fast the transition is and the redshift of the transition $z_t$. For reference in the next sections, the precise relation between $\Omega_t$ and the corresponding redshift $z_t$ is shown in Figure 5.4 giving a monotonic relationship between $z_t$ and $\Omega_t$.

Looking now at the slope of the EoS curve in Figure 5.1, this is given by

$$c_s^2 = \frac{\rho_\Lambda}{\rho_s} \frac{1 - \tanh^2 \left( \frac{\rho - \rho_t}{\rho_s} \right)}{1 - \tanh \left( \frac{\rho_\Lambda - \rho_t}{\rho_s} \right)},$$

(5.12)

representing the adiabatic speed of sound $c_s^2 = \dot{\rho}/\dot{\rho}$ of the UDM. Figure 5.2 shows how the dominant components of the Jeans length evolve with time for $\rho_t/\rho_\Lambda = 5$ and $\rho_s/\rho_\Lambda = 1$.

$c_s^2$ attains its maximum value

$$c_{s,\text{max}}^2 \bigg|_{\rho_s} = \frac{\rho_\Lambda/\rho_s}{1 - \tanh \left( \frac{\rho_\Lambda - \rho_t}{\rho_s} \right)}$$

(5.13)

at the flex point $\rho_X = \rho_t$, clearly showing that smaller values of $\rho_s/\rho_\Lambda$ make the transition faster (see the dashed line in Figure 5.3). On the other hand, models with large
Figure 5.3: The EoS $w(z)$ for four representative values of $\rho_s/\rho_\Lambda$ and transition at $\rho_t/\rho_\Lambda = 5 (\Omega_t \simeq 3.5)$. The thin horizontal solid lines respectively represent pure CDM ($w = 0$) and the boundary $w = -1/3$ between acceleration and deceleration, the solid line represents the total $w$ of $\Lambda$CDM. For the smaller value (dashed line) of $\rho_s/\rho_\Lambda$ the transition is fast and the model reaches the $\Lambda$CDM phase before the start of acceleration. For larger $\rho_s/\rho_\Lambda$ the transitions is slower (dot-dashed and dotted lines), acceleration starts later than in $\Lambda$CDM, and the $\Lambda$CDM phase is approached later, or not at all. If $\rho_s/\rho_\Lambda$ is large enough (long dashed line), the UDM model is close to $\Lambda$CDM at all times.
Figure 5.4: The mapping between $z_t$ and $\Omega_t$.

enough $\rho_s/\rho_\Lambda$ have a background evolution very close to that of $\Lambda$CDM at all times (see the long dashed line in Figure 5.3), with $c_s^2 \simeq 0$. These “slow transition” models are indistinguishable from $\Lambda$CDM, hence we mostly focus on models with a fast transition.

More precisely, in the fast transition models we consider, $c_s^2 \simeq 0$ well before and after the transition, in the early CDM-like (EdS) phase and in the late $\Lambda$CDM phase, which implies a vanishingly small Jeans scale $\lambda_J$ (Equation 5.9) and, consequently, an evolution of perturbations during these epochs which is much like in the usual matter era and in $\Lambda$CDM. However, it follows from (Equation 5.12) that during the fast transition the speed of sound can be very large, with potentially disruptive effects on the evolution of perturbations during this time. However, the analysis carried out in Piattella et al. (2010) shows that the Jeans scale during the transition can be kept small if the transition is fast.

In models with $\rho_t \ll \rho_s$, a slow transition, we have $c_s^2|_{\text{max}} \sim \rho_\Lambda/\rho_s \sim 0$, from Equation 5.13, and these models are close to a $\Lambda$CDM at all times. Instead for $\rho_t \gg \rho_s$, a fast transition, we have two subcases: a) when $\rho_\Lambda \approx 2\rho_s$, we get $c_s^2|_{\text{max}} \sim \rho_\Lambda/2\rho_s < 1$; b) when $\rho_\Lambda \gtrsim 2\rho_s$, we get $c_s^2|_{\text{max}} \sim \rho_\Lambda/2\rho_s > 1$. The latter subcase may in principle imply superluminal perturbations: fortunately, a-causal effects can be avoided if the transition is sufficiently fast. Indeed, it was shown in (Piattella et al., 2010) that with a fast transition the predicted CMB and linear matter power spectra do not display large differences from
Figure 5.5: The evolution of the energy densities of UDM and radiation in two UDM models, compared with the evolution of CDM and radiation in an EdS model, and with the total density of dark components and radiation in a $\Lambda$CDM model. In both models $\Omega_s = 10^{-8}$, with $\Omega_t = 2$ in the top panel and $\Omega_t = 5$ in the bottom panel. Clearly, for the latter the matter-radiation equality is close to that in $\Lambda$CDM, while for $\Omega_t = 2$ it is close to that in the EdS model.
those computed in the ΛCDM model. In addition, with a fast transition the Jeans length $\lambda_J$ remains small at all times, except for negligibly short periods, giving an acceptable matter power spectrum. Thus, within the limits of the linear analysis here and in (Piattella et al., 2010), no superluminal effects have been found in the UDM models considered.

A very fast transition, or in the opposite case a slow transition, is not enough to produce viable models if the transition occurs at too low a redshift. In this case, see Figure 5.5, for a given $\Omega_\Lambda$ the evolution is too close to that of an EdS for too long, and the matter-radiation equality (indicated by the cross over of the $\Omega_r$ line (red dotted line) and the $\Omega_{UDM}$ line (black solid line)) occurs too early with respect to that of ΛCDM. For given initial conditions, this results in an increase in amplitude of perturbations below the matter-radiation equality scale.

The main effects of varying the values of $\Omega_s$ and $\Omega_t$ are summarised in Figure 5.6 for the matter power spectrum and in Figure 5.7 for the CMB power spectrum. Figure 5.6 shows that as we increase $\Omega_s$ (top of figure) the power spectrum tends towards that of ΛCDM independent of the value of $\Omega_t$ chosen. The central part of the figure shows regions in the $\Omega_t$, $\Omega_s$ parameter space where $k_J$ is too small, which results in an oscillating potential on scales smaller than $k_J$. However we are most interested in fast transition models, small $\Omega_s$ (bottom of figure), where the power spectrum tends towards ΛCDM for larger $\Omega_t$. As said above, a small $\Omega_t$ results in too much small scale power due to the late transition and the early matter-radiation equality. This causes the change in the amplitude of the peaks in the CMB power spectrum for small $\Omega_t$ in Figure 5.7, and also affects the angular diameter distance to the CMB, as the Universe is matter dominated for longer.

At this point a relation can be used to investigate the maximum length of time a transition can occur for and remain undetected by for instance DES, so we define $\alpha$ as the interval of the time in which the transition takes place normalised by the present Hubble time, $H_0$, where $\alpha$ is given by

$$\alpha = H \left. \frac{p}{p_t} \right|_{\text{max}} = \frac{-w}{3 c_s^2 (1 + w)} \left|_{\text{max}} \right. \simeq \frac{2 \rho_s / \rho_t}{3 (2 - \rho_\Lambda / \rho_t)}$$ (5.14)

where again $H = \frac{\dot{a}}{a} / H_0$. At this point, we impose that for $\alpha \leq \alpha_{fast} \ll 1$ we have a fast transition regime, where $\alpha_{fast}$ is the maximum amount of time allowed for the transition to go undetected by our surveys.
Figure 5.6: Power spectra with different $\Omega_t$ and $\Omega_s$, with other parameters set by WMAP7 (solid black) and WMAP7 $\Lambda$CDM power spectrum (grey dashed).
CHAPTER 5. UNIFIED DARK MATTER MODELS

5.2 Results

Since dark matter and dark energy are a single component in this model, we examine the total matter power spectrum and must look at the combination $\Omega_m \sigma_8$ when comparing $\sigma_8$ between $\Lambda$CDM and UDM models.

We will consider the use of the combined shear correlation function using the form of the shear power spectrum in Equation 4.24 where $\Omega_m(a)$ is replaced with $\Omega_{\text{tot}}(a) = H(a)^2$ to put constraints on the UDM model parameters. We use the galaxy distributions defined in Chapter 3 and work out the covariance matrix due to sample variance and shape noise in a similar way to that calculated in Chapter 3, apart from restricting ourselves to linear scales (i.e. $\theta \geq 30'$), using the Horizon simulation (Teyssier et al., 2009) in order to obtain our weak lensing results.

We then used the modified CosmoMC code used in Chapter 4 to calculate the predicted weak lensing signal and CMB power spectrum from the UDM total power spectra, which were calculated using a modified CAMB. This is modified to include the UDM equation of state (Equation 5.10) and speed of sound (Equation 5.12). We then compare the CAMB output to the results obtained for a fiducial $\Lambda$CDM total power spectrum using the parameters measured by WMAP7.

In this analysis we use 8 $\theta$ bins and 3 redshift bins from $z = 0.3$ to $z = 1.5$ as used in Chapter 3. We obtain CosmoMC results from varying $\Omega_b, \Omega_l, \Omega_s, \Omega_\Lambda, h, \sigma_8$ and $n_s$. 

Figure 5.7: The CMB power spectrum for $\Omega_s = 10^{-3}$ and various values of $\Omega_l$. 

...
Figure 5.8: Predicted constraints on UDM parameters from the shear correlation with DES errors (green), Euclid errors (blue) and from the TT power spectrum with Planck errors (red).
Figure 5.9: Best fit $\alpha$ for predicted constraints with DES (green), Euclid (blue) and Planck (red).
Figure 5.8 shows the constraints obtained from CosmoMC for parameters for the UDM model with DES, Euclid and Planck surveys. It should be noted that for these plots $\theta$, the ratio of the sound horizon at decoupling to the angular diameter distance to decoupling, and $\tau$, the reionisation optical depth, are fixed. Weak lensing is not very sensitive to these parameters however including these parameters could reduce the Planck constraint. There is no upper bound on $\Omega_t$ and no lower bound on $\Omega_s$, which is due to the UDM model tending towards $\Lambda$CDM in these regimes. However we can put a lower bound on $\Omega_t$ at a 95% confidence level of 3.1 for DES and 4.0 for Euclid and an upper bound on $\Omega_s$ at a 95% confidence level of $2 \times 10^{-3}$ for both DES and Euclid. Table 5.1 shows all marginalised parameters for DES and Euclid for these models. Comparing the best fit of the usual cosmological parameters to those used in the fiducial model shows the parameters are equal within the 68% confidence limits.

It is clear from Figure 5.8 that weak lensing observations put tighter constraints on $\Omega_s$, and that CMB observations put tighter constraints on $\Omega_t$. This means that weak lensing is more sensitive to the speed of transition, while it is not affected by the time the transition occurs provided it doesn’t occur in the low redshift range it directly probes. The reason for this can be seen by looking at Figure 5.6, where it is clear that an $\Omega_s$ that is too large results in an oscillating power spectrum, which weak lensing is sensitive to. However the CMB is more greatly affected by the time of transition, since if the transition happens later then the Universe looks like an EdS universe longer, directly affecting the distance to the CMB and the initial amplitude of the power spectrum, which are the main constraints from the CMB on these models.

It is also interesting to note that Euclid improves the weak lensing constraint on $\Omega_t$, but does not improve the $\Omega_s$ constraint, despite the slightly deeper redshift distribution and much improved statistics. This is due to the $\Omega_s$ constraint coming from pushing the oscillations in the power spectrum to scales past the linear regime we are probing, therefore a full non-linear model could improve constraints.

Here we have made predictions for a high redshift probe, the CMB, and a low redshift probe, weak lensing. However Figures 5.10, 5.11 and 5.12 all show features that do not appear in $\Lambda$CDM. Figure 5.10 shows differences between the $\Lambda$CDM $D_A$ and the UDM $D_A$ of up to $\sim 10\%$ using DES errors, up to $\sim 5\%$ for Euclid, and only up to $\sim 1\%$ error for Planck. Figures 5.11 and 5.12 show an even larger possible difference between the $\Lambda$CDM $\delta$ and the UDM $\delta$ with bumps for some when the transition occurred.

However these results are difficult to compare, as while they have similar values for $\Omega_t$, each survey has a different best fit $\Omega_b$, $\Omega_\Lambda$, $h$, $\sigma_8$ and $n_s$ for a specific $\Omega_t$ and $\alpha$ combination. Therefore results from combining the probes will be important.

If the combined results give similar results, then in order to constrain these models further a mid redshift range probe, that could probe the mid redshift transitions directly
would provide the best constraints, as transitions at high redshifts lead to models that always look like $\Lambda$CDM so are of little interest. Possible mid-redshift probes include the radio telescopes LOFAR and SKA since these probe out to $z \sim 4$.

We also demonstrate in Figure 5.9 that Equation 5.14 correctly predicts the relationship between the bounds of $\Omega_s$ and $\Omega_t$ for the fast transition regime for the weak lensing observations. Here we see that $\alpha_{\text{fast}} = 5 \times 10^{-6}$ fits the 95% confidence limit for DES and Euclid; however for Planck this is a poor fit since the slope is too steep. This is the same relationship we saw in Figure 5.6 and enables us to estimate how fast the transition must be in order for it to be within the observational constraints.

## 5.3 Conclusions

In this work we have shown constraints on UDM fast transition models using weak lensing, with DES and Euclid, and the CMB, with Planck. We predict that these observations should be able to constrain the transition redshift to $z_t > 5$ at a 95% confidence level and the maximum time the transition can take to $< 5 \times 10^{-6}/H_0$ at a 95% confidence level for a $\Lambda$CDM fiducial.

We have outlined the physics of UDM models and discussed how the parameters in the fast transition UDM model alter the evolution of the background and growth.

We have shown that these models can only be significantly constrained by using probes at a variety of redshifts, since the main effect of the transition redshift is the change in the redshift of matter-radiation equality, and therefore the Hubble expansion and the amplitude of the power spectrum at high redshift. The main effect of the speed of transition is on the Jeans length, the effect of which we see at low redshift, so we need to constrain both the high redshift physics and the late $z$ power spectrum.

Therefore this kind of adiabatic UDM model could evade the “no-go theorem” of Sandvik et al. (2004) who, studying the generalised Chaplygin gas UDM models, showed that this broad class must have an almost constant negative pressure at all times in order to satisfy observational constraints, making these models in practice indistinguishable from the $\Lambda$CDM model (see also Pietrobon et al., 2008 and Piattella et al., 2010). However a combined weak lensing and CMB analysis will give better constraints on this.
Figure 5.10: Difference in $D_A$ for $\alpha = 5 \times 10^{-6}$ for DES best fit (top), Euclid best fit (middle) and Planck best fit (bottom) and for different values of $\Omega_t$, which we have plotted to see what angular diameter distances are allowed along the $\alpha = 5 \times 10^{-6}$ line.
Figure 5.11: Difference in $\delta(k = 0.02)$ for $\alpha = 5 \times 10^{-6}$ for DES best fit (top), Euclid best fit (bottom) for different values of $\Omega_t$, which we have plotted to see what values of $\delta$ are allowed along the $\alpha = 5 \times 10^{-6}$ line.
Figure 5.12: Same as Figure 5.11 but with $k=0.2$
### Table 5.1: Marginalised parameters for DES and Euclid surveys with $1\sigma$ errors.

<table>
<thead>
<tr>
<th>Survey</th>
<th>$\Omega_t$ (95% CL)</th>
<th>$\Omega_s$ (95% CL)</th>
<th>$H_0$</th>
<th>$\Omega_\Lambda$</th>
<th>$\Omega_m\sigma_8$</th>
<th>$n_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>DES</td>
<td>3.1 \ (&gt; 3.1)</td>
<td>$2 \times 10^{-3}$</td>
<td>$75 \pm 9$</td>
<td>$0.765 \pm 0.032$</td>
<td>$0.212 \pm 0.015$</td>
<td>$0.956 \pm 0.106$</td>
</tr>
<tr>
<td>Euclid</td>
<td>4.0 \ (&gt; 4.0)</td>
<td>$2 \times 10^{-3}$</td>
<td>$73 \pm 7$</td>
<td>$0.772 \pm 0.017$</td>
<td>$0.210 \pm 0.010$</td>
<td>$0.961 \pm 0.017$</td>
</tr>
<tr>
<td>Planck</td>
<td>67 \ (&gt; 67)</td>
<td>$5 \times 10^{-2}$</td>
<td>$70 \pm 1$</td>
<td>$0.726 \pm 0.006$</td>
<td>$0.224 \pm 0.022$</td>
<td>$0.959 \pm 0.003$</td>
</tr>
</tbody>
</table>
Chapter 6

A general relativistic model of weak lensing

Forthcoming wide field lensing surveys have brought about the possibility of doing precision cosmology with weak lensing. This however brings its own problems, since it brings into question how well we understand lensing systematics and the effects of our approximations used in our models in Section 2.4. The main question we are interested in is this: If we found deviations from the concordance model of cosmology are these due to the underlying cosmology, or the way we model lensing?

We have analytical models for linear scales (FRW) and very non-linear scales (virialised structures), however there is no analytical model that connects these two regimes. This leads us to question what is happening on these intermediate scales, and also how changing our model of these scales affects our observations.

Another component of our model, for which the possibility of having a direct effect on lensing has been debated by many papers, is the cosmological constant. Rindler and Ishak (2007); Ishak (2008); Ishak et al. (2008) argue that there is an effect; however this uses a static Kottler metric as used by Lake (2002) who found no such effect. Other papers that have investigated this effect are Kantowski et al. (2009) which looks at a Swiss cheese model with a Kottler metric embedded in a FRW background and finds an effect, and Simpson et al. (2008) which looks at the perturbed FRW usually used in lensing and finds no such effect.

In this chapter I am interested in trying to find a watertight GR model for galaxy and cluster lenses to investigate the effect of how well the perturbed FRW models lensing, how the physics at intermediate scales affects our observations and how \( \Lambda \) effects lensing, whether directly or indirectly. The paradigm I will examine is a single virialised galaxy or cluster in an expanding background, which I will describe in Section 6.3; however first I will look at the results from the Kottler metric in Section 6.1 and the perturbed
FRW metric in Section 6.2, and discuss the merits and flaws of each of the models and how we can possibly overcome these. In Section 6.3 I will investigate the possibility of using a Lemaitre-Tolman-Bondi (LTB) dust model to model a lens in an expanding FRW background and show that we can only model a virialised structure if pressure is introduced and we use a generalised form of the LTB, which is investigated in Section 6.4. The lensing quantities we use for this analysis are detailed in Section 6.5 and the results for these models will be shown in Section 6.6.

6.1 Kottler Metric

Many papers that have argued for the direct effect of the cosmological constant (Rindler and Ishak, 2007; Ishak, 2008; Ishak et al., 2008) have used the Kottler metric (Kottler, 1918; Weyl, 1919), which introduces a \( \Lambda \) term into the Schwarzschild metric (see Section 2.3), by including \( \Lambda \) in the usual Schwarzschild metric EFEs, as follows

\[
ds^2 = \left(1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2\right) dt^2 - \left(1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right) \tag{6.1}\]

By comparing with the Minkowski metric, we can see that proper time interval \( d\tau \) and \( dt \) are related by \( \left(1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2\right)^{1/2} \), however unlike the Schwarzschild in Section 2.3 at large \( r \) we do not obtain \( t \rightarrow \tau \), but in instead \( t \rightarrow \frac{\tau}{r} \sqrt{\frac{2}{\Lambda}} \). The proper distance interval \( d\tilde{r} \) is related to \( dr \) by \( \left(1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2\right)^{-1/2} \), so again at large distances \( r \) is not equal to the proper distance and instead \( \tilde{r} \rightarrow \sqrt{\frac{\Lambda}{3}} \sqrt{\frac{2}{\Lambda}} \), which looks like a spherical geometry; however the angular part of the metric is the same as for the Minkowski and Schwarzschild metric so we can see that \( r \) is the angular diameter distance.

All quantities are dimensionless since distance \( (r) \) is normalised by \( c/H_0 = 3000 h^{-1}\text{Mpc} \), time \( (t) \) is normalised by \( 1/H_0 = 10^{10} h^{-1}\text{y} \) and \( M(r) \) is normalised by \( c^3/GH_0 = 10^{23} M_\odot \). I will use these normalisations throughout this chapter.

When only looking in the plane \( \theta = \pi/2 \) the null geodesic condition, \( ds^2 = 0 \), gives a form for \( \dot{r} \)

\[
\dot{r} = \left[ \left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)^2 \dot{t}^2 - \left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right) r^2 \dot{\phi}^2 \right]^{1/2} \tag{6.2}
\]
where dot denotes $\frac{d}{d\sigma}$ where $\sigma$ is some affine parameter. The geodesic equations, Equation 2.2, give the forms for $\dot{t}$ and $\dot{\phi}$

$$\dot{t} = k \left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)^{-1}$$  \hfill (6.3)\nonumber

$$\dot{\phi} = \frac{J}{r^2}$$  \hfill (6.4)\nonumber

Dividing Equation 6.4 by Equation 6.2 gives how $\phi$ varies with $r$

$$\frac{d\phi}{dr} = \frac{1}{r^2} \left[\frac{1}{b^2} - \frac{1}{r^2} \left(1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\right)\right]^{-1/2}$$  \hfill (6.5)\nonumber

where $b = J/k$. Expanding this in terms of $\frac{2M}{r}$ to first order gives the following bend angle $\hat{\alpha}$ as shown in Figure 6.1

$$\hat{\alpha} = \frac{4M}{b} \sqrt{1 + \frac{\Lambda}{3}} b^2$$  \hfill (6.6)\nonumber

where the first term is the usual bend angle in a Schwarzschild spacetime shown in figure 6.1 and the additional term is due to the presence of a cosmological constant.

However this model is static with a cosmological constant and a point mass. This is not a very realistic model of a lens since in a universe full of material such as ours, the cosmological constant causes the spacetime to expand, which it does not here as the
spacetime is static. In addition we know that lenses are not point masses but rather are extended objects.

6.2 Perturbed FRW

Simpson et al. (2008) used a perturbed FRW to look at the effect of $\Lambda$ on lensing. They compared the Kottler to the perturbed FRW and found an expression for the Kottler metric components, $f(r) = 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}$, in terms of the perturbed quantity $\Phi$.

We start with the perturbed FRW in spherical polar coordinates for a flat spacetime given by

$$ds^2 = (1 + 2\Phi)dt^2 - (1 - 2\Phi)a^2 d\chi^2 - \chi^2(d\phi^2 + \sin^2 \theta d\phi^2)$$  \hspace{1cm} (6.7)

and then transform our coordinates to get this metric in terms of the proper radial distance $\tilde{r}$ and remove terms in $\chi$ by comparing the perturbed FRW with the Minkowski metric from which we obtain $\tilde{r} \approx (1 - \Phi)a\chi$ to first order in $\Phi$. Differentiating this w.r.t. $\sigma$, squaring, and taking $\Phi$ to first order again gives the radial component of the line element

$$a^2(1 - 2\Phi)d\chi^2 = [(1 + \chi \Phi_{,\sigma})dr - (a_{,\sigma}\chi(1 - \Phi) - a\chi \Phi_{,\tau} + a_{,\tau}\chi^2 \Phi_{,r})dt]^2$$ \hspace{1cm} (6.8)

Transforming the $t$ coordinate, so there are no cross terms, gives to first order

$$f(r) = 1 - \frac{2r}{a} \Phi' - H^2 r^2$$ \hspace{1cm} (6.9)

where it has been assumed that $a_{,\tau}\chi \ll 1$.

Substituting the usual form for the Hubble parameter $H^2 = \frac{2M}{R^3} + \frac{\Lambda}{3}$, where $R = aR_v$ is the proper radius of the vacuole, the form for $f(r)$ results in the cosmological constant term cancelling, then $\Phi$ is given by

$$\Phi = -\frac{M}{r} - \frac{M r^2}{2 R^3} + \frac{3M}{2R}$$  \hspace{1cm} (6.10)

Since there is no $\Lambda$ term in the potential this demonstrates that in this model $\Lambda$ has no direct effect on lensing, as the bend angle only depends $\Phi$. It is argued that because the Kottler metric is static, and therefore has no Hubble term, the $\Lambda$ term in the bend angle is purely a gauge artefact.

However this model is perturbative and only uses first order terms in $\Phi$, as used in the usual weak lensing derivation (see Section 2.4.2), and therefore an approximate solution, not an exact solution, which may not be accurate enough for precision cosmology.
CHAPTER 6. A GENERAL RELATIVISTIC MODEL OF WEAK LENSING

Considering these approximations and the problems of the Kottler metric in Section 6.1 we look at trying to model gravitational lensing with a cosmological constant with an expanding background and a static, extended mass in the centre, all within one continuous spacetime with no need to match boundaries as I will discuss in Section 6.3.

6.3 LTB dust model

I now examine the LTB dust model (Lemaître, 1933; Tolman, 1934; Bondi, 1947), which is a spherically symmetric model that allows us to model an overdensity within a FRW background using a continuous spacetime with no need to match boundaries. It should be noted that we are not trying to model a void (as mentioned in Section 1.9.3), but in fact a spherical overdensity in the centre of the spacetime, where the observer is not at the centre.

The metric can be derived by considering a spherically symmetric spacetime in which the source in the EFEs is a perfect fluid, rotation is necessarily zero so a comoving-synchronous coordinate system can be used and the metric has the form (Plebanski and Krasinski, 2006)

\[ ds^2 = \alpha(r, t)^2 dt^2 - \beta(r, t)^2 dr^2 - R(r, t)^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]  (6.11)

To derive the Einstein equations (Equation 1.1) we can use the energy momentum tensor for a perfect fluid given by Equation 1.12 where in our comoving coordinate system \(|u^\mu| = \delta^\mu_0 / \sqrt{g_{\mu\nu}}\), where \(\delta^\mu_0\) is the Kronecker delta, which gives

\[ G_{tt}^t = \frac{2R_{tt}}{\beta^2 R} + \frac{2\beta_t R_t}{\beta^3 R} + \frac{2\beta_r R_r}{\beta^3 R} + \frac{1}{R^2} + \frac{R_t^2}{\alpha^2 R^2} - \frac{R_r^2}{\beta^2 R^2} = 3\Omega_m + \Lambda \]  (6.12a)

\[ G_{rr}^r = -\frac{2\alpha_r R_r}{\alpha R} + \frac{2\alpha_t R_t}{\beta R} = 0 \]  (6.12b)

\[ G_{\theta\theta}^\theta = -\frac{\alpha_r R_r}{\alpha^3 R} + \frac{R_{tt}}{\alpha^2 R} - \frac{\alpha_r R_r}{\alpha^2 R^2} + \frac{\alpha_{rr} R_{rr}}{\alpha^3 R} - \frac{\beta_r R_r}{\beta^3 R} + \frac{\beta_t R_t}{\beta^2 R} + \frac{\alpha_r \beta_r}{\beta^3 R} - \frac{\alpha_{rr} \beta_r}{\alpha^2 R^2} + \frac{\beta_{rt}}{\alpha R} - \frac{\alpha_{rr} \beta_r}{\alpha R} = -3P - \Lambda \]  (6.12d)
and the energy momentum conservation equations give

\[ \frac{\Omega_{,t}}{\Omega_m + P} + \frac{\beta_{,t}}{\beta} + 2\frac{R_{,t}}{R} = 0 \]  
(6.13a)

\[ P_{,r} + \frac{\alpha_{,r}(\Omega_m + P)}{\alpha} = 0 \]  
(6.13b)

Since we are interested in a dust model \( P = 0 \) and Equation 6.13b reduces to

\[ \frac{\alpha_{,r}\Omega_m}{\alpha} = 0 \]  
(6.14)

so assuming \( \Omega \neq 0 \), as this would just be an empty spacetime, gives \( \alpha_{,r} = 0 \) leaving \( \alpha(t) \), which can be set to \( \alpha = 1 \) by transforming the time coordinate. Using this with Equation 6.12b gives

\[ -R_{,rt} + \frac{\beta_{,t}R_r}{\beta} = 0 \Rightarrow \left( \frac{R_r}{\beta} \right)_{,t} = 0 \]  
(6.15)

so assuming \( R_r \neq 0 \), which I will assume throughout this chapter then this can be integrated to obtain

\[ \beta^2 = \frac{R^2_r}{1 + E(r)} \]  
(6.16)

where \( E(r) \) is the constant of integration.

This gives the line element for a dust model LTB as

\[ ds^2 = dt^2 - \frac{R(r, t)^2}{1 + E(r)}dr^2 - R(r, t)^2(d\theta^2 + \sin^2 \theta d\phi^2) \]  
(6.17)

Comparing this to the FRW metric we can see that \( dt = d\tau, ad\chi = \frac{R(r, t, \tau)}{\sqrt{1 + E(r)}}dr \), so \( r \) is a comoving distance and the angular part shows that \( R \) is the angular diameter distance, where these two distances are no longer necessarily related by \( a(t) \). The function \( E(r) \) here can be related to the curvature of space or the kinetic energy of freely falling particles at radius \( r \). The set up of the coordinate system for this model is shown in Figure 6.2, where the coordinates \( R, \theta \) and \( \phi \) will describe the ray position, so these will be used to calculate the lensing quantities.

We can further constrain the form of our model by looking at the remaining EFEs. Substituting in the form for \( \alpha \) and \( \beta \) into 6.13a gives

\[ R_{,rt} = \frac{-2R_{,t}R_{,r}}{R} \]  
(6.18)
CHAPTER 6. A GENERAL RELATIVISTIC MODEL OF WEAK LENSING

Substituting $\alpha$ and $\beta$ into the EFEs shows that the $rr$ component gives the temporal evolution of the radially dependent scale factor $R(r, t)$

$$R_{,tt} = \frac{1}{2} \left( \Lambda R - \frac{R^2}{R} + \frac{E}{R} \right)$$

(6.19)

and the $tt$ component of the EFEs gives the radial evolution of $R$

$$R_{,r} = -\frac{E_{,r} R}{3\Omega_m R^2 + \Lambda r^2 + E + 3R^2_{,t}}$$

(6.20)

where we can integrate Equation 6.19 to find a Friedmann like evolution equation

$$R^2_{,t} = \frac{2M(r)}{R} + \frac{1}{3} \Lambda R^2 + E$$

(6.21)

where I have used the definition of the Misner-Sharp mass (Misner and Sharp, 1964)

$$M_{,r} = \frac{3}{2} \Omega_m R^2 R_{,r}$$

(6.22)

These equations leave $E(r)$ as the only free function.

Throughout this work all distances ($r$ and $R$) are normalised by $c/H_0 = 3000h^{-1}\text{Mpc}$, time, $t$, is normalised by $1/H_0 = 10^{10}h^{-1}\text{y}$, $M(r)$ is normalised by $c^3/GH_0 = 10^{23}\text{M}_\odot$, $\Omega_m = \rho_m/\rho_c$, which we shall denote $\Omega$ for the remainder of this chapter, where $\rho_c = 3H_0^2/8\pi G$ and $P$ is normalised by $\rho_c c^2$. We can also write each of these terms in terms of the density parameters $\Omega_m$ and $\Omega_\Lambda$, where $\Lambda = 3\Omega_\Lambda$.

**Simplification for a point mass**

This spacetime can be simplified to model the Schwarzschild metric when M is a point mass, $E=0$ and $\Lambda=0$, giving the usual bend angle found for a point mass. We can also represent the Schwarzschild metric using the Novikov coordinates.

Novikov coordinates can be seen as representing observers who are freely falling into a black hole. The coordinates are chosen such that these free falling observers are always at rest representing comoving coordinates in their frame. Novikov gives a comoving radial coordinate, $r$, to each test particle as it emerges from the singularity and it keeps that coordinate throughout. This radial coordinate is given by

$$r = \sqrt{\frac{R_{\text{max}}}{2M} - 1}$$

(6.23)
where $R_{\text{max}}$ is the maximum $R(r, t)$ at $t = 0$ and $M$ is the mass at the centre (Misner et al., 1975). Using the standard LTB metric with

$$E(r) = -\frac{1}{r^2 + 1} = -\frac{2M}{R_{\text{max}}}$$  \hspace{1cm} (6.24)

allows us to write the static Schwarzschild using the standard LTB form

$$ds^2 = dt^2 - \frac{R(r, t)^2}{1 - \frac{2M}{R_{\text{max}}}} dr^2 - R(r, t)^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$  \hspace{1cm} (6.25)

with the initial conditions for $r$ set by Equation 6.23. The description of a light ray bent by a Schwarzschild mass in Novikov coordinates is shown in Figure 6.3, showing how this choice of $r$ results in a light ray with a spike when the ray comes within close proximity of the mass. This spike can be understood by the fact that observers nearer the black hole will fall in more quickly, so a geodesic in these coordinates appears to have a spike near the mass. This cautions us about the interpretation of $r$ in the LTB, at small $r$. On the other hand $R$ behaves like an angular diameter distance at all $r$ and $t$ so I will choose this coordinate to calculate the lensing observables.

While the Schwarzschild model is an interesting test case for our model, we are mainly interested in extended masses as seen in the Universe.
Figure 6.3: Comparison of numerics for Schwarzschild (dotted line) for Novikov (solid line) coordinates, where $x = r \cos \theta \sin \phi$ and $y = r \sin \theta \sin \phi$ in Mpc/h.

**Simplification for a homogeneous and isotropic spacetime**

The model can also be simplified to model the FRW metric where $\Omega$ and $E$ do not depend on $r$, so $R(r, t) \rightarrow a(t)r$, $E \rightarrow -kr^2$ and Equation 6.21 reduces to the usual Friedmann equation. At large distances the full model will tend to the FRW metric.

$$ds^2 = dt^2 - \frac{(a(t)r)^2}{1 - kr^2} dr^2 - (a(t)r)^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$  

$$= dt^2 - \frac{a(t)^2}{1 - kr^2} dr^2 - a(t)^2 r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$  \hspace{1cm} (6.26)

The advantage of the LTB metric is being able to combine these extremes into one spacetime, and we are going to investigate how altering each of the matter/energy components changes the lensing we observe.
Full density profile

We are interested in using a physically motivated galaxy/cluster model for our mass distribution using the NFW profile (Navarro et al., 1996).

\[ \Omega_m = \frac{\delta_c}{(r/r_s)(1 + r/r_s)^2} \]  

(6.27)

where \( c \) is the halo concentration and \( \delta_c \) is the characteristic overdensity of the halo, given by

\[ \delta_c = \frac{200}{3} \frac{c^3}{(\log(1 + c) - c/(1 + c))} \]  

(6.28)

and \( r_s = r_{200}/c \) is a characteristic radius, where \( r_{200} \) is the radius at which the mean density of an object at radius \( r_{200} \) from its centre is 200 times the critical density giving \( M_{200} = 200 \rho_c (4\pi/3) r_{200}^3 \). Figure 6.4 shows how changing \( M_{200} \) and \( c \) alters the density profile. These values of \( c \) are typical for galaxies and clusters.

![Figure 6.4](image-url)

Figure 6.4: This shows how the NFW density profile changes with mass, \( M_{200} = 10^{12} M_\odot \) (blue lines) and \( M_{200} = 10^{14} M_\odot \) (green lines), and with increasing concentration with lighter to darker lines, where \( c = [1, 5, 10, 50] \).
CHAPTER 6. A GENERAL RELATIVISTIC MODEL OF WEAK LENSING

We will use this with a constant density at large $r$ to create our full density profile, describing an NFW in a FRW background.

**Numerical integration of geodesic equations**

To find the light paths in our model we need to solve the null geodesic condition $ds^2 = 0$

$$i^2 - \frac{R_r^2}{1 + E} r^2 - R^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) = 0$$

(6.29)

and from substituting the metric into Equation 2.2 we obtain the geodesic equations

$$\ddot{i} + \frac{R_r R_{rt}}{1 + E} i^2 + R R_t (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) = 0$$

(6.30a)

$$\ddot{i} + 2 \frac{R_r}{R_r^r} \dot{i} + \left( \frac{E_r}{R_r} - \frac{E_r}{2(1 + E)} \right) \dot{i}^2 - R(1 + E) \frac{R_r}{R_r} (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) = 0$$

(6.30b)

$$\ddot{\theta} + 2 \frac{R_t}{R} \dot{\theta} + 2 \frac{R_r}{R} \dot{r} \dot{\theta} - \sin \theta \cos \theta \dot{\phi}^2 = 0$$

(6.30c)

$$\ddot{\phi} + 2 \frac{R_t}{R} \dot{\phi} + 2 \frac{R_r}{R} \dot{r} \dot{\phi} + 2 \frac{\cos \theta}{\sin \theta} \dot{\theta} \dot{\phi} = 0$$

(6.30d)

The geodesic equations (Equations 6.30a, 6.30b, 6.30c and 6.30d) are numerically integrated simultaneously with the spacetime quantities (Equations 6.18, 6.20, 6.22 and 6.19). The spacetime quantities are calculated at the same time to reduce numerical error due to interpolation and are integrated in $\sigma$ to reduce running time.

I use an Adams-Bashforth-Moulton solver to integrate the geodesic equations and spacetime quantities, which is a multistep method of solving ODEs (Press et al., 2007). Instead of just using the information from the previous step, as in a Runge-Kutta solver, to find the next step the method uses the information over several previous steps, increasing the accuracy and efficiency of the algorithm. It is a predictor corrector method, so it makes a prediction about the value of the next step and then a corrector refines the prediction. This solver uses a predictor based on

$$y(x_{k+1}) = y(x_k) + \int_{x_k}^{x_{k+1}} f(x, y(x)) dx$$

(6.31)

where the function $f(x, y(x))$ uses a Lagrange polynomial approximation based on $n$ previous steps. The integration above with this form for $f(x, y(x))$ gives the predicted value of the next step. The correction to this predicted value is found by using a second Lagrange polynomial based on $n - 1$ previous steps and the predicted step. This method
can also predict the error in each step allowing stringent error tolerances to be set and has adaptive step size, so can take large steps over regions of little evolution and small steps over areas where the quantities vary greatly, increasing its speed compared to fixed step solvers. The results for this model are shown in Section 6.6.

While this model satisfies our conditions for modelling an overdensity in an FRW background in a continuous spacetime it fails to keep the centre static with the components in the model, since $E(r)$ is the only free component and cannot be chosen to satisfy both $R_{,tt} = 0 \Rightarrow E = -\Lambda r^2$ (Equation 6.19) and $R_{,t} = 0 \Rightarrow E = -\frac{\Lambda}{3}r^2 - \frac{2M}{r}$ (Equation 6.21). Therefore we will look at a generalised form of the LTB model with pressure.

6.4 Generalised LTB model with pressure

We are now going to examine the generalised LTB model (Derived by Lasky and Lun, 2006, using the ADM equations), where we use the same general metric that we had in Equation 6.11 and the same EFEs except we will not set $P = 0$.

Using the conservation of the energy momentum tensor the $r$ component, Equation 6.13b, gives a condition for $\alpha$

$$\alpha_{,r} = -\alpha \frac{P_r}{\Omega_m + P}$$

and substituting in the Misner Sharp mass given by Equation 6.22 into the $t$ component of $T^\mu_{\nu;\mu} = 0$ gives

$$M_{,rt} = -\frac{3}{2} R_{,rt} R^2 P - 3 R_{,r} R_{,t} P R - \frac{3}{2} R_{,t} R^2 P_r$$

which we can integrate w.r.t. $t$ to give the temporal evolution of $M$

$$M_{,t} = -\frac{3}{2} P R^2 R_{,t}$$

Since $P = 0$ is also a solution of this metric we know that $\beta^2 = R_{,r}^2/(1 + E(r))$ in this limit, so we postulate that $\beta^2 = R_{,r}^2/(1 + E(r, t))$ and using this in Equation 6.12b gives

$$E_{,t} = 2 \frac{\alpha_{,t} R_{,t} (1 + E)}{\alpha R_{,r}}$$

which shows that in the absence of pressure $E_{,t} = 0$ and therefore $E(r, t) \rightarrow E(r)$ as required in the zero pressure case.
Therefore the line element has the form
\[ ds^2 = \alpha(r, t)^2 dt^2 - \frac{R(r, t)^2}{1 + E(r, t)} dr^2 - R(r, t)^2 (d\theta^2 + \sin^2 \theta d\phi^2) \] (6.36)

This model is able to keep the centre static with the pressure component.

Since we want the pressure to keep the centre static \( P(r, t) \to P(r) \) and \( \Omega(r, t) \to \Omega(r) \). This means that \( \alpha(r, t) \to \alpha(r) \), since \( \alpha \) is only dependent on \( \Omega \) and \( P \). We will use this simplification throughout this work.

Using the rest of the EFEs we can find constraints on how the angular diameter distance \( R \) evolves in space and time. We will use these equations to evolve the components of the model and \( R \), which we require to solve the null geodesics.

From Equation 6.12c we find the \( t \) evolution of \( R \)
\[ R_{,tt} = -\frac{3\alpha^2 R P}{2} - \frac{R^2}{2R} + \frac{\alpha^2 \Lambda R}{2} + \frac{\alpha^2 E}{2R} + \frac{\alpha \alpha \alpha(1 + E)}{R} \] (6.37)

and using Equations 6.35 and 6.34 and substituting these into Equation 6.37 we can write (assuming \( R_{,t} \neq 0 \))
\[ R_{,t}^2 = \alpha^2 \left( \frac{2M(r, t)}{R} + \frac{1}{3} \Lambda R^2 + E(r, t) \right) \] (6.38)

Equating the \( t \) derivative of Equation 6.22 and the \( r \) derivative of Equation 6.34 we can show
\[ R_{,rt} = R_{,t} \left( \frac{\alpha_{,r}}{\alpha} - 2 \frac{R_{,r}}{R} \right) \] (6.39)

Substituting Equations 6.35 and 6.39 into 6.12a and using Equation 6.22 we find
\[ R_{,r} = -\frac{\alpha^2}{3\alpha^2 \Omega m R^2 + \alpha^2 \Lambda R^2 + 3R_{,t}^2 + \alpha^2 E} E R \] (6.40)

Since we want the spacetime to have a static centre, \( 0 \leq r \leq r_{\text{cluster}} \), we set \( R_{,tt} = 0 \) and \( R_{,t} = 0 \), giving \( R = r \) and \( R_{,r} = 1 \). Substituting these into Equations 6.37 and 6.38 and rearranging gives constraints on the pressure (from Equation 6.37) and curvature (from Equation 6.38) in this static region which are
\[ P_{,r} = \frac{\Omega_{\text{static}} + P}{2(1 + E_{\text{static}})^r} (\Lambda r^2 - 3P r^2 + E_{\text{static}}) \] (6.41)
and

\[ E_{\text{static}} = -\frac{2M_{\text{static}}}{r} - \frac{\Lambda r^2}{3} \]  
\(6.42\)

These are the usual Tolman-Oppenheimer-Volkoff (TOV) equations (Tolman, 1939; Oppenheimer and Volkoff, 1939) for a static stellar interior, so using these constraints throughout would reduce the generalised LTB to the TOV model. However we are only concerned with keeping the centre static, as we want an expanding universe at large scales. Therefore we will solve Equation 6.41 for the pressure for the density distribution we want to keep static, \(\Omega_{\text{static}}\), with \(E_{\text{static}}\) given in Equation 6.42. We will then use the full density distribution \(\Omega_m = \Omega_{\text{static}} + \Omega_{\text{evolving}}\), with this \(P\) derived from the static case, and allow for radial freedom in \(E\) again so \(E = E_{\text{static}} + E_{\text{evolving}}\) for the evolution of \(M, R, E\) and the geodesics.

Since we want all evolution equations in terms of the affine parameter \(\sigma\) to optimise the speed and accuracy of the numerical integration we want to write them in the form where the evolution of the pressure \(P(r)\) is given by

\[ \dot{P} = P_r \dot{r} \]  
\(6.43\)

where \(P_r(r)\) is given by Equation 6.41.

The evolution of \(\alpha(r)\) is given by

\[ \dot{\alpha} = \alpha_r \dot{r} \]  
\(6.44\)

where \(\alpha_r(r)\) is given by Equation 6.32.

The evolution of \(M(r, t)\) is given by

\[ \dot{M} = M_r \dot{r} + M_t \dot{t} \]  
\(6.45\)

where \(M_r(r, t)\) is given by Equation 6.22 and \(M_t(r, t)\) is given by Equation 6.34.

The evolution of \(E(r, t)\) is given by

\[ \dot{E} = E_r \dot{r} + E_t \dot{t} \]  
\(6.46\)

where \(E_r(r, t)\) is given by our choice of form of the radial dependence of \(E\) and \(E_t(r, t)\) is given by Equation 6.35.

Finally the evolution of \(R(r, t)\) is given by

\[ \dot{R} = R_r \dot{r} + R_t \dot{t} \]  
\(6.47\)
where
\[ \dot{R}_{,t} = R_{,rt} \dot{r} + R_{,tt} \dot{t} \] (6.48)
and
\[ \dot{R}_{,r} = R_{,rr} \dot{r} + R_{,rt} \dot{t} \] (6.49)
where \( R_{,tt}(r, t) \) is given by Equation 6.37 and \( R_{,rt}(r, t) \) is given by Equation 6.39.

**Null Geodesics**

As with the LTB model we need to solve the null geodesic condition
\[ \alpha^2 \dot{t}^2 - \frac{R_{,r}^2}{1 + E} \dot{r}^2 - R^2(\dot{\theta}^2 \sin^2 \theta \dot{\phi}^2) = 0 \] (6.50)
and substituting the metric into Equation 2.2 gives the geodesic equations
\[ \ddot{t} + 2 \frac{\alpha}{\alpha^2} \dot{t} \dot{r} + \left( \frac{R_{,rt} R_{,r}}{1 + E} - \frac{R_{,r}^2 E_{,t}}{2(1 + E)^2} \right) \dot{r}^2 + \frac{R R_{,tt}}{\alpha^2} \left( \dot{\theta}^2 \sin^2 \theta \dot{\phi}^2 \right) = 0 \] (6.51a)
\[ \ddot{r} + \frac{\alpha \alpha_{,r}(1 + E)}{R_{,r}^2} \dot{r}^2 + \left( 2 \frac{R_{,rt}}{R_{,r}} - \frac{E_{,t}}{1 + E} \right) \dot{t} \dot{r} + \left( \frac{R_{,rr}}{R_{,r}} - \frac{E_{,r}}{2(1 + E)} \right) \dot{r}^2 \right. \\
\left. - \frac{R(1 + E)}{R_{,r}} \left( \dot{\theta}^2 \sin^2 \theta \dot{\phi}^2 \right) = 0 \] (6.51b)
\[ \ddot{\theta} + 2 \frac{R_{,t}}{R} t \dot{\theta} + 2 \frac{R_{,r}}{R} r \dot{\phi} - \sin \theta \cos \theta \dot{\phi}^2 = 0 \] (6.51c)
\[ \ddot{\phi} + 2 \frac{R_{,t}}{R} t \dot{\phi} + 2 \frac{R_{,r}}{R} r \dot{\theta} + 2 \frac{\cos \theta}{\sin \theta} \dot{\phi} = 0 \] (6.51d)

I use the same integration method used in Section 6.3, but now I solve Equations 6.43, 6.44, 6.45, 6.46, 6.47, 6.48, 6.49, 6.51a, 6.51b, 6.51c and 6.51d simultaneously to find the light trajectories through the model and calculate the lensing quantities.
6.5 Lensing Quantities

In order to calculate the lensing quantities we will follow the trajectory of several light rays and use these to calculate the bend angle, shear and convergence. Figure 6.5 shows the set up and propagation of several light rays used in our analysis, where we start with several light rays at a single point and propagate them backwards in time using the geodesic equations, and using the final and initial positions of these light rays we directly measure lensing quantities in our model.

![Diagram showing the propagation of several light rays from a single point and using these light rays to directly measure lensing quantities in our model, where \( y = R \sin \theta \sin \phi \) and \( z = R \cos \phi \) in Mpc/h.](image)

The bend angle is given in the Figure 6.1 and we define the ellipticity as \( \epsilon = (a^2 - b^2)/(a^2 + b^2) \), where \( a \) is the major axis and \( b \) the minor axis of our bundle. We compare this to the ellipticity expected using the usual lensing formula for a mass \( M \)

\[
\epsilon = 2 \frac{\Delta M}{\xi^2} \frac{D_{ls}D_{ls}^2}{D_lD_s} \quad \text{(6.52)}
\]
We will test our model with well known results such as the Schwarzschild model, Tolman-Oppenheimer-Volkoff model and the FRW model to test its robustness and then extend it to the full case.

6.6 Results

Here I present results for an LTB without pressure. The results for the pressure case are ongoing.

In order to test the numerical accuracy of the geodesic and spacetime integration several test cases were carried out, where we could compare the numerically integrated results to the analytical results. One of these tests was for an LTB with a point mass. Figure 6.6 demonstrates the accuracy of our numerical integration in measuring the bend angle in an LTB reduced to the Schwarzschild form compared to the true Schwarzschild bend angle calculated analytically, where our error < 1%.

Another test case was to check that when reduced to the form \( R(r, t) = a(t)r \) and \( E = 0 \), giving the same form as the flat FRW, that the LTB gave no bend angle in the comoving coordinate \( r \); however the light ray is bent in \( R \) due to the background evolution. This is found to be true to similar numerical accuracy as shown with the Schwarzschild case.

Figure 6.7 shows how using an NFW instead of a point mass with \( M = M_{200} \) can result in a larger bend angle at large \( \theta \) due to the tails of the density distribution. The two curves could be made identical by cutting the distribution at \( r_{200} \), but it is not clear that a cut here is the most suitable physical option or what form the cut should take. This means much care should be taken in these intermediate scale regions, where there are no physical models to constrain our density distribution.

This was then extended to include an NFW in a FRW background as shown in Figure 6.8, where the difference between the bend angle for an NFW with FRW background and the bend angle obtained in the same FRW background with no NFW is presented. This plot does appear to show a dependence on \( \Lambda \); however the NFW in this model is not static, so the presence of \( \Lambda \) would alter the expansion of the central mass and therefore cause a change in the mass distribution which would affect the bend angle. This will be studied further in the future with the LTB with pressure model, so the central mass can be kept static and therefore not affected by the presence of \( \Lambda \). Nevertheless this highlights the sensitivity of lensing to regions that contain mass and are not static, such as areas of gravitational infall on the edges of clusters.

Figure 6.9 also shows how the form of \( E(r) \) can alter the bend angle, where \( E(r) \) can be considered to be a curvature perturbation. It can be seen that at large angles \( E \)
Figure 6.6: Comparison of numerics for Schwarzschild (blue dashed line) and LTB with point mass (red solid line).

is constrained by the need for a flat FRW background, however adding a bump in $E$ at $\sim 10\, Mpc$, or using the usual form for the FRW $\Omega_k r^2$, can lead to very different bend angles to that of $E = 0$ for all $r$, demonstrating how a radially dependent curvature can produce a lensing signal. However since this change in $E$ will result in a change in the evolution of $R$ it will also change the mass within $R$, so the signal from $E$ and $M$ cannot be disentangled.

While these results demonstrate some of the ways our model choice can affect the bend angle, a model with pressure, which is the basis for our current and future work, would allow a virialised structure and therefore give a model which is closer to the types of structures we observe in the Universe.

### 6.7 Conclusions

In this chapter I have questioned whether our current model of gravitational lensing is going to be accurate enough for the precision cosmology we hope to do with forthcoming
surveys. I therefore look at how to investigate some of the least well modelled parts of the lensing model, such as how the physics at intermediate scales affects lensing, and whether $\Lambda$ has a direct effect on lensing.

I examined two of the previous models used to model the $\Lambda$ lensing effect and discussed their possible pitfalls. From these considerations we chose to use an Lemaitre-Tolman-Bondi metric for our lens model, since this spacetime allows one to have an overdensity in an expanding universe using a continuous spacetime. However this model does not allow the central overdensity to be virialised, so we also examined a generalised form of the LTB with pressure.

The lensing quantities were calculated by propagating the geodesic equations and spacetime evolution equations backwards in time. The path of the light rays was then used to calculate the lensing observables.

This model can be simplified to represent a Schwarzschild spacetime and a FRW spacetime, which provided useful test cases for our numerics, and showed our numerical results for these spacetimes to be within $<1\%$ of the analytical result expected.
I then showed that adding a perturbation in the curvature $E(r)$ alters the bend angle, implying lensing is sensitive to curvature perturbations as well as mass perturbations.

I also investigated how introducing a cosmological constant altered the bend angle, but while the results did seem to show a dependence on $\Lambda$ it is not clear that this effect was not due to the expansion of the central mass. Therefore to investigate this effect fully the LTB model with pressure is required.

Figure 6.8: Difference between bend angle for an NFW with FRW background and the bend angle (in the $R$ coordinate) obtained in the same FRW background with no NFW for different values of $\theta$ and $\Omega_\Lambda$
Figure 6.9: This figure shows how the inclusion of a radially varying $E$ can affect the bend angle, where we have included a Gaussian bump at $\sim 10\text{Mpc}$ in the form of $E(r)$ for the red dotted line, used usual form for the FRW $\Omega_k r^2$ for the green dashed line, and used $E = 0$ for all $r$ for the blue solid line.
Chapter 7

Conclusions

This thesis has looked at testing cosmological models with gravitational lensing, as well as testing our models of gravitational lensing.

In Chapter 1 I gave an overview of the current state of theoretical and observational cosmology. In Section 1.1, I looked at how the EFEs can be derived from the assumption that gravity is a consequence of spacetime geometry, and how Riemannian geometry can describe the curvature of spacetime.

I then examined in Sections 1.2 and 1.3 the concordance model of cosmology using the cosmological principle, showing that the only possible metric in a homogeneous and isotropic spacetime is the FRW metric. Using the EFEs with this metric showed how a FRW spacetime and its constituents evolve with time in Section 1.4. From this the redshift can be derived in Section 1.5 and the distance measures, in Section 1.6, used throughout observational cosmology. However in lensing we are interested in inhomogeneities as these are what cause lensing, so I derived the growth equations for perturbations on a FRW background in Section 1.7.1. I also examined how the Meszaros effect, pressure, photon diffusion and free streaming damp perturbations inside the horizon. On small scales, structures can no longer be approximated by linear theory so non-linear models must be used, which I discussed in Section 1.7.2.

Lastly I examined current observational evidence for the concordance model in Section 1.8 including the cosmic microwave background, baryon acoustic oscillations, redshift space distortions and supernovae, which all provide compelling evidence in favour of the concordance model. I then looked at possible alternatives to $\Lambda$CDM in Section 1.9, some of which I investigated in later chapters.

In Chapter 2 I looked at gravitational lensing. First I looked at the geodesic equation in Section 2.1 and why light travels along geodesics. This is the equation that all lensing observations are built around. I then examined the geometry of a lens system in Section 2.2, allowing us to relate a ray’s final position to its initial position via the bend angle.
I calculated the bend angle expected for a point mass in the Schwarzschild metric in Section 2.3 and showed that the bend angle relates to the mass of the system.

I then moved onto weak lensing and derived the form for the geodesic deviation equation in Section 2.4.1, and showed how the deviation of geodesics depends on the Riemann tensor. In Section 2.4.2 I decomposed the Riemann tensor into its component parts and showed how these provide the source of shear and convergence. I also calculated the optical tidal matrix for a perturbed FRW and showed how the weak lensing quantities, convergence, shear and rotation, are related to the gradients of the potential. These quantities were then integrated along the line of sight to find the overall convergence and shear for a light ray. Since shear and convergence are small effects (∼1% distortion in ellipticity) we use the correlation of many sources to obtain a measurable signal, as shown in Section 2.4.3. This resulting formula was used throughout this thesis to find the shear correlation of different cosmologies.

Finally in Section 2.4.4 I looked at some of the systematics that must be dealt with to obtain a clean lensing signal, including the PSF, noise on the CCD and charge transfer effects. I then looked at some previous observational results and some of the forthcoming lensing surveys in Section 2.5. The lensing quantities derived in this chapter were used throughout this thesis.

In Chapter 3 I looked at how we can include nonlinear scales in our predictions for weak lensing in modified gravities. I did this through two different approaches, using specific models and using a parameterisation. First, in Section 3.1, I considered the DGP and $f(R)$ models, together with dark energy models with the same expansion history, and derived the growth and Hubble expansion for both models. These were used to calculate the power spectrum and geometric quantities required for the shear correlation function. I also examined how DGP and $f(R)$ models must be screened on small scales to fit solar system observations and the methods used to do this.

I then took a more empirical approach to small scales in Section 3.2 where I used the requirement that gravity is close to GR on small scales, to estimate the non-linear power for these models, and used an interpolation between the unscreened non-linear modified gravity power spectrum and a non-linear GR power spectrum, both using Halofit, with the same background. I then calculated weak lensing statistics, showing their behaviour as a function of scale and redshift, and presented predictions for measurement accuracy with future lensing surveys, taking into account cosmic variance and galaxy shape noise, shown in Section 3.3. I demonstrated the improved discriminatory power of weak lensing for testing modified gravities once the nonlinear power spectrum contribution has been included, and how not including the GR asymptote can lead to an overestimation of the discriminatory power of lensing on modified gravities.
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In Section 3.4, I took a less model specific approach and investigated the parameterisation of the non-linear power spectrum using the interpolation formula and the growth factor $\gamma$. I examined the ability of future lensing surveys to constrain this parameterisation of the non-linear power spectrum and showed that even including these extra parameters still resulted in better constraints from the non-linear result than the linear.

In Chapter 4 I presented non-linear weak lensing predictions for coupled dark energy models using the CoDECS simulations. These models modify $\Lambda$CDM by assuming that there exists a coupling between the dark components of the model. In Section 4.1, I derived the EFEs and energy momentum conservation equations for an FRW perturbed by a small scalar potential and used these to calculate the evolution of the potential and density perturbations. I then modified the energy momentum conservation equations to include a coupling between dark matter and dark energy, in Section 4.2, and derived the modified growth and velocity equations using these.

I investigated two potentials, an exponential potential and a super gravity potential, and a constant coupling and an evolving coupling. While the growth equation can be used to calculate the linear power spectrum, the non-linear cannot be calculated using an analytical model, so we used the CoDECS simulations to estimate the non-linear power.

In Section 4.2.2 I discussed how the usual $P_\kappa$ is modified for models with $\Omega_m(a) \not\propto a^{-3}$ and how I modified CosmoMC to obtain cross correlated tomographic weak lensing predictions for a given power spectrum and geometry. I also discussed the merits of MCMC approaches over a grid-based method (used in Chapter 3) and the various methods used. Using this modified CosmoMC I calculated the shear correlation function and used error covariance expected for these models, calculated from the Horizon simulations, for forthcoming ground-based (such as DES) and space-based (Euclid) weak lensing surveys. From this I obtained predictions for the discriminatory power of a ground-based survey similar to DES and a space-based survey such as Euclid in distinguishing between $\Lambda$CDM and coupled dark energy models, which I presented in Section 4.2.3. These results showed that using the non-linear lensing signal we can discriminate between $\Lambda$CDM and exponential constant coupling models with $\beta_0 \geq 0.1$ at 99.994% confidence level with a DES-like survey, and $\beta_0 \geq 0.05$ at 99.99994% confidence level with Euclid. I also demonstrated that estimating the coupled dark energy models’ non-linear power spectrum, using the $\Lambda$CDM Halofit fitting formula, results in biases in the shear correlation function that exceed the survey errors.

In Chapter 5 I examined unified dark matter models with a fast transition, which are cosmological models where the dark matter and dark energy are represented by a single fluid where the transition in the equation of state of this dark fluid is fast. In Section 5.1 I derived the background and growth equations from a perturbed FRW metric and showed how the density evolution depends on the equation of state and the sound speed
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of the dark fluid. I showed how the Jeans length is altered for these models, and how its form motivates us to look at fast transition models. The fast transition model I looked at was a dark fluid with a barotropic equation of state of the form a hyperbolic tangent, to obtain a fast transition, dependent on three parameters, the speed of transition, the time of transition and the effective \( \Lambda \) density and I showed the overall effects of these parameters on the total power spectrum.

In Section 5.2 I presented the constraints on the UDM parameters using the estimated shear correlation with the error covariance expected for DES and Euclid and the CMB temperature power spectrum with the error covariance expected for Planck. These constraints showed that in order to constrain both the speed and time of the transition we need to combine high redshift, CMB observations, and low redshift, weak lensing observations, since they are sensitive to different effects. Therefore for a \( \Lambda \)CDM fiducial, Planck constrains \( z_t > 5 \) at a 95% confidence level, while DES and Euclid constrain the maximum time the transition can take to \( < 5 \times 10^{-6}/H_0 \) at a 95% confidence level.

Lastly in Chapter 6 I investigated how our choice of model for weak lensing affects our observations. This is especially important for forthcoming surveys where the possibility of precision cosmology with weak lensing will be upon us. Therefore we’d like to know how much our model and our assumptions affect our constraints, so we can determine whether our constraints are due to the underlying cosmology or our model. I am particularly interested in how the intermediate scales are modelled, as we have no physical analytical model for this, and whether, as has been suggested recently, the cosmological constant has a direct effect on lensing.

In Sections 6.1 and 6.2 I looked at two of the models that have been used to model the effect of \( \Lambda \). However both of these models cannot describe the dynamics of all scales for a virialised structure in an expanding spacetime. This led to the use of an LTB model, first with only dust in Section 6.3, and later with pressure in Section 6.4. In this model we can have a continuous spacetime, which describes an FRW on large scales and a virialised halo on small scales, with the presence of a cosmological constant. I then investigated how weak lensing observables are affected by our choice of cosmological model. The results so far are inconclusive, as the bend angle in a dust LTB shows a possible dependence of \( \Lambda \), after removal of the geometrical effect; however it is not possible to have a virialised structure in this model, so the continuous collapse of the mass could affect these results. Therefore I am now investigating the use of pressure in the LTB to keep the centre static.

In summary this thesis has looked at the discriminatory power of forthcoming weak lensing surveys for dark energy and modified gravity alternatives to the cosmological constant, and has shown the importance of using accurate cosmological models for both the concordance cosmology and alternatives.
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