Primordial Perturbations from Early Universe Cosmology

by

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Declaration

Whilst registered as a candidate for the above degree, I have not been registered for any other research award. The results and conclusions embodied in this thesis are the work of the named candidate and have not been submitted for any other academic award.
Para os meus pais, Fátima e Zé, para as minhas irmãs, Ana e Gabriela, e para a minha tia Fernanda.
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Abstract

The very early universe is where we expect the observed primordial perturbations in the cosmic microwave background to have originated. In this thesis we study isocurvature field fluctuations during inflation and ekpyrotic contraction as sources of the primordial curvature perturbations.

We start by introducing concepts of modern cosmology followed by an overview of early universe cosmology. After, we introduce perturbation theory and how to compute perturbations from early universe models.

After reviewing all fundamental concepts necessary for this thesis, we estimate large-scale curvature perturbations from isocurvature fluctuations in the waterfall field during hybrid inflation, in addition to the usual inflaton field perturbations. The tachyonic instability at the end of this inflation model leads to an explosive growth of super-Hubble scale perturbations, but they retain the steep blue spectrum characteristic of vacuum fluctuations in a massive field during inflation. We extend the usual $\delta N$ formalism to include the essential role of small fluctuations when estimating the large-scale curvature perturbation.

The following two chapters study perturbations within the curvaton proposal. Firstly, we consider how non-Gaussianity of the primordial density perturbation and the amplitude of gravitational waves from inflation can be used to determine parameters of the curvaton scenario for the origin of structure. We show that in the simplest quadratic model, where the curvaton evolves as a free scalar field, measurement of the bispectrum relative to the power spectrum, $f_{NL}$, and the tensor-to-scalar ratio can determine both the expectation value of the curvaton field during inflation and its dimensionless decay rate relative to the curvaton mass. We show how these predictions are altered by the introduction of self-interactions. In the following chapter, we then characterise the primordial perturbations produced due to both inflaton and curvaton fluctuations. We show how observational bounds on non-linearity parameters and the tensor-scalar ratio can be used to constrain curvaton and inflaton parameters.
The final research presented in this thesis, considers a simple model of cosmological collapse driven by canonical fields with exponential potentials. We generalise the two-field ekpyrotic collapse to consider non-orthogonal potentials and give the general condition for isocurvature field fluctuations to have a slightly red spectrum of perturbations as required by current observations. However a red spectrum of fluctuations implies that the two-field ekpyrotic phase must have a finite duration and requires a preceding phase which sets the initial conditions for what otherwise appears to be a fine-tuned trajectory in the phase space.

We end this thesis with some concluding remarks and comments on possible future work.
Preface

The work presented in this thesis was mainly carried out at the Institute of Cosmology and Gravitation, University of Portsmouth, United Kingdom. Some of the work presented in Chapter 4 was carried out at the Yukawa Institute for Theoretical Physics, University of Kyoto, Japan (January-March 2010).

The following chapters are based on published work:


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Chapter 1

Preamble

In a place free of light pollution we can see a clear night sky dotted with light. Since the dawn of humanity we have tried to understand the observed celestial sphere. In the early stages of civilisation, Cosmogony would be the only available explanation for the observed sky. Myths about constellations were common and the dots of light would live in concentric spheres with their centre in the earth. In medieval Europe the outer sphere would be the place where heaven and god would live. The sky was seen as created in the beginning and immutable.

The first evidence for European thinkers that it wasn’t the case was given by the planets in the solar system. With his heliocentric model, Copernicus took the earth away from the centre of the universe to put the Sun in its place. At that time the universe was most of the Solar System plus the outer sphere. The Copernican Principle states that the Earth, or us humans in a broader sense, are not in a special place in the universe. Later, in 1785, William Herschel was the first to attempt to describe the shape of the Milky Way, by counting the number of stars in the sky. In the 1920s a great debate between Harlow Shapley and Heber Curtis took place concerning the nature of the galaxy. Their discussion was about some observed nebulae, like Andromeda, whether they were part of the Milky Way or not. In the same decade Edwin Hubble was able to obtain better resolution images of Andromeda and probe the existence of other galaxies. Similarly by observing the galaxies’ redshifts he was able to discover the expansion of the universe.

With the development of General Relativity (GR) such observations supported the fact that the universe is well described by the Friedmann-Roberston-Walker metric. Later, in the 1940s, Alpher and Gamow proposed the Big Bang Nucleosynthesis, based on the idea of a primordial hot universe. The prediction of a remnant radiation from the Hot Big Bang model, the cosmic microwave background (CMB), gave the final proof for what is nowadays the standard model of cosmology. In 1965, Penzias and Wilson observed the CMB, the oldest radiation in the universe, formed 400,000 years after the big bang.
With more observations being done, the universe seems statistically isotropic [1]. In other words, the universe looks the same regardless of the direction we are looking at. We observe stars, clusters of stars and other astrophysical objects, galaxies and clusters of galaxies but on large scales the distribution of matter is statistically homogeneous [2]. This means that beyond some distance, all volumes of a given size have the same average density. We can summarise the three fundamental principles of physical cosmology as:

**Copernican Principle** - We are not in a special location in the Universe;

**Statistical Isotropy** - the Universe looks the same independently of the direction;

**Statistical Homogeneity** - on large scales the energy density of the universe is homogeneous.

Nonetheless, on smaller scales the universe shows structure. We observe it nowadays just by the existence of galaxies and clusters of galaxies. But we also observe it in the temperature fluctuations in the CMB. The origin of primordial perturbations, of the seeds of the observed structure, is not fully understood. It is within this context that the work carried out in this thesis was produced. As we will see later on, the paradigm of inflation is probably the best way to describe the physics of the early universe. Such a paradigm encompasses a zoology of theories which can be very well motivated. It is arguable if the paradigm may or may not be falsifiable but future observations will definitely give an indication of which theories are allowed. In this thesis we study primordial perturbations from inflationary models. We also look at alternative theories for the origin of structure.

The outline is as follows:

- **Chapter 2** gives a basic introduction to cosmology. We also introduce the FRW metric and study scalar fields in cosmology. With those we set the basis to review models of the early universe;

- **Chapter 3** reviews perturbation theory including the $\delta N$-formalism and its application to the simplest early universe models;

- **Chapter 4** studies primordial perturbations from the waterfall field in hybrid inflation. It also presents an extension to the $\delta N$-formalism;

- **Chapter 5** studies the simplest curvaton model. In this chapter we constrain fundamental parameters of the model with future observations;

- **Chapter 6** develops a similar study as of chapter 5 but considers both the inflaton and the curvaton as contributors to the primordial power spectrum of curvature perturbations;
• Chapter 7 presents and studies a phenomenological extension of the new ekpyrotic model;

• Chapter 8 states the conclusions of the research done, presents future possible work and discusses prospects of observational constraints on the models studied.

In the end of this thesis, in Appendix A, we derive the standard result for the variance of a field in a de Sitter space-time.

1.1 Notation and Units

We will use the metric signature \((-, +, +, +)\). We will use the Einstein summation convention. The greek indices \((\mu, \nu, \ldots)\) stand for the time and spatial coordinates while indices with latin letters \((i, j \ldots)\) will stand for spatial coordinates only. Capital latin letters will refer to different fields.

We will set the speed of light to unity, \(c = 1\). We will also set the Planck constant to one, \(\hbar = 1\). Then energy density and mass density are then equivalent. Therefore we will measure quantities in particle physics units of \(eV\). It will be useful to measure temperature in \(eV\) units as well. To do so we will consider the Boltzmann constant equal to 1, \(k_B = 1\).

The reduced Planck mass is defined as

\[
m_{Pl} = \sqrt{\frac{\hbar c}{8\pi G}} = 2.436 \times 10^{18}\text{GeV}.
\] (1.1)

Excluding the first sections of the introduction we will replace Newton’s constant \(G\) by the reduced Planck mass. In subsection 2.4.2 and chapter 7 we set \(m_{Pl} = 1\) for simplicity.
Chapter 2

Introduction to Early Universe Cosmology

In this first chapter we will revise the fundamental ideas and equations in modern cosmology. We will revisit the dynamics of the universe and the standard metric for cosmology, the Friedmann-Robertson-Walker (FRW) metric. Then, we will review scalar fields in a FRW background and statistics of random fields. We will follow with the study of fields within the context of inflation and pre-big bang scenarios.

2.1 Dynamics of the universe

We can derive the fundamental equations of cosmology from first principles like thermodynamics and Newtonian dynamics.

Let's consider a universe homogeneously filled with dust particles. Dust is a pressure-less fluid and for the time being let's discard linear and thermal momentum. We start by considering a shell of radius $a$ with spherical symmetry. This infinitesimal shell has a volume element of $dV = 4\pi a^2 da$. From Gauss' theorem we know that the inner volume will not feel any gravity from the shell. On the other hand the shell itself will experience a gravitational force from the particles in the inner volume. Since the energy density $\rho$ is homogenous the inner mass is $M = (4\pi/3)a^3\rho$, which is constant. From Newton's second law a particle in the shell will feel a gravitational acceleration

$$\ddot{a} = -\frac{GM}{a^2}, \quad (2.1)$$

where $G$ is Newton's constant. We can multiply by $\dot{a}$ on both sides and integrate. The constant of integration is taken to be $-K/2$, where later on $K$ is going to be understood
as the curvature of the universe. Then, we have

\[ \frac{\dot{a}^2}{2} = \frac{GM}{a} - \frac{K}{2} = \frac{4\pi G}{3} \rho a^2 - \frac{K}{2}, \]  

(2.2)

which gives

\[ H^2 \equiv \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{K}{a^2}. \]  

(2.3)

This is the Friedmann Equation. Although we derived it in a Newtonian regime this equation is properly obtained from the 0-0 component Einstein equations for a perfect fluid in a FRW universe [3]. The proper interpretation of the equation is slightly different from the heuristic argument given to derive it above. The Hubble parameter \( H \) measures the expansion rate of the universe, \( a \) is the scale factor that gives the expansion, \( \rho \) is the energy density of the universe and is the sum of the energy densities of each independent component of the universe and \( K \) is the curvature of the universe.

The energy density of an individual fluid component evolves with the expansion. To determine its evolution equation let us consider the first law of thermodynamics,

\[ dE = -p \, dV + T \, dS, \]

(2.4)

where \( E \) is the energy, \( p \) the pressure, \( V \) the volume, \( T \) the temperature and \( S \) the entropy of the universe. We consider the expansion of the universe to be an adiabatic process, i.e., \( T \, dS = 0 \). Then \( dE = -p \, dV \). On the other hand if we take the differential of \( E = \rho V \) we get

\[ dE = \rho \, dV + V \, d\rho \Leftrightarrow d\rho + (p + \rho) \frac{dV}{V} = 0. \]  

(2.5)

Note that \( dV/V = 3d\alpha/a \). If we consider the time variation of the infinitesimal elements we arrive at the continuity equation

\[ \dot{\rho} = -3H \, (\rho + p). \]  

(2.6)

Another way of obtaining this equation is using the covariant conservation of the energy-momentum tensor, \( \nabla_{\mu} T^{\mu 0} = 0 \), for a perfect fluid [3]. It is common to write the continuity equation in terms of the energy density and the equation of state

\[ w \equiv \frac{p}{\rho}. \]  

(2.7)

For most purposes in cosmology, matter velocities are cosmologically irrelevant, hence we take matter to be pressure-less, i.e., \( w = 0 \). On the other hand, radiation or a relativistic fluid has \( w = 1/3 \). A cosmological constant has \( w = -1 \) [4].
The third important equation governing the dynamics of the universe is a consequence of the Friedman equation (2.3) and the continuity equation (2.6). If we take the time derivative of $a^2H^2$ and use the continuity equation one arrives, after some algebra, at the Raychaudhuri equation,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3p).$$  

(2.8)

This equation gives the acceleration of the expansion. One can also derive at this equation using the $i-i$ component of the Einstein equations and the Friedmann equation [3].

### 2.2 Standard FRW Cosmology

To properly study the universe we require General Relativity to relate the geometry of the universe with its energy content. In standard cosmology we consider the universe filled with matter ($w = 0$), radiation ($w = 1/3$) and cosmological constant $\Lambda$ ($w = -1$). When each component dominates the universe we will have different cosmological eras.

#### 2.2.1 FRW metric

The exact solution of the Einstein equations that describes an expanding, homogeneous and isotropic universe is the FRW metric. It was independently discovered firstly by Friedmann in the Soviet Union in 1922 and 1924 [5], and later by Lemaître [6] in Leuven, Belgium in 1927. The standard form of the metric was proposed in 1935 by Robertson [7] in the US and Walker [8] in the UK. They proved that this is the only spatially homogeneous and isotropic space-time.

The most general form of the FRW background metric, $g_{\mu\nu}$, is given by

$$ds^2 = -dt^2 + a(t) \left[ \frac{dr^2}{1 - Kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right].$$  

(2.9)

Note that the metric is written in polar coordinates $r$, $\theta$ and $\varphi$. On spatially homogeneous hyper-surfaces the time $t$ corresponds to cosmic time. Note that we are working with units of $c = 1$. This generic form allows for a flat universe ($K = 0$), open universe ($K < 0$) and a closed universe ($K > 0$). Here $a(t)$ is the scale factor. Hereafter we will only consider a flat universe, i.e., $K = 0$.

This metric is a solution of the Einstein Equations,

$$G_{\mu\nu} = 16\pi GT_{\mu\nu},$$  

(2.10)
where $T_{\mu\nu}$ is the energy-momentum tensor of a perfect fluid, i.e.,

$$T^\nu_{\mu} = (\rho + p) u^\mu u^\nu + p \delta^\nu_{\mu}. \quad (2.11)$$

The 4-velocity of the fluid is given by $u_\mu$. The $0-0$ component of Eq. (2.10) gives the Friedmann equation (2.3), while the $i-i$ component leads to Raychaudhuri Eq. (2.8). Momentum conservation and energy conservation (Eq. (2.6)) are obtained from the conservation of the energy-momentum tensor, $\nabla_\mu T^{\mu\nu} = 0$.

### 2.2.2 Cosmological Eras

Neglecting for the moment the history of the very early universe one can identify three main epochs in the history of the universe: Radiation Domination, Matter Domination and Late-time acceleration or Cosmological Constant (-like) Domination. First let’s consider that a fluid with equation of state $w$ dominates the universe. From the continuity equation (2.6) we find that the energy density of a fluid with equation of state $w$ scales as

$$\rho_i = \rho_{i,0} \left( \frac{a_0}{a} \right)^{3(1+w)}. \quad (2.12)$$

The subscripts $i$ stands for an individual component and 0 for a fixed time that we usually take to be the present time. Then, integrating the Friedmann Eq. (2.3) assuming $K = 0$, one determines how the scale factor depends on cosmic time, i.e.,

$$a(t) = a_0 \left( \frac{t}{t_0} \right)^{\frac{2}{3(1+w)}}. \quad (2.13)$$

One should note that for $w = -1$ the solution is exponential. For $w \neq -1$, the Hubble parameter is given by

$$H = \frac{2}{3(1+w)} \frac{1}{t}. \quad (2.14)$$

It is also convenient to define the density parameter,

$$\Omega_i \equiv \frac{\rho_i}{\rho_{\text{critical}}}, \quad (2.15)$$

where $\rho_{\text{critical}} = 3H^2/8\pi G$ is the critical energy density for a flat universe. For a flat universe we have $\sum_i \Omega_i = 1$. 

Radiation Era

The radiation fluid density (2.12) for \( w = 1/3 \) scales with \( a^{-4} \), then we expect it to dominate the universe at early times. Hence the universe energy density scales as

\[
\rho_r = \rho_{r,0} \left( \frac{a_0}{a} \right)^4 ,
\]

and the scale factor (2.13)

\[
a(t) = a_0 \left( \frac{t}{t_0} \right)^{1/2} .
\]

It is during the radiation era that Big-Bang Nucleosynthesis occurs. As the temperature drops quarks bind to make protons and neutrons. These are in thermal equilibrium via weak interaction. Soon after neutrinos decouple the weak interaction freezes out and the ratio between neutrons and proton becomes constant. During this time the light nuclei are formed [3, 4].

Matter Era

The matter energy density dilutes, in the expanding universe, proportional to \( a^{-3} \). Then, at some time \( t_{eq} \) it will overtake radiation and become the dominant component of the universe. For matter the equation of state is given by \( w = 0 \). Then from (2.12)

\[
\rho_m = \rho_{m,0} \left( \frac{a_0}{a} \right)^3 ,
\]

and the scale factor (2.13) gives

\[
a(t) = a_0 \left( \frac{t}{t_0} \right)^{2/3} .
\]

Soon after matter-radiation equality the temperature falls to \( \sim 1eV \) and recombination happens. The first atoms are formed and photons are no longer in thermal equilibrium with matter. This is the time when the Cosmic Microwave Background (CMB) is formed. After this process the universe is filled with clouds of neutral hydrogen and helium. Is also during the matter era that structures start to form. As the universe expands these become proto-galaxies that merge due to hierarchal clustering [9]. As galaxies and the first stars develop they re-ionise the universe [9].

\( \Lambda \)-domination

The late-time acceleration occurs when the cosmological constant energy density, or some other form of dark energy, comes to dominate over matter and radiation. It is a
negative pressure fluid. For a cosmological constant, its equation of state is \( w = -1 \). The energy density becomes constant

\[
\rho_\Lambda = \rho_{\Lambda,0}.
\]  

(2.20)

This gives rise to an exponential expansion of the universe

\[
a(t) = a_0 e^{H_0(t-t_0)},
\]  

(2.21)

with a constant expansion rate \( H_0 = \sqrt{8\pi G \rho_\Lambda / 3} \).

If matter density is negligible, the present epoch is well approximated by the previous equations. The universe is not only expanding but it is accelerating, \( \ddot{a} > 0 \).

2.3 Scalar fields in Cosmology

Let’s consider the following action for a canonical scalar field \( \phi(\vec{x}) \)

\[
S = \int dx^4 \sqrt{-g} \left( -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right).
\]  

(2.22)

For a FRW unperturbed flat (\( K = 0 \)) metric, the determinant of the metric \( g_{\mu\nu} \) is \( g = -a^6 \). The first term in brackets corresponds to the kinetic energy of the field while \( V(\phi) \) is the potential energy. The given low-energy action is the one that is going to be used to study scalar fields throughout this thesis. It is valid for an effective field theory that will give the shape of the potential [10]. From the variational principle \( \delta S / \delta \phi = 0 \) we obtain for a test field in an FRW universe the field evolution equation

\[
\ddot{\phi} + 3H \dot{\phi} - \frac{\nabla^2 \phi}{a^2} + \frac{dV}{d\phi} = 0,
\]  

(2.23)

where \( \nabla^2 \equiv \delta^{ij} \partial_i \partial_j \) is the Laplacian. If we vary the action with respect to the metric we get the energy-momentum tensor

\[
T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}}.
\]  

(2.24)

For a canonical scalar field it is

\[
T^\mu_\nu = g^{\mu\sigma} \partial_\sigma \phi \partial_\nu \phi - \delta^\mu_\nu \left[ \frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + V(\phi) \right].
\]  

(2.25)
In an FRW cosmology the background scalar field is going to be homogeneous and isotropic, i.e., $\phi = \phi(t)$. From Eq. (2.11) we know that $T^0_0 = -\rho$, thus

$$\rho = \frac{1}{2} \dot{\phi}^2 + V(\phi).$$

(2.26)

Similarly, $T^i_i = p$, then

$$p = \frac{1}{2} \dot{\phi}^2 - V(\phi).$$

(2.27)

### 2.3.1 Statistics of random fields

Scalar fields are very important in cosmology either to study models of the very early universe or to study statistical properties of observable quantities, like temperature. A powerful tool to study stochastic properties are Fourier expansions. For most purposes in cosmology the Fourier expansion should be done in a box of the size of the observable universe. We will take the limit where the box size goes to infinity. In that case the expansion becomes an integral. We will make this approximation for convenience. A powerful tool to study stochastic properties are Fourier expansions. For most purposes in cosmology the Fourier expansion should be done in a box size of the order of the observable universe $H_0^{-1}$. The first reason to do so, concerns the determination of averages that we wish to compare with the observations [11]. If we take a box size much bigger than the observable universe the volume averages and the stochastic properties of the ensemble may vary. When we sample the universe with a box of a given size $R$, smaller than the observable universe, we consider that we are in a typical region of the universe, i.e., the stochastic properties are translational and rotational invariant. This may not be the case if we Fourier expand in a box much bigger than $H_0^{-1}$ where an unobservable large scale fluctuation alter the average value of a quantity. Nonetheless we will assume that it is not the case. Furthermore, the physics beyond the horizon may vary, as speculated within string theory landscape [12]. In so far, there is no observational evidence for different physics beyond the observable universe, hence we will assume that physics are the same outside the horizon. Under such assumptions, and for mathematical convenience, we will take the limit where the box size goes to infinity. In that case the Fourier series becomes a Fourier integral.

Let’s consider a generic random field $\phi(\vec{x})$. Its Fourier and inverse Fourier transforms are going to be defined as

$$\phi_{\vec{k}} = \int d^3 x e^{i \vec{k} \cdot \vec{x}} \phi(\vec{x})$$

(2.28)
\[ \phi(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k} \cdot \vec{x}} \phi_k \]  
(2.29)

where variables with arrows are vector quantities.

In Fourier space, we define the lowest correlation functions as

\[ \langle \phi_{\vec{k}_1} \phi_{\vec{k}_2} \rangle = (2\pi)^3 P_\phi(k_1) \delta^3(\vec{k}_1 + \vec{k}_2), \]  
(2.30)

\[ \langle \phi_{\vec{k}_1} \phi_{\vec{k}_2} \phi_{\vec{k}_3} \rangle = (2\pi)^3 B_\phi(k_1, k_2, k_3) \delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3), \]  
(2.31)

\[ \langle \phi_{\vec{k}_1} \phi_{\vec{k}_2} \phi_{\vec{k}_3} \phi_{\vec{k}_4} \rangle = (2\pi)^3 T_\phi(k_1, k_2, k_3, k_4) \delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4). \]  
(2.32)

The function \( P_\phi(k_1) \) is the power spectrum, \( B_\phi(k_1, k_2, k_3) \) the bispectrum and the trispectrum is \( T_\phi(k_1, k_2, k_3, k_4) \). \( \delta^3(\sum_i \vec{k}_i) \) is the Dirac-delta. Due to statistical isotropy and homogeneity the N-point functions depend only on the absolute value of the wave-vectors. We are only interested in connected correlation functions. A N-point correlation function is said to be connected if it is linear in only one Dirac-delta of the form \( \delta^3(\sum_i \vec{k}_i) \). The two-point function measures, in real space, the probability excess, with respect to a random distribution, of finding two fluctuations of the density field, i.e., galaxies, at a given distance from each other. On the other hand, higher-order correlation functions measure modulation in the power spectrum [13].

The variance of a field is the real-space two-point function of the field at the same coordinate, i.e,

\[ \sigma_\phi^2 = \langle \phi(\vec{x})^2 \rangle = \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} e^{-i\vec{k}_1 \cdot \vec{x}} e^{-i\vec{k}_2 \cdot \vec{x}} \langle \phi_{\vec{k}_1} \phi_{\vec{k}_2} \rangle \]  
(2.33)

\[ = \int \frac{d^3k_1}{(2\pi)^3} 4\pi k_2^2 dk_2 e^{-i\vec{x} \cdot (\vec{k}_1 + \vec{k}_2)} P_\phi(k_1) \delta^3(\vec{k}_1 + \vec{k}_2) \]  
(2.34)

\[ = \int \frac{dk_1 k_1^3}{k_1^2} P_\phi(k_1). \]  
(2.35)

We then define the dimensionless power spectrum \( P_\phi(k) \) as

\[ P_\phi(k) = \frac{k^3}{2\pi^2} P_\phi(k). \]  
(2.36)

It is common to speak about scale invariant dimensionless power spectrum, i.e., \( P_\phi(k) \propto k^0 \). This is equivalent to state that the power spectrum has an inverse cubic dependence on the Fourier mode, i.e., \( P_\phi(k) \propto k^{-3} \).

For Gaussian random fields all higher-order (higher than 2) connected correlation functions vanish. This is no longer the case for non-Gaussian fields. Anything that isn’t Gaussian is, by definition, non-Gaussian.
We are often interested in local non-Gaussian fields that can be described by expansion [14]

\[ \Phi(\vec{x}) = \sum_i c_i (\phi(\vec{x})^i - \langle \phi(\vec{x})^i \rangle) \]  

(2.37)

where \( \phi(\vec{x}) \) is a Gaussian random field and \( c_i \) are constants. Since \( \phi \) is a Gaussian field we have \( \langle \phi^{2i+1} \rangle = 0 \). We can write Eq.(2.37) as

\[ \Phi(\vec{x}) = c_1 \phi(\vec{x}) + c_2 (\phi(\vec{x})^2 - \langle \phi(\vec{x})^2 \rangle) + c_3 \phi(\vec{x})^3 + \ldots \]  

(2.38)

For this thesis we will truncate the expansion (2.37) at third order. The Fourier transform of \( \Phi(\vec{x}) \) is given by (2.28) and can be expressed as

\[ \Phi_{\vec{k}} = c_1 \phi_{\vec{k}} + c_2 ((\phi \star \phi)_{\vec{k}} - (2\pi)^3 \sigma_{\phi}^2 \delta^3(\vec{k})) + c_3 (\phi \star \phi \star \phi)_{\vec{k}}. \]  

(2.39)

The convolution \( \star \) is defined by

\[ (\phi \star \phi)_{\vec{k}} \equiv \int \frac{d^3k_1}{(2\pi)^3} \phi_{\vec{k}_1} \phi_{\vec{k} - \vec{k}_1}, \]  

(2.40)

\[ (\phi \star \phi \star \phi)_{\vec{k}} \equiv \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \phi_{\vec{k}_1} \phi_{\vec{k}_2} \phi_{\vec{k} - \vec{k}_1 - \vec{k}_2}. \]  

(2.41)

It is easy to check that at lowest order in \( \phi \) the power spectrum of \( \Phi \) is

\[ P_{\Phi}(k) = c_1^2 P_{\phi}(k). \]  

(2.42)

To compute the 3-point function of \( \Phi_{\vec{k}}, \langle \Phi_{\vec{k}_1} \Phi_{\vec{k}_2} \Phi_{\vec{k}_3} \rangle \), we need to use Eq. (2.39) up to second order \( \phi \) and the fact that the 3-point function for a Gaussian field vanishes. Then we can write the lowest order bispectrum in terms of the power spectrum as

\[ B(k_1, k_2, k_3) = 2c_2 c_1^2 \left( P_{\phi}(k_1) P_{\phi}(k_2) + P_{\phi}(k_1) P_{\phi}(k_3) + P_{\phi}(k_2) P_{\phi}(k_3) \right). \]  

(2.43)

The 4-point function \( \langle \Phi_{\vec{k}_1} \Phi_{\vec{k}_2} \Phi_{\vec{k}_3} \Phi_{\vec{k}_4} \rangle \) will have contributions from disconnected and connected diagrams. Furthermore, \( \langle \phi_{\vec{k}_1} \phi_{\vec{k}_2} \phi_{\vec{k}_3} \phi_{\vec{k}_4} \rangle \phi_{\vec{k}_5} = 0 \), since for a Gaussian field its 4-point function is a product of disconnected pairs. To arrive to the trispectrum one need to use the expansion in Eq. (2.39) up to third order. After lengthy algebra, the trispectrum is given by

\[ T_{\phi}(k_1, k_2, k_3, k_4) = 6c_3 c_1^3 \left[ P_{\phi}(k_2) P_{\phi}(k_3) P_{\phi}(k_4) + 3 \text{Perm.} \right] + 4c_2^2 c_1^2 \left[ P_{\phi}(k_1) P_{\phi}(k_3) P_{\phi}(k_4) + 11 \text{Perm.} \right], \]  

(2.44)
where $k_{13} = \left| \vec{k}_1 + \vec{k}_3 \right|$. 

## 2.4 Models of the Early Universe

Conformal time, $\tau$, is a useful coordinate to study the motion of photons in a FRW metric. In conformal time photons move as in Minkowski space, i.e., along $45^\circ$ lines in the $\tau - x$ plane, as exemplified in Figure 2.1. The definition of conformal time is given by

$$
d\tau = \frac{dt}{a} \iff \tau = \tau_i + \int_0^t \frac{dt'}{a(t')}.
$$

In (2.45) in a radiation or matter dominated universe the integral is convergent and we set $\tau_i = \tau(0) = 0$. For convenience let’s set $a_0 = 1$ and $t_0 = (3H_0/2)^{-1}$, the current cosmic time. Note that we are implicitly assuming matter domination to set the constants. In a conformal frame the comoving distance travelled by photons is $l = \tau$. The proper physical distance since the Big Bang is

$$
D = a(t)l = a(\tau)\tau.
$$

The surface of last scattering happened at $a_{ls} \sim 10^{-3}[4]$, i.e., when the scale factor was a thousand times smaller than it is nowadays. Let us consider two points at the surface of last scattering, A and B as in Figure 2.1. Both points are at a conformal distance $\tau_{ls}$ from the big bang surface. Let us imagine point B sends a light ray at the big bang surface that is received by point A at the surface of last scattering. Since light rays travel in $45^\circ$ we have that the distance between the two points is $l_{ls} = \tau_{ls}$. Hence, at the last scattering surface, point A and B are causally connected, and the comoving size of the causal horizon is $l_{ls}$. For an observer today the size of the causal horizon of a point in the surface of last scattering is $D_{ls} = a_0 l_{ls} = \tau_{ls}$. Similarly the size of our causal horizon is $D_0 = l_0 = \tau_0$, which is also the distance to the Big Bang surface. When we look at the sky we observe angular distances. Observationally speaking the last scattering and big bang surfaces are concentric spherical surfaces centered on us (point O), as exemplified in Figure 2.2 (where we projected the 3D sphere into a circle). The two points, A and B, at the surface of last scattering will be seen with an angular separation that we call $\theta$. Therefore $\theta$ corresponds to the causally connected region in the surface of last scattering. From Figure 2.2 we can see that the points A, B and O form an approximately right angle for $\tau_0 \gg \tau_{ls}$. Hence we can approximate the geometrical relationship using trigonometry.
Figure 2.1: Diagram exemplifying the comoving causal structure between two points at the last scattering surface. For simplicity we only considered one spatial dimension, the horizontal axis. In vertical axes we have conformal time $\tau$. 
Figure 2.2: Diagram showing the geometrical relations between the size of the causal horizon and the distance to the surface of last scattering. The point O represents the observer today while the points A and B correspond to the border of a causally connected region in the CMB, as given by Fig. 2.1.
\[ \tan (\theta) = \frac{\tau_{ls}}{\tau_0 - \tau_{ls}} \approx \frac{\tau_{ls}}{\tau_0}. \] (2.47)

Since \( \tau_0 \gg \tau_{ls} \) we can just approximate \( \tau_0 - \tau_{ls} \approx \tau_0 \). We can also take a small angle approximation, i.e., \( \tan(\theta) \approx \theta \). After last scattering the universe is matter dominated. If we rewrite the integral in Eq. (2.45) in terms of the scale factor we find

\[ \tau = \int_0^a \frac{da'}{Ha'^2} = \frac{1}{H_0} \int_0^a \frac{da'}{a'^{1/2}} = \frac{2a^{1/2}}{H_0}, \] (2.48)

where we used the fact that during matter era \( H = H_0/a^{3/2} \). Thus the angle \( \theta \) is

\[ \theta \approx \frac{\tau_{ls}}{\tau_0} = \sqrt{a_{ls}}. \] (2.49)

Since last scattering happened at \( a_{ls} \sim 10^{-3} \) we find \( \theta \approx 0.03 \text{rad} = 1.8^\circ \). This means that two points in the CMB would be causally connected if they were observed today within an angle of roughly \( 2^\circ \). The CMB is observed to have a homogeneous temperature of \( 2.725 \, \text{K} \) across the whole sky. Hence one can ask: if not all regions were causally connected why is the CMB so homogeneous? If it wasn’t within the same causal horizon how could the universe thermalise? This is generically stated as the horizon problem.

One could appeal to initial conditions to solve the problem but how could we set initial conditions in the initial singularity?

Another puzzle in cosmology is the flatness problem. Using Eq. (2.15) the total density parameter is

\[ \Omega_T = \frac{\rho_T}{3m_{Pl}^2 H^2}. \] (2.50)

If we rewrite the Friedmann equation (2.3) in terms of the total density parameter \( \Omega_T \), the scale factor and the curvature we have

\[ \Omega_T - 1 = \frac{K}{a^2 H^2}. \] (2.51)

If the expansion is dominated by the energy content of the universe then \( H^2 \propto \rho \) and from Eq. (2.12) one finds that \( (aH)^2 \propto a^{-(1+3w)} \). If \( w < -1/3 \) as the universe expands and \( \Omega_T \) approaches 1, i.e., flatness. On the other hand, if \( w > -1/3 \), as for matter or radiation, \( \Omega_T \) diverges from one. If the universe was radiation and matter dominated for most of its lifetime it should be moving away from flatness. From observations the curvature is constrained to \( |\Omega_{K,0}| = |\Omega_T - 1|_0 \lesssim 10^{-2} [15] \). From Eq. (2.51) one can see that the curvature today and in the Planck era are related by

\[ \frac{|\Omega_T - 1|_{Pl}}{|\Omega_T - 1|_0} = \left( \frac{a_0 H_0}{a_{Pl} H_{Pl}} \right)^2 \] (2.52)
To have a simply rough estimate of the relevance of the curvature in the Planck epoch we assuming radiation domination \((H \propto a^{-2})\) until that early time. We then find that the curvature was even less important, i.e.,

\[
|\Omega_T - 1|_{Pl} \sim |\Omega_T - 1|_0 \left( \frac{a_{Pl}}{a_0} \right)^2 \sim |\Omega_T - 1|_0 \frac{t_{Pl}}{t_0} \simeq 10^{-62},
\]

where the age of the universe is \(t_0 = 13.76 \pm 0.11\) Gyr [15] and the Planck time is \(t_{Pl} \simeq 5.4 \times 10^{-44}\) s [4]. This is probably an exaggerated extrapolation but gives us a hint that we should expect the universe to be flatter at earlier times. So, why is the universe so flat, or started so close to flatness? One could claim that our universe has a flat geometry \(K = 0\) from the beginning, but one would like to have the least parameters to fit and therefore have a theory such that an observable flat universe results from any curvature.

Both the flatness and horizon problems can be addressed in the same way. For simplicity we will only consider the horizon problem. The easiest way to understand the problem is to say that the universe hasn’t had enough time to thermalise. Hence we can solve the horizon problem if the surface of last scattering was causally connected in the past. We require that \(\tau_{ls} > \tau_0 - \tau_{ls}\). If we recall the definition of conformal time we can rewrite it as

\[
\tau = \int_0^a \frac{d\tilde{a}}{a^2 H}.
\]

If \(w\) is constant then \(a^2 H \propto \tilde{a}^{1 - 3w}/2\). If \(\ddot{a} > 0\) then, from Eq. 2.8, the equation of state of the universe obeys \(w < -1/3\) and the integral is divergent as \(a \to 0\). Therefore we require a period with \(\dot{a} > 0\) in the early universe to solve the horizon problem in an expanding universe.

We will consider two models of the early universe: Inflationary paradigm and Pre-Big Bang Scenarios.

To introduce the horizon problem we assumed that the FRW metric is a valid description of the universe till the Big Bang surface. Such assumption is not fully valid since we know already that General Relativity is not a good description of small scale physics. At high energies, as the ones that we would have in the very early universe, quantum physics are relevant. So far we do not have a Quantum Theory of Gravity, hence we expect new physics to rule the universe at such early stages. The assumption of an initial singularity comes from extrapolations of classical GR equations until \(t = 0\). To extrapolate till such early times we are assuming that the theory of GR is not modified at high energies. Therefore an initial singularity may not exist, being a fluke of assuming that GR, as we know it, remains valid at such small scales. We will not discuss such issues in this thesis. We will consider inflation as a mechanism to solve the horizon problem as described previously. Nonetheless, the horizon problem is, in a broader sense, a problem about initial
conditions. Some authors argue that the subset of initial conditions that evolve into our observable universe without a period of cosmological inflation is bigger than the subset with a period of inflation. In that sense inflation is "unnatural" and less likely to happen. We will not consider such issues in this thesis. We will understand the horizon problem as described previously and under such premises consider inflation as a solution. More discussion about the subject can be found in [16, 17, 18]

2.4.1 Inflation

Inflation is a period in the early universe of accelerated expansion ($\ddot{a} > 0$). It therefore solves the horizon and flatness problems. It can be modeled by a de-Sitter phase of the universe, i.e., $H$ constant. In this case the scale factor grows exponentially with time as

$$a(t) = a_\ast e^{Ht}.$$  \hfill (2.55)

If the Hubble parameter is constant during inflation then so is the energy density. Hence, from the continuity equation, we require that the fluid driving the expansion has equation of state $w \simeq -1$.

Before proceeding it is interesting to determine how much expansion from inflation is required to solve the Horizon problem. From the previous analyses we require that the amount of conformal time before last scattering to be bigger than the conformal time since last scattering. We will do a rough approximation and require that the duration of inflation, $\Delta \tau_{inf}$, to be bigger than the subsequent conformal time during radiation era until today, $\Delta \tau_{after}$. The amount of conformal time after inflation is roughly given by

$$\Delta \tau_{after} = \int_{a_{inf}}^{a_0} \frac{d\tilde{a}}{\tilde{a}^2 H} = \frac{1}{H_{inf} a_{inf}^2} \int_{a_{inf}}^{a_0} d\tilde{a} \sim \frac{1}{H_{inf} a_{inf}^2}.$$ \hfill (2.56)

This is a rough estimate since we considered radiation domination until today with $H = H_{inf} (a_{inf}/a)^2$. The subscript $inf$ stands for the end of inflation. We take $a_{inf} \ll a_0$. The duration of inflation is

$$\Delta \tau_{inf} = \int_{a_i}^{a_{inf}} \frac{d\tilde{a}}{\tilde{a}^2 H} = \frac{1}{H_{inf}} \int_{a_i}^{a_{inf}} \frac{d\tilde{a}}{\tilde{a}^2} \sim \frac{1}{H_{inf} a_i}.$$ \hfill (2.57)

The subscript $i$ stands for the beginning of inflation. Hence the requirement $\Delta \tau_{inf} > \Delta \tau_{after}$ is equivalent to

$$\frac{a_{inf}}{a_i} > \frac{1}{a_{inf}}.$$ \hfill (2.58)
We know that temperature scales roughly with $a^{-1}$ then

$$\frac{1}{a_{inf}} = \frac{T_{inf}}{T_0} = \frac{10^{16}}{10^{-3}} \text{ GeV} \sim 10^{28},$$

(2.59)

where $T_0 \simeq 10^{-3} \text{ eV}$ is the temperature of the CMB today and we assumed that re-heating happened at $T_{inf} = 10^{16} \text{ GeV}$ [4]. Then the number of e-folds of inflation needed to solve the horizon problem is $N = \ln \frac{a_{inf}}{a_i} \sim 64$. This is a rough estimate of the minimal amount of inflation.

**Standard Model of Inflation**

The standard model of inflation considers a homogeneous scalar field as the inflaton. As described in 2.3 scalar fields will have

$$w = \frac{p}{\rho} = \frac{1}{2} \dot{\phi}^2 - V(\phi) \frac{1}{2} \dot{\phi}^2 + V(\phi).$$

(2.60)

If the potential energy of the field is much bigger than the kinetic energy, $V(\phi) \gg \dot{\phi}^2/2$, then the equation of state approaches $-1$. Then the Friedmann equation reads

$$H^2 \simeq \frac{1}{3m_{Pl}^2} V(\phi).$$

(2.61)

We also require inflation to last long enough. Hence, the variation of potential energy with time must be bigger than the variation of kinetic energy with time. This translates into $V_{,\phi} \gg \ddot{\phi}$, where $V_{,\phi} = dV/d\phi$. Hence we can write the field evolution equation as

$$3H \dot{\phi} \simeq -V_{,\phi}.$$  

(2.62)

During inflation the Hubble parameter is taken to be roughly constant. Therefore it is important to measure how it changes. We define

$$\epsilon \equiv -\frac{\dot{H}}{H^2}.$$  

(2.63)

This is the first slow-roll parameter and we require $\epsilon < 1$ to have inflation. It is also conventional to express the slow-roll approximation in terms of the field slow-roll parameters

$$\epsilon_\phi \equiv \frac{1}{2} m_{Pl}^2 \left( \frac{V_{,\phi}}{V} \right)^2,$$

(2.64)
\[ \eta_{\phi \phi} = m_{Pl}^2 \frac{V_{,\phi \phi}}{V}, \quad (2.65) \]

being much less than 1, i.e., \( \epsilon_{\phi}, |\eta_{\phi \phi}| \ll 1 \). Note that for single field slow-roll inflation \( \epsilon \simeq \epsilon_{\phi} \), but this is not necessarily the case in general. Inflation will end when \( \epsilon \geq 1 \). The number of e-folds from inflation is given by

\[ N \equiv \int_{t_i}^{t_{end}} H dt \simeq \frac{1}{m_{Pl}^2} \int_{\phi_i}^{\phi_{end}} \frac{d\phi}{\sqrt{2\epsilon_{\phi}}}, \quad (2.66) \]

where we used the slow-roll approximation for the final expression. At the end of inflation, the inflaton decays into radiation via reheating. Once inflation ends radiation era starts and the standard model of cosmology follows.

The simplest model of inflation is chaotic inflation given by the simplest potential for a scalar field, a quadratic potential,

\[ V(\phi) = \frac{1}{2} m^2 \phi^2, \quad (2.67) \]

where \( m \) is the mass of the scalar field. To obey the slow-roll condition we require \( \phi \gg m_{Pl} \) and \( m^2 \ll H^2 \). Once the slow-roll conditions are violated the decay of the field starts. If \( m \gg H \) then from Eq. (2.23) we can see that the homogeneous field will oscillate with an angular frequency \( m \). It will behave like a pressure-less fluid, \( \langle p \rangle = 0 \), and its average energy density will be, \( \langle \rho_{osc} \rangle = m^2 \phi^2_{osc}/2 \). This will be the energy density of radiation in the beginning of the radiation era assuming that the inflaton decays quickly into radiation.

### 2.4.2 Pre-Big Bang models

Pre-Big Bang models are alternatives to inflation. In these models the Big Bang is not the beginning of time, it is rather a bounce from a contracting phase of a previous universe to the current expanding one. In these models the puzzles in cosmology are solved during a contracting phase prior to the bounce. This contraction may be embedded in a broader picture of the universe. One proposal states that the universe is cyclical [19]. In the braneworld picture we live on a brane that has a partner anti-brane. Their separation is described by a scalar field. The late time acceleration of the universe is required in these models to dilute matter, black holes and entropy. Then the scalar field with a negative potential inverts the expansion to contraction. The universe undergoes an ekpyrotic collapse where structures are seeded. A kinetic contraction phase afterwards will lead the universe to the Big Crunch. After the Big Bounce we recover standard cosmological eras and BBN, decoupling and the formation of structures in the universe. The main ideas of
the Ekpyrotic and Cyclic universe can be found in [19]. In this subsection we will use units with \( m_{Pl} = 1 \).

**Ekpyrotic Contraction**

The Ekpyrotic contraction is driven by a canonical homogenous scalar field with a potential of the form

\[
V(\phi) = -V_0 e^{-c\phi},
\]

(2.68)

with \( c \gg 1 \). A solution of Eqs. (2.3) and (2.23) is a power law solution for the scale factor

\[
a(t) = (-t)^p,
\]

(2.69)

where \( p = 2/c^2 \) and the scaling solution for the field

\[
\phi = \frac{2}{c} \ln(-t) - \frac{1}{c} \ln \left( \frac{2(1-3p)}{c^2 V_0} \right).
\]

(2.70)

We see that as time increases towards the crunch the field values (2.70) become more and more negative. In a similar manner the potential (2.68) becomes more and more negative. This is a runaway solution of the field. Since \( c \gg 1 \) (or equivalently \( p \ll 1 \)) the contraction is very slow. The scale factor is nearly constant and the Hubble parameter is increasing rapidly

\[
H(t) = \frac{p}{t}.
\]

(2.71)

Note that from Eq. 2.60 one finds \( w_\phi = 2/(3p) - 1 \), then for slow contraction \( p \ll 1 \) we have \( w_\phi \gg 1 \). This is the defining feature of ekpyrotic contraction, a stiff fluid coming from a scalar field falling down a steep negative potential.

To understand why we require a stiff fluid, let us consider the Friedmann equation for a universe filled with radiation \( R \), matter \( m \), anisotropies, Cosmological constant \( \Lambda \), and a scalar field \( \phi \) [19],

\[
H^2 = \frac{1}{3} \left( \frac{\rho_R}{a^4} + \frac{\rho_m}{a^3} + \rho_\Lambda + \frac{\rho_{anisotropies}}{a^6} + \frac{\rho_\phi}{a^3(1+w)} \right).
\]

(2.72)

In a contracting phase, the fluid which scales more strongly with the expansion, i.e., the one with the highest exponent in the scale factor in denominator, comes to dominate the universe. Then, for the scalar field to dominate the dynamics of the universe we require \( w_\phi > 1 \). From Eq. (2.60) we see that a field with a negative potential gives \( w_\phi > 1 \). For ekpyrotic contraction we have from (2.12), (2.68) and (2.70) that \( w_\phi = c^2/2 - 1 \), then the condition \( w_\phi > 1 \) requires \( c > \sqrt{6} \) which is satisfied by the model for \( c \gg 1 \).

From previous analyses we require at least 60 \( e \)-folds to solve the flatness and horizon
problems. In other words $aH$ needs to grow 60 $e$-folds. Since $a$ is roughly constant during the contraction we find from Eq. (2.71)

$$t_{\text{beg}} \leq e^{60} t_{\text{end}}.$$  \hspace{1cm} (2.73)

Following [19] we will consider that the end of ekpyrosis occurs at $t_{\text{end}} \approx -10^3 m_{Pl}^{-1}$. From the scaling solution (2.70) we see that $V \propto t^{-2}$. Then

$$t_{\text{beg}} = \sqrt{\frac{V_{\text{end}}}{V_{\text{beg}}} t_{\text{end}}}.$$  \hspace{1cm} (2.74)

If the ekpyrotic phase starts at today’s dark energy scale and finishes at the GUT scale we have $\sqrt{V_{\text{end}}/V_{\text{beg}}} \sim e^{130}$, i.e., 130 $e$-folds to solve the flatness problem. Intuitively one can see why it also solves the horizon problem. In the contracting phase $aH$ grows so the causal horizon shrinks. Hence, by the end of the collapse causally connected patches of the universe will become disconnected. After the bounce these patches will have similar conditions since they were causally connected prior to the collapse.

In these models the energy (kinetic and potential) contained in the field will source the radiation and matter in the subsequent universe.

**New ekpyrotic**

As we will see in Chapter 3, the ekpyrotic contraction driven by a single scalar field does not reproduce the observed primordial density power spectrum [20]. The new ekpyrotic scenario [21, 22, 23] introduces a second scalar field. In fact other scalar fields can be introduced but effectively we will always have a field space direction that drives the contraction and isocurvature directions that can produce an almost scale invariant power spectrum.

In the presence of multiple fields the Friedmann equation reads

$$3H^2 = V(\phi_1, \phi_2, ...) + \frac{1}{2} \sum_i \phi_i^2.$$  \hspace{1cm} (2.75)

The scale factor still has a power law dependence in time given by

$$a(t) = (-t)^p,$$  \hspace{1cm} (2.76)

where $p \ll 1$ in order to have a slow contraction. The potential for the 2-field system is given by [21, 22, 23]

$$V(\phi_1, \phi_2) = -V_1 e^{-c_1 \phi_1} - V_2 e^{-c_2 \phi_2}.$$  \hspace{1cm} (2.77)
The coefficients $c_1$ and $c_2$ are constants as well as the amplitudes of the potentials, $V_1$ and $V_2$. One can perform a rotation in field space and define an adiabatic direction, $\sigma$, and an isocurvature direction $\chi$ [24, 25, 26, 27], i.e.,

$$\sigma = \frac{c_2 \phi_1 + c_1 \phi_2}{\sqrt{c_1^2 + c_2^2}}, \quad \chi = \frac{c_1 \phi_1 - c_2 \phi_2}{\sqrt{c_1^2 + c_2^2}}. \quad (2.78)$$

Then, the potential takes the form

$$V(\sigma, \chi) = -U(\chi)e^{-c\sigma}, \quad (2.79)$$

where

$$U(\chi) = V_1e^{-(c_1/c_2)\chi} + V_2e^{(c_2/c_1)\chi}. \quad (2.80)$$

Now $c$ is defined as

$$\frac{1}{c^2} \equiv \sum_i \frac{1}{c_i^2}. \quad (2.81)$$

The potential $U(\chi)$ has a minimum at $\chi_0$. Close to the minimum we can expand (2.80) about $\chi_0$:

$$U(\chi) \simeq U_0 \left(1 + \frac{c^2}{2} (\chi - \chi_0)^2 + \frac{c^3}{6} \left(\frac{c_2}{c_1} - \frac{c_1}{c_2}\right) (\chi - \chi_0)^3 + \ldots\right). \quad (2.82)$$

The field $\sigma$ is the adiabatic mode while $\chi$ is the isocurvature mode. If we take the isocurvature field to be at its minimum the general solution for $\sigma$ is the scaling solution

$$\sigma = \frac{2}{c} \ln(-t) - \frac{1}{c} \ln \left(\frac{2(1 - 3p)}{c^2 U_0}\right). \quad (2.83)$$

The value of $\chi$ stays close to the minimum until a tachyonic transition happens after which we recover single field ekpyrotic collapse [27].

### 2.5 Summary

In this introduction chapter we firstly revised the relevant quantities to study the dynamics of the universe. It is governed by the Friedmann equation (2.3), the continuity equation (2.6) and the Raychaundhuri equation (2.8). These equations determine the evolution of the scale factor with time and how the energy density of different components of the universe evolve with the scale factor. We also revisited the FRW metric which is a good approximation of the universe on large scales. Similarly we recalled the energy-momentum tensor of a perfect fluid since it is a good description of the background...
universe. We then looked at the 3 main cosmological eras in the universe: radiation domination, dust and late-time acceleration.

In this thesis we will study models of the early universe that are described by scalar fields. Hence, we revised the basics of scalar fields in a FRW universe. We derived its evolution equation and its energy momentum tensor. We also studied the statistics of random fields. This is going to be important later on when we need to compute $N$-point functions of scalar quantities of relevant observational interest.

We finished this chapter with models of the early universe. We started revising cosmological problems with the standard big bang cosmology. We then introduced inflation, a model of the very early universe, that solves the cosmological problems and, as we will see later in chapter 3, naturally sources primordial perturbations in the energy density of the universe. Firstly we gave a rough estimate for how inflation solves the horizon and flatness problems and then we went through the standard picture of slow-roll single field inflation. In the end of section 2.4 we presented a pre-big bang alternative to inflation.
Chapter 3

Primordial Cosmological Perturbations

The universe we observe today has galaxies and clusters of galaxies. These are density perturbations within the homogenous background universe. The FRW cosmology only describes the background evolution of the universe. To study the density perturbations that lead to galaxies and the large scale structure we need to consider metric perturbations to FRW. Similarly, a good model of the early universe should provide a source of primordial perturbations. The models of the early universe are well described by homogeneous scalar fields. We will perturb these field in the following manner

$$\phi(t,x) = \bar{\phi}(t) + \delta\phi(t,x). \quad (3.1)$$

The unperturbed background part $\bar{\phi}$ will be responsible for the dynamics of the particular early universe model. On the other hand the field perturbation $\delta\phi$ will source fluctuations in the density field. In a similar manner we split the local density field into an unperturbed background quantity and and a density perturbation

$$\rho(t,x) = \bar{\rho}(t) + \delta\rho(t,x). \quad (3.2)$$

We also need to consider perturbations of the metric. The metric will be split into FRW unperturbated background and metric perturbation, following [28]

$$g_{\mu\nu}(t,x) = g^{FRW}_{\mu\nu}(t) + \delta g_{\mu\nu}(t,x), \quad (3.3)$$

as described in the next subsection.
3.1 Scalar Metric Perturbations and Gauge invariant quantities

The line element of a general linearly perturbed FRW metric is [28]

\[
    ds^2 = -(1 + 2\varphi)dt^2 + 2a(\partial_i B - S_i)dt dx^i + \\
    a^2 \left[ (1 - 2\psi) \delta_{ij} + 2\partial_{ij} E + 2\partial_i F_j + h_{ij} \right] dx^i dx^j. \tag{3.4}
\]

The \( \partial_i \) denote spatial partial derivatives \( \partial/\partial x^i \). These metric perturbations are split into scalar, vector and tensor parts. In total we have ten degrees of freedom, 4 coming from the scalar perturbations, another 4 from the transverse vector perturbations and 2 from the transverse trace free tensor perturbation. For the purposes of this thesis we are only interested on scalar and tensor perturbations so we will neglect vector components, \( S_i \) and \( F_i \). In any case, first-order vector perturbations are constrained to be zero for scalar field cosmologies [29]. As it will be discussed in this subsection, after gauge fixing we are left with with two scalar degrees of freedom. These, in the presence of matter source will propagate as it will be clearer in subsection 3.2. The tensor perturbations \( h_{ij} \) are propagating degrees of freedom even in the absence of sources and will be discussed in subsection 3.5. At linear order, this decomposition leads to decoupled Einstein equations for each component. Hence we can treat them independently. The quantity \( \psi \) is generically referred to as the curvature perturbation since the intrinsic spatial Ricci scalar curvature in constant time hyper-surfaces is

\[
    (^{(3)}R) = \frac{4}{a^2} \delta^{ij} \partial_i \partial_j \psi. \tag{3.5}
\]

The Einstein equations still have some freedom for gauge/coordinate choice. Since we are only interested in scalar quantities let’s consider the infinitesimal gauge/coordinate transformation

\[
    (\tilde{t}, \tilde{x}^i) = (t, x^i) + \xi^\alpha \tag{3.6}
\]

where \( \xi^\alpha = (\xi^0, \delta^{ij} \partial_i \xi) \). Under such transformations we expect scalar quantities to be independent of the coordinates. In other words they only depend on the point on the manifold and not on the coordinates given to the manifold. Let’s consider a general scalar quantity \( f(t, x) \). Hence \( \tilde{f}(\tilde{t}, \tilde{x}) = f(t, x) \). Each can be decomposed into a background quantity and a perturbation, i.e., \( f(t, x) = \bar{f}(t, x) + \delta f(t, x) \). Since we are considering an infinitesimal transformation, to first order \( \tilde{f}(t, x) - \bar{f}(\tilde{t}, \tilde{x}) = -\bar{f}_{,\alpha} \xi^\alpha = \bar{f}_{,\alpha} \xi^\alpha \). Therefore we find that perturbations of a scalar quantity transform

\[
    \delta f(\tilde{t}, \tilde{x}) = \delta f(t, x) - \tilde{f}_{,\alpha} \xi^\alpha, \tag{3.7}
\]
under infinitesimal gauge transformations. Examples are scalar field and density perturbations. For simplicity we will drop the coordinate dependence, where a tilde quantity will implicitly depend on tilde coordinates. A scalar field perturbation will transform as

\[ \tilde{\delta}\phi(\tilde{t}, \tilde{x}) = \delta\phi(t, x) - \dot{\phi} \xi^0. \]  

Note that the homogeneous part of the scalar field only depends on time. Similarly the density perturbation follows the transformation

\[ \tilde{\delta}\rho = \delta\rho - \dot{\bar{\rho}} \xi^0. \]  

In the case of a tensor the gauge transformation will change the tensor in two ways. One comes from the coordinate change itself, i.e.,

\[ \tilde{g}_{\alpha\beta} = \frac{dx^\mu}{d\tilde{x}^\alpha} \frac{dx^\nu}{d\tilde{x}^\beta} g_{\mu\nu}. \]  

Then using the split into background and perturbed quantities and noting that \( dx^\mu/d\tilde{x}^\alpha \simeq \delta^\mu_{\alpha} - \xi^\mu_{\alpha} \) and \( \tilde{g}_{\alpha\beta} \simeq \bar{g}_{\alpha\beta} + \bar{g}_{\alpha\beta,\gamma} \xi^\gamma \) we find that the metric perturbation transforms as

\[ \delta\tilde{g}_{\mu\nu} = \delta g_{\mu\nu} - \bar{g}_{\mu\nu,\gamma} \xi^\gamma - \bar{g}_{\mu\gamma} \xi^\nu - \bar{g}_{\gamma\nu} \xi^\mu. \]  

We are particularly interested in how the curvature perturbation changes. From the previous equation one can find that \( \psi \) transforms as

\[ \tilde{\psi} = \psi + H\xi^0. \]  

By looking at Eqs. (3.9) and (3.12) on can see that the quantity

\[ \zeta \equiv -\psi - \frac{H}{\bar{\rho}} \delta\rho, \]  

is gauge invariant. \( \zeta \) is the curvature perturbation in uniform density hyper-surfaces and was first defined by Bardeen, Steinhardt and Turner [30]. It can be shown that on super horizon scales and in the absence of non-adiabatic perturbations \( \dot{\zeta} = 0 \) [31]. Therefore, \( \zeta \) is a very useful quantity to use to describe the curvature perturbation.

Another useful gauge invariant quantity is [32]

\[ \delta\phi_\psi \equiv \delta\phi + \frac{\dot{\phi}}{H} \psi. \]
This will be an important gauge invariant variable to study the evolution of field perturbations, since it coincides with the field fluctuation in a flat gauge. Note that for single field slow-roll inflation
\[\dot{\rho} \simeq V_\phi \dot{\phi} \quad \text{and} \quad \delta \rho \simeq V_\phi \delta \phi.\]
Then in this regime, \(\delta \phi_\psi\) is related with \(\zeta\) by a simple rescaling
\[\zeta \simeq -\frac{H}{\dot{\phi}} \delta \phi_\psi. \quad (3.15)\]

3.2 Perturbed Einstein Equations and the wave equation for field perturbations

The perturbed Einstein equation
\[\delta G_{\mu\nu} = \frac{1}{m_{Pl}^2} \delta T_{\mu\nu} \quad (3.16)\]
enable us to relate the metric perturbations with the matter perturbations. The perturbed energy-momentum tensor \(\delta T^\mu_\nu\) is related with the energy density perturbation \(\delta \rho\), the pressure perturbation \(\delta p\) and with the momentum perturbation \(\delta q_i\) by [28, 29]
\[\delta T^0_0 = -\delta \rho, \quad \delta T^0_i = \delta q_i, \quad \delta T^i_j = 3 \delta p. \quad (3.17)\]

Since we are only considering scalar perturbations \(\delta q_i = \delta q_i\), where \(\delta q\) is the scalar momentum potential. The energy and momentum constraints [33]
\[\frac{\delta \rho}{m_{Pl}^2} = 3H \left(\dot{\psi} + H \varphi\right) + \frac{k^2}{a^2} \left[\psi + H \left(a^2 \dot{E} - aB\right)\right], \quad (3.18)\]
\[\frac{\delta q}{m_{Pl}^2} = \dot{\psi} + H \varphi. \quad (3.19)\]

Note that these equations are given in Fourier space. Hereafter we will consider all quantities are in Fourier space therefore to simplify notation we will drop the conventional \(k\) subscript in the variables. The conservation of the energy-momentum tensor, \(\nabla_\mu T^\mu_0 = 0\), gives at first order the evolution equation for the density perturbation [28],
\[\dot{\delta \rho} + 3H (\delta \rho + \delta p) = \frac{k^2}{a^2} \dot{\delta q} + (\rho + p) \left[3 \dot{\psi} + \frac{k^2}{a^2} \left(a^2 \dot{E} - aB\right)\right]. \quad (3.20)\]

We are interested in scalar field perturbations in a perturbed FRW. Let’s consider a multiple scalar field theory minimally coupled to gravity
\[S = \int dx^4 \sqrt{-g} \left[-\frac{1}{2} \sum_I g^{\mu\nu} \partial_\mu \phi_I \partial_\nu \phi_I - V(\phi_1, \ldots, \phi_N)\right]. \quad (3.21)\]
Then the total energy, pressure and momentum perturbations are related with field fluctuations by [28]

\[ \delta \rho = \sum_I \left[ \dot{\phi}_I \left( \delta \phi_I - \dot{\phi}_I \varphi \right) + V_I \delta \phi_I \right], \quad (3.22) \]
\[ \delta p = \sum_I \left[ \dot{\phi}_I \left( \delta \phi_I - \dot{\phi}_I \varphi \right) - V_I \delta \phi_I \right], \quad (3.23) \]
\[ \delta q = - \sum_I \dot{\phi}_I \delta \phi_I. \quad (3.24) \]

Therefore using Eq. (3.20) and the previous relations we arrive to the wave equation for first order field fluctuations

\[ \ddot{\delta \phi}_I + 3H \dot{\delta \phi}_I + \frac{k^2}{a^2} \delta \phi_I + \sum_J V_{IJ} \delta \phi_J = -2V_I A + \dot{\phi}_I \left[ \dot{A} + 3 \psi + \frac{k^2}{a^2} \left( a^2 \dot{E} - aB \right) \right]. \quad (3.25) \]

In a spatially flat gauge (\( \psi = 0 \)) we have that \( \delta \phi_\psi = \delta \phi \). Then if we use the Einstein equations to eliminate the metric perturbation we find [28]

\[ \ddot{\delta \phi}_\psi + 3H \dot{\delta \phi}_\psi + \frac{k^2}{a^2} \delta \phi_\psi + \sum_J \left[ V_{IJ} \frac{1}{a^3 m^2_{pl}} \frac{d}{dt} \left( \frac{a^3 \dot{\phi}_I \dot{\phi}_J}{H} \right) \right] \delta \phi_{\psi J} = 0. \quad (3.26) \]

### 3.3 \( \delta N \) Formalism

The \( \delta N \) formalism is a very powerful tool to study linear and non-linear perturbations from the early universe. The total curvature perturbation on uniform density hypersurfaces can be re-interpreted as the perturbation in the local integrated expansion. In other words, \( \zeta \) is given by the perturbation in the integrated expansion to a uniform total density hyper-surface from a flat initial hyper-surface [34]. In Figure 3.1 we illustrate the \( \delta N \) formalism with a geometrical diagram. Mathematically we write

\[ \zeta = \delta N. \quad (3.27) \]

We can generalise the definition of \( \zeta \) (Eq.(3.13)) to each component of the universe \( \zeta_I \). Hence it is useful to distinguish them and relate them. One can show that on uniform total energy density hyper-surfaces [35]

\[ \zeta_I = \delta N + \frac{1}{3} \int_{\tilde{\rho}_I}^{\rho_I} \frac{d\tilde{\rho}_I}{\tilde{\rho}_I + \tilde{\rho}_I}, \quad (3.28) \]

where \( \rho_I \) is the local energy density of the fluid and \( \tilde{\rho}_I \) is the homogeneous energy density of the fluid. Then, if the equation of state \( w_I \) is constant we can write the local energy
density of a fluid in a uniform total energy hyper-surface as [36]

\[ \rho_I = \bar{\rho}_I e^{3(1+w)(\zeta_I - \zeta)}. \]  

(3.29)

For an adiabatic perturbation the uniform density hyper-surfaces coincide with the uniform total energy density hyper-surfaces, then \( \zeta_I = \zeta \). It is conventional to consider the perturbations in the radiation fluid as adiabatic, thus until the curvature perturbation in the radiation fluid is sourced \( \zeta = 0 \). For an isocurvature perturbation its local energy density will always be perturbed in a total uniform density hyper surface and is "modulated" by the entropy perturbation. We define the entropy perturbation of a non-adiabatic fluid with respect to the radiation fluid as [28]

\[ S_{I\gamma} = 3 (\zeta_I - \zeta_\gamma). \]

(3.30)

Another powerful use of the \( \delta N \) formalism arises within the separate universe approach [34, 37]. On large scales, and when gradient terms and anisotropies are negli-
gible, the local energy density, pressure, expansion, etc, evolve like in a homogeneous FRW background \cite{34, 37}. Hence, two different patches of the universe will evolve as two independent FRW universes. In the final hyper-surface the curvature perturbation that comes from inhomogeneities in the local integrated expansion, occurs due to local field fluctuations in the initial flat hyper-surface. Therefore we can expand $\zeta$ in terms of initial field fluctuations as \cite{38}

$$
\zeta = \sum_A N_A \delta \phi_A + \frac{1}{2} \sum_{AB} N_{AB} \delta \phi_A \delta \phi_B + \frac{1}{6} \sum_{ABC} N_{ABC} \delta \phi_A \delta \phi_B \delta \phi_C + \ldots ,
$$

(3.31)

where $N_A = \partial N/\partial \phi_A$, $N_{AB} = \partial^2 N/\partial \phi_A \partial \phi_B$ and $N_{ABC} = \partial^3 N/\partial \phi_A \partial \phi_B \partial \phi_C$.

### 3.4 Primordial Power Spectrum of Curvature Perturbations and Non-Linear Perturbations

Using the $\delta N$ formalism one can easily compute the primordial power spectrum of curvature perturbations. From subsection 2.3.1 one can find the power spectrum of zeta is at leading order

$$
P_\zeta = \sum_A N_A^2 \mathcal{P}_{\delta \phi_A} ,
$$

(3.32)

where we consider that the fields do not have field interactions. Gravitational interactions give slow-roll corrections \cite{34} that we will not consider at linear order. When we observe the power spectrum we are not only interested in its amplitude but also how it depends on comoving scales. One defines the spectral index as

$$
n_\zeta - 1 \equiv \frac{d \ln P_\zeta}{d \ln k} .
$$

(3.33)

We say that the spectrum is scale invariant when the tilt, $n_\zeta$ is equal to one. This is a general definition since we can and will extend it to entropy perturbations. For adiabatic perturbations, $\zeta$ is conserved on large scales \cite{35}, therefore once the modes leave the Hubble radius the power spectrum of curvature perturbations and its scale dependences became constant. Then for adiabatic perturbations we measure the spectral index at horizon exit $k = aH$. For adiabatic field fluctuations, and assuming that the integrated expansion is scale invariant $d \ln N_A/d \ln k = 0$, we have from Eq. (3.32) that

$$
n_\zeta - 1 = \frac{1}{P_\zeta} \sum_A N_A^2 \mathcal{P}_{\delta \phi_A} (n_{\delta \phi_A} - 1) ,
$$

(3.34)
where we defined \( n_{\delta \phi A} - 1 \equiv d \ln P_{\delta \phi A}/d \ln k \). Note that Eq. (3.33) is to be determined at a fixed time which we usually take to be when \( \zeta \) becomes a conserved quantity. The running of the power spectrum is defined as

\[
\alpha_\zeta \equiv \frac{dn_\zeta}{d \ln k}.
\]

(3.35)

The 3-point function is the first measure of non-linearities in the curvature perturbation. In this thesis we are only interested in the local shape of non-Gaussianity. Other shapes are also possible and more details can be found in [39]. We can define the local type non-Gaussianity as

\[
f_{NL}(k_1, k_2, k_3) \equiv \frac{5}{6} \frac{B_\zeta(k_1, k_2, k_3)}{P_\zeta(k_1)P_\zeta(k_2) + P_\zeta(k_1)P_\zeta(k_3) + P_\zeta(k_2)P_\zeta(k_3)}.
\]

(3.36)

This type of non-linearity is nearly momentum independent. It is also called squeezed non-Gaussianity. The delta N formalism (3.31) is an expansion of the form of Eq. (2.37), hence one finds at leading order

\[
f_{NL} = \frac{5}{6} \frac{\sum_{AB} N_A N_B N_{AB}}{(\sum_A N_A^2)^2}.
\]

(3.37)

Similarly one finds that the trispectrum non-linear parameters, appearing in the 4-point function, are [40]

\[
g_{NL} = \frac{25}{54} \frac{\sum_{ABC} N_A N_B N_C N_{ABC}}{(\sum_A N_A^2)^3},
\]

(3.38)

\[
\tau_{NL} = \frac{\sum_{ABC} N_A N_B N_{AC} N_{BC}}{(\sum_A N_A^2)^3}.
\]

(3.39)

There is a very interesting result relating \( f_{NL} \) and \( \tau_{NL} \) first derived in [41]. Let’s consider two vectors \( W_A = N_A \) and \( V_A = \sum_B N_B \bar{N}_{BA} \). Note that

\[
f_{NL} = \frac{5}{6} \frac{W.V}{(W.W)^2},
\]

(3.40)

\[
\tau_{NL} = \frac{V.V}{(W.W)^3}.
\]

(3.41)

It follows from the Cauchy-Schwarz inequality, \((W.V)^2 \leq (W.W)(V.V)\), that

\[
\tau_{NL} \geq \frac{36}{25} f_{NL}^2.
\]

(3.42)
Equality happens when the two vectors are collinear, i.e., when \( N_A \) is an eigenvector of \( N_{AB} \).

### 3.5 Tensor Perturbations

The tensor perturbations \( h_{ij} \) defined in Eq. (3.4) are by definition traceless, \( h_{ij}^t = 0 \), and transverse, \( \partial^i h_{ij} = 0 \). This gives four constraint equations. Because of these conditions the tensor modes are immediately gauge invariant. Since the metric is symmetric \( h_{ij} \) has 6 independent component, but with the 4 constraint equations we are left with two spatial degrees of freedom only. We split the tensor perturbations into scalar amplitude \( h(t) \) that only depends on time and a spatial part that only depends on position and includes the two spatial degrees of freedom. We will use the eigenmodes of the spacial Laplacian, \( \nabla^2 e_{ij} = -k^2 e_{ij} \), with comoving wavenumber \( k \) as the two spatial degrees of freedom. Then we have

\[
h_{ij} = h(t)e_{ij}^{(t)}(x) + \bar{h}(t)e_{ij}^{(x)}(x). \tag{3.43}
\]

Tensor perturbations do not have any constraint equation. They are purely gravitational degrees of freedom. In other words they are gravitational waves. The wave equation for the amplitude of the tensor modes is [28]

\[
\ddot{h} + 3H\dot{h} + \frac{k^2}{a^2}h = 0, \tag{3.44}
\]

which is equivalent to Eq. (3.26) for a massless field in an unperturbed FRW universe.

The power spectrum of gravitational waves is given by [28]

\[
\mathcal{P}_T = 2 \frac{k^3}{2\pi^2} |h|^2. \tag{3.45}
\]

The factor of two comes from the fact that we have two polarisations of the gravitational waves. The spectral index of the tensor power spectrum is defined as

\[
n_T \equiv \frac{d\ln \mathcal{P}_T}{d\ln k}. \tag{3.46}
\]

When we study a model it is conventional to define the tensor-to-scalar ratio \( r_T \) which is defined by the ratio between the power in the tensor modes and the power in the scalar curvature perturbation, i.e.,

\[
r_T \equiv \frac{\mathcal{P}_T}{\mathcal{P}_\zeta}. \tag{3.47}
\]
3.6 Perturbations in single field slow-roll inflation

During single field inflation Eq. (3.26) simplifies to

$$\ddot{\delta \phi} + 3H \dot{\delta \phi} + \frac{k^2}{a^2} \delta \phi + \left[ V_{\phi \phi} - \frac{1}{a^3 m_{Pl}^2} \frac{d}{dt} \left( \frac{a^3 \dot{\phi}^2}{H} \right) \right] \delta \phi = 0 . \quad (3.48)$$

where we have taken a flat gauge. The term with $m_{Pl}^{-2}$ indicates gravitational coupling of the field perturbations. Introducing the new variable $v = a \delta \phi$ we find, to first order in the slow-roll parameters, the wave equation

$$v'' + \left[ k^2 - a^2 H^2 \left( 2 + 5 \epsilon_{\phi} - 3 \eta_{\phi \phi} \right) \right] v = 0 , \quad (3.49)$$

where primes denote derivatives with respect to conformal time. To first order in the slow-roll parameters and ignoring time derivatives of the slow-roll parameters one can find [28]

$$aH \simeq - \frac{1}{(1 - \epsilon) \tau} . \quad (3.50)$$

We then write Eq. (3.49) as [28]

$$v'' + \left[ k^2 - \frac{\nu^2 - 1/4}{\tau^2} \right] v = 0 , \quad (3.51)$$

where

$$\nu \simeq \frac{3}{2} + 3 \epsilon_{\phi} - \eta_{\phi \phi} . \quad (3.52)$$

The general solution of Eq. (3.49) is

$$v(\tau) = \frac{\sqrt{\pi}}{2 \sqrt{k}} \sqrt{-k \tau} e^{(1 + 2 \nu) \frac{\tau}{4}} \left[ C_1(k) H_{\nu}^{(1)}(-k \tau) + C_2(k) H_{\nu}^{(2)}(-k \tau) \right] . \quad (3.53)$$

The functions $H_{\nu}^{(1,2)}(-k \tau)$ are the Hankel functions, or Bessel functions of 3rd kind, defined by

$$H_{\nu}^{(1)}(x) = J_{\nu}(x) + i Y_{\nu}(x) \quad (3.54)$$
$$H_{\nu}^{(2)}(x) = J_{\nu}(x) - i Y_{\nu}(x) \quad (3.55)$$

where $J_{\nu}(x)$ is the Bessel function of first kind and $Y_{\nu}(x)$ is the Bessel function of second kind. The normalisation is picked such that the mode functions obey to the Wronskian equal to $-i$, i.e.,

$$v(\tau)v^*(\tau) - v^*(\tau)v'(\tau) = -i . \quad (3.56)$$
We wish to recover Minkowski vacuum state in the asymptotic past, \(-k\tau \rightarrow \infty\), i.e., \(v(\tau) = e^{-i k\tau}/\sqrt{2k}\). Note that in such limit \(H_{\nu}^{(1)}(-k\tau) \rightarrow \sqrt{2/(\pi k\tau)}e^{-ik\tau-\frac{(1+2\nu)}{4}}\). Therefore we set \(C_1(k) = 1\) and \(C_2(k) = 0\).

The power spectrum of field perturbations is given by

\[
P_{\delta\phi} = \frac{k^3}{2\pi^2} \frac{v(\tau)v^*(\tau)}{a^2}.
\] (3.57)

We are interested in the power spectrum of field perturbations on large scales at late times, i.e., when \(k\tau \rightarrow 0^-\). In this limit \(H_{\nu}^{(1)}(-k\tau) \rightarrow -(i/\pi)\Gamma(\nu)(-k\tau/2)^{-\nu}\), where \(\Gamma(\nu)\) is the gamma function. Then

\[
v(\tau) = -i \frac{2^{\nu-1}}{\sqrt{k\pi}} \Gamma(\nu) (-k\tau)^{1/2-\nu}.
\] (3.58)

Therefore the power spectrum of field perturbations in this limit is

\[
P_{\delta\phi} = \frac{2^{2\nu-3}}{\pi^3} \Gamma^2(\nu)(1 - \epsilon_{\phi})^2 H^2 (-k\tau)^{-2\nu}.
\] (3.59)

Hence, at zeroth order in slow-roll parameter the amplitude of the power spectrum of field perturbations at horizon exit is given by

\[
P_{\delta\phi} = \left(\frac{H_*}{2\pi}\right)^2,
\] (3.60)

where we use the fact that \(\Gamma(3/2) = \sqrt{\pi}/2\). The subscript \(*\) stands for when the modes leave the horizon, i.e., \(k = aH\).

In single field slow-roll inflation the field fluctuations are related to the curvature perturbation on uniform density hyper-surfaces via Eq. (3.15). Hence the power spectrum of \(\zeta\) is given by

\[
P_{\zeta} = \left(\frac{H}{\dot{\phi}}\right)^2 P_{\delta\phi}.
\] (3.61)

As we have seen on super-horizon scales \(\zeta\) is conserved in the absence of non-adiabatic pressure and the power spectrum remains constant. I.e., for \(k \gg aH, H/\dot{\phi} \rightarrow \text{constant}\), then one can measure the power spectrum at horizon exit. Intuitively one can understand where the curvature perturbations come from. During inflation the quantum field fluctuations are stretched due to the expansion. Once they leave the horizon the modes freeze out. Using the slow-roll approximation of finds that the amplitude of the power spectrum
of the curvature perturbation in uniform density hyper-surfaces $\zeta$ is

$$P_\zeta \simeq \frac{1}{2m_{Pl}^2\epsilon_\phi}\left(\frac{H_*}{2\pi}\right)^2\bigg|_{k=aH}$$

(3.62)

These perturbations are transferred to radiation once inflation ends and reheats the universe.

The spectral index (3.33) $n_\zeta$ is given from Eqs. (3.61) and (3.59) by

$$n_\zeta - 1 = 3 - 2\nu .$$

(3.63)

Then, from (3.52), we find to leading order in the slow roll parameters we have [42]

$$n_\zeta - 1 = -6\epsilon_\phi + 2\eta_{\phi\phi} .$$

(3.64)

Since the slow-roll parameters are much smaller than unity the power spectrum is nearly scale-invariant. The running of the power spectrum is given by [42]

$$\alpha_\zeta = -16\epsilon_*\eta_{\phi\phi} + 24\epsilon_*^2 + 2\xi_{\phi\phi}$$

(3.65)

where $\xi_{AB}^2 = (\partial^4 V/\partial \phi_A \partial \phi_B)/9H^4$.

We can use the $\delta N$ formalism to estimate the non-Gaussianities coming from single field slow-roll inflation. Note that

$$N_\phi = m_{Pl}^{-2} \frac{V}{V_\phi},$$

(3.66)

then

$$N_{\phi\phi} = m_{Pl}^{-2} \left(1 - \frac{VV_{\phi\phi}}{V_\phi^2}\right).$$

(3.67)

Therefore using Eq. (3.37) we find

$$f_{NL} = \frac{5}{6} (2\epsilon_\phi - \eta_{\phi\phi}).$$

(3.68)

For this result we considered the field perturbations to be Gaussian. If the field perturbations are non-Gaussian this result will have corrections [42].

Inflation will also excite tensor modes. Doing the transformation $u = ahm_{Pl}/2$ and using conformal time, Eq. (3.44) becomes

$$u'' + [k^2 - a^2H^2 (2 - \epsilon_\phi)] u = 0 .$$

(3.69)
This is similar to Eq. (3.49). One can define a tensor index $\nu_T$ by

$$(aH)^2(2 - \epsilon_\phi) \approx \frac{\nu_T^2 - 1/4}{\tau^2}. \quad (3.70)$$

with $\nu_T \approx 3/2 + \epsilon_\phi$. The solution is similar to Eq. (3.49) since we require the same Minkowski vacuum state at early times and the normalisation of the wave functions to the same as (3.53). Then we have

$$P_T = \frac{8}{m_{Pl}^2} \frac{2^{2\nu_T - 3}}{\pi^3} \Gamma^2(\nu_T)(1 - \epsilon_\phi)^2 H^2 (-k \tau)^{3 - 2\nu_T}. \quad (3.71)$$

Hence to lowest order in slow-roll the power spectrum of the tensor perturbations is

$$P_T \approx \frac{8}{m_{Pl}^2} \left( \frac{H_*}{2\pi} \right)^2 \left| \frac{k}{aH} \right|^2. \quad (3.72)$$

As for the scalar power spectrum it is straightforward to determine the tilt of the tensor power spectrum from Eq (3.71),

$$n_T = 3 - 2\nu_T \approx -2\epsilon_\phi. \quad (3.73)$$

For single field inflation the tensor-to-scalar ratio (3.47) is just

$$r_T = 16\epsilon_\phi. \quad (3.74)$$

### 3.7 Isocurvature perturbations during ekpyrotic collapse

During ekpyrotic contraction the fields present will have quantum fluctuations on small scales and early times as in inflation. During collapse the fact that the horizon shrinks makes the modes leave the horizon and freeze out. Perturbations in the adiabatic mode in the new ekpyrotic model and the “old” ekpyrotic are exactly the same. In the case of the new ekpyrotic the presence of an additional scalar field means that we have another degree of freedom which will also source non-Gaussianities.

The adiabatic field does not produce the correct spectrum of curvature perturbations, since $n_\zeta - 1 = 2/(1 - p)$ [20], therefore for $p \ll 1$ the power spectrum is not nearly scale invariant as observed in the CMB. The fact that the “old” Ekpyrotic model does not reproduce the observed power spectrum led to the introduction to a second scalar field. As seen in subsection 2.4.2 with two scalar fields one can define an adiabatic mode and
an isocurvature mode. The perturbations in the isocurvature field obey to [27]

\[
\ddot{\delta \chi} + 3H \dot{\delta \chi} + \left( \frac{k^2}{a^2} + m^2_\chi \right) \delta \chi = 0,
\]

(3.75)

where

\[
m^2_\chi = -\frac{2(1 - 3p)}{\ell^2}.
\]

(3.76)

It is convenient to work in conformal time defined by \(dt = ad\tau\). For a power-law solution \(a = (-t)^p\) we have the relation

\[
aH = \frac{1}{(\epsilon - 1)\tau},
\]

(3.77)

where the fast-roll parameter is

\[
\epsilon = \frac{1}{p} = \frac{c^2}{2}.
\]

(3.78)

Performing the transformation \(v = a\delta \chi\) one finds

\[
v'' + \left[ k^2 - \frac{a''}{a} + a^2 m^2_\chi \right] v = 0,
\]

(3.79)

where prime denote derivatives with respect to conformal time \(\tau\). This is similar to the wave equation for inflation perturbations. From Eq. (3.77) we have

\[
\frac{a''}{a} = -(c^2/2 - 2)a^2 H^2 = -\frac{2c^2 - 8}{(c^2 - 2)^2\tau^2}.
\]

(3.80)

One should also note that

\[
a^2 m^2_\chi = -\frac{2c^2}{\tau^2} \frac{(c^2 - 6)}{(c^2 - 2)^2}.
\]

(3.81)

Thus for the isocurvature field we write

\[
v'' + \left[ k^2 - \frac{1}{\tau^2} \frac{8 + 2c^4 - 14c^2}{(c^2 - 2)^2} \right] v = 0.
\]

(3.82)

Using the Bunch-Davis vacuum state to normalise the amplitude of fluctuations at early times \((k\tau \to -\infty)\) one finds

\[
v(\tau) = \frac{\sqrt{\pi}}{2} e^{-i\frac{\nu}{2}(\nu + \frac{1}{2})} (-k\tau)^{1/2} H^{(1)}_\nu(-k\tau),
\]

(3.83)

with

\[
\nu^2 = \frac{9}{4} - \frac{6c^2}{(c^2 - 2)^2}.
\]

(3.84)
In the fast-roll limit $c^2 \gg 1$, $\nu \simeq 3/2 - 2/c^2$. On large scales $k \rightarrow 0$, the Hankel function is approximately equal to $-(i/\pi)\Gamma(\nu)(-k\tau/2)^{-\nu}$, where $\Gamma(\nu)$ is the gamma function. Then the power spectrum for the isocurvature field perturbations in this theory is

$$P_{\delta\chi} = \frac{k^3}{2\pi^2} \frac{v(\tau)v^*(\tau)}{a^2} = C_{\nu} k^2 (-k\tau)^1 - 2\nu,$$  \hspace{1cm} (3.85)

where

$$C_{\nu} = 2^{\nu-3/2} \frac{\Gamma(\nu)}{\pi^{3/2}}.$$  \hspace{1cm} (3.86)

To lowest order $\nu = 3/2$, thus the amplitude of the power spectrum is [43]

$$P_{\delta\chi} \simeq \frac{c^4}{4} \left( \frac{H}{2\pi} \right)^2.$$  \hspace{1cm} (3.87)

One can easily determine the scale dependence of the power spectrum of $\chi$-field perturbations using Eq. (3.85) and Eq. (3.84) in the fast roll limit, i.e.,

$$n_{\delta\chi} - 1 = 3 - 2\nu = \frac{4}{c^2}.$$  \hspace{1cm} (3.88)

In the fast-roll limit $c \gg 1$ the power spectrum is nearly scale invariant although it is blue. Observations [15] favour a nearly scale invariant red spectrum but still allow for a slightly blue spectrum. Since the spectral index is constant there is no running of the power spectrum $\alpha_{\delta\chi} = 0$.

The isocurvature field will also source non-Gaussian perturbations as the $\chi$ field does not stay Gaussian throughout the ekpyrosis phase. Here we will reproduce the calculation done in [44]. First let’s take a look at the evolution of the $\chi$ field on large scales including cubic self interactions

$$\ddot{\chi} + 3H \dot{\chi} + m^2 \chi = -\frac{m^2 \tilde{c}}{2} \chi^2 + O(\chi^3),$$  \hspace{1cm} (3.89)

where

$$\tilde{c} = c \left( \frac{c_2}{c_1} - \frac{c_1}{c_2} \right).$$  \hspace{1cm} (3.90)

First let’s look at the homogeneous part of Eq. (3.89). Then the linear part of $\chi$ obeys

$$\ddot{\chi}_L + 3H \dot{\chi}_L + m^2 \chi_L = 0.$$  \hspace{1cm} (3.91)
Let’s start with the Ansatz $\chi_L \propto t^{-\beta}$. Doing so one gets for the growing mode

$$\beta = -\frac{1}{2} \left( 1 - \frac{6}{c^2} - \sqrt{\left(1 - \frac{6}{c^2}\right)^2 + 8 \left(1 - \frac{6}{c^2}\right)} \right).$$

(3.92)

In the fast-roll limit

$$\beta \simeq 1 - \frac{2}{c^2}.$$  

(3.93)

To solve Eq. (3.89) we follow [44] and take a perturbative approach. If we try the Ansatz for the growing mode solution for $\chi$ given by

$$\chi = \chi_L + \frac{1}{2} C \chi^2_L + \mathcal{O}(\chi^3_L).$$

(3.94)

Solving Eq. (3.89) order by order we find, to zero order, that $C = \tilde{c}/2$. One should note that this perturbative expansion is only valid for $C \chi \ll 1$.

Let $\chi_L = \alpha H^2$. The following procedure is described in [44] for $\beta = 1$. The constant parameter $\alpha$ distinguishes different trajectories of $\chi_L$. Then

$$\chi = \alpha H^2 + \frac{1}{2} C (\alpha H^2)^2.$$  

(3.95)

On constant $H$ hyper-surfaces the perturbations of $\chi$ correspond to perturbations of the parameter $\alpha$. Then we can write

$$\delta \chi = \left( H^2 + C H^2 \alpha \right) \delta \alpha + \left( \frac{C}{2} H^2 \right) (\delta \alpha)^2.$$  

(3.96)

One should note that $\delta \alpha = \delta \chi L H^{-\beta}$. Since $\delta \chi L$ is Gaussian to high accuracy then $\delta \alpha$ is also Gaussian. Then the parameter $\alpha$ is a good Gaussian variable to work with and computes non-linearities. Therefore we do the $\delta N$ expansion in terms of the parameter $\alpha$,

$$\delta N = N_\alpha \delta \alpha + \frac{1}{2} N_{\alpha \alpha} (\delta \alpha)^2 + \ldots.$$  

(3.97)

To determine how $N$ depends on $\alpha$ we follow the calculation done in [44]. Let assume that at an instant $t = t_T$ there is an instantaneous transition from multi-field scaling solution to single-field $\phi_2$-dominated scaling solution, i.e., instantaneously the trajectory in field space changes, and the new adiabatic field direction follows an exponential potential, i.e., $V(\phi_2) \propto e^{-c_2 \phi_2}$. Then

$$N \equiv \int_{t_T}^{t_F} H dt = \int_{t_T}^{t_T} H dt + \int_{t_T}^{t_F} H dt,$$  

(3.98)
\[ \begin{align*}
= & \int_{t_r}^{t_T} \frac{2}{c^2 I} dt + \int_{t_r}^{t_T} \frac{2}{c^2 I} dt, \\
= & -\frac{2}{c^2_1} \ln |H_T| + \text{constant},
\end{align*} \]

(3.99)

(3.100)

where the subscript \( T \) stand for the transition time and \( 1/c^2_1 = 1/c^2 - 1/c^2_2 \). To see how \( H_T \) varies with \( \alpha \) we invert Eq.(3.95) to get

\[ \alpha = \frac{\chi}{H^\beta} \left( 1 - \frac{1}{2} C \chi \right). \]

(3.101)

Therefore at transition time, on \( \chi = \chi_T \) hyper-surfaces, we have \( \alpha \propto H_T^{-\beta} \). Hence

\[ \delta N = \frac{2}{c^2_1 \beta} \frac{\delta \alpha}{\alpha} - \frac{1}{c^2_1 \beta} \left( \frac{\delta \alpha}{\alpha} \right)^2 + \ldots, \]

(3.102)

or alternatively

\[ \begin{align*}
N_\alpha &= \frac{2}{c^2_1 \beta} \frac{1}{\alpha}, \\
N_{\alpha \alpha} &= -\frac{2}{c^2_1 \beta} \frac{1}{\alpha^2}.
\end{align*} \]

(3.103)

(3.104)

Hence from the \( \delta N \) expansion in terms of \( \alpha \) in Eq. (3.97) and the definition of \( f_{NL} \) (Eq. (3.37)) we have

\[ f_{NL} = \frac{5}{6} \frac{N_{\alpha \alpha}}{N_\alpha^2}, \]

(3.105)

which gives in the case of study

\[ f_{NL} = -\frac{5\beta}{12} c^2_1. \]

(3.106)

Observations still allow negative values of local \( f_{NL} \) although for \( f_{NL} \neq 0 \) a positive value is preferred. At linear order there is a perfect cancelation of \( \chi \) contributions to \( f_{NL} \) similar to the one found in [44].

### 3.8 Summary and Current Observational Constraints

In this chapter we reviewed perturbation theory and applied it to early universe models. We started by splitting quantities into background parts independent of the position and perturbations about the homogeneous background. Firstly we looked at perturbations in the FRW metric and defined useful gauge invariant quantities to study the curvature
perturbation. We then used the perturbed Einstein equations to relate the metric perturbations with the energy density, momentum and pressure perturbations. We combined those with field perturbations to arrive to a perturbation evolution equation (3.26). We continued by reviewing the $\delta N$ formalism that allows us to relate the curvature perturbation on uniform total density hyper-surfaces at a given time with initial field fluctuations.

After, we defined the power spectrum of curvature perturbations and its scale dependence. We also defined the non-linear parameters arising from the 3 and 4 point functions. We then studied tensor perturbations and defined the tensor-to-scalar ratio.

We computed relevant observational quantities from single field slow-roll inflation and from isocurvature perturbations in ekpyrotic collapse.

In this thesis we wish to test models for the origin of primordial perturbations against observations using linear and non-linear perturbations in the curvature perturbation and tensor perturbations. The main observables coming from the linear part of the curvature perturbation are the amplitude of the power spectrum $P_{\zeta}$ and its scale dependence $n_{\zeta}$ and $\alpha_{\zeta}$. The tensor modes are tested using the tensor-to-scalar ratio $r_T$. In this thesis we only studied models that produce local type non-Gaussianity. We will use the current bounds to test the early universe models.

The state of the art of current observations is given by the latest WMAP 7 results [15]. On table 3.1 we summarise the relevant observational quantities.

<table>
<thead>
<tr>
<th>Observable quantity</th>
<th>Measure/Bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>Power Spectrum Amplitude</td>
<td>$P_{\zeta} = (2.430 \pm 0.091) \times 10^{-9}$ (68%CL)</td>
</tr>
<tr>
<td>Tilt (no running, no tensors)</td>
<td>$n_{\zeta} = 0.968 \pm 0.012$ (68%CL)</td>
</tr>
<tr>
<td>Running (no tensors)</td>
<td>$\alpha_{\zeta} = -0.022 \pm 0.020$ (68%CL)</td>
</tr>
<tr>
<td>Tensor-to-scalar ratio (no running)</td>
<td>$r_T &lt; 0.24$ (95%CL)</td>
</tr>
<tr>
<td>Local non-Gaussianity</td>
<td>$-10 &lt; f_{NL} &lt; 74$ (95%CL)</td>
</tr>
</tbody>
</table>

Table 3.1: Observational quantities relevant for this thesis.
Chapter 4

Large-scale perturbations from the waterfall field in hybrid inflation

Inflation is our most successful theory for explaining the initial conditions required for the hot Big Bang cosmology. In particular, primordial density perturbations can be produced from initial quantum fluctuations that are stretched by the accelerated expansion up to super-Hubble scales to become the large-scale structure of the Universe today. This mechanism can give rise to an almost scale-invariant power spectrum, as observed in the cosmic microwave background [15].

Hybrid inflation, driven by the energy density of a false vacuum state which is destabilised when a slow-rolling field reaches a critical value, was originally proposed by Linde [45] and subsequently analysed by Linde [46] and many others [47, 48]. It has proved to be particularly successful for realising inflation in supersymmetric models of particle physics [48, 49, 50, 51]. In hybrid models there is another field, usually called the waterfall field, which is trapped in a false vacuum state until the instability is triggered. Nonetheless the vacuum fluctuations of this field are usually neglected since the field is massive when scales corresponding to large-scale structure of our observable universe leave the horizon. Quantum fluctuations of a massive scalar field, with mass much larger than the Hubble scale, remain over-damped even on super-Hubble scales, corresponding to decaying, oscillating mode functions and thus a steep blue power spectrum.

However, it was suggested [52, 53] that non-adiabatic large-scale perturbations in the waterfall field could play an important role, in particular leading to non-Gaussian curvature perturbations [52], since these long wavelength modes experience the most rapid growth due to the tachyonic instability in the waterfall field at the end of inflation. It is known that a slow transition during hybrid inflation (allowing inflation to continue for some period after the tachyonic instability), could lead to large curvature perturbations on scales which leave the horizon around the time of the instability [54, 55]. But it is
difficult to model the large-scale primordial curvature perturbations through the phase transition where perturbations about the classical background necessarily become large.

In this chapter we will start by revisiting the hybrid inflation model. It follows a review on perturbations in the inflaton and the waterfall field. Then we will reconsider the issue of modelling the primordial curvature perturbation on large scales and its evolution through the tachyonic instability, using the $\delta N$-formalism [34, 35, 37, 56]. We stress the essential role of small, Hubble-scale field perturbations at the end of inflation in determining the local integrated expansion $N = \int H dt$ (2.66), in parts of the universe with different values of the waterfall field averaged on large, super-Hubble scales. We show that the variance of the waterfall field becomes dominated by Hubble-scale perturbations when the tachyonic instability begins, rapidly leading to the end of inflation. The duration of inflation is shown to be independent of the large-scale field at first-order, simply due to the symmetry of the potential. The curvature perturbation due to long-wavelength modes of the waterfall field are shown to be suppressed due to the steep blue spectrum of the waterfall field fluctuations, similar to the case of false vacuum inflation supported by thermal corrections [57]. We will finish this chapter by studying the parameter dependence of the waterfall field power spectrum.

### 4.1 Hybrid Inflation model

The original hybrid inflation model [45, 46, 48] is described by a slowly rolling inflaton field, $\phi$, and the waterfall field, $\chi$, with a potential energy

$$V(\phi, \chi) = \left(M^2 - \frac{\sqrt{\lambda}}{2} \chi^2\right)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{2} \gamma \phi^2 \chi^2 .$$

(4.1)

The second term in the potential is the mass term for the inflaton field. When $\chi = 0$ the potential simplifies to chaotic inflation potential with false vacuum energy $V = M^4$. The last term is the coupling between the inflaton field and the waterfall field and it plays the role of terminating the period of inflation by triggering the tachyonic transition. The first term in the potential is a Mexican hat potential for the waterfall field with the false vacuum at $\chi = 0$ and true vacuum at $\chi^2 = 2M^2/\sqrt{\lambda}$ when $\phi = 0$.

The effective mass of the waterfall field in the false vacuum state is

$$m^2_{\chi}(\phi) = \gamma \left(\phi^2 - \phi_c^2\right) .$$

(4.2)

where we define $\phi_c^2 \equiv 2\lambda^{1/2}M^2/\gamma$. Thus the false vacuum is stabilised for $\phi^2 > \phi_c^2$, while for $\phi^2 < \phi_c^2$ there is a tachyonic instability. Note that during the period of inflation we require the inflation field to be effectively massless, i.e., $\eta_0 = m^2 m_{\text{Pl}}^2 / M^4 \ll 1$.
Note that the simple potential (4.1) with a real scalar field, $\chi$, has two discrete minima at $\chi = \pm 2M/\sqrt{\lambda}$. Thus regions which settle into different true vacuum states are separated by domain walls at late times. However vacuum states with higher-dimensional vacuum manifolds may have cosmic strings, monopoles or no topologically stable defects. We will neglect the formation of cosmic defects while noting their presence could have important cosmological consequences in particular hybrid models [48, 55].

**Background Evolution of the fields**

In a spatially-flat FRW cosmology the evolution equations for the background fields are (Eq. 2.23)

\begin{align*}
\ddot{\phi} + 3H\dot{\phi} &= -(m^2 + \gamma \chi^2)\phi, \quad (4.3) \\
\ddot{\chi} + 3H\dot{\chi} &= (2\sqrt{\lambda}M^2 - \gamma \phi^2 - \lambda \chi^2)\chi, \quad (4.4)
\end{align*}

where the Friedmann equation (Eq. 2.3) gives the Hubble rate

\[ H^2 = \frac{1}{3m^2_{Pl}} \left( V(\phi, \chi) + \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \dot{\chi}^2 \right). \quad (4.5) \]

Initially we assume $\phi > \phi_c$ so that the $\chi$ field is held in the false vacuum and we have the background solution $\langle \chi \rangle = 0$. For simplicity we will work in the vacuum-dominated regime [48] where we can neglect the energy density of the $\phi$ field upon the Hubble expansion, $H \approx$ constant,

\[ a = H^{-1}_c \exp \left[ H_c (t - t_c) \right] \text{ where } H^2_c = \frac{M^4}{3m^2_{Pl}}. \quad (4.6) \]

To first order $t_c$ is the transition time. Then the early-time solution ($t < t_c$) for Eq. (4.3) yields

\[ \phi = \phi_c \exp \left[ -rH_c (t - t_c) \right] \quad (4.7) \]

with

\[ r = \frac{3}{2} - \sqrt{\frac{9}{4} - \frac{m^2}{H^2_c}}. \quad (4.8) \]

The $\phi$ field slow-rolls for $m \ll H$ until the critical point, $t = t_c$, when the tachyonic instability is triggered. In our numerical examples we choose $r = 0.1$. One can verify that this is consistent with vacuum-dominated regime (4.6) so long as $\gamma$ is not extremely small [48].
Until the tachyonic instability is attained one considers $\chi = 0$. This is no longer the case after the transition. For $\phi < \phi_c$ the term $2\sqrt{\lambda}M^2$ in Eq. (4.4) is dominant, then we can simplify the evolution equation of $\chi$ to

$$\ddot{\chi} + 3H\dot{\chi} = 2\sqrt{\lambda}M^2\chi.$$ \hfill{(4.9)}

The growing mode solution is [55]

$$\chi = \chi_e \exp \left[ sH_c (t - t_e) \right], \hfill{(4.10)}$$

with

$$s = -\frac{3}{2} + \sqrt{\frac{9}{4} + \frac{2\sqrt{\lambda}M^2}{H_c}}. \hfill{(4.11)}$$

The subscript $e$ indicates the end of inflation. Assuming that the period of inflation is terminated when $\dot{\chi}^2 \sim V$ then $\chi_e \approx 2m_{Pl}^2/s(s + 1)$ [55]. Note that the tachyonic instability may add a few e-folds of inflation. In the following we will evaluate numerically the number of e-folds.

### 4.2 Structure from quantum field fluctuations

Quantum fluctuations in the slow-rolling $\phi$ field (coupled to scalar metric perturbations) lead to an almost scale-invariant spectrum of adiabatic curvature perturbations on super-Hubble scales during inflation [58]. These correspond to local perturbations in the evolution along the classical $\chi = 0$ background solution. They give rise to effectively Gaussian primordial curvature perturbations usually considered in hybrid inflation models [48]. By contrast quantum fluctuations in the waterfall field correspond to isocurvature field perturbations during inflation decoupled from both $\phi$-field and metric perturbations at linear order [26]. In this section we will review perturbations in the inflaton field and study isocurvature perturbations from the waterfall field. We will use the $\delta N$ formalism to study numerically the evolution of isocurvature perturbations during the tachyonic transition.

The relevant quantum fluctuations in both fields are generated before the tachyonic transition, i.e., when we can take $\langle \chi \rangle = 0$. 

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4.2.1 Inflaton perturbations

The evolution equation for the linear inflaton perturbation, with comoving scale $k$, is

$$\ddot{\delta \phi}_k + 3H_e \dot{\delta \phi}_k + \left( \frac{k^2}{a^2} + \alpha H_e^2 \right) \delta \phi_k = 0,$$  \hspace{1cm} (4.12)

where $\alpha$ is the ratio between the inflaton mass and the scale of inflation,

$$\alpha \equiv \frac{m^2}{H_e^2}.$$

Note that Eq. (4.12) is similar to Eq. (3.26) where we just consider the effect of the potential in the field perturbations. The waterfall field perturbations do not affect this calculation since we take the classical value of $\chi$ to be zero. We know already that $\alpha \ll 1$ for inflation to occur. It is more convenient to use the canonically normalised variable $v \equiv a \delta \phi$. Substituting it into Eq. (4.12) we obtain a wave equation

$$v''_k + \left( k^2 - \frac{2 - \alpha}{\eta^2} \right) v_k = 0,$$  \hspace{1cm} (4.14)

where primes denote derivatives with respect to conformal time, $\eta = \int dt/a$ (2.45). Note that in de Sitter (4.6) the conformal time is given by

$$\eta = - \frac{1}{aH_e} = - \exp \left[ -H_e (t - t_c) \right].$$  \hspace{1cm} (4.15)

We have chosen the normalisation for the scale factor in Eq. (4.6) such that $aH = 1$ and $\eta = -1$ when $t = t_c$, i.e., Fourier modes with $k = 1$ leave the Hubble horizon when $\phi = \phi_c$ and the tachyonic instability is triggered.

The mass of the $\phi$-field is constant therefore Eq. (4.14) has an exact solution [55],

$$v(\eta)_k = \frac{\sqrt{\pi}}{2\sqrt{k}} e^{i(1-r)\pi/2} (-k\eta)^{1/2} H^{(1)}_{3/2-r} (-k\eta).$$  \hspace{1cm} (4.16)

The function $H^{(1)}_{3/2-r}$ is the first Hankel function of index $3/2 - r$. This solution has the correct flat space early-time limit $-k\eta \to -\infty$, the plane wave $e^{i k \eta / \sqrt{2k}}$. The late time solution, $-k\eta \to 0$

$$v(\eta)_k = \frac{C(r)}{\sqrt{2k}} e^{i(1-r)\pi/2} (-k\eta)^{-r-1},$$  \hspace{1cm} (4.17)

where

$$C(r) = 2^{-r} \frac{\Gamma(3/2 - r)}{\Gamma(3/2)}.$$

(4.18)
The power spectrum for the canonically normalised variable \( v_k \) is given by
\[
P_v = \frac{k^3}{2\pi^2} \left| v_k \right|^2. \tag{4.19}
\]

Then, the amplitude of the power spectrum at horizon crossing is given by
\[
P_{\delta\phi} = C(r)^2 \left( \frac{H_c}{2\pi} \right)^2, \tag{4.20}
\]
where to linear order \( C(0) = 1 \). Note that this calculation does not take into account the tachyonic instability. This is a good approximation since the large-scale modes, \( k \ll 1 \), left the Hubble-horizon long before the instability has happened at \( \eta = -1 \).

### 4.2.2 Waterfall-field perturbations

Linear \( \chi \)-field perturbations, with comoving wavenumber \( k \), obey the evolution equation (Eq. (3.26) neglecting the effect of metric perturbations)
\[
\ddot{\delta\chi}_k + 3H_c\dot{\delta\chi}_k + \left( \frac{k^2}{a^2} - \beta H_c^2 + \gamma \phi^2 \right) \delta\chi_k = 0. \tag{4.21}
\]

where the bare tachyonic mass of the waterfall field relative to the inflationary Hubble scale is given by
\[
\beta = 2\sqrt{\lambda M^2 H_c^2}. \tag{4.22}
\]
We assume \( \beta \gg 1 \) so that the time-scale associated with the tachyonic instability is much less than a Hubble time and inflation ends soon after the instability begins. If \( \beta \) is of order unity or less then there is the possibility of an extended period of slow-roll inflation and associated large metric perturbations on scales associated with the transition [54, 55]. In our numerical solutions we choose \( \beta = 100 \).

If we substitute the solution for \( \phi \) (Eq. (4.7)), and rewrite the evolution equation (4.21) in term of the canonically quantised variable, \( u = a\delta\chi \), then we obtain a wave equation with time-dependent mass
\[
u_k'' + \left[ k^2 + \mu^2(\eta) \right] u_k = 0 \quad \text{where} \quad \mu^2(\eta) \equiv \frac{\beta (|\eta|^{2r} - 1) - 2}{\eta^2}. \tag{4.23}
\]

For \( k = 0 \) we find an analytic solution [55]
\[
u_0 = (-\eta)^{1/2} \left[ C J_{-\nu} \left( \frac{\sqrt{\beta}(-\eta)^r}{r} \right) + D J_{\nu} \left( \frac{\sqrt{\beta}(-\eta)^r}{r} \right) \right], \tag{4.24}
\]
where $r$ is defined in Eq. (4.8) and

$$\nu = \frac{\sqrt{\frac{9}{4} + \beta}}{r}. \quad (4.25)$$

In Figure 4.1 we show a typical solution for $u_0(\eta)$.

For all modes with a finite $k$ the time-dependent mass-squared in Eq. (4.23) is positive, $\mu^2(\eta) \to \beta|\eta|^{-2(1-r)}$, but negligible at early times where $k\eta \to -\infty$. We thus normalise the mode functions at early time to the quantum vacuum for a free field, $u_k \propto e^{-ik\eta/\sqrt{2k}}$. The time-dependent mass-squared reaches a maximum

$$\mu^2_{\text{max}} = \frac{(\beta + 2)r}{1 - r} \left( \frac{\beta (1 - r)}{\beta + 2} \right)^{1/r} \sim \beta r \quad \text{when } \eta_{\text{max}} = \left( \frac{\beta + 2}{\beta (1 - r)} \right)^{1/2r} \sim -1,$$

and then tends to minus infinity at late times $\mu^2(\eta) \to -\beta(2)/\eta^2$ as $\eta \to 0$. At late times we have the asymptotic solution

$$u_k \propto (-\eta)^{-s - 1} \quad (4.27)$$

corresponding to the tachyonic growing mode solution $\chi = \chi_c \exp [sH(t - t_c)]$ with

$$s = \sqrt{\frac{9}{4} + \beta} - \frac{3}{2}. \quad (4.28)$$

The amplitudes of the mode functions, $|u_k^2(\eta)|$ for different values of $k$ are shown in Figure 4.2.

The power spectrum is defined as

$$P_u = \frac{k^3}{2\pi^2} |u_k^2|. \quad (4.29)$$

Modes with $k \ll 1$ leave the Hubble-horizon well before the instability, when $|\eta| \gg 1$. At these early times the time-dependent mass term in Eq. (4.23) is positive with $\mu^2(\eta) \simeq \beta|\eta|^{-2(1-r)} \gg k^2$, so that the mode function is suppressed on super-Hubble scales and we find $|u_k^2| \propto |\eta|^{2(1-r)} \propto a^{-2(1-r)}$. Normalising the initial state to the quantum vacuum for a free field then leads to a steep blue spectrum on super-Hubble scales with $P_\chi \propto k^3$, which is even steeper than the quantum vacuum for a free field on the smallest scales, $P_\chi \propto k^2$. There are no classical perturbations on super-Hubble scales before the transition.

Modes with $k \gg 1$ remain within the Hubble horizon until after the tachyonic instability is triggered. Even after $\phi = \phi_c$, gradient terms stabilise short-wavelength modes
Figure 4.1: The amplitude of the infinite wavelength solution, $u_0$, given in Eq. (4.24), plotted as a function of conformal time, $\eta$, for $r = 0.1$ and $\beta = 100$. 
Figure 4.2: The amplitude of the mode functions, $|u_k^2|$, normalised with respect to their final value, as a function of conformal time, $\eta$, for $r = 0.1$ and $\beta = 100$. $k = 1$ corresponds to modes leaving the Hubble-horizon when $\phi = \phi_c$. The black dashed line indicates the expected late time behaviour, Eq. (4.27), as $\phi \to 0$. 
with \( k^2 + \mu^2(\eta) > 0 \) and longer-wavelength modes begin to grow first. As a result we find that the power spectrum for the waterfall field begins to peak on scales of order the Hubble horizon at the transition, \( k \sim 1 \). This is clearly shown in Figure 4.3, where we fix \( \beta = 100 \) and \( r = 0.1 \).

4.2.3 Applications of \( \delta N \) Formalism

In order to calculate the primordial curvature perturbation, \( \zeta \), due to fluctuations in the waterfall field we can calculate the perturbed expansion, \( \zeta \equiv \delta N \equiv \delta(\ln a) \) \([34, 35, 37, 56, 59]\), from an initial spatially-flat hyper-surface to a final uniform-density hyper-surface \([29]\). For adiabatic perturbations in the \( \phi \)-field during the slow-roll phase this yields the standard first-order result (Eq. (3.15))

\[
\zeta_\phi = -\frac{H\delta\phi}{\dot{\phi}},
\]

where \( H\delta\phi/\dot{\phi} \) can be evaluated on spatially-flat hyper-surfaces any time after Hubble exit for each Fourier mode. Perturbations in the waterfall field are non-adiabatic field perturbations and hence we must follow their effect on the expansion through the tachyonic transition from false to true vacuum at the end of inflation. At first order we can simply add these independent contributions to the primordial curvature perturbation from both fields

\[
\zeta = \zeta_\phi + \zeta_\chi.
\]

The usual “separate universe” approach \([37]\) is to calculate the classical expansion for different initial values of the field assuming a locally homogeneous and isotropic (i.e., FRW) cosmology, assuming the long-wavelength behaviour is independent of much shorter wavelength modes. However in the hybrid inflation model the classical solution assuming homogeneous fields is liable to give an incorrect estimate of the duration of the transition. Indeed the classical background trajectory assumed in section 4.2.2 was \( \chi = 0 \) for which, classically, inflation never ends and \( N = \ln(a) \to \infty \) as \( t \to \infty \) and \( \phi \to 0 \) in Eqs. (4.6) and (4.7).

The dashed line in Figure 4.4 shows the classical expansion, \( N_f(\chi_*) = N(\chi_* \to \chi_f) \) from an initial value of the waterfall field close to the transition, \( \chi_* \), to a given final value, \( \chi_f \), for our chosen values of \( r \) and \( \beta \) using the linearised equations of motion. In particular this shows the singular behaviour of the classical solution for \( N_f(\chi_*) \) about \( \chi_* = 0 \).
Figure 4.3: The power spectrum of $u_k$ as a function of the comoving wavenumber, $k$, relative to the Hubble-horizon at the transition. We show the power spectrum at four different times to show its time evolution. On large scales, $k \ll 1$ the power is suppressed before the transition, $\eta < \eta_*$, but then grows rapidly due to the tachyonic instability, for $\eta > \eta_*$, where $\eta_* \approx -1$. 
Figure 4.4: The integrated expansion from an initial time $\eta_* = -1$ to a final density corresponding to $\chi_f = 10^5 H_c$ as a function of the average value of the waterfall field, $\chi_L$, averaged on some super-Hubble scale, $L \gg |\eta_*|$. The upper dashed line shows the classical solution to the linearised equation (4.21) in the long-wavelength limit, $k = 0$, i.e., assuming a homogeneous field $\chi_* = \chi_L$. The lower dashed line shows the classical solution to the non-linear FRW equations (4.3-4.5). The solid lines shows the average expansion in each case integrated over a Gaussian distribution for the Hubble-scale field, $\chi_*$, with average value $\chi_L$ and variance $\sigma_*^2 \approx 0.02 H^2$. Both lines are obtained using $r = 0.1$ and $\beta = 100$. 

\[
\begin{array}{c}
\text{Figure 4.4: The integrated expansion from an initial time } \eta_* = -1 \text{ to a final density corresponding to } \chi_f = 10^5 H_c \text{ as a function of the average value of the waterfall field, } \chi_L, \text{ averaged on some super-Hubble scale, } L \gg |\eta_*|. \text{ The upper dashed line shows the classical solution to the linearised equation (4.21) in the long-wavelength limit, } k = 0, \text{ i.e., assuming a homogeneous field } \chi_* = \chi_L. \text{ The lower dashed line shows the classical solution to the non-linear FRW equations (4.3-4.5). The solid lines shows the average expansion in each case integrated over a Gaussian distribution for the Hubble-scale field, } \chi_*, \text{ with average value } \chi_L \text{ and variance } \sigma_*^2 \approx 0.02 H^2. \text{ Both lines are obtained using } r = 0.1 \text{ and } \beta = 100. \\
\end{array}
\]
Small scale quantum fluctuations play an essential part in the transition as they are amplified by the tachyonic transition. The variance of the \( \chi \) field averaged on some scale, \( L = 2\pi/k_L > (aH)^{-1} \), is given by integrating over all longer wavelengths

\[
\langle \delta \chi^2 \rangle_L = \frac{1}{a^2} \int_0^{k_L} P_u(k) d\ln k .
\]  

(4.32)

On super-Hubble scales we have a steep blue spectrum \( P_u(k) \propto k^3 \) and hence \( \langle \delta \chi^2 \rangle_L = P_u(k_L)/3a^2 \). We see clearly from figure 4.3 that the variance peaks on scales of order the Hubble scale at the transition. Thus we need to include these Hubble-scale modes in our estimate of the background expansion, \( \langle N \rangle_L \), and the perturbation, \( \zeta = \delta \langle N \rangle_L \), due to longer wavelength modes.

We can always split a Gaussian random field, such as the initial \( \chi \)-field fluctuations at the transition, into a long-wavelength part and a short-wavelength part:

\[
\chi^*(x) = \chi_L(x) + \chi_S(x) = \int_0^{k_{\text{split}}} d^3k \chi_k e^{-ikx} + \int_{k_{\text{split}}}^{k_{\text{UV}}} d^3k \chi_k e^{-ikx} .
\]  

(4.33)

In this case we will identify \( k_{\text{split}} \) with the peak of the spectrum, around Hubble scale at the transition, \( k_* \). In a Gaussian field the long and short wavelengths modes are uncorrelated. Thus we may think of the Hubble-scale fluctuations as a statistically homogeneous distribution regularising the divergence of the classical solution for the long wavelength field \( N(\chi_L) \), as illustrated in Figure 4.4.

A precise calculation of the expansion through the transition would require a full non-linear numerical simulation of the inhomogeneous quantum fields on a lattice. We will instead make two important simplifications which will enable us to make a simple numerical estimate and which we expect to capture the essential physics. Firstly, we will treat the fluctuations as classical on Hubble scales, shortly after the transition. This becomes valid due to the tachyonic growth of the mode function, but is only marginal at the transition. And secondly, we will use the separate universe assumption, neglecting spatial gradients on these scales, which again is only marginal at the transition. Modes leaving the Hubble horizon at the transition will continue to grow relative to the Hubble scale as inflation continues, but eventually will return within the Hubble scale as inflation ends. The separate universe assumption on these scales is only marginally valid and for a limited period, but this is precisely the period we wish to study.

Using these assumptions we can then estimate the expansion, \( N_f(\chi_s) \), in a Hubble-scale patch from an initial value, \( \chi_s \), around the time of the transition. However, in

\footnote{Note that we introduce an ultra-violet cut-off, \( k_{\text{UV}} \), at small scales to avoid the UV-divergence of the fluctuations as \( k \to \infty \). Modes with \( k^2/a^2 \gg 2\sqrt{\Lambda M^2} \) do not experience any tachyonic growth and remain in the quantum vacuum state, hence we do not include them in \( \chi_S \).}
a much larger super-Hubble scale region with an average value for the waterfall field, $\chi_L$, we sample many Hubble-scale patches and integrating over a large volume can be replaced by an integral over a Gaussian distribution for the initial values, $\chi_*$, given a local background value, $\chi_L$. This is an extension of the usual separate universe picture on large scales which incorporates small scale variance, $\sigma^2_* = \langle \chi_S^2 \rangle_*$, uncorrelated with the long-wavelength field.

Thus we obtain

$$\langle N_f \rangle_L = \int_{-\infty}^{\infty} d\chi_* N_f(\chi_*) P(\chi_*|\sigma_*, \chi_L), \quad (4.34)$$

where $P(\chi_*|\sigma_*, \chi_L)$ is the Gaussian probability distribution for the local value of the waterfall field in a Hubble-size region at time $t_*$ soon after $\phi = \phi_c$,

$$P(\chi_*|\sigma_*, \chi_L) = \frac{1}{\sqrt{2\pi\sigma_*}} \exp \left( -\frac{(\chi_* - \chi_L)^2}{2\sigma^2_*} \right), \quad (4.35)$$
given the average value on the larger scale, $\chi_L$, and the variance of the field on Hubble scales, $\sigma^2_* \simeq P_u(k_*)/a^2_*$, due to smaller scale fluctuations, $\chi_S(x)$.

The expression (4.34) for $\langle N_f \rangle_L$ gives us the expansion as a smooth function of the large-scale field $\chi_L$, and thus the curvature perturbation can be expanded about the background solution $\chi_L = 0$ as a function of the perturbation $\delta \chi_L$

$$\zeta_\chi = \frac{d\langle N_f \rangle_L}{d\chi_L} \bigg|_{\chi_L=0} \delta \chi_L + \frac{1}{2} \frac{d^2\langle N_f \rangle_L}{d\chi^2_L} \bigg|_{\chi_L=0} \delta \chi_L^2 + \ldots, \quad (4.36)$$

where, given the Gaussian probability distribution Eq. (4.35), we have

$$\frac{d\langle N_f \rangle_L}{d\chi_L} \bigg|_{\chi_L=0} = \frac{1}{\sqrt{2\pi\sigma_*}} \int_{-\infty}^{\infty} d\chi_* N_f(\chi_*) \frac{\chi_*}{\sigma_*^2} \exp \left( -\frac{\chi_*^2}{2\sigma^2_*} \right), \quad (4.37)$$

$$\frac{d^2\langle N_f \rangle_L}{d\chi^2_L} \bigg|_{\chi_L=0} = \frac{1}{\sqrt{2\pi\sigma_*}} \int_{-\infty}^{\infty} d\chi_* N_f(\chi_*) \frac{\chi_*^2 - \sigma_*^2}{\sigma^4_*} \exp \left( -\frac{\chi_*^2}{2\sigma^2_*} \right). \quad (4.38)$$

From the symmetry of the system (4.1) under $\chi \to -\chi$ we have $N_f(\chi_*) = N_f(-\chi_*)$ and hence we immediately see from Eq. (4.37) that

$$\frac{d\langle N_f \rangle_L}{d\chi_L} \bigg|_{\chi_L=0} = 0. \quad (4.39)$$

Thus there is no linear contribution to the primordial curvature perturbation from large-scale fluctuations in the waterfall field. The leading order contribution to the primordial
curvature perturbation from fluctuations in the waterfall field on large scales will be second order, and hence non-Gaussian \[52, 60\].

**Linearised solution**

Firstly, we obtain numerical solutions for the classical evolution, \(N_f(\chi_*)\), from an initial value of the waterfall field, \(\chi_*\) when \(\phi = \phi_c\), to a final value \(\chi_f \gg \chi_*\) using the linearised equation of motion (4.21) for \(\delta \chi_k = \chi_*\), where we work in the long-wavelength limit and set \(k = 0\). As expected, the classical evolution \(N_f(\chi_L)\), shown by the dashed line in figure 4.4, is singular in the limit \(\chi_L \to 0\). By contrast, the solid line shows how the classical singularity at \(\chi_L = 0\) is regularised by the quantum dispersion of the local value, \(\chi_*\), once we include smaller Hubble-scale modes, yielding a finite value for \(\langle N_f \rangle_L\) as \(\chi_L \to 0\).

We can obtain a rough analytic estimate for \(N_f(\chi_L)\) by noting that at sufficiently late times after the transition, \(\phi \to 0\) and we expect \(\chi\) to have the late-time behaviour, \(\chi \propto (-\eta)^{-s}\) given in Eq. (4.27). Therefore, we expect \(N_f\) and \(\chi_f/\chi_*\) to be approximately given by

\[
N_f = \frac{1}{s} \ln \frac{\chi_f}{\chi_*} + \text{const.} \quad (4.40)
\]

Note that this assumes, \(\phi \ll \phi_c\), by which point the linear approximation for \(\chi\) is expected to have broken down. Nonetheless, substituting Eq. (4.40) into Eq. (4.34) allows us to give an analytic estimate

\[
\langle N_f \rangle_L = N_* + \frac{1}{2s} \ln \left(\frac{2\sigma_*^2}{\chi_f^2}\right) + \ln \frac{2 + \gamma_{\text{EM}}/2}{\sqrt{2s}} - \frac{1}{2\sqrt{2s}} \frac{\delta \chi_L^2}{\sigma_*^2} + O \left(\frac{\delta \chi_L^4}{\sigma_*^4}\right) \quad (4.41)
\]

where \(\gamma_{\text{EM}} = 0.57721\) is the Euler-Mascheroni constant. The term \(N_*\) comes from integrating the constant term in Eq. (4.40). The other terms come from the integration of the first term in Eq. (4.40). To see this we substitute Eq. (4.40) and the Gaussian probability distribution function Eq. (4.35) into Eq. (4.34),

\[
\langle N_f \rangle_L = \int d\chi_* \frac{1}{2s} \ln \left(\frac{\chi_f^2}{\chi_*^2}\right) \frac{1}{\sqrt{2\pi\sigma_*}} e^{-\frac{(\chi_* - \chi_f)^2}{2\sigma_*^2}} \quad (4.42)
\]

\[
= -\frac{1}{2s} \int d\chi_* \left( \ln \left(\frac{\chi_*^2}{2\sigma_*^2}\right) - \ln \left(\frac{2\sigma_*^2}{\chi_f^2}\right) \right) \frac{1}{\sqrt{2\pi\sigma_*}} e^{-\frac{(\chi_* - \chi_f)^2}{2\sigma_*^2}}. \quad (4.43)
\]

We rewrote the natural logarithm so that the second term in the integral does not depend on the integrand and hence can be integrate over to give the second term in Eq. (4.41). To estimate the integral of the first term we need to expand the exponential of the Gaussian distribution in powers of \(\chi_L/\sigma_*\). Note that the integration should be performed from \(-\chi_f\)
to $\chi_f$. But due to the exponential suppression we can take the integral from $-\infty$ to $+\infty$, and therefore the odd integrands vanish. The expansion reads

$$\frac{1}{\sqrt{2\pi s}} \left(1 - \frac{\chi_L^2}{2\sigma_s^2}\right) \int_{0}^{\infty} \frac{d\chi_s}{\sigma_s} \ln \left(\frac{\chi_s^2}{2\sigma_s^2}\right) e^{-\frac{\chi_s^2}{2\sigma_s^2}} \left(1 + \frac{1}{2} \frac{\chi_L^2 \chi_s^2}{\sigma_s^2 \sigma_s^2} + \mathcal{O}\left(\frac{\chi_L^4}{\sigma_s^4}\right)\right).$$

To make the calculation more easy to visualise let us do the substitution $y = \chi_s / \sqrt{2\sigma_s}$. Note that we can write the logarithm as the following limit:

$$\ln y^2 = \lim_{\alpha \to 0} \frac{y^{2\alpha} - 1}{\alpha}. \quad (4.45)$$

Using such expression one can check the relation

$$\int_{0}^{\infty} \ln y^2 e^{-y^2} dy = \lim_{\alpha \to 0} \frac{1}{2\alpha} \int_{0}^{\infty} dx (x^{\alpha - 1/2} - x^{-1/2}) e^{-x}$$

$$= \lim_{\alpha \to 0} \frac{1}{2\alpha} \left[\Gamma\left(\frac{1}{2} + \alpha\right) - \Gamma\left(\frac{1}{2}\right)\right] \quad (4.47)$$

$$= \frac{1}{2} \Gamma'\left(\frac{1}{2}\right) \quad (4.48)$$

Similarly one can check that

$$\int_{0}^{\infty} dy \ln(y^2) y^2 e^{-y^2} = \lim_{\alpha \to 0} \int_{0}^{\infty} dy \frac{y^{2\alpha} - 1}{\alpha} y^2 e^{-y^2} \quad (4.49)$$

$$= \lim_{\alpha \to 0} \int_{0}^{\infty} dy \frac{y^{2\alpha + 2} - y^2}{\alpha} e^{-y^2} \quad (4.50)$$

$$= \lim_{\alpha \to 0} \frac{\Gamma(3/2 + \alpha) - \Gamma(3/2)}{2\alpha} \quad (4.51)$$

$$= \frac{1}{2} \Gamma'\left(\frac{3}{2}\right) \quad (4.52)$$

The derivative of the Gamma function can be expressed in terms of the Gamma function and the Digamma function $\Psi(x)$ using the formula

$$\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}. \quad (4.53)$$

Using the known results $\Gamma(1/2) = \sqrt{\pi}/2$, $\Gamma(3/2) = \sqrt{\pi}/2$, $\Psi(1/2) = -2 \ln 2 - \gamma$ and $\Psi(3/2) = \Psi(1/2) + 2$ one obtains the last two terms in Eq. (4.41).
For our chosen parameters, \( r = 0.1 \) and \( \beta = 100 \), we have \( \sigma_*^2 \simeq P_u/a_*^2 \simeq (H_c/2\pi)^2 \), and we set \( \chi_f = 10^5 H_c \). The analytic approximation (4.41) then yields

\[
\langle N_f \rangle_L \simeq \text{const.} - 1.62 \frac{\delta \chi_L^2}{H_c^2} + \mathcal{O} \left( \frac{\delta \chi_L^4}{H_c^4} \right).
\]  
(4.54)

Performing the integration numerically in the linearised approximation one obtains

\[
\langle N_f \rangle_L \simeq \text{const.} - 4.62 \frac{\delta \chi_L^2}{H_c^2} + \mathcal{O} \left( \frac{\delta \chi_L^4}{H_c^4} \right).
\]  
(4.55)

So the analytic approximation (4.40), gives a result roughly a factor of 3 away from the numerical result in the linear analysis.

**Non-linear FRW solution**

As we have neglected gradient terms by setting \( k = 0 \) in the linear equation for \( \chi(N) \), we can in fact solve the full non-linear equations in the homogeneous limit, treating the local Hubble-scale patches as separate universes [37] obeying the FRW equations (4.3–4.5) in order to determine the coupled evolution of \( \chi \) and \( \phi \), and hence \( N \), starting from \( \chi = \chi_* \) when \( \phi = \phi_c \). In this case we must also specify the energy scale of inflation. In Figure 4.4 we show the classical solution \( N_f(\chi_L) \) as well as the averaged \( \langle N_f \rangle_L \) including the small scale dispersion, where we have set \( M \simeq 10^{15} \) GeV and hence \( H_c \simeq 10^{11} \) GeV. The results are qualitatively similar to those obtained from the linear approximation. Non-linearities, in particular, the \( \chi \)-dependent effective mass for the \( \phi \) field, leads to a slightly more rapid transition, especially for larger initial values of \( \chi_L \), and we find

\[
\langle N_f \rangle_L \simeq \text{const.} - 6.59 \frac{\delta \chi_L^2}{H_c^2} + \mathcal{O} \left( \frac{\delta \chi_L^4}{H_c^4} \right).
\]  
(4.56)

Note that in our non-linear, separate universe solutions we have calculated \( N_f \) up to a final fixed density, and have verified that in practice this coincides with a final value for the waterfall field \( \chi_f \simeq 10^5 H_c \). For our chosen parameter values the universe is still inflating at this final time, with slow-roll parameter \( \epsilon \equiv -\dot{H}/H^2 \simeq 0.001 \). Thus our initial comoving Hubble-scale at \( \eta_* \) is still larger than the final Hubble scale. But the slow-roll parameter is growing quickly and inflation ends soon afterwards \( (N_{\text{end}} \simeq N_f + 0.4) \).

The curvature perturbation due to large-scale perturbations in the waterfall field is second-order, and suppressed relative to the Hubble-scale variance of the field at the
transition. To see this let us express $\zeta$ as

$$\zeta \simeq \text{const.} + \mathcal{O}(1) \frac{\delta \chi^2}{H_c^4}. \quad (4.57)$$

Then the 2-point function for $\zeta$ is

$$\langle \zeta_k \zeta_{k'} \rangle = (2\pi)^3 \frac{2\pi^2}{k^3} \mathcal{P}_{\zeta \chi}(k) \delta^3(k + k') \simeq \frac{1}{H_c^4} \langle (\delta \chi^2_L)_k (\delta \chi^2_L)_{k'} \rangle, \quad (4.58)$$

where

$$\langle (\delta \chi^2_L)_k \rangle = \int \frac{d^3k''}{(2\pi)^3} \delta \chi_{Lk-k''} \delta \chi_{Lk''}. \quad (4.59)$$

The correlation function present in Eq. (4.58) can be written as a double mode integral of a 4-point function of $\delta \chi$, i.e.,

$$\langle (\delta \chi^2_L)_k (\delta \chi^2_L)_{k'} \rangle = \int \frac{d^3k''}{(2\pi)^2} \frac{d^3k'''}{(2\pi)^2} \langle \delta \chi_{Lk-k''} \delta \chi_{Lk''} \delta \chi_{Lk'-k'''} \delta \chi_{Lk'''} \rangle. \quad (4.60)$$

The field $\delta \chi$ is Gaussian, therefore we can use Wick’s Theorem to expand the 4-point function in terms of 2-point functions, i.e.,

$$\langle \delta \chi_{Lk-k''} \delta \chi_{Lk''} \delta \chi_{Lk'-k'''} \delta \chi_{Lk'''} \rangle = \langle \delta \chi_{Lk-k''} \delta \chi_{Lk''} \rangle \langle \delta \chi_{Lk'-k'''} \delta \chi_{Lk'''} \rangle + \text{Permutations},$$

$$= 2 (2\pi)^6 \frac{2\pi^2}{k_{L3}} \mathcal{P}_{\delta \chi_L}(k'') \frac{2\pi^2}{k_{L3}} \mathcal{P}_{\delta \chi_L}(k''') \times \delta^3(k - k'' + k''') \delta^3(k'' + k' - k'''). \quad (4.61)$$

Substituting the previous result into Eq. (4.60) and integrating over $k'''$ one gets

$$\langle (\delta \chi^2_L)_k (\delta \chi^2_L)_{k'} \rangle = 2 \int d^3k'' \frac{2\pi^2}{k_{L3}} \mathcal{P}_{\delta \chi_L}(k'') \frac{2\pi^2}{|k'' - k|^3} \mathcal{P}_{\delta \chi_L}(|k'' - k|) \delta^3(k + k'). \quad (4.62)$$

If we identify $\mathcal{P}_{\delta \chi_L}(k)$ as

$$\mathcal{P}_{\delta \chi_L}(k) = k^3 \int d^3k' \frac{\mathcal{P}_{\delta \chi_L}(k') \mathcal{P}_{\delta \chi_L}(|k' - k'|)}{|k' - k'|^3}, \quad (4.63)$$

then, up to numerical coefficients of order 1, one has

$$\mathcal{P}_{\zeta \chi}(k) \simeq H_c^{-4} \mathcal{P}_{\delta \chi^2_L}(k). \quad (4.64)$$

Given the steep blue spectrum of the super-Hubble perturbations in the waterfall field, $\mathcal{P}_{\delta \chi_L}(k) = \mathcal{P}_u(k)/a^2 \propto k^3$ (which implies a white spectrum in the standard terminology
since $P_{\delta\chi L}(k) = 2\pi^2 P_{\delta\chi L}(k)/k^3 \propto \text{const.}$, we thus conclude that the spectrum of the resulting primordial curvature perturbation, $\zeta$, on super-Hubble scales is also blue for $k \ll k^*$:

$$P_{\zeta}(k) \sim H_c^{-4}P_{\delta\chi L}(k^*)P_{\delta\chi L}(k) \sim (k/k^*)^3.$$  \hspace{1cm} (4.65)

Assuming cosmological scales leave the Hubble-horizon around 40 e-folds before the end of inflation we have $(k_{\text{cmb}}/k^*)^3 \sim 10^{-54}$.

### 4.3 Parameter dependence

In the previous sections, we studied the evolution of waterfall field perturbation through the tachyonic instability for a fixed $r = 0.1$ and $\beta = 100$. In this section we will study the parameter dependence of the power spectrum of the waterfall field soon after the beginning of the tachyonic transition.

The waterfall field passes from massive field to a tachyonic field thought a period which it is effectively massless. In reality at $\phi = \phi_c$, the waterfall field is massless as we can see from Eq. (4.2). We know that effectively massless fields give rise to almost scale invariant power spectra during inflation. The field becomes effectively massless when the mass drops below the Hubble scale, i.e.,

$$m_{\chi}^2 = \frac{9}{4}H_c^2 = \gamma \left( \phi(t)^2 - \phi_c^2 \right).$$  \hspace{1cm} (4.66)

Using the background solution of $\phi$ (Eq. (4.7)) and the scale factor (4.6), the definition of $\beta$ (4.22) and recalling that $\phi_c^2 \equiv 2\lambda^{1/2}M^2/\gamma$ we can estimate the scale at which the power spectrum starts to flatten out. It is given by

$$k_{\text{flat}} = \left( \frac{9}{4\beta} + 1 \right)^{-\frac{1}{2\eta}}.$$  \hspace{1cm} (4.67)

This defines the transition from a $k^3$ dependence to a flat shape of the power spectrum of field perturbations.

In figure 4.5 we plot the power spectrum of $u$ normalised on scales $k = 10^{-9}$. Figure 4.5 is constructed by determining the power 2 e-folds after the tachyonic transition ($\eta = -0.1$), for a fixed $r$ ($r = 0.001$) and varying $\beta$. We notice two main features with the change of $\beta$. Firstly, the transition scale from the peak region of the power spectrum to small scale quantum fluctuation dominated power spectrum decreases (bigger $k$) as the ratio of the bare $\chi$-field mass increases with respect to the Hubble scale (higher $\beta$). Similarly, as $\beta$ increases the peak of the power becomes more sharply defined. Mathematically this corresponds to the fact that $k_{\text{flat}} \rightarrow 1$ as $\beta$ grows. Conversely, low $\beta$
Figure 4.5: In the $y$-axes is plotted the power spectrum of $u$ normalised on $k = 1 \times 10^{-9}$. On $x$-axis we plot the logarithm of cosmological scales. In this plot we determine the power for $\eta = -0.1$ and $r = 0.001$ and vary $\beta$.

Figure 4.6: In the $y$-axes is plotted the power spectrum of $u$ normalised on $k = 1 \times 10^{-9}$. On $x$-axis we plot the logarithm of cosmological scales. In this plot we determine the power for $\eta = -0.1$ and $\beta = 1000$ and vary $r$.  

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extends the region for which the waterfall field is effectively massless. One can also study the validity of Eq. (4.67) in the Fig. 4.5. For instance for $\beta = 10$, $k_{\text{flat}} \simeq 10^{-44}$. This is the reason why there is no region for $\beta = 10$ where $P \propto k^3$. On the other hand for $\beta = 10000$, $k_{\text{flat}} \simeq 0.9$. Eq. (4.67) is a good approximation for the transition scale from $k^3$ dependence of the power to the peak of the power spectrum. One should note that we require $\beta \gg 1$ to have the tachyonic instability that terminates inflation. If $\beta$ is order unity, or below, we start having a second period of inflation along the waterfall direction, or Hilltop inflation [61].

The effect of the variation of $r$ is shown in Figure 4.6. The first thing to notice is that when $r \to 0$ then $k_{\text{flat}} \to 0$. Conversely as $r$ approaches 1, $k_{\text{flat}}$ gets closer to Hubble scales (depending on $\beta$ as well). With the decrease of $r$, one observes a relative growth of the peak with respect to the small scale quantum fluctuations.

### 4.4 Summary

We have reviewed the model of hybrid inflation. We studied the background evolution of the inflaton field and of the waterfall field. We computed the power spectrum of field fluctuations for both fields. For the waterfall field we studied its field fluctuations during slow-roll inflation and through the tachyonic instability at the end of inflation. We observed an exponential growth of the waterfall field fluctuations due to the transition. We then estimated the primordial curvature perturbation produced by fluctuations in the waterfall field during hybrid inflation. We have calculated linear perturbations about the classical background trajectory, $\chi = 0$, during slow-roll inflation and then studied how these affect the primordial curvature perturbation when the waterfall field is released and inflation comes to an end. To do this, we have used an extension of the usual $\delta N$- formalism that identifies the primordial curvature perturbation with the perturbation in the local expansion on a uniform-density hyper-surface, $\zeta = \delta N$. We have considered the effect of fluctuations below the Hubble scale to regularise divergences present in the classical equations. We numerically solved the Friedmann equation and the evolution equation of waterfall field fluctuations using the separate universe picture. We concluded that the power spectrum of curvature perturbations from the waterfall field is quadratic in the power spectrum of waterfall field fluctuations and has a highly suppressed contribution. We finished this chapter by studying the parameter dependence of our analyses.

After the work described in this chapter was done several authors studied the same and similar issues in the model. In [62] the authors went through a detailed calculation of the contribution of the waterfall field to the curvature perturbation at the end of hybrid inflation. Furthermore, they studied non-Gaussianities finding a negligible bispectrum.
The authors of [63] explored the parameter space of the hybrid inflation model paying attention to the dynamics of the waterfall field and curvature perturbations from its field fluctuations. The author of [64] and [65] uses the formation of primordial black holes to constrain the contribution $\zeta_\chi$ to the curvature perturbation. Others [66, 67] relaxed the condition of a massive waterfall field.
Chapter 5

Non-Gaussianity and Gravitational Waves from Quadratic and Self-interacting Curvaton

As reviewed in 2.4, inflation solves the horizon problem, the flatness problem and the monopole problem. Furthermore, it gives a simple way to source primordial perturbations from quantum vacuum fluctuations. Any light scalar field during a period of inflation with an almost constant Hubble expansion acquires an almost scale-invariant power spectrum of fluctuations that could be the origin of primordial density perturbations [28, 42].

The curvaton is one such field which is only weakly coupled and hence decays on a time-scale much longer than the duration of inflation [68, 69, 70, 71, 72]. Its lightness enables the field to acquire super-Hubble perturbations from vacuum fluctuations during inflation. When it decays into radiation some time after inflation has ended, its decay can source the perturbations in the radiation density of the universe, and all other species in thermal equilibrium [73, 74]. More fundamental theories predict light degrees of freedom other than the inflaton in the early universe. An example of it is that a curvaton could be a Peccei-Quinn field [75] or a Pseudo-Nanbu-Goldstone boson [76]. Other authors arrived to a quadratic curvaton potential from string theory [77]. Others [78] have identified a quadratic curvaton with polynomial corrections to a MSSM flat direction. A common potential for the curvaton is a quadratic potential which can have higher order self-interactions. Another possible potential is a cosine potential, i.e., a PNGB. Then, understanding the curvaton mechanism with different potentials can, in principle, help understanding such more fundamental theories that predict them.

One of the distinctive predictions of the curvaton scenario for the origin of structure is the possibility of non-Gaussianity in the distribution of the primordial density perturbations [38, 79, 80]. Treating the curvaton as a pressureless fluid one can estimate
the resulting non-Gaussianity either analytically by treating the decay of the curvaton as instantaneous [38, 74, 79], or numerically [81, 82], showing that the non-Gaussianity parameter $f_{NL}$ (Eq. (3.36)) becomes large when the curvaton density at the decay time becomes small.

The non-linear evolution of the field before it decays can also contribute to the non-Gaussianity of the final density perturbation. The authors of [83, 84, 85, 86, 87, 88] looked at the effect of polynomial corrections to the quadratic curvaton potential. In some cases the curvaton density can be significantly subdominant at decay and still yield small $f_{NL}$ [85]. For small values of $f_{NL}$, the non-Gaussianity can instead be probed by the trispectrum parameter, $g_{NL}$.

Primordial gravitational waves on super-Hubble scales are also present since they are an inevitable byproduct at some level of an inflationary expansion. Sizable non-Gaussianity can, on its own, rule out single-field slow roll inflation since it is not capable of sourcing significant non-Gaussianity [89]. Local non-Gaussianity alone favors models that predict it, like the curvaton, but may not be enough to distinguish between them [90]. Furthermore it does not give tight constraints on curvaton model parameters. But non-Gaussianity together with gravitational waves can place stringent bound on the model parameters and consistency relationships between them [91]. Nakayama et al [92] studied the effects of the entropy released by the decay of a curvaton field with a quadratic potential on the spectrum of gravitational waves that are already sub-horizon scale at the decay and consider the possibilities of future direct detection experiments, such as BBO or DECIGO, to constrain the parameter space. In this chapter we restrict our attention to gravitational waves on super-Hubble scales when the curvaton decays which are not affected by the decay, and consider self-interactions of the curvaton field in addition to the quadratic potential [88, 93]. This includes scales which contribute to the observed CMB anisotropies, where the power in gravitational waves is typically given by the tensor-to-scalar ratio for the primordial metric perturbations, $r_T$.

In this chapter we will investigate how non-Gaussianity and gravitational waves provide constraints on curvaton model parameters. For any value of the curvaton model parameters we can obtain the observed amplitude of primordial density perturbations on large scales by adjusting the Hubble scale of inflation, which we assume to be an independent parameter in the curvaton model. However observational constraints on the tensor-to-scalar ratio places an upper bound on the inflationary Hubble scale, while non-Gaussianity constrains the remaining model parameters.

We numerically solve the evolution of the curvaton field in a homogeneous radiation-dominated era after inflation allowing for non-linear evolution of the curvaton field due to both explicit self-interaction terms in the potential and the self-gravity of the curvaton. In particular we consider quadratic and non-quadratic potentials which reduce to a quadratic
potential about the minimum with self-interaction terms governed by a characteristic mass scale, corresponding to cosine or hyperbolic-cosine potentials. Cosine potentials arise for PNGB axion fields and are often considered as candidate curvaton fields \[94, 95, 96, 97, 98\]. The hyperbolic cosine is representative of a potential where self-interaction terms become large beyond a characteristic scale. In each case we show how the non-linearity parameter \(f_{NL}\) and tensor-to-scalar ratio, \(r_T\), can be used to determine model parameters.

In Section 5.1 we review the perturbations generated during inflation and how these are transferred to the primordial density perturbation in the curvaton scenario. In Section 5.2 we present the numerical results of our study for three different curvaton potentials.

### 5.1 Inflationary perturbations in the curvaton scenario

In the curvaton scenario, initial quantum fluctuations in the curvaton field, \(\chi\), during a period of inflation at very early times give rise to the primordial density perturbation in the subsequent radiation-dominated universe some time after inflation and after the curvaton field has decayed into radiation, e.g., the density perturbation in the epoch of primordial nucleosynthesis. This primordial density perturbation is conveniently characterised by the gauge-invariant variable, \(\zeta\), corresponding to the curvature perturbation on uniform-density hypersurfaces \[29\].

Throughout this chapter we will use the \(\delta N\) formalism \[34, 37, 38, 59\] (see section 3.3) to compute the primordial density perturbation in terms of the perturbation in the local integrated expansion, \(N\), from an initial spatially-flat hypersurface during inflation, to a uniform-density hypersurface in the radiation-dominated era

\[
\zeta = \delta N = N'\delta \chi_* + \frac{1}{2}N''\delta \chi_*^2 + \ldots \tag{5.1}
\]

where \(\delta \chi_* = \chi_* - \langle \chi_* \rangle\) and primes denote derivatives with respect to \(\chi_*\), the local value of the curvaton during inflation.

Quantum fluctuations of a weakly-coupled field on super-Hubble scales \((k/a \ll H)\) during slow-roll inflation is well described by a Gaussian random field with two-point function

\[
\langle \chi_{\vec{k}_1}\chi_{\vec{k}_2} \rangle = (2\pi)^3P_\chi(k_1)\delta^3(\vec{k}_1 + \vec{k}_2) \tag{5.2}
\]

We define the dimensionless power spectrum \(P_\chi(k)\) as

\[
P_\chi(k) = \frac{k^3}{2\pi^2}P_\chi(k) \tag{5.3}
\]
The power spectrum of curvature perturbations is thus given, at leading order, by (3.32)

\[ P_\zeta(k) = N''^2 P_{\delta\chi}(k). \]  

(5.4)

and we define the spectral index as (3.34)

\[ n_\zeta - 1 \equiv \frac{d \ln P_\zeta}{d \ln k}, \]

(5.5)

and the running of the spectral index as 3.35

\[ \alpha_\zeta \equiv \frac{d \ln |n_\zeta - 1|}{d \ln k}. \]

(5.6)

The connected higher-order correlation functions are suppressed for a weakly-coupled scalar field during slow-roll inflation, but non-linearities in the dependence of \( N \) and hence \( \zeta \) on the initial curvaton value in Eq. (5.1) can lead to significant non-Gaussianity of the higher-order correlation functions, in particular the bispectrum

\[ \langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \zeta_{\vec{k}_3} \rangle = (2\pi)^3 B_\zeta(k_1, k_2, k_3) \delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3). \]

(5.7)

The bispectrum is commonly expressed in terms of the dimensionless non-linearity parameter, \( f_{\text{NL}} \), such that

\[ B_\zeta(k_1, k_2, k_3) = \frac{6}{5} f_{\text{NL}} [P_\zeta(k_1)P_\zeta(k_2) + P_\zeta(k_1)P_\zeta(k_3) + P_\zeta(k_2)P_\zeta(k_3)]. \]

(5.8)

If the initial field perturbations, \( \delta\chi^* \), correspond to a Gaussian random field then it follows from Eq. (5.1) that \( f_{\text{NL}} \) is independent of the wavenumbers, \( k_i \), and is given by (3.37)

\[ f_{\text{NL}} = \frac{5}{6} \frac{N''}{N'^2}. \]

(5.9)

In practice non-linear evolution of the field can lead to non-Gaussianity of the field perturbations on large scales and a weak scale dependence of \( f_{\text{NL}} \) [99, 100, 101].

Current bounds from the CMB on local-type non-Gaussianity require \(-10 < f_{\text{NL}} < 74\) [15]. Large-scale structure surveys lead to similar bounds [102].

### 5.1.1 Isocurvature field perturbations during inflation

Perturbations of an isocurvature field, whose fluctuations have negligible effect on the total energy density, can be evolved in an unperturbed FRW background and obey the
wave equation
\[ \ddot{\delta \chi} + 3H \dot{\delta \chi} + \left( \frac{k^2}{a^2} + m^2_\chi \right) \delta \chi = 0, \tag{5.10} \]
where the effective mass-squared is given by \( m^2_\chi = \partial^2 V / \partial \chi^2 \). During any period of accelerated expansion quantum vacuum fluctuations on small sub-Hubble scales (comoving wavenumber \( k > aH \)) are swept up to super-Hubble scales (\( k < aH \)). For a light scalar field, \( \chi \), with effective mass much less than the Hubble rate during inflation (\( m^2_\chi \ast \ll H^2_\ast \)) the power spectrum of fluctuations at Hubble exit is given by
\[ P_\chi \ast \simeq \left( \frac{H_\ast}{2\pi} \right)^2 \text{ for } k = a_\ast H_\ast. \tag{5.11} \]
On super-Hubble scales the spatial gradients can be neglected and the overdamped evolution (5.10) for a light field is given by
\[ H^{-1} \dot{\delta \chi} \simeq -\eta_{\chi \chi} \delta \chi. \tag{5.12} \]
where we define the dimensionless mass parameter
\[ \eta_{\chi \chi} = \frac{m^2_\chi}{3H^2}. \tag{5.13} \]
Combined with the time-dependence of the Hubble rate in Eq. (5.11), given by the slow-roll parameter \( \epsilon \equiv -\dot{H}/H^2 \), this leads to a scale-dependence at any given time of the field fluctuations on super-Hubble scales \([71, 103]\)
\[ \Delta n_\chi \equiv \frac{d}{d \ln k} \ln P_\chi \simeq -2\epsilon + 2\eta_{\chi \chi}. \tag{5.14} \]
which is small during slow-roll inflation, \( \epsilon \ll 1 \), for light fields with \( |\eta_{\chi \chi}| \ll 1 \).

Self-interaction terms in the curvaton potential during inflation only modify the predictions for the power spectrum and spectral tilt beyond these leading order results in the slow-roll approximation. However they do lead to time-dependence of the effective mass of the \( \chi \) field, so that the effective mass appearing in the expression for the spectral tilt may differ from that when the curvaton oscillates about the minimum of its potential some time after inflation. In particular the effective mass-squared during inflation could be negative, leading to a negative tilt, \( \Delta n_\chi < 0 \), even if \( \epsilon \) is very small.

The time-dependence of both \( \epsilon \) and \( \eta_{\chi \chi} \)
\[ H^{-1} \dot{\eta}_{\chi \chi} \simeq 2\epsilon \eta_{\chi \chi} - \xi^2_{\chi \phi}, \tag{5.15} \]
\[ H^{-1} \dot{\epsilon} \simeq -2\epsilon (\eta_{\phi \phi} - 2\epsilon), \tag{5.16} \]
during slow-roll inflation driven by an inflaton field with dimensionless mass $\eta_{\phi\phi} = V_{\phi\phi}/3H^2$ and $\xi_{\chi\phi}^2 = (\partial^4 V/\partial^3 \chi \partial \phi)/9H^4$, gives rise to a running of the spectral index in Eq. (5.14) [104]

$$\alpha_\chi \equiv \frac{d \Delta n_\chi}{d \ln k} \simeq 4\epsilon (-2\epsilon + \eta_{\phi\phi} + \eta_{\chi\chi}) - 2\xi_{\chi\phi}^2 , \quad (5.17)$$

In the following we shall make the usual assumption that the curvaton has no explicit interaction with the inflaton, so that $\xi_{\chi\phi} = 0$ and the running is second-order in slow-roll parameters and expected to be very small. Note, however, that in the curvaton scenario the tensor-to-scalar ratio and spectral tilt do not directly constrain the slow-roll parameters $\epsilon$ and $\eta_{\phi\phi}$ as in single-inflaton-field inflation, so they could be relatively large.

### 5.1.2 Transfer to curvaton density

In the curvaton scenario, these super-Hubble fluctuations in a weakly-coupled field whose energy density is negligible during inflation generates the observed primordial curvature perturbation, $\zeta$, after inflation if the curvaton comes to contribute a non-negligible fraction of the total energy density after inflation.

As the curvaton density becomes non-negligible one must include the backreaction of the field fluctuations on the spacetime curvature. However on super-Hubble scales, $k \ll aH$, where spatial gradients and anisotropic shear become negligible we can model the non-linear evolution of the field in terms of locally FRW dynamics [105]. In the following we will employ this “separate universe” picture [37] and we have

$$\ddot{\chi}_L + 3H_L \dot{\chi}_L + V'_\chi_L \simeq 0 ,$$

$$H_L^2 \simeq \frac{8\pi G}{3} \left( V_L + \frac{1}{2} \chi_L^2 \right) . \quad (5.18)$$

where $\chi_L = \chi + \delta \chi$, $H_L$, $V_L$ and $V'\chi_L$ denote the field, Hubble rate, potential and potential gradient smoothed on some intermediate scale $(aH)^{-1} \ll L < k^{-1}$, and dots denote derivatives with respect to the local proper time.

Once the Hubble rate drops below the effective mass scale, the long-wavelength modes of the field, $\chi_L$, oscillate about the minimum of the potential. Any scalar field with finite mass has a potential which can be approximated by a quadratic sufficiently close to its minimum, and the effective equation of state, averaged over several oscillation times, becomes that of a pressureless fluid

$$\rho_\chi = \frac{1}{2} m^2 \chi^2 + \frac{1}{2} \chi_L^2 \propto a^{-3} . \quad (5.19)$$
Thus the energy density of the curvaton grows relative to radiation, \( \rho_\gamma \propto a^{-4} \). The curvaton must eventually decay if it is to transfer its inhomogeneous density into a perturbation of the radiation density. We assume a slow, perturbative decay of the curvaton at a fixed decay rate, \( \Gamma_\chi \ll m_\chi \) (though we note that oscillating fields can also undergo a non-perturbative decay, or partial decay at earlier times [106, 107]).

We will numerically solve for the evolution of the curvaton field until it begins oscillating and determine its subsequent energy density. In order to follow the subsequent evolution and eventual decay of the curvaton density on time scales, \( \sim \Gamma_\chi^{-1} \), much longer than the oscillation time, \( \sim m_\chi^{-1} \), we adopt the results of Ref. [108].

Once the curvaton field behaves as a pressureless fluid, one can show that phase-space trajectory is determined by the dimensionless parameter [108, 109]

\[
p = \lim_{\Gamma_\chi/H \to 0} \frac{\Omega_\chi}{\Gamma_\chi} \sqrt{\frac{H}{\Gamma_\chi}}.
\]

(5.20)

In practice one can only treat the curvaton field as a pressureless fluid once it has begun to oscillate about the minimum of its potential. Taking the density of the curvaton when it begins to oscillate, \( \rho_{\chi,\text{osc}} \simeq m_\chi^2 \chi_{\text{osc}}^2 / 2 \) in Eq. (5.20), we can estimate \( p \) as [71]

\[
p \simeq p_{\text{LW}} = \frac{\chi_{\text{osc}}^2}{6m_{\text{Pl}}^2} \sqrt{\frac{m_\chi}{\Gamma_\chi}}.
\]

(5.21)

where the subscript “osc” denotes the time for which \( H_{\text{osc}} = m_\chi \) and \( m_{\text{Pl}} \equiv (8\pi G)^{-1/2} \simeq 2.43 \times 10^{18} \text{GeV} \) is the reduced Planck mass. Although the actual time when the curvaton begins oscillating is also not precisely defined this need not be a problem as \( \Omega_\chi \sqrt{H/\Gamma_\chi} \) is a constant while the curvaton is sub-dominant at early times, since \( \Omega_\chi \propto a \propto t^{1/2} \) and \( H \propto t^{-1} \) for a pressureless fluid in a radiation dominated era, and we simply require \( \chi_{\text{osc}}^2 / 6m_{\text{Pl}}^2 \ll 1 \).

However, Eq.(5.20) only estimates \( p \) in terms of the curvaton field value when the curvaton starts oscillating and we have assumed it has a quadratic potential at this time. More generally, to allow for self-interactions of the curvaton field that could lead to non-linear evolution after inflation and could still be significant when the curvaton begins to oscillate we define a transfer function for the field \( \chi_{\text{osc}} = g(\chi_*) \) [82] such that

\[
p = \frac{g^2(\chi_*)}{6m_{\text{Pl}}^2} \sqrt{\frac{m_\chi}{\Gamma_\chi}}.
\]

(5.22)

in order to relate the density of curvaton at late times, as it oscillates about the minimum of its potential, to the value of the curvaton field during inflation, \( \chi_* \).


5.1.3 Transfer to primordial perturbation

The amplitude of the resulting primordial curvature perturbation depends both on the perturbation in the curvaton density, \( \delta \rho_\chi / \rho_\chi \), and the energy density in the curvaton field when it decays. To first-order in the perturbations we write

\[
\zeta = R_\chi \left( \frac{\delta \rho_\chi}{3\rho_\chi} \right)_{osc} = R_\chi \frac{\delta p}{3p}.
\]

(5.23)

where \( 0 < R_\chi < 1 \) is a dimensionless efficiency parameter related to the fraction of the total energy density in the curvaton field when it decays into radiation. Using the separate universe picture, we take derivatives of the same function \( g(\chi_*) \) defined in terms of the homogeneous background fields in Eq. (5.22) to determine the linear density perturbation and higher-order perturbations in terms of the field perturbations during inflation. We thus have the transfer function for linear curvaton field perturbations during inflation into the primordial curvature perturbation

\[
\zeta = R_\chi \frac{1}{3} \frac{p'\delta \chi_*}{\rho} = R_\chi \frac{2}{3} \frac{g'(\chi_*)}{g} \frac{\delta \chi_*}{\chi_*}.
\]

(5.24)

where primes denote derivatives with respect to \( \chi_* \).

Modelling the transfer of energy from the curvaton field to the primordial radiation by a sudden decay at a fixed value of \( H_{\text{decay}} = \Gamma_\chi \) gives the transfer parameter [71, 74]

\[
R_\chi \approx \left[ \frac{3\rho_\chi}{4\rho_{\text{total}} - \rho_\chi} \right]_{\text{decay}}.
\]

(5.25)

However this expression is of limited use if we want to predict the primordial curvature perturbation in terms of the inflationary value of the curvaton field and its perturbations because this expression refers to the curvaton density at the decay time. The curvaton density changes with time and the decay time is not precisely defined since the decay happens over a finite period of time around \( H \sim \Gamma_\chi \).

More generally, the transfer parameter, \( R_\chi \) in Eq. (5.23), is a smooth function of the phase-space parameter \( p \) defined in Eq. (5.20). One can determine \( R_\chi \) as a function of \( p \) numerically, which gives the analytic approximation [109]

\[
R_\chi(p) \approx 1 - \left( 1 + \frac{0.924}{1.24} p \right)^{-1.24}.
\]

(5.26)

A distinctive feature of the curvaton scenario is the possibility that the primordial curvature perturbation may have a significantly non-Gaussian distribution even if the curvaton field itself is well described by a Gaussian distribution. This is due primarily
to the fact that the energy density of a massive scalar field when it oscillates about the
minimum of its potential is a quadratic function of the field. Simply assuming a linear
transfer (5.23) from a quadratic curvaton density to radiation yields \[74\]
\[\zeta = \frac{R_{\chi}}{3} \left( \frac{2\chi \delta \chi + \delta \chi^2}{\chi^2} \right), \quad (5.27)\]
and hence a primordial bispectrum of local form [14] characterised by the dimensionless
parameter
\[f_{NL} = \frac{5}{4R_{\chi}}. \quad (5.28)\]
This provides a good estimate of the non-Gaussianity for a quadratic curvaton with Gaus-
sian distribution when \(f_{NL} \gg 1\).

Incorporating the full non-linear transfer for a quadratic curvaton density while as-
suming the curvaton field has a Gaussian distribution at a sudden decay, yields corrections
of order unity \[38, 79, 82\]
\[f_{NL} \approx \frac{5}{4R_{\chi}} - \frac{5}{3} - \frac{5R_{\chi}}{6}. \quad (5.29)\]
Numerical studies \[81, 82\] confirm that this sudden-decay formula for \(f_{NL}(R_{\chi})\) repre-
sents an excellent approximation to the actual exponential decay, \(n_{\chi} \propto e^{-R_{\chi}t/a^3}\), where
we take \(R_{\chi}\) in Eq. (5.29) to be the linear transfer efficiency defined by Eq. (5.23). In
particular we find the robust result \(f_{NL} \geq -5/4\) for any value of \(R_{\chi}\).

More generally, if we allow for possible non-linear evolution of the local curvaton
field after Hubble-exit through the function \(g(\chi^*)\) defined in Eq. (5.22), and allow for
possible variation of the transfer parameter \(R_{\chi}\) with the value of the curvaton field (but
still take the curvaton fluctuations to be Gaussian at Hubble-exit) then we have \[82\]
\[f_{NL} = \frac{5}{4R_{\chi}} \left[ \left( 1 + \frac{gg''}{g'^2} \right) + \frac{R_{\chi} \left( g'g'' - 2R_{\chi} \right)}{R_{\chi}} \right]. \quad (5.30)\]
This expression follows directly from Eq. (3.37) when we take \(N' = \frac{2}{3}R_{\chi} \frac{g'^2}{g}\).

If we adopt the sudden-decay approximation for \(R_{\chi}(p)\) then Eq. (5.30) reduces to \[38\]
\[f_{NL} \approx \frac{5}{4R_{\chi}} \left( 1 + \frac{g''g}{g'^2} \right) - \frac{5}{3} - \frac{5R_{\chi}}{6}. \quad (5.31)\]

5.1.4 Metric perturbations during inflation

In most studies of the curvaton scenario it is assumed that the amplitude of scalar or met-
ric perturbations generated during inflation are completely negligible. Indeed the original
motivation for the study of the curvaton was to show that it was possible for fluctuations
in a field other than the inflaton to completely dominate the primordial curvature perturbation. However gravitational waves describe the free oscillations of the metric tensor, independent (at first order) of the matter perturbations, and some amplitude of fluctuations on super-Hubble scales is inevitably generated during an accelerated expansion. The resulting power spectrum of tensor metric perturbations is given by

$$\mathcal{P}_T = \frac{8}{m_{Pl}^2} \left( \frac{H_*}{2\pi} \right)^2. \quad (5.32)$$

The power spectrum of primordial gravitational waves if they can be observed by future CMB experiments, such as CMBPol [110], would give a direct measurement of the energy scale of inflation and hence the Hubble rate, $H_\ast$. In practice the amplitude of gravitational waves is usually expressed relative to the observed primordial curvature perturbation as the tensor-to-scalar ratio

$$r_T \equiv \frac{\mathcal{P}_T}{\mathcal{P}_\zeta} \simeq 8.1 \times 10^7 \left( \frac{H_*}{m_{Pl}} \right)^2 = 0.14 \times \left( \frac{H_*}{10^{14}\text{GeV}} \right)^2. \quad (5.33)$$

Current observational bounds from CMB anisotropies are partially degenerate with bounds on the spectral index and dependent on theoretical priors, but can be used give $r_T < 0.24$ [15]. Bounds from the power spectrum of the B-mode polarisation of the CMB are less model dependent and require $r_T < 0.72$ [111].

The tensor perturbations are massless and the scale dependence of the spectrum after Hubble-exit (5.32) is simply due to the time dependence of the Hubble rate:

$$n_T = -2\epsilon. \quad (5.34)$$

Thus the tilt of the gravitational wave spectrum on very large scales today gives a direct measurement of the equation of state during inflation, $w = -1 + (2\epsilon/3)$.

If inflation is driven by a light inflaton field, $\phi$, this inflaton field also inevitably acquires a spectrum of fluctuations during the accelerated expansion, $\mathcal{P}_{\phi_*} = (H/2\pi)^2$. These adiabatic field perturbations [26] correspond to a curvature perturbation at Hubble-exit during inflation

$$\mathcal{P}_{\zeta_*} = \left( \frac{H}{\dot{\phi}} \right)_s^2 \mathcal{P}_{\phi_*} = \frac{1}{16\epsilon} \mathcal{P}_T. \quad (5.35)$$

The scale-dependence of the tensor spectrum (5.34) together with the time-dependence of $\epsilon$ during inflation, given in Eq. (5.16), leads to a scale dependence of the curvature perturbation from adiabatic perturbations

$$n_{\zeta_*} - 1 = -6\epsilon + 2\eta_{\phi\phi}, \quad (5.36)$$
where the dimensionless inflaton mass parameter is $\eta_{\phi} = m_{\phi}^2/3H^2$. Note that the primordial curvature perturbation due to canonical inflaton field perturbations is effectively Gaussian with $|f_{NL}|_s \ll 1$ suppressed by slow-roll parameters [89].

In the presence of a curvaton field, the adiabatic perturbations during inflation represent only a lower bound on the primordial curvature perturbation and one should add the uncorrelated contributions to the primordial curvature perturbation from both the curvaton field (5.24) and the inflaton field (5.35):

$$P_\zeta = \left( \frac{2g'R_x}{3g} \right)^2 P_X + \frac{1}{16\epsilon} P_T. \quad (5.37)$$

For example, if the spectral tilt of the primordial curvature perturbation from a very light curvaton field (5.14) is $n_X - 1 \approx -0.03$ and primarily due to the time-dependence of the Hubble rate during inflation, $n_X - 1 \approx n_T \approx -2\epsilon$, then we have $16\epsilon \approx 16 \times 0.015 = 0.24$ and hence $P_{\zeta s} \approx 4P_T$. Hence $P_{\zeta s} \ll P_\zeta$ for $r_T \ll 0.3$.

In this chapter we will assume $\epsilon$ is large enough that the inflaton contribution to the primordial curvature perturbation can be neglected even if the primordial tensor perturbations are potentially observable. We will study in Chapter 6 the curvaton model when we include inflation contributions to the primordial power spectrum.

### 5.1.5 Lyth Bound on $H_*$ from $f_{NL}$

The curvaton starts oscillating about its minimum when $H \sim m_X$. During coherent oscillations one expects the curvaton energy density to behave like matter, i.e., $\rho_X \propto a^{-3}$. For a quadratic potential we have $\rho_X = m_X^2 \langle \chi^2 \rangle$. At the beginning of the oscillations the critical energy density of the universe reads $\rho = 3m_{Pl}^2m_X^2$. Then the density parameter of the curvaton when oscillations start is

$$\Omega_{\chi,osc} = \frac{\chi_{osc}^2}{3m_{Pl}^2}. \quad (5.38)$$

This immediately gives us an upper bound on value of the field,

$$\chi_{osc} < \sqrt{3}m_{Pl}, \quad (5.39)$$

since $\Omega_{\chi,osc} < 1$.

The Hubble parameter keeps decreasing as time goes by and eventually it attains the value $H \sim \Gamma_\chi \sim (3m_{Pl})^{-1}T^2_{dec}$ when the curvaton decays, assuming a sudden decay.
The curvaton is subdominant, then the universe is radiation dominated, thus $H \propto a^{-2}$. So
\[
\frac{m_X}{(3m_{Pl})^{-1}T_{dec}^2} = \left(\frac{a_{dec}}{a_{osc}}\right)^2.
\] (5.40)

After oscillations start, the curvaton behaves like a non relativistic fluid implying that
\[
\Omega_{\chi, dec} = \Omega_{\chi, osc}\left(\frac{a_{dec}}{a_{osc}}\right)^2 \approx \frac{\chi_{osc}^2(3m_Xm_{Pl})^{1/2}}{3m_{Pl}T_{dec}}.
\] (5.41)

Note that this is precisely the definition of $p$ (5.21). Although one expects the curvaton to be over-damped, the inequality $\chi_{osc} < \chi_*$ always holds, thus
\[
p \leq \frac{\chi_*^2(3m_Xm_{Pl})^{1/2}}{3m_{Pl}T_{dec}}.
\] (5.42)

For the simplest curvaton with linear evolution during the radiation era, the final power spectrum transferred into radiation is given by
\[
P_\zeta = R_X^2 \left(\frac{H_*}{2\pi M_X}\right)^2.
\] (5.43)

From WMAP7 [15] we know that $P_\zeta \simeq 2.5 \times 10^{-9}$. Hence we find
\[
\frac{p}{R_X} < 3.3 \times 10^6 \left(\frac{m_X}{H_*}\right)^{1/2} \frac{R_X H_*^{5/2}}{T_{dec} m_{Pl}^{3/2}}.
\] (5.44)

When $p \gg 1$ the value of $R_X$ must be saturated, so $R_X \rightarrow 1$. On the other hand when $p \ll 1$ then one has $R_X \sim p$. One expects $R_X$ to be a non-decreasing function of $p$, therefore the inequality $p/R_X > 1$ always holds. Using this fact in eq. (5.42) one obtains a lower bound on the inflation scale
\[
H_* > 7.8 \times 10^7 \text{GeV} \left(\frac{H_*}{m_X}\right)^{1/5} \left(\frac{T_{dec}}{1 \text{MeV}}\right)^{2/5} \left(\frac{f_{nl}}{50}\right)^{2/5}.
\] (5.45)

where $R_X$ was substituted by $f_{nl}$ using eq. (5.28) in the linear regime when $R_X \ll 1$. To recover the Big Bang Nucleosynthesis the curvaton has to decouple before neutrinos so $T_{dec} \gtrsim 1 \text{ MeV}$. It is effectively massless during inflation meaning $m_X < H_*$. This condition is crucial for the curvaton to produce the primordial density fluctuations. Using these bounds one arrives to Lyth bound [112], that gives a lower bound to inflation scale,
\[
H_* \gtrsim 7.8 \times 10^7 \text{GeV} \left(\frac{f_{nl}}{50}\right)^{2/5}.
\] (5.46)
5.2 Numerical results

In our numerical analysis we have used the separate universe equations (5.18) to evolve the local value of $\chi_L$ for long-wavelength perturbations of the curvaton field. This incorporates both the non-linear self-interactions included in the potential of the curvaton, $V(\chi_L)$, and non-linearity of the gravitational coupling through the dependence of the Hubble expansion rate on the curvaton field kinetic and potential energy density in the Friedmann equation.

We do not solve for the curvaton field evolution during inflation or during (p)reheating at the end of inflation since this would be model dependent. Instead we start the evolution with a radiation density such that the initial Hubble rate is much larger than the effective mass of the curvaton, consistent with our assumption that the initial value of the curvaton field is effectively the same as its value at the end of inflation, $\chi_*$. We evolve the curvaton until it begins to oscillate in the minimum of its potential and can accurately be described as a pressureless fluid, in order to exploit earlier work which used a fluid model to study the linear [109] and non-linear [82] transfer of the curvaton perturbation to radiation and hence the primordial curvature perturbation. Thus we evolve the curvaton field until $\rho_\chi \propto a^{-3}$. Note that this may be sometime after the time when $H = m_\chi$ since the curvaton potential may have significant non-quadratic corrections at this time.

We need to be able to determine the dimensionless parameter $p$ defined in Eq. (5.20) which determines the transfer parameter $R_\chi(p)$. To do so we identify

$$p = \sqrt{\frac{m_\chi}{\Gamma_\chi}} p_{\text{FW}},$$

(5.47)

where

$$p_{\text{FW}} \equiv \Omega_\chi (1 - \Omega_\chi)^{-3/4} \sqrt{\frac{H}{m_\chi}},$$

(5.48)

is constant for a pressureless fluid, $\chi$, plus radiation. It is straightforward to check that Eq. (5.47) coincides with the definition of $p$ given in Eq. (5.20), which is evaluated in the early time limit, $\Omega_\chi \to 0$. The advantage of our variable $p_{\text{FW}}$ is that it can evaluated at late times, so long as the curvaton decay is negligible, $\Gamma_\chi \ll H$, whereas at early times the curvaton field may never actually evolve like a pressureless fluid and we may not have a well-defined early time limit for $\Omega_\chi \sqrt{H/\Gamma_\chi}$. 

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In our numerical code following the curvaton field evolution we use Eq. (5.18) with the rescaled time variable \( \tau = m_\chi t \), implicitly setting \( \Gamma_\chi = 0 \), such that
\[
\chi'' + 3h\chi' + \frac{V_\chi}{m_\chi^2} = 0, \tag{5.49}
\]
\[
h^2 = \frac{8\pi}{3m_{Pl}^2} \left( \frac{\rho_\chi}{m_\chi^2} + \frac{V}{m_\chi^2} + \frac{1}{2} \chi'^2 \right). \tag{5.50}
\]
Note that \( m_\chi = V_{\chi \chi}(\chi = 0) \). For a quadratic potential we have \( V_\chi/m_\chi^2 = \chi \) and \( V/m_\chi^2 = \chi^2/2 \) and hence the evolution of \( \chi(\tau) \) is independent of \( m_\chi \). We evolve the curvaton field from an initial value \( \chi_i = \chi_* \) when \( H_i^2 = 100V_{\chi \chi} \). This is consistent with the usual assumption that the curvaton is a late-decaying field with \( \Gamma_\chi \ll m_\chi \). We are then able to determine \( p_{\text{FW}}(\chi_*) \) which approaches a constant as the curvaton density approaches that of a pressureless fluid at late times. We then obtain the actual parameter \( p \) in Eq. (5.47) for a finite decay rate, by multiplying by a finite value of \( \sqrt{m_\chi/\Gamma_\chi} \). Thus the parameter \( p \) is a function of \( \chi_* \) and \( m_\chi/\Gamma_\chi \), but not \( m_\chi \) and \( \Gamma_\chi \) separately.

We use the previously determined [109] transfer function \( R_\chi(p) \) given by Eq. (5.26). From Eq. (5.11) and (5.24) we then have
\[
P_\zeta = \left( \frac{p'R_\chi(p)}{3p} \right)^2 \left( \frac{H_*}{2\pi} \right)^2. \tag{5.51}
\]
Normalising the amplitude of the primordial power spectrum to match the observed value on CMB scales, \( P_\zeta \simeq 2.5 \times 10^{-9} \) [15], then fixes the amplitude of vacuum fluctuations of the curvaton field during inflation and hence the scale of inflation
\[
H_* = 9.4 \times 10^{-4} \left( \frac{p}{p'R_\chi(p)m_{Pl}} \right) m_{Pl}. \tag{5.52}
\]
or, equivalently, the tensor-scalar ratio
\[
r_T = 72 \left( \frac{p}{p'R_\chi(p)m_{Pl}} \right)^2. \tag{5.53}
\]

The non-linearity parameter, \( f_{\text{NL}} \), is given by Eq. (5.30). Note that for \( r_T \) we must determine not only \( p \) but also its first derivative, \( p' \), with respect to the initial field value, \( \chi_* \). For the non-linearity parameter, \( f_{\text{NL}} \), we also need the second derivative, \( p'' \), and to describe higher-order non-Gaussianity we would need higher derivatives. In terms of the parameter \( p \), Eq. (5.30) becomes
\[
f_{\text{NL}} = \frac{5}{2R_\chi} \left[ pp'' + \frac{R_\chi'}{p'} - 1 \right]. \tag{5.54}
\]
5.2.1 Quadratic curvaton

We show the results in Figure 5.1 and 5.2 for a quadratic curvaton potential. In this case we are able to compare our numerical result against an exact analytic expression while the curvaton density remains negligible during the radiation-dominated era. In this case the curvaton field is given by

\[ \chi = \frac{\pi \chi_*}{2^{5/4} \Gamma(3/4)} \frac{J_{1/4}(m_\chi t)}{(m_\chi t)^{1/4}}. \]  

where \( J_{1/4}(m_\chi t) \) is the Bessel function of the first kind of order 1/4. This has the asymptotic solution \( \chi \approx 1.023 \chi_* \cos(m_\chi t - 3\pi/8)/(m_\chi t)^{3/4} \), and substituting this into Eq. (5.20) gives

\[ p \approx 1.046 \sqrt{\frac{m_\chi \chi_*^2}{\Gamma m_{Pl}^2}}. \]  

We see from Figure 5.1 that Eq. (5.56) gives an excellent approximation to the numerical results for \( \chi_* \ll m_{Pl} \).

Contour plots are given in Figure 5.2 for the non-linearity parameter, \( f_{NL} \), and the inflation Hubble scale, \( H_* \), (and hence tensor-scalar ratio, \( r_T \)) for a non-self-interacting curvaton with a quadratic potential.

Given that the analytic result for \( p(\chi_*) \) given in Eq. (5.56) is an excellent approximation, except for \( \chi_* \sim m_{Pl} \), we deduce that \( \chi_{osc} = g(\chi_*) \) defined by Eq. (5.21) is a linear function \( g(\chi_*) \approx \sqrt{2} \chi_* \). Thus the non-linearity parameter \( f_{NL} \) is given in terms of \( R_\chi \) in Eq. (5.29). We have two regimes for the transfer function \( R_\chi(p) \) given by Eq. (5.26). For \( \chi_* \gg (\Gamma/\chi_{osc})^{1/4} m_{Pl} \) we have \( p \gg 1 \) and hence \( R_\chi \approx 1 \), while for \( \chi_* \ll (\Gamma/\chi_{osc})^{1/4} m_{Pl} \) we have \( p \ll 1 \) and hence \( R_\chi \approx 0.924 p \). Thus we find from Eq. (5.29)

\[ f_{NL} \approx \begin{cases} 
-5/4 & \text{for } \chi_* \gg (\Gamma/\chi_{osc})^{1/4} m_{Pl} \\
3.9 \sqrt{\frac{\Gamma_{Pl}^2}{m_{\chi}^2 \chi_*^2}} & \text{for } \chi_* \ll (\Gamma/\chi_{osc})^{1/4} m_{Pl} 
\end{cases}. \]  

Potentially observable levels of non-Gaussianity (5 < \( f_{NL} < 100 \)) are found in a band of parameter space

\[ \chi_* \approx (1 - 4) \times 10^{17} \text{ GeV} \left( \frac{\Gamma_\chi}{10^{-6} m_\chi} \right)^{1/4}. \]  

The degeneracy between values of \( \chi_* \) and \( \Gamma_\chi/m_\chi \) which would be consistent with the same value of \( f_{NL} \) is broken by a measurement of the scalar to tensor-ratio, \( r_T \). Substituting the approximation (5.56) in Eq. (6.58). We have

\[ H_* \approx 4.7 \times 10^{-4} \frac{\chi_*}{R_\chi(p)}, \]  

\[ 79 \]
Figure 5.1: Dimensionless curvaton parameter $p_{FW}$, defined in Eq. (5.48) as a function of initial curvaton field value, $\chi_*$, for three different potentials: quadratic potential (dotted blue line), cosine potential with $f = 10^{18}$GeV (upper red dashed line) and hyperbolic cosine potential with $f = 10^{18}$Gev (lower green dot-dashed line). For comparison, the solid black line shows $\chi_*^2 / 3 m_{Pl}^2$, which provides an excellent approximation for $\chi_* \ll m_{Pl}$. 
Figure 5.2: Contour plots showing observational predictions for a curvaton field with quadratic potential as a function of the dimensionless decay rate, $\log_{10}(m_\chi/\Gamma_\chi)$, and the initial value of the curvaton, $\log_{10}(\chi_*/\text{GeV})$. Top: Contour lines for the non-Gaussianity parameter $f_{NL}$ (in blue). The dotted black lines correspond to Eq. (5.28). Middle: Contour lines for inflationary Hubble scale, $\log_{10}(H_*/\text{GeV})$. The plotted contour lines correspond to $H_* = 10^{13}, 10^{14}, 10^{15}$ GeV. The black dotted lines correspond to the 2 limits of Eq. (5.60). Bottom: Contour lines for both the non-Gaussianity parameter, $f_{NL}$, (blue thick solid line) and tensor-scalar ratio, $r_T$, (red dotted line).
This yields two simple expressions for $H_*$ according to whether $p \gg 1$ and hence $R_\chi \simeq 1$ or $p \ll 1$ and hence $R_\chi \simeq 0.924 p$. We thus have

$$H_* \simeq \begin{cases} 
4.7 \times 10^{-4} \chi_* & \text{for } \chi_* \gg (\Gamma_\chi/m_\chi)^{1/4} m_{Pl} \\
1.5 \times 10^{-3} \sqrt{\frac{\Gamma_\chi m_\chi^2}{\chi_*}} & \text{for } \chi_* \ll (\Gamma_\chi/m_\chi)^{1/4} m_{Pl} \end{cases}$$

(5.60)

Even a conservative bound on the tensor-scalar ratio such as $r_T < 1$ thus places important bounds on the curvaton model parameters. Firstly there is the model-independent bound on the inflation Hubble scale, $H_* < 2.7 \times 10^{14}$ GeV. In the case of a quadratic curvaton potential this imposes an upper bound on the value of the curvaton during inflation

$$\chi_* < 5.7 \times 10^{17} \text{ GeV},$$

which is consistent with $\chi_* < m_{Pl}$ required to use the analytic approximation (5.56). We also find an upper bound on the dimensionless decay rate

$$\frac{\Gamma_\chi}{m_\chi} < 0.023 \left( \frac{\chi_*}{m_{Pl}} \right)^2,$$

(5.62)

and in any case $\Gamma_\chi < 10^{-3} m_\chi$. For example, for a TeV mass curvaton [113] we require $\Gamma_\chi < 1$ GeV. More generally, if we require the curvaton to decay before primordial nucleosynthesis at a temperature of order 1 MeV, we require $\Gamma_\chi > H_{\text{BBN}}$ and hence $m_\chi > 10^3 H_{\text{BBN}}$. On the other hand if the curvaton decays before decoupling of the lightest supersymmetric particle at a temperature of order 10 GeV, we require $\Gamma_\chi > 10^{-17}$ GeV and hence $m_\chi > 10^{-14}$ GeV.

Bounds on the curvaton decay rate due to gravitational wave bounds were also studied in Ref. [92], who also considered the case where that curvaton oscillations begin immediately after inflation has ended at $H < m_\chi$.

We note that bounds on the tensor-scalar ratio rule out large regions of parameter space that would otherwise give rise to large non-Gaussianity.

A simultaneous measurement of primordial non-Gaussianity, $f_{NL}$, and primordial gravitational waves, $r_T$, for a non-self-interacting curvaton field with quadratic potential would determine both the energy scale of inflation, $H_*$, and the expectation value of the curvaton, $\chi_*$. It would also determine the dimensionless decay rate $\Gamma_\chi/m_\chi$, but not the absolute value of the mass and decay rate separately. More optimistically, if the gravitational amplitude was large enough to determine the tensor tilt, $n_T$ and hence $\epsilon$, the scale dependence of the scalar spectrum would determine the curvaton mass:

$$m_\chi^2 = 3 \eta_\chi H_*^2 \simeq \frac{3}{2}(n_\zeta - 1 - n_T) \frac{r_T}{2.0 \times 10^2 m_{Pl}^2}.$$  

(5.63)
However once $\epsilon_*$ is known then from Eq. (5.35) we also know the curvature perturbation due to inflaton perturbations during slow-roll inflation: $P_{\zeta_*} = (r_T/16\epsilon)P_C$. If $\epsilon_* \leq 0.02$, as is commonly assumed, then our assumption that the inflaton perturbations are negligible is no longer valid for $r_T \sim 0.3$. In this case we need to consider a mixed inflaton-curvaton model. This inflaton-curvaton model has a much richer phenomenology [114, 115, 116, 117, 118] and we leave the study of the combined non-Gaussianity and gravitational wave bounds in this scenario to Chapter 6. Otherwise we must assume $\epsilon_*$ is sufficiently large that the inflaton-generated perturbations remain negligible.

5.2.2 Self-interacting curvaton

We have seen that non-linear field evolution due to gravitational back-reaction of the curvaton field with a quadratic potential is limited to large initial values $\chi_* \sim m_{Pl}$ which are incompatible with bounds on the tensor-scalar ratio in the curvaton scenario with a quadratic potential. However significant non-linear field evolution may arise from self-interactions of the curvaton field, due to deviations from a purely quadratic potential. Polynomial self-interaction terms of the form $V_{\text{int}} \propto \chi^n$ where $n \geq 4$ have been shown to have a large effect on observational predictions in some regions of parameter space [85, 87, 88].

Rather than choose a monomial correction term, we choose a functional form that leads to significant corrections at a specified mass scale. In particular we are motivated by axion type potentials where the curvaton field has a natural range, $f$. Thus we consider a cosine-type potential, with a smaller mass effective mass for $\chi_* \sim f$ and a hyperbolic-cosine potential which has a much larger mass for $\chi_* \sim f$. In both cases the corrections lead to a finite range $\chi_* \sim f$ for the initial curvaton field.

Cosine potential

We consider an axion-type potential for a weakly-broken $U(1)$-symmetry[71, 76]

$$V(\chi) = M^4 \left( 1 - \cos \left( \frac{\chi}{f} \right) \right) \approx \frac{1}{2} m^2_{\chi} \chi^2 - \frac{1}{24} \frac{m^2_{\chi} \chi^4}{f^2} + \ldots ,$$  \hspace{1cm} (5.64)

where $m^2_{\chi} = M^4/f^2 \ll M^2$ and we have an additional model parameter corresponding to the mass scale $f \gg M$ which determines the relative importance of self-interaction terms at a given curvaton field value. It also determines a natural expectation value for the curvaton field, $\chi_* \sim f$. In the following we assume $f < m_{Pl}$.

In Figure 5.1 we show the numerical solution for $p_{\text{FW}}$ as a function of $\chi_*$, corresponding to $p$ for a fixed value of $m_{\chi}/\Gamma_{\chi}$. As expected we see that for $\chi_* \ll f$ we recover
the analytic result (5.56) as the potential is effectively quadratic and self-interactions have a negligible effect. For larger values of $\chi^*$, the potential becomes flatter than the corresponding quadratic potential and we see that $p_{FW}$, and hence $p$, can become much larger than would be obtained for a quadratic correction. Note that the potential (5.64) is periodic and we can identify $p_{FW}(\chi^* + \pi f/2) = p_{FW}(\pi f/2 - \chi^*)$.

We show numerical predictions for the non-Gaussianity parameter, $f_{NL}$, and the tensor-scalar ratio, in Figure 5.3. Non-linear evolution of the field becomes important for $\chi^* \sim f$. In particular we see that an upper bound on the tensor-scalar ratio no longer places an upper bound on the decay rate $\Gamma^*/m_\chi$ as we approach the top of the potential, i.e., as $\chi^* \rightarrow \pi f$.

Modest, positive values of the non-linearity parameter, $1 < f_{NL} < 10$, become possible even if the curvaton dominates the energy density when it decays ($p > 1$) if $\chi^* > 2.5f$, but we never find very large values of $f_{NL} > 100$. Because $g'' > 0$ in Eq. (5.31) we have $f_{NL} > -5/4$, as in the case of a quadratic potential, and we never find large negative values of $f_{NL}$ for a cosine-type potential.

**Hyperbolic-cosine potential**

Non-linearity of the cosine potential (5.64) yields a flat potential with small effective mass during inflation for $\chi^* \sim f$. To consider the effect of self-interactions leading to a larger effective mass we consider a hyperbolic cosine potential which becomes an exponential function of the curvaton field at large field values, as may be expected due to supergravity corrections.

$$V(\chi) = M^4 \left( \cosh \left( \frac{\chi}{f} \right) - 1 \right) \simeq \frac{1}{2} m_\chi^2 \chi^2 + \frac{1}{24} \frac{m_\chi^2}{f^2} \chi^4 + \ldots .$$ (5.65)

In the following we assume $f < m_{Pl}$. As in the case of the cosine potential, this also yields a natural range for $\chi^* \sim f$. In the case of a hyperbolic potential, the field becomes heavy relative to the Hubble scale and evolves rapidly for values of $\chi^*$ much larger than $f$. In particular the requirement that the curvaton have an effective mass less than $0.1 H$ at the start of our numerical solutions imposes the constraint $\chi^* < 5f$.

In Figure 5.1 we show the numerical solution for $p$ as a function of $\chi^*$ for a fixed value of $m/\Gamma$. As expected we see that for $\chi^* \ll f$ we recover the analytic result (5.56) when the potential is effectively quadratic. However for the a hyperbolic potential we see that due to the steeper potential the effective energy density when the curvaton decays, determined by the parameter $p$, becomes less than the quadratic case for $\chi^* \sim f$.

We show numerical predictions for the non-Gaussianity, $f_{NL}$, and the inflation Hubble scale, $H_*$, (and hence the tensor-scalar ratio) in Figure 5.4. The non-linear correction
Figure 5.3: Contour plots showing observational predictions for a curvaton field with a cosine type potential. The three plots show, from top to bottom, observational parameters for cosine potentials with $f = 10^{17}$ GeV, $f = 10^{16}$ GeV and $f = 10^{15}$ GeV, respectively, as a function of the dimensionless decay rate, $\log_{10}(m_\chi/\Gamma_\chi)$, and the initial value of the curvaton, $\chi^*/$GeV. Thick solid blue contour lines show bispectrum amplitude, $f_{\text{NL}}$, decreasing from left to right. Dotted red contour lines show the tensor-scalar ratio $r_T$, also decreasing from left to right.
Figure 5.4: Contour plots showing observational predictions for a curvaton field with a hyperbolic-cosine-type potential. The three plots show, from top to bottom, observational parameters for potentials with $f = 10^{17}\text{GeV}$, $f = 10^{16}\text{GeV}$ and $f = 10^{15}\text{GeV}$, respectively, as a function of the dimensionless decay rate, $\log_{10}(m_{\chi}/\Gamma_{\chi})$, and the initial value of the curvaton, $\chi_*/\text{GeV}$. Thick solid blue contour lines show bispectrum amplitude, $f_{\text{NL}}$, increasing from top to bottom. Dotted red contour lines show the tensor-scalar ratio, $r_T$, decreasing from left to right.
$g''g/g^2$ in Eq. (5.31) becomes negative for $\chi_* \sim f$ and we can obtain large negative values of $f_{NL}$.

However we find that the bound on the tensor-scalar plays an important role. Regions of parameter space which yield large negative $f_{NL}$ also give large tensor-scalar ratios. In regions where $p \gg 1$ and the curvaton dominates when it decays we have $R_\chi \sim 1$ and both the tensor-scalar ratio and the non-linearity parameters become functions solely of $\chi_*$. In this regime, we have, from Eqs. (5.53) and (5.31)

\begin{align}
    r_T &\approx \frac{9}{2} \left( \frac{g}{g' m_{Pl}} \right)^2, \quad (5.66) \\
    f_{NL} &\approx \frac{5}{4} \left( \frac{g''g}{g'^2} \right), \quad (5.67)
\end{align}

which are both clearly functions of $\chi_*$. Indeed formally we can eliminate $g(\chi_*)$ and its derivatives in order to write

\begin{equation}
    f_{NL} \approx -\sqrt{\frac{25}{12}} m_{Pl} \left( \sqrt{r_T} \right)', \quad (5.68)
\end{equation}

Hence the contours of equal values of both $r_T$ and $f_{NL}$ become horizontal on the right-hand-side of Figure 5.4. For example, with $f = 10^{16}$ GeV a weak bound on the tensor-scalar ratio of $r_T < 1$ requires $f_{NL} > -1000$. A stronger bound $r_T < 0.1$ requires $f_{NL} > -100$.

Of course $(\sqrt{r_T})'$ is not an observable parameter, but if we assume that $\sqrt{r_T}$ is a smooth function of $\chi_*/f$ we can estimate $(\sqrt{r_T})' \sim (\sqrt{r_T})/f$ and hence

\begin{equation}
    f_{NL} \sim -\frac{m_{Pl}}{f} \sqrt{r_T}. \quad (5.69)
\end{equation}

This semi-empirical relation appears to hold for sufficiently small $\Gamma/m$ and it would be interesting to see if this is also the case for the polynomial correction terms [85, 87].

Unlike the case of a cosine-type potential we still have a strict upper bound on the decay rate, as in the case of a purely quadratic potential. Thus, although there are regions of parameter space for $\chi_* \sim f$, where the non-Gaussianity can be small even if the curvaton is subdominant when it decays, $p \ll 1$, we find that these regions correspond to large values for the tensor-scalar ratio and are excluded by bounds on primordial gravitational waves.
5.2.3 Stochastic approach for a quadratic curvaton

For the cosine potential (5.64) we may expect the curvaton VEV in different patches of the Universe, $\chi_*$, to be distributed uniformly in the interval between $0 \leq \chi_* \leq \pi f$. We assume that inflation lasts long enough for the quantum fluctuations to lead a random walk that explores the whole potential (and we identify $\chi_* \leftrightarrow -\chi_*$ and $\chi_* \leftrightarrow \chi_* + 2\pi$). Similarly for the hyperbolic-cosine we may expect the curvaton VEV in different patches of the Universe, $\chi_*$, to be approximately uniformly distributed in the interval between $0 \leq \chi_* \lesssim \pi f$ as for much larger values the potential becomes steep and the classical evolution dominates over the random walk.

For the quadratic potential there is no obvious scale for the curvaton VEV, unless we take a stochastic approach for the distribution of the VEV and assume that inflation lasted long enough so that we find a Gaussian distribution with variance [119] (see appendix A)

$$\langle \chi_*^2 \rangle = \frac{3}{8\pi^2 m^2_\chi} H^4. \quad (5.70)$$

As a rough approximation let us take $\chi_* \simeq \sqrt{\langle \chi_*^2 \rangle}$. Another way of writing this equation is

$$\left( \frac{\chi_*}{m_{Pl}} \right)^2 \simeq \frac{r_T P_{\zeta}}{16\eta_{\chi \chi}}. \quad (5.71)$$

One can use the previous equation to test the consistency of an effectively massless curvaton. In Figure 5.5 we draw contour lines for $\eta_{\chi \chi}$ in the $\log m_\chi/\Gamma_\chi - \log \chi_*$ plane. The current bounds give $f_{NL} \lesssim 100$ which is consistent with a massless curvaton as seen in Figure 5.5. Hence we conclude that the assumption of the subdominant curvaton during inflation is only violated in already excluded regions in parameter space.

The stochastic approach can also be used to disentangle the dependence of the observational quantities on the ratio $m_\chi/\Gamma_\chi$. From equation (5.70) one finds

$$\Gamma_\chi = \sqrt{\frac{3}{8\pi^2 \chi_* m_\chi/\Gamma_\chi}} H^2. \quad (5.72)$$

Note that we determine $H_*$ from the constraint equation (5.51). In Figure 5.6 we draw contour lines of $\Gamma_\chi$ in the $\log m_\chi/\Gamma_\chi - \log \chi_*$ plane. We can see that the current bounds on $f_{NL}$ and $r_T$ are consistent with a late decaying curvaton.

5.3 Summary

The curvaton model [71] is an alternative inflationary model for the origin of structure in the universe. It is also a good candidate to explain primordial perturbations that devi-
Figure 5.5: On the vertical axes we have the $\log(\chi^*/\text{GeV})$ of the curvaton field and on the horizontal axes we have $\log(m_{\chi}/\Gamma_{\chi})$. The thick green contour lines correspond to the values of $\eta_{\chi\chi}$. In blue we plot the contour line $f_{NL} = 100$.

Figure 5.6: On the vertical axes we have the $\log(\chi^*/\text{GeV})$ of the curvaton field and on the horizontal axes we have $\log(m_{\chi}/\Gamma_{\chi})$. The thick black contour lines correspond to $\log(\Gamma_{\chi}/\text{GeV})$. In blue we plot the contour line $f_{NL} = 100$ and in red the contour line $r_T = 0.24$. 
ate from a pure Gaussian distribution. The curvaton is a light, late decaying scalar field that lives in the early universe and acquires a Gaussian spectrum of field perturbations during inflation. After inflation, when it decays, it transfers its perturbations into radiation density fluctuations. In this chapter the curvaton was taken as the only source of primordial perturbations. In this regime we reviewed the predictions of the model for the amplitude of the power spectrum and its tilt and running. Similarly we computed the non-linearities present in the 3-point function. The distribution of density perturbations may acquire non-linearities via different ways: either by gravitational evolution on super-horizon scales prior to decay or by inefficient decay into radiation, or both. Furthermore, deviations from a quadratic potential for the curvaton field, or other interaction terms, will add extra non-linearities.

In this chapter we have investigated the numerical evolution of a curvaton field from its overdamped regime after inflation until it decays into radiation. We have shown how measurement of both the non-linearity parameter, $f_{NL}$, and the tensor-to-scalar ratio, $r_T$, provide complementary constraints on the model parameters. To evaluate the predictions of the model we used analytical approaches when possible and numerical solutions otherwise. Using a $\delta N$ approach we linked the fluid description of the decay with the scalar field description prior to coherent oscillations. We then used theoretical bounds and observations to narrow down parameter space of the curvaton model, for different potentials, namely the VEV of the curvaton $\chi^*$ and the duration of decay $m_\chi/\Gamma_\chi$. The normalisation of the power spectrum $P_\zeta$ was the only observational constraint imposed while the tensor-to-scalar ratio $r_T$ and the value of local non-Gaussianity $f_{NL}$ were forecasted for different model parameters. We did this for three different curvaton potentials: the quadratic potential, axion-like cosine potentials and hyperbolic potentials. For a quadratic potential we arrived to analytical approximations for $f_{NL}$ and $H_*$ in two different regimes: an inefficient decay, $R_\chi \ll 1$; and for a very efficient transfer of curvature perturbations $R_\chi \sim 1$. Using the current bounds on $r_T$ and $f_{NL}$ we obtained lower bounds on the duration of coherent oscillations of the curvaton. Considering self-interactions only alters the constraints when the interactions become important ($\chi^* \sim f$). As expected both the cosine and the hyperbolic potentials recover the quadratic regime when $\chi^* \ll f$.

If we take a stochastic approach for the distribution of the curvaton field values we find the variance of the Gaussian distribution function to be Eq. 5.70. Approximating the field value by the square-root of the variance we checked that for the observationally allowed parameter space is consistent with a late decaying and effectively massless scalar field.
Chapter 6

Primordial non-Gaussianity from mixed inflaton-curvaton perturbations

The standard model of inflation is described by a light scalar field $\phi$, the inflaton, whose slow-roll controls the potential energy that drives the accelerated expansion. When inflation ends the inflaton potential energy rapidly decays into radiation. Any light scalar field during slow-roll inflation will acquire a nearly scale-invariant spectrum of perturbations at Hubble exit, $k = aH$, and any such field can, in principle, source structure in the Universe. As we have seen in Chapter 5, the curvaton, $\chi$, is a light, weakly-coupled field during inflation whose energy density is negligible during inflation, but if the field remains weakly coupled at the end of inflation, its energy density can grow relative to radiation after inflation, and perturbations in the curvaton field can lead to primordial density perturbations when the curvaton decays into radiation [69, 70, 71, 72, 74, 120, 121]. A distinctive feature of the curvaton model is that it can leave behind significant local-type non-Gaussianity in the primordial density distribution, $|f_{NL}| \gg 1$.

In chapter 5 [92, 122] have used primordial non-Gaussianity, characterised by the non-linearity parameter $f_{NL}$, and the tensor-to-scalar ratio, $r_T$, to constrain curvaton model parameters. In the simplest model of a curvaton with a quadratic potential, the amplitude of primordial density perturbations, together with a measurement of $f_{NL}$ and $r_T$ would fix the energy scale of inflation, the initial curvaton vacuum expectation value (VEV) and the dimensionless curvaton decay rate. A self-interacting curvaton would introduce additional model parameters that could be fixed by scale-dependent $f_{NL}$ and higher-order correlators. These studies focussed solely on density perturbations coming from the curvaton field. But if inflation is driven by a slow-roll inflaton, then there are inevitably fluctuations in the inflaton field too which lead to some level of density...
perturbations when inflation ends and the inflaton energy decays into radiation. Infla-
ton perturbations are adiabatic and thus lead to constant curvature perturbations, \( \zeta \), on super-Hubble scales, whose local-type non-Gaussianity is always small, \( |f_{\text{NL}}| < \mathcal{O}(1) \).

In this chapter we consider the possibility that both fields contribute for the primordial density perturbation. In particular we note the possibility that while the inflaton contribution to the primordial power spectrum may dominate over that from the curvaton, the curvaton can nonetheless source significant non-Gaussianity. Bartolo and Liddle \[123\] were the first to consider in what regime the linear curvaton or inflaton field perturbations would dominate the primordial power spectrum in a simple model of two massive scalar fields. Ichikawa et al \[117\] also considered non-Gaussianity and the tensor-scalar ratio in several classes of inflation models, including chaotic, hybrid, and new inflation. In this paper we will evaluate the relative contribution of curvaton field fluctuations to the primordial power spectrum for a general inflaton potential. The contribution of adiabatic inflaton perturbations is given relative to the tensor power spectrum by the slow-roll parameter, \( \epsilon^\star \). Here we define an analogous parameter, \( \epsilon_c \), describing the contribution of curvaton perturbations relative to the tensor power spectrum. The second inflaton slow-roll parameter, \( \eta \), then affects only the scale-dependence of the power spectrum.

Very recently Kobayashi and Takahashi have considered the scale-dependence of both the power spectrum and the non-Gaussianity including mixed inflaton-curvaton models \[124\]. Note that Langlois and Vernizzi \[114\] studied linear adiabatic and isocurvature density perturbations allowing for a general inflaton potential, which was then extended \[36\] to second- and higher-order to study non-Gaussianity in both adiabatic and residual isocurvature density perturbations after inflation and their correlation. In the following we assume that all species are in thermal equilibrium after the curvaton decays and thus there are no residual isocurvature modes \[125, 126\].

In Section 6.1 we will briefly review the origin of density perturbations coming from both inflaton and curvaton field perturbations, and the spectrum of tensor metric perturbations (gravitational waves). We will review both the linear transfer of field perturbations into radiation, and the non-linear transfer for curvaton perturbations at second- and third-order in the field perturbations which can give rise to a non-vanishing primordial bispectrum and trispectrum. In Section 6.2 we present our results based on numerical solutions of the curvaton field evolution after inflation \[122\] and previous numerical studies of curvaton decay based on a fluid description of the curvaton at late times \[81, 82, 108, 109\]. We focus primarily on the simplest curvaton model with a quadratic potential with a fixed mass, \( m_\chi \). In this case significant non-Gaussianity arises only when the curvaton is sub-dominant when it decays, \( R_\chi \ll 1 \) and we derive a consistency relation between the bispectrum and trispectrum parameters which holds even in the mixed inflaton+curvaton case. The third-order non-linearity parameter, \( g_{\text{NL}} \), remains
small, $g_{NL} \ll f_{NL}^2$, even in the mixed scenarios. Scale-dependence of $f_{NL}$ may distinguish the mixed inflaton+curvaton model from the curvaton limit for a quadratic curvaton potential. We also examine self-interacting curvaton models, including a cosine potential, which introduce an additional mass scale, $f$, where self-interaction terms become important for $\chi_* \sim f$. Self-interactions can produce significant non-Gaussianity even if the curvaton dominates when it decays, $R_\chi \simeq 1$. Self-interacting curvatons produce large third-order non-linearity parameter, $g_{NL}$, as well as scale-dependent $f_{NL}$.

6.1 Density perturbations from inflation

6.1.1 Background evolution

We take both the inflaton and the curvaton to be in a slow-roll regime during inflation but assume that the Friedmann equation is dominated by the inflaton potential energy

$$H^2 \simeq \frac{V(\phi)}{3m_{Pl}^2},$$

where $m_{Pl}$ is the reduced Planck mass, $m_{Pl}^{-2} = 8\pi G_N$.

We define the slow-roll parameters

$$\epsilon \equiv -\frac{\dot{H}}{H^2},$$

and

$$\epsilon_A \equiv \frac{1}{2} m_{Pl}^2 \left( \frac{V_A}{V} \right)^2,$$

$$\eta_{AB} \equiv \frac{m_{Pl}^2 V_{AB}}{V},$$

where $V_A \equiv \partial V/\partial A$. Note that in the slow roll approximation $\epsilon \simeq \sum A \epsilon_A \ll 1$ and in the curvaton scenario we assume that $\epsilon_\chi \ll \epsilon_\phi$ so that $\epsilon \simeq \epsilon_\phi$. We also assume the fields are decoupled so that $\eta_{\phi\chi} = 0$.

6.1.2 Perturbations during inflation

During inflation, any light scalar fields (with effective mass less than the Hubble scale, $|\eta| < 1$) acquire a spectrum of perturbations due to vacuum fluctuations on sub-Hubble
scales being stretched up to super-Hubble scales by the accelerated expansion. In particular the curvaton and inflaton field perturbations on spatially-flat hypersurfaces at Hubble exit have a power spectrum

$$\mathcal{P}_{\delta \phi^*} \simeq \mathcal{P}_{\delta \chi^*} \simeq \left( \frac{H_*}{2\pi} \right)^2,$$

where we neglect slow-roll corrections, including the cross-correlation between inflaton and curvaton perturbations [127].

Since the inflaton determines the energy density during inflation, inflaton field perturbations on spatially-flat hypersurfaces, $\delta \phi$, correspond to adiabatic curvature perturbations on uniform-density hypersurfaces at Hubble exit, $\zeta^* = -(H \delta \phi/\dot{\phi})_*$ and hence we have

$$\mathcal{P}_{\zeta^*} = \mathcal{P}_{\zeta^\phi} \simeq \frac{1}{2 m_{Pl} \epsilon_*} \left( \frac{H_*}{2\pi} \right)^2.$$  (6.6)

On the other hand curvaton fluctuations are isocurvature field perturbations during inflation, $\epsilon_\chi \ll \epsilon_\phi$, and remain effectively frozen, $\dot{\chi} \simeq 0$, and hence are gauge-invariant during inflation. In particular we can identify curvaton field perturbations on spatially flat hypersurfaces with relative entropy perturbations [26]

$$S_{\chi^*} \propto \left( \delta \chi - \frac{\ddot{\chi}}{\dot{\phi}} \delta \phi \right)_* \simeq \delta \chi_*.$$  (6.7)

The power spectrum of free gravitational waves (tensor metric perturbations), like light scalar fields, only depends on the inflation scale at Hubble exit

$$\mathcal{P}_{T^*} = \frac{8}{m_{Pl}^2} \left( \frac{H_*}{2\pi} \right)^2.$$  (6.8)

The tensor-to-scalar ratio is defined by

$$r_T \equiv \frac{\mathcal{P}_T}{\mathcal{P}_\zeta}.$$  (6.9)

Both the adiabatic curvature perturbation and the tensor perturbations remain constant on super-Hubble scales, so we have a tensor-scalar ratio during inflation [103]

$$\left( \frac{\mathcal{P}_T}{\mathcal{P}_{\zeta^\phi}} \right)_* = 16 \epsilon_*.$$  (6.10)

The tensor spectral index is due solely to the variation of the Hubble scale during inflation

$$n_T \equiv \frac{d \ln \mathcal{P}_T}{d \ln k} = -2 \epsilon_*.$$  (6.11)
However the inflaton field and curvaton field evolve on super-Hubble scales due to their effective mass, and gravitational coupling for the inflaton field, so their final spectral tilts, prior to curvaton decay, are given to leading order by [103]

\[ n_\phi - 1 \equiv \frac{d \ln P_\phi}{d \ln k} = -6\epsilon_* + 2\eta_{\phi\phi}, \]  
\[ n_\chi - 1 \equiv \frac{d \ln P_\chi}{d \ln k} = -2\epsilon_* + 2\eta_{\chi\chi}. \]  

(6.12)  
(6.13)

### 6.1.3 End of inflation and after

At the end of inflation the inflaton decays completely into radiation transferring its curvature perturbation to the radiation, \( \zeta_R = \zeta_\phi \). We assume that reheating or preheating does not alter the power spectrum of the adiabatic density perturbation on large scales, nor does it alter the fluctuations of the curvaton field on large (super-Hubble) scales.

The curvaton stays in an over-damped regime until the Hubble rate drops to \( H \simeq m_\chi \).

At this point the curvaton starts oscillating and behaves like a pressureless matter fluid. (We will not consider the possibility of the curvaton driving a second period of inflation [128], i.e., we assume \( \chi_* < m_{Pl} \).) Once the curvaton starts evolving like a pressureless fluid we can write its local energy density on uniform-total-density hypersurfaces, \( \rho_\chi \), in terms of its homogeneous value, \( \bar{\rho}_\chi \), and the inhomogeneous entropy perturbation [35, 36]

\[ \rho_\chi = \bar{\rho}_\chi e^{3(\zeta_\chi - \zeta_\phi)} = \bar{\rho}_\chi e^{S_\chi}, \]  
\[ \text{where } S_\chi \equiv 3 (\zeta_\chi - \zeta_\phi) \text{ is the non-adiabatic part of the curvaton perturbation. One should note that in the standard curvaton scenario one takes } \zeta_\chi \gg \zeta_\phi, \text{ hence } S_\chi \sim 3\zeta_\chi. \]  

In the mixed inflaton-curvaton case this may no longer hold, therefore the quantity to use is \( S_\chi \). When the expansion rate drops to \( H \sim m_\chi \) the curvaton starts oscillating in the bottom of its potential, behaving like a pressureless, non-interacting fluid. At later times, but before the curvaton decays, the potential of the curvaton field can be well approximated by a quadratic potential and its time-averaged energy density can be described by

\[ \rho_\chi = \frac{1}{2} m_\chi^2 |\chi|^2. \]  
\[ \text{(6.15)} \]

One can use Eq. (6.14) to determine the relation between the entropy perturbations of the curvaton and its field fluctuations during inflation. In the beginning of oscillation we have

\[ \bar{\rho}_\chi e^{S_\chi} = \frac{1}{2} m_\chi^2 \chi_{osc}^2, \]  
\[ \text{(6.16)} \]

where \( \bar{\rho}_\chi = m_\chi^2 \bar{\chi}_{osc}^2/2 \). Note that the subscript \( osc \) stands for beginning of oscillations. Let’s define \( \bar{\chi}_{osc} \equiv g(\chi_*) \) where \( g \) accounts for non-linear evolution of the field between
inflation and oscillations [82]. If the curvaton potential is quadratic and we can neglect the self-gravity of the curvaton, we expect linear evolution. On the other hand if it is not quadratic throughout all evolution we need to correct the field perturbations. It is convenient to expand $\chi_{\text{osc}}$ in terms of field perturbations during inflation, $\delta\chi_{*}$, i.e.,

$$
\chi_{\text{osc}} \simeq g + g'\delta\chi_{*} + \frac{1}{2} g''\delta\chi_{*}^2 + \ldots ,
$$

(6.17)

where primes denote derivatives with respect to $\chi_{*}$. Expanding both sides of Eq. (6.16) up to second order we find that the curvaton entropy perturbation is

$$
S_{\chi} = 2\frac{g'}{g}\delta\chi_{*} + \left[ \frac{g''}{g} - \left( \frac{g'}{g} \right)^2 \right] \delta\chi_{*}^2 + \mathcal{O}(\delta\chi_{*}^3).
$$

(6.18)

One should note that for a non-interacting, isocurvature field, $\delta\chi_{*}$ is a Gaussian random field. Therefore we can separate the curvaton entropy perturbation in a Gaussian linear part and second order term as

$$
S_{\chi} = S_G + \frac{1}{4} \left( \frac{gg''}{g^2} - 1 \right) S_G^2
$$

(6.19)

where

$$
S_G \equiv 2\frac{g'}{g}\delta\chi_{*}.
$$

(6.20)

Hence, using Eqs. (6.5) and (6.20), the power spectrum of entropy perturbations in the curvaton is, at leading order, given by

$$
P_{S_{\chi}} = P_{S_G} = 4 \left( \frac{g'}{g} \right)^2 \left( \frac{H_*}{2\pi} \right)^2.
$$

(6.21)

### 6.1.4 Transfer of linear perturbations

The curvaton decays into radiation when $H \simeq \Gamma_{\chi}$. We will consider that the curvaton (and the inflaton) decay prior to CDM freeze-out. Therefore we won’t consider any residual isocurvature perturbations after curvaton decay [125].

The primordial density perturbation produced by curvaton decay can be estimated using the sudden decay approximation [71]. This assumes that the curvaton happens instantaneously on the total-uniform-density hypersurface $H = \Gamma_{\chi}$. Before the curvaton decays, $\zeta_R = \zeta_{\phi}$. Therefore we know that on the sudden decay hypersurface we have

$$
\rho_\gamma = \bar{\rho}_\gamma e^{4(\zeta_{\phi} - \zeta)}, \quad \rho_\chi = \bar{\rho}_\chi e^{3(\zeta_{\chi} - \zeta)}.
$$

(6.22)
At sudden decay we have that the final radiation energy density \( \bar{\rho} = \rho_\gamma + \rho_\chi \). Hence after decay we have

\[
(1 - \Omega_\chi) e^{4(\zeta_\phi - \zeta)} + \Omega_\chi e^{3(\zeta_\chi - \zeta)} = 1.
\] (6.23)

After the decay we have a constant curvature perturbation on super-Hubble scales. Expanding Eq. (6.23) to first order, we have

\[
\zeta = R_\chi \zeta_\chi + (1 - R_\chi) \zeta_\phi,
\] (6.24)

\[
= \zeta_\phi + \frac{R_\chi}{3} S_\chi.
\] (6.25)

where [71, 74]

\[
R_\chi = \frac{3\Omega_\chi}{4 - \Omega_\chi}_{\text{dec}}.
\] (6.26)

Since the adiabatic inflaton field perturbations and the isocurvature curvaton field fluctuations (6.5) are uncorrelated, the power spectrum of the total primordial curvature perturbations, after curvaton decay, is given by

\[
\mathcal{P}_\zeta = \mathcal{P}_{\zeta_\phi} + \frac{R_\chi^2}{9} \mathcal{P}_{S_\chi},
\] (6.27)

Following Eq. (6.6), and using Eq. (6.21), we can write this as

\[
\mathcal{P}_\zeta = \frac{1}{2m_{Pl}^2} \left( \frac{1}{\epsilon_\ast} + \frac{1}{\epsilon_c} \right) \left( \frac{H_s}{2\pi} \right)^2.
\] (6.28)

where we define a quantity

\[
\epsilon_c \equiv \frac{9}{8} \left( \frac{g}{g^m_{Pl}} \right)^2 \frac{1}{R_\chi^2}.
\] (6.29)

The curvaton contribution to the primordial power spectrum corresponds to

\[
(2m_{Pl}^2 \epsilon_c)^{-1} \left( \frac{H_s}{2\pi} \right)^2;
\] (6.30)

i.e., \( \epsilon_c \) plays the same role for the curvaton contribution to the final power spectrum as \( \epsilon_\ast \) does for \( \mathcal{P}_{\zeta_\phi} \) in Eq. (6.6). Thus \( \epsilon_c \) marks the critical value of \( \epsilon_\ast \) between inflaton-domination of the primordial power spectrum and curvaton-domination of the power spectrum. It follows that we can write

\[
\omega_\chi \equiv \frac{\epsilon_\ast}{\epsilon_\ast + \epsilon_c}.
\] (6.31)
$w_\chi$ can be seen as the function that weighs the curvaton contribution to the final power spectrum. For $\epsilon_* \gg \epsilon_c$ the curvaton is the dominant contributor to scalar perturbations and $w_\chi \simeq 1$. In the opposite regime, $\epsilon_* \ll \epsilon_c$, the inflaton dominates the primordial power spectrum and $w_\chi \ll 1$.

The spectral index of the primordial power spectrum is then given by

$$n_\zeta - 1 \equiv \frac{d \ln P_\zeta}{d \ln k} = w_\chi (n_\chi - 1) + (1 - w_\chi)(n_\phi - 1). \quad (6.32)$$

Substituting the tilts (6.12) and (6.13) for each field in (6.32) we have [103, 114]

$$n_\zeta - 1 = -2 \epsilon_* + 2 \eta_\chi w_\chi + (1 - w_\chi)(-4 \epsilon_* + 2 \eta_\phi) \cdot \quad (6.33)$$

The running of the power spectrum, assuming slow roll inflation and neglecting curvaton-inflaton interactions, is given by

$$\alpha_\zeta \equiv \frac{d n_\zeta}{d \ln k} = w_\chi \alpha_\chi + (1 - w_\chi)\alpha_\phi + w_\chi (1 - w_\chi) (n_\chi - n_\phi)^2 \cdot \quad (6.34)$$

with [42, 104]

$$\alpha_\phi = 16 \epsilon_* \eta_\phi - 24 \epsilon_*^2 - 2 \epsilon_\phi^2, \quad (6.35)$$

$$\alpha_\chi = 4 \epsilon_* (-2 \epsilon_* + \eta_\phi + \eta_\chi), \quad (6.36)$$

where $\xi_\phi^2 = (\partial^4 V/\partial \phi^4)/9H^4$.

### 6.1.5 Non-linearities

Using the $\delta N$ formalism we identify the non-linear curvature perturbation, $\zeta$, with the perturbed expansion up to a final uniform-density hypersurface, $N$, as a function of the local field values on super-Hubble scales during inflation [38]

$$\zeta = \sum_A N_A \delta \varphi_A + \frac{1}{2} \sum_{A,B} N_{AB} \delta \varphi_A \delta \varphi_B + \frac{1}{6} \sum_{A,B,C} N_{ABC} \delta \varphi_A \delta \varphi_B \delta \varphi_C + \ldots, \quad (6.37)$$

where $N_A \equiv d N/d \phi_A$ and $N_{AB} \equiv d^2 N/d \phi_A d \phi_B$, etc. We define the first non-Gaussianity parameter in terms of the amplitude of quadratic contribution to $\zeta$ relative to the linear terms:

$$f_{NL} = \frac{5}{6} \sum_{AB} N_A N_B N_{AB}, \quad (6.38)$$

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If we consider terms in Eq. (6.23) up to second order we find [36]

\[
\zeta = R_\chi \zeta_\chi + (1 - R_\chi) \zeta_\phi + \frac{R_\chi(1 - R_\chi)(3 + R_\chi)}{2} (\zeta_\chi - \zeta_\phi)^2 \quad (6.39)
\]

\[
= \zeta_\phi + \frac{R_\chi}{3} S_\chi + \frac{R_\chi(1 - R_\chi)(3 + R_\chi)}{18} S_\chi^2. \quad (6.40)
\]

Plugging (6.19) into (6.40) we find

\[
\zeta = \zeta_\phi + \frac{R_\chi}{3} S_G + \frac{R_\chi}{18} \left[ 3 \left( 1 + \frac{g g''}{g'^2} \right) - 2 R_\chi - R_\chi^2 \right] S_G^2. \quad (6.41)
\]

It is straightforward to see from Eqs. (6.6), (6.21) and (6.27) that

\[
N_\chi = R_\chi \frac{2g'}{3g}, \quad (6.42)
\]

\[
N_\phi = \frac{1}{\sqrt{2\epsilon_* m^2_{Pl}}}. \quad (6.43)
\]

We only need to consider the linear terms from the inflaton since \( N_{\phi\phi} \ll (N_\phi)^2 \) and \( N_{\phi\chi} = 0 \). Then, the first non-Gaussian parameter (6.38) for curvaton plus inflaton simplifies to [36]

\[
f_{NL} = \frac{5}{6} \frac{N_{\chi\chi}}{N_\chi^2} w_\chi^2. \quad (6.44)
\]

It follows directly from Eq. (6.41) that \( f_{NL} \) is given by

\[
f_{NL} = \left[ \frac{5}{4R_\chi} \left( 1 + \frac{g g''}{g'^2} \right) - \frac{5}{3} - \frac{5}{6} R_\chi \right] w_\chi^2. \quad (6.45)
\]

Note that taking the derivative of Eq. (6.42) with respect to \( \chi_* \) we get

\[
N_{\chi\chi} = \frac{2}{3} R_\chi \left( \frac{g''}{g} - \frac{g'^2}{g^2} \right) + R_\chi \frac{2g'}{3g}. \quad (6.46)
\]

Then, using Eq. (6.44) we find the general expression for \( f_{NL} \) in the mixed curvaton-inflaton scenario

\[
f_{NL} = \left[ \frac{5}{4R_\chi} \left( 1 + \frac{g'g}{g'^2} \right) + \frac{5}{4} \frac{R_\chi g'/g' - 2R_\chi}{R_\chi^2} \right] w_\chi^2. \quad (6.47)
\]

Comparing this with Eq. (6.45) obtained in the sudden-decay case we have [82]

\[
R_\chi \frac{g'}{g'} = 2R_\chi - \frac{4}{3} R_\chi^2 - \frac{2}{3} R_\chi^3. \quad (6.48)
\]
The third order non-linear parameters are $g_{NL}$ and $\tau_{NL}$. They are defined by \[38, 40\]

$$g_{NL} \equiv \frac{25}{54} \sum_{ABC} N_A N_B N_C \frac{N_{ABC}}{\sum_{AB} N_A N_B \delta_{AB}}^3,$$  \hfill (6.49)

$$\tau_{NL} \equiv \sum_{ABC} N_A N_B N_A C N_B C \frac{N_{ABC}}{\sum_{AB} N_A N_B \delta_{AB}}^3.$$

\hfill (6.50)

For the inflaton+curvaton case Eqs. (6.49) and (6.50) reduce to

$$g_{NL} = \frac{25}{54} \frac{N_{XXX}}{N^3_x} w^3_x,$$  \hfill (6.51)

$$\tau_{NL} = \frac{N^2_x}{N^3_x} w^3_x.$$  \hfill (6.52)

Taking the third derivative of $N$ with respect to $\chi$, we find

$$N_{XXX} = \frac{2}{3} R_{X,dec} \left( \frac{g'''}{g} - 3 \frac{g'' g'}{g^2} + 2 \frac{g''^2}{g^3} \right) + \frac{4}{3} \frac{R'_{X,dec}}{R_X^3} \left( \frac{g''}{g} - \frac{g' g^2}{g^2} \right) + \frac{R''_{X,dec}}{R_X^3} \frac{2 g'}{3g}.$$  \hfill (6.53)

Substituting Eqs. (6.42) and (6.53) into Eq. (6.51) we get

$$g_{NL} = \frac{25}{24} \left[ \frac{1}{R_X^2} \left( \frac{g'''}{g} - 3 \frac{g'' g'}{g^2} + 2 \right) + \frac{2 R'_{X,dec}}{R_X^3} \left( \frac{g''}{g} - \frac{g' g^2}{g^2} \right) + \frac{R''_{X,dec}}{R_X^3} \frac{g'}{g} \right] w^3_x.$$  \hfill (6.54)

Using Eq. (6.48) for the sudden-decay approximation we can eliminate the derivatives of $R_X$ to obtain \[82\]

$$g_{NL} = \frac{25}{54} \left[ \frac{9}{4R_X^2} \left( \frac{g'''}{g} + 3 \frac{g'' g'}{g^2} \right) - \frac{9}{R_X} \left( 1 + \frac{g''}{g g'} \right) + \frac{1}{2} \left( 1 - 9 \frac{g''}{g g'} \right) + 10 R_X + 3 R_X^2 \right] w^3_x.$$  \hfill (6.55)

Equations (6.47) and (6.54) do not rely on the sudden decay approximation. Nonetheless Ref. \[82\] showed that the sudden-decay formulas (6.45) and (6.55) do give a good fit to $f_{NL}(R_X)$ and $g_{NL}(R_X)$ from the full numerical solution with continuous decay. For example, $f_{NL}(R_X)$ is accurate to 1% for $f_{NL} > 60$. Nonetheless in the following we will use Eq. (6.47) and (6.54).

Note that from Eqs. (6.44) and (6.52) we have

$$\tau_{NL} = \frac{1}{w_X} \left( \frac{6}{5} f_{NL} \right)^2.$$  \hfill (6.56)

The inequality $\tau_{NL} \geq (6 f_{NL}/5)^2$ is an important test of non-Gaussianity the mixed curvaton+inflaton scenario and multi-field scenarios in general \[41\] with equality only in the curvaton limit, $w_X \to 1$. 

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6.2 Numerical results

We now wish to compute observable quantities such as $f_{\text{NL}}$, $g_{\text{NL}}$ and $r_T$ for different model parameters. We assume that the curvaton is effectively frozen during inflation, which should be a good approximation while the effective curvaton mass is much less than the inflationary Hubble scale, i.e., $\eta_{XX} \ll 1$. We will then numerically solve for the local evolution of the curvaton field during the radiation- and, possibly, curvaton-dominated epochs after inflation, until the curvaton starts oscillating in the minimum of its potential and behaves like a pressureless fluid, but before it decays. We allow approximately $10^3$ oscillations, i.e., we assume sufficiently slow decay, $\Gamma_\chi/m_\chi < 10^{-3}$, consistent with the hypothesis that the curvaton is weakly coupled to other fields.

We numerically solve the Klein-Gordon equation for $\chi$ prior to decay

$$\ddot{\chi} + 3H\dot{\chi} + V_\chi = 0,$$

where the Friedmann equation takes the form

$$H^2 = \frac{1}{3m_{Pl}^2} (\rho_\gamma + \rho_\chi),$$

and the curvaton density is given by $\rho_\chi = \dot{\chi}^2/2 + V(\chi)$. This allow us to take different potentials for the curvaton field.

The initial conditions to solve Eq. (6.57) are $\chi_i \approx \chi_*$ and $\dot{\chi}_i \approx -V_\chi/3H_*$, since the curvaton is slow-rolling down its potential. We take the universe to be radiation dominated initially, $\rho_{\gamma,i} \gg \rho_{X,i}$ and hence $H_*^2 \gg V(\chi_*)/3m_{Pl}^2$. In practice we set $\rho_{\gamma,i} = 10^{41}m_\chi^2 \text{GeV}^2$ in our numerical solutions, where $m_\chi$ is the mass of the curvaton at late times. This ensures that $H_* > 100m_\chi$ and $\rho_{\gamma,i} \gg m_\chi^2/3m_{Pl}^2$ for any $\chi_i < m_{Pl}$.

Once the curvaton starts oscillating we can compute the quantity 

$$p_{FW} \equiv \Omega_\chi (1 - \Omega_\chi)^{-3/4} \sqrt{\frac{H}{m_\chi}},$$

which becomes constant during oscillations as $\rho_\chi \rightarrow \text{const}/a^3$ and we have $g^2 = \chi_{osc}^2 \propto p_{FW}$. In this way we connect the scalar field description of the curvaton with a fluid description which has previously been used to numerically study the decay of the curvaton [81, 82, 108, 109] and has been used in Chapter 5. Following [122] we can compute the efficiency parameter $R_\chi$ using the fitting formula [109]

$$R_\chi(p_{FW}) \approx 1 - \left(1 + \frac{0.924}{1.24} \sqrt{\frac{m_\chi^2/p_{FW}}{\Gamma_\chi}}\right)^{-1.24}.$$
We find that, for a given curvaton potential, $p_{FW}$ is dependent only of the initial curvaton field value, $\chi_i$. Therefore $R_\chi$ is dependent only upon the initial curvaton field value and the dimensionless decay rate, $\Gamma_\chi/m_\chi$.

Going beyond Chapter 5 [122] we will include inflaton perturbations in addition to curvaton field perturbations in our computation of the primordial density perturbation. However the inflaton perturbations represent adiabatic perturbations on super-Hubble scales, i.e., local perturbations along the same background trajectory [26], and they can be treated independently of the curvaton field perturbations. Looking at the total power spectrum, Eq. (6.27) we see that we have gained an extra free degree of freedom, $\epsilon_*$, with respect to the purely curvaton limit ($\epsilon_* \gg \epsilon_c$). Therefore our free parameters will be $\Gamma_\chi/m_\chi$, $\chi_*$ and $\epsilon_*$, for the quadratic curvaton [122]. Going beyond the quadratic curvaton potential we will consider models where self-interactions lead to the potential becoming flatter or steeper beyond a characteristic mass scale, $f_i$, introducing one new parameter in addition to the curvaton mass about the minimum of its potential.

WMAP 7 [15] gives $P_\zeta \simeq 2.43 \times 10^{-9}$ for the amplitude of the power spectrum of curvature perturbations. We will use this observational constraint to fix the inflationary scale. Since the power spectrum (6.27) is proportional to $H^2$ we can find a value of $H_*$ that gives the correct power for any values of the other curvaton model parameters. Using Eqs. (6.28) and (6.31) we arrive to the constraint equation

$$H_* = 2\sqrt{2\pi}m_{Pl}\left(\epsilon_*^{-1} + \epsilon_c^{-1}\right)^{-1/2} P_\zeta^{1/2}, \quad (6.61)$$

where the value of $\epsilon_c$ is determined numerically via the formula

$$\epsilon_c = \frac{9}{2}\left(\frac{p_{FW}}{p_{FW}^\prime m_{Pl} R_\chi}\right)^2. \quad (6.62)$$

The tensor-to-scalar ratio is then

$$r_T = 16w_\chi \epsilon_* = 16\epsilon_*(1 - w_\chi). \quad (6.63)$$

### 6.2.1 Quadratic potential

The simplest potential consistent with a curvaton scenario is a quadratic potential

$$V(\chi) = \frac{1}{2}m_\chi^2 \chi^2, \quad (6.64)$$

and we will see that it is also a good description of the behavior in more general cases when the curvaton is sufficiently close to the minimum of its potential.
We start by studying the relative contribution of the curvaton to the scalar power spectrum characterised by parameter $\epsilon_c$ defined in Eq. (6.29). In Figure 6.1 we plot $\epsilon_c$ as a function of the initial curvaton VEV, $\chi_*$, and the dimensionless decay time $m/\Gamma$.

While the curvaton remains subdominant in the radiation era, we have an analytic solution for the curvaton field [114, 117] and we find [122]

$$p_{FW} \simeq \frac{1.046\chi_*^2}{3m_{Pl}^2}. \quad (6.65)$$

We can clearly identify the two analytic regimes in Figure 6.1

$$\epsilon_c \simeq \begin{cases} 1.125 \left( \frac{\chi_*}{m_{Pl}} \right)^2 & \text{for } R_\chi \simeq 1, \\ 10.8 \frac{\Gamma_\chi}{m_\chi} \left( \frac{m_{Pl}}{\chi_*} \right)^2 & \text{for } R_\chi \ll 1, \end{cases} \quad (6.66)$$

corresponding to the straight lines in Figure 6.1.

If we now include the contribution from inflaton perturbations to the total scalar perturbation, we can identify 3 regimes of interest which depend on the value of $\epsilon_*$ for a given $\epsilon_c$:

1. The curvaton limit corresponds to $\epsilon_c \ll \epsilon_*$. In this case most of the structure in the universe comes from the curvaton, i.e., $w_\chi \simeq 1$ from Eq. (6.31). This case has been studied in Chapter 5 [122] and in most of the curvaton literature. We can identify

![Figure 6.1: The plot show contours of $\epsilon_c$ defined in Eq. 6.29 as a function of the curvaton parameters, $\chi_*$, the curvaton VEV, and $m_\chi/\Gamma_\chi$ the dimensionless decay time, for the quadratic curvaton potential, Eq. (6.64). Curvaton perturbations dominate the primordial scalar power spectrum for $\epsilon_* \gg \epsilon_c$.](image)
this limit in Figure 6.1 for a fixed value of \( \epsilon_* \) as the region inside the contours towards the right of the plot, i.e., for long decay times (\( \Gamma_\chi \ll m_\chi \)).

From Eq. (6.63) we have in the curvaton limit

\[
\mathcal{r}_T \simeq 16 \epsilon_c.
\] (6.67)

Therefore upper bounds on the tensor-to-scalar ratio place constraints on \( \epsilon_c \) but do not directly constrain \( \epsilon_* \) since \( \epsilon_* \gg \epsilon_c \).

In the curvaton limit, \( w_\chi \simeq 1 \), and assuming an effectively massless curvaton, \( \eta_\chi \ll 1 \), then Eq. (6.33) gives a red spectral tilt, \( n_\zeta - 1 \simeq -2 \epsilon_* \). In this limit the tilt gives a measurement of the first slow roll parameter, \( \epsilon_* \). Consider a fiducial value \( n_\zeta \simeq 0.96 \) consistent with WMAP7 [15]. For this value of \( \epsilon_* \simeq 0.02 \) we can identify the curvaton limit with the region to the right of the contour \( \epsilon_c = 0.02 \) in Figure 6.1.

2. The second limit of interest is \( \epsilon_c \gg \epsilon_* \). This is the case for which curvaton perturbations are sub-leading in the scalar power spectrum, i.e., \( w_\chi \ll 1 \) in Eq. (6.31). These regions correspond to a parameter range where the decay happens too fast (bottom left of the plot), or the curvaton VEV is too big (top) suppressing the curvaton power spectrum. The region \( \epsilon_c \gtrsim 1 \), in Fig. 6.1, will always be in this inflaton dominated limit in slow-roll inflation since \( \epsilon_* \ll 1 \).

The tensor-scalar ratio \( \mathcal{r}_T \) directly constrains the slow-roll parameter \( \epsilon_* \) in the this limit. From Eq. (6.63) we have

\[
\mathcal{r}_T \simeq 16 \epsilon_* .
\] (6.68)

The spectral tilt of the primordial scalar power spectrum (6.33) is determined by the usual inflaton slow-roll parameters, \( n_\zeta - 1 \simeq -6 \epsilon_* + 2 \eta_\phi \phi \) for \( w_\chi \ll 1 \).

In this limit the presence of the curvaton may still be important to as a source of primordial non-Gaussianity or residual isocurvature perturbations after the curvaton decays [36].

3. The third region of parameter space corresponds to \( \epsilon_* \sim \epsilon_c \) which corresponds to a mixed scenario. In this case the tensor-scalar ratio (6.63) no longer directly constrains \( \epsilon_* \) or \( \epsilon_c \) but the combination

\[
\mathcal{r}_T = \frac{16 \epsilon_c \epsilon_*}{\epsilon_c + \epsilon_*}.
\] (6.69)
For example, an observed tensor-scalar ratio, \( r_T \), places a lower bound on the slow-roll parameter, \( \epsilon_* \geq r_T / 16 \).

In Figure 6.2 we show contour plots for the non-Gaussianity parameter, \( f_{NL} \), and the tensor-scalar ratio, \( r_T \), for the case \( \epsilon_* = 0.02 \). The thin black dotted line is the contour line \( \epsilon_c = 0.02 \) which marks the borderline between the region (1) described above with curvaton-dominated primordial power spectrum and region (2), inflaton-dominated. The curvaton limit, region (1), lies to the right of the \( \epsilon_c = 0.02 \) contour.

We also plot the current observational upper bound on the tensor-scalar ratio, \( r_T \lesssim 0.24 \) [15]. For a given value of \( \epsilon_* \), the contours of the tensor-to-scalar ratio follow the contours of \( \epsilon_c \) plotted in Figure 6.1, as expected from Eq. (6.69). However, rather then growing without bound as \( \epsilon_c \) becomes large, as happens if we consider only the curvaton perturbations [122], in the presence of a finite \( \epsilon_* \) the tensor-scalar ratio saturates with \( r_T \to 16 \epsilon_* / \epsilon_c \) in region (2) where \( \epsilon_c \gg \epsilon_* \). For \( \epsilon_* = 0.02 \), for example, the tensor-scalar ratio is bounded so that \( r_T \leq 0.32 \).

Similarly the inflaton’s (Gaussian) contribution to the primordial scalar power spectrum suppresses the non-linearity parameter \( f_{NL} \) for \( \epsilon_c > \epsilon_* \) in region (2). We see that the largest values for \( f_{NL} \) occur in region (3), near the boundary between the curvaton- and inflaton-dominated power spectra, where \( \epsilon_c \simeq \epsilon_* \). In the absence of any inflaton perturbations \( (w_{\chi} = 1) \), the non-Gaussianity continues to grow without bound as \( \chi_*/m_{Pl} \to 0 \) for a fixed value of \( m_{\chi}/\Gamma_{\chi} \) [122]. But \( \epsilon_* \) also becomes large as \( \chi_*/m_{Pl} \to 0 \) and therefore the inflaton perturbations dominate the primordial power spectrum. From Eq. (6.47) we see that \( f_{NL} \) is suppressed by an additional factor \( w_{\chi} \simeq \epsilon_*^2 / \epsilon_c^2 \) and we have

\[
f_{NL} \simeq 5 \frac{\epsilon_*^2}{4 R_{\chi} \epsilon_c^2} \simeq 0.038 \epsilon_*^2 \left( \frac{m_{\chi}}{\Gamma_{\chi}} \right)^{3/2} \left( \frac{\chi_*}{m_{Pl}} \right)^2,
\]

which is suppressed as \( \chi_*/m_{Pl} \to 0 \) for a given \( m_{\chi}/\Gamma_{\chi} \).

If we demand a lower bound on the non-Gaussian parameter, \( f_{NL} > 10 \) for example, this places an upper bound on the curvaton VEV, \( \chi_* < 1.2 \times 10^{16} \) GeV for \( \epsilon_* = 0.02 \), but also a lower bound on the decay rate \( \Gamma_{\chi} < 3 \times 10^{-8} m_{\chi} \).

Figure 6.3 is similar to Figure 6.2 but corresponds to a larger slow-roll parameter \( \epsilon_* = 0.1 \). The thin black dotted line separating the inflaton- and curvaton-dominated regions here corresponds to \( \epsilon_c = 0.1 \). We see that for larger values of \( \epsilon_* \) the parameter regime (1) corresponding to the curvaton limit extends to smaller values of \( m_{\chi}/\Gamma_{\chi} \) and larger \( \chi_* \), permitting larger values of \( f_{NL} \).

On the other hand observational bounds on the tensor-scalar ratio now place more severe constraints on the allowed parameter values. If we put a lower bound on the non-Gaussian parameter, \( f_{NL} > 10 \) for example, this places an upper bound on the curvaton
Figure 6.2: The plot shows contour lines for the non-linear parameter $f_{NL}$ (blue lines) and the tensor-to-scalar ratio $r_T$ (thick red lines) as a function of the curvaton parameters, $\chi^*$ and $m_\chi/\Gamma_\chi$, for the quadratic curvaton potential, Eq. (6.64), and a fixed value of the inflation slow-roll parameter, $\epsilon_*=0.02$. The black broken line corresponds to $\epsilon_c=0.02$.

Figure 6.3: The plot shows contour lines for the non-linear parameter $f_{NL}$ (blue lines) and the tensor-to-scalar ratio $r_T$ (thick red lines) as a function of the curvaton parameters, $\chi^*$ and $m_\chi/\Gamma_\chi$, for the quadratic curvaton potential, Eq. (6.64), and a fixed value of the inflation slow-roll parameter, $\epsilon_*=0.1$. The black broken line corresponds to $\epsilon_c=0.1$. 
VEV, \( \chi_* < 2.5 \times 10^{16} \) GeV, and a lower bound on the decay rate, \( \Gamma_\chi < 2 \times 10^{-7} m_\chi \), for \( \epsilon_* = 0.1 \) and \( r_T < 0.24 \).

The entire inflaton dominated region (2) is excluded by observational bounds on the tensor-scalar ratio for such a large value of \( \epsilon_* \). On the other hand \( \epsilon_* = 0.1 \) is allowed in much of the curvaton dominated region (1).

Note however that such a large value of \( \epsilon_* \) requires a similar positive value of \( \eta_{\chi \chi} \), tuned such that the spectral tilt remains small in the curvaton limit, \( |n_\zeta - 1| \simeq 2|\eta_{\chi \chi} - \epsilon_*| < 0.1 \). Note that inflaton mass, \( \eta_{\phi \phi} \), does not affect the spectral tilt in the curvaton limit so the inflaton mass could be of order the Hubble scale without producing a large spectral tilt, but it does affect the running of the spectral index. The running (6.34) in the curvaton limit \( w_\chi \simeq 1 \) is \( \alpha_\zeta \simeq \alpha_{S_\chi} \) which is of the same order of magnitude as the spectral index for \( \eta_{\phi \phi} \sim 1 \).

### 6.2.2 Self-interacting potential

**Cosine potential**

We consider an axion-type potential for a weakly broken \( U(1) \)-symmetry \( (f \gg M) \) \cite{71, 96, 97, 98, 76}

\[
V(\chi) = M^4 \left[ 1 - \cos \left( \frac{\chi}{f} \right) \right].
\]

(6.71)

For \( \chi_* \ll f \) the effective potential reduces to the quadratic potential (6.64) with \( m_\chi \equiv M^2/f \), but the cosine potential has self-interaction terms which become significant for \( \chi_* \sim f \). By symmetry it is enough to consider the range \( 0 \leq \chi_* / f \leq \pi \) for the curvaton VEV during inflation.

Figure 6.4 shows the parameter \( \epsilon_c \) defined in Eq. (6.29), which determines the contribution of the curvaton to the scalar power spectrum for a given value of the inflationary energy scale, \( H_* \), as a function of \( \chi_* \) and \( m_\chi / \Gamma_\chi \) for a cosine potential with \( f = 10^{16} \) GeV. For \( \chi_* \ll f \) we recover the previous results for the quadratic potential shown in Figure 6.1. (Note that the y-axis is linear in Figure 6.4 but logarithmic in Figure 6.1). For values of \( \chi_* > f \) the higher-order terms in the potential reduce the potential gradient and hence slow-down the evolution of \( \chi \). The curvaton has a larger density when it decays than it would have done for the same initial VEV in a quadratic potential. Thus \( R_\chi \) increases and \( \epsilon_c \) decreases relative to the same parameter values in the quadratic potential. In particular this increases the parameter range for which the curvaton dominates the primordial power spectrum, \( \epsilon_c > \epsilon_* \), relative to the quadratic case.

In Figure 6.5 we show the non-Gaussianity parameter, \( f_{NL} \), and the tensor-scalar ratio, \( r_T \), for different curvaton parameter values and a fixed slow-roll parameter, \( \epsilon_* = 0.02 \). Bounds on the tensor-scalar ratio no longer place a lower bound on the decay time,
Figure 6.4: The plot show contours of $\epsilon_c$ defined in Eq. 6.29 as a function of the curvaton parameters, $\chi_*$, the curvaton VEV, and $m_\chi/\Gamma_\chi$ the dimensionless decay time for the cosine curvaton potential, Eq. (6.71), with $f = 10^{16}$ GeV. Curvaton perturbations dominate the primordial scalar power spectrum for $\epsilon_* \gg \epsilon_c$.

Figure 6.5: The plot show contour lines for the non-linear parameter $f_{\text{NL}}$ (blue lines) and the tensor-to-scalar ratio $r_T$ (thick red lines) as a function of the curvaton parameters, $\chi_*$ and $m_\chi/\Gamma_\chi$, for the cosine curvaton potential, Eq. (6.71), with $f = 10^{16}$ GeV, and a fixed value of the inflation slow-roll parameter, $\epsilon_* = 0.02$. The black broken line corresponds to $\epsilon_c = 0.02$. 

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\( \frac{m_\chi}{\Gamma_\chi} \), as the tensor-scalar ratio becomes small when \( \epsilon_c \) is large for \( \chi_* \sim \pi f \) where the curvaton VEV is close to the maximum of the cosine potential. Large positive values of \( f_{NL} \) also become possible for \( \chi_* \sim \pi f \) due to the non-linear evolution of the curvaton field, even though \( R_\chi \approx 1 \).

For \( \chi_* \sim \pi f \) and \( R_\chi \approx 1 \) we have from Eq. (6.47)

\[
f_{NL} \approx \frac{5}{4} \left( \frac{g'' g}{g'^2} \right) \omega^2, \tag{6.72}
\]

and from Eq. (6.54)

\[
g_{NL} = \frac{25}{24} \left( \frac{g'' g^3}{g'^3} - 3 \frac{g'' g}{g'^2} \right) \omega^3. \tag{6.73}
\]

**Hyperbolic-cosine potential**

We also consider a hyperbolic-cosine potential

\[
V(\chi) = M^4 \left[ \cosh \left( \frac{\chi}{f} \right) - 1 \right]. \tag{6.74}
\]

For \( \chi_* \ll f \) the effective potential reduces to the quadratic potential (6.64) with \( m_\chi \equiv M^2/f \). Self-interaction terms which become significant for \( \chi_* \sim f \) and for \( \chi_* \gg f \) the curvaton field becomes massive during inflation and evolves rapidly to smaller values, hence we will assume \( \chi_* \lesssim f \) in our discussion.

We start by studying the relative contribution of the curvaton to the scalar power spectrum characterised by parameter \( \epsilon_c \) defined in Eq. (6.29). Figure 6.6 shows \( \epsilon_c \) as a function of \( \chi_* \) and \( m_\chi/\Gamma_\chi \) for a hyperbolic-cosine potential with \( f = 10^{16} \) GeV. Again, for \( \chi_* \ll f \) we recover the previous results for the quadratic potential shown in Figure 6.1. For values of \( \chi_* > f \) the higher-order terms in the potential increase the potential gradient and speed up the evolution of \( \chi \) relative to the quadratic potential. The curvaton has a smaller density when it decays than it would have done and thus \( R_\chi \) increases and \( \epsilon_c \) decreases relative to the same parameter values in the quadratic potential. This decreases the parameter range for which the curvaton dominates the primordial power spectrum, \( \epsilon_c > \epsilon_* \), relative to the quadratic case.

In Figure 6.7 we show the non-Gaussianity parameter, \( f_{NL} \), and the tensor-scalar ratio, \( r_T \), for different curvaton parameter values and a fixed slow-roll parameter, \( \epsilon_* = 0.02 \). As in the case of a quadratic potential bounds on the tensor-scalar ratio place a lower bound on the decay time, \( m_\chi/\Gamma_\chi \) if \( r_T < 16 \epsilon_* \).

Large negative values of \( f_{NL} \) are in principle possible due to the non-linear evolution of the curvaton field for \( \chi_* \gg f \). However, just as in the case of positive \( f_{NL} \) for the quadratic potential, extremely large values are not possible for finite \( \epsilon_* \) since \( \epsilon_c \) becomes
Figure 6.6: The plot show contours of $\epsilon_c$ defined in Eq. 6.29 as a function of the curvaton parameters, $\chi_s$, the curvaton VEV, and $m_\chi/\Gamma_\chi$, the dimensionless decay time for the hyperbolic-cosine curvaton potential, Eq. (6.74), with $f = 10^{16}$ GeV. Curvaton perturbations dominate the primordial scalar power spectrum for $\epsilon_s \gg \epsilon_c$.

Figure 6.7: The plot show contour lines for the non-linear parameter $f_{NL}$ (blue lines) and the tensor-to-scalar ratio $r_T$ (thick red lines) as a function of the curvaton parameters, $\chi_s$ and $m_\chi/\Gamma_\chi$, for the hyperbolic-cosine curvaton potential, Eq. (6.74), with $f = 10^{16}$ GeV, and a fixed value of the inflation slow-roll parameter, $\epsilon_* = 0.02$. The black broken line corresponds to $\epsilon_c = 0.02$. 
small for \( \chi_* \gg f \) and hence \( w_\chi \to 0 \) and \( f_{\text{NL}} \to 0 \) for \( \chi*/f \to +\infty \). The maximum value of \( f_{\text{NL}} \) (for sufficiently small decay rates, such that \( R_\chi \simeq 1 \)) occurs when we have \( \epsilon_c \sim \epsilon_* \), i.e., at the boundary of curvaton and inflaton limits.

### 6.3 Summary

In Chapter 5 [122] we have shown how observables such as the tensor-scalar ratio, \( r_T \), and non-linearity parameter, \( f_{\text{NL}} \), are related to curvaton model parameters, specifically the curvaton VEV, \( \chi_* \), and the dimensionless decay rate, \( \Gamma_\chi/m_\chi \), assuming that the curvaton is the only source of primordial density perturbations. In this chapter we have allowed for the presence of primordial perturbations due to adiabatic inflaton field fluctuations in addition to isocurvature curvaton field fluctuations during inflation. Hence, we rederived the curvature perturbation after the curvaton decay. We then computed the final radiation power spectrum of curvature perturbations and its scale dependence relating them with the inflaton and curvaton power spectrum and scale dependence. We also computed the non-linear parameters \( f_{\text{NL}}, \tau_{\text{NL}} \) and \( g_{\text{NL}} \) and the tensor-to-scalar ratio in the mixed inflaton-curvaton scenario.

We then solved numerically the evolution equation of the curvaton field during radiation domination as described in chapter 5. The inclusion of inflaton perturbations introduces an additional model parameter, the slow-roll parameter \( \epsilon_* \), which determines the primordial power spectrum due to inflaton field fluctuations relative to the tensor power spectrum. We have constructed an equivalent parameter, \( \epsilon_c \), which determines the primordial power spectrum due to curvaton field fluctuations relative to the tensor power spectrum. For \( \epsilon_c \ll \epsilon_* \) the curvaton fluctuations dominate the primordial scalar power spectrum, \( w_\chi \simeq 1 \), and we recover the results of chapter 5 [122]. For \( \epsilon_c \gg \epsilon_* \) the inflaton fluctuations dominate the primordial scalar power spectrum, \( w_\chi \ll 1 \), although the curvaton can still source non-Gaussianities. We presented plots of \( \epsilon_c \) for the three different potentials of study: quadratic, cosine and hyperbolic cosine. For the quadratic potential we obtained analytical estimates for \( \epsilon_c \) in terms of \( \chi_* \) and \( \Gamma_\chi/m_\chi \) for the regimes \( R_\chi \ll 1 \) and \( R_\chi \simeq 1 \).

In practice we have presented two-dimensional contour plots of the tensor-to-scalar ratio, \( r_T \), and non-linearity parameter, \( f_{\text{NL}} \), as functions of \( \chi_* \) and \( \Gamma_\chi/m_\chi \) for fixed values of \( \epsilon_* \). We have shown that a curvaton can produced detectable non-Gaussianity and/or gravitational waves for a range of model parameters, even allowing for the presence of inflaton perturbations. For a small slow-roll parameter, \( \epsilon_* < \epsilon_c \), very large values of the non-linearity parameters are suppressed (\( f_{\text{NL}} \propto (\epsilon_*/\epsilon_c)^2 \), \( g_{\text{NL}} \propto (\epsilon_*/\epsilon_c)^3 \), etc).
Nonetheless $f_{\text{NL}} > 10$ may still be produced for $\epsilon_* < \epsilon_c$ when $R_\chi \ll 1$ or in the presence of self-interactions and non-linear curvaton field evolution, $|g^r g/g|^2 \gg 1$.

By allowing the curvaton and the inflaton to contribute to the final curvature perturbation in the radiation we obtained different bounds from those in chapter 5. Previous regions of parameter space that were excluded due to high values of $f_{\text{NL}}$ now become allowed since they correspond to inflaton dominated regions that highly suppress non-Gaussianities. Then, comparing chapter 5 with chapter 6 one concludes that the analyses in chapter 5 is too restrictive. I.e., assuming curvaton domination of the power spectrum may lead to unreasonable constrains to the parameter space. Relaxing such assumption allows to do a more realistic study of the curvaton parameter space and of which regions are allowed.
Chapter 7

Tilted Ekpyrosis

Understanding the origin of structure in our Universe is one of the biggest challenges in modern cosmology. An inflationary expansion in the very early universe has become the standard explanation, addressing the flatness and the horizon problems as well as seeding an almost scale-invariant, nearly Gaussian distribution of inhomogeneous perturbations about a Friedmann-Robertson-Walker space-time [42, 119]. Nonetheless, it is interesting to ask if there are alternative scenarios that can source primordial perturbations consistent with current observations. As we have noted in section 3.8, we require primordial density perturbations which are well-described by a power spectrum $P_\zeta(k) \propto k^{n_\zeta - 1}$ where $0.944 < n_\zeta < 0.992$ [15] at 95%CL, and the distribution must be sufficiently Gaussian, such that the amplitude of the bispectrum with respect to the square of the power spectrum, given by the non-linearity parameter $f_{NL}$, is constrained to be $-10 < f_{NL} < 74$ [15] for local-type non-Gaussianity [129].

Pre-Big Bang models offer a possible alternative where the comoving Hubble-horizon shrinks during a collapse phase, generating a distribution of classical fluctuations on super-Hubble scales [130, 131]. One of such model is an ekpyrotic collapse prior to the Big Bang [19, 132, 133] where a canonical scalar field with a steep, negative potential energy drives the contraction. In sections 2.4.2 and 3.7 we have seen that the potential for this field has a scale-invariant form and leads to a power-law collapse and a power-law power spectrum of fluctuations. All collapse models face a challenge to connect this runaway collapse to a decelerated expansion, but in any case this single-field model predicts a steep blue spectrum of adiabatic density perturbations, $n_\zeta \approx 3$ [20], in contradiction with observations.

As shown in chapter 3, an almost scale-invariant distribution of perturbations can be realised in the new ekpyrotic scenario [21, 22, 23] by considering a multi-field system [24]. Each field has its own steep, negative potential (see sections 2.4.2 and 3.7). The adiabatic fluctuations have a steep blue spectrum as before, but isocurvature fluctuations...
can also source the primordial density perturbation. The isocurvature field spectral tilt is given by $n_{\chi} - 1 \simeq 4/c^2$ during an ekpyrotic contraction with $c^2 \gg 1$; therefore the power spectrum can be nearly scale invariant.

An essential feature of this two-field model is that the power-law solution with $p = 2/c^2$ is unstable; there is a tachyonic instability since the effective mass-squared of the $\chi$ field is negative [27]. Such an instability is necessary to achieve an almost scale-invariant spectrum. Quantum fluctuations on the Hubble scale have a power spectrum $P_\chi \simeq (c^4/4) (H/2\pi)^2$ which grows rapidly during collapse, therefore the power spectrum on larger scales must also experience a rapid growth, proportional to $H^2$, in order to keep pace with the growing power on the shrinking Hubble scale.

This raises the question of how the universe started sufficiently close to this unstable solution, which we will return to later. However, the tachyonic instability does provide a mechanism to convert isocurvature field fluctuations into density perturbations [43]. The growth of the $\chi$ field leads to a change from the two-field solution with $p = 2/c^2$ to a single-field solution with either $p = 2/c_1^2$ or $p = 2/c_2^2$. The corresponding change in the local equation of state, controlled by the local value of the $\chi$ field, leads to a density perturbation, $\zeta \propto \delta \chi$ [43]. Other mechanisms have also been proposed which could convert the isocurvature field fluctuations to density perturbations including a kinetic conversion due to an abrupt change in the field trajectory after the ekpyrotic phase [21] or a curvaton-type conversion due to modulated reheating in an expanding phase following the bounce [134]. In any case any linear process preserves the scale dependence of the power spectrum and we have $n_{\zeta} = n_\chi$. Note however that the power spectrum is slightly blue, $n_\zeta > 1$, in tension with current observations [15].

Non-linearity in the evolution of perturbations also provides important constraints on the model. The tachyonic conversion of isocurvature field fluctuations into density perturbations leads to local-type non-Gaussianity (Eq. 3.106) characterised by the non-linearity parameter [44] $f_{\text{NL}} = -(5/12)c^2_I$ for $I = 1, 2$. Given that we must have $c^2_I > c^2$ this implies $f_{\text{NL}} < (-5/3)(n_\zeta - 1)^{-1}$, e.g., if $n_\zeta - 1 < 0.01$ we require $f_{\text{NL}} < -100$, in contradiction with observations. Alternative conversion processes can lead to model-dependent results for non-Gaussianity and in particular the kinetic conversion can lead to $f_{\text{NL}} \sim \pm c$ which may be compatible with observational constraints given above.

In this chapter we will look at consequences of simple generalisations of the new ekpyrotic scenario to include non-orthogonal potentials and how this alters the predicted distribution of super-Hubble perturbations and the problem of initial conditions. We will also try to address the issues with initial conditions.
7.1 Tilting the Ekpyrotic Potential

We will consider $n$ canonical scalar fields, $\phi = (\phi_1, \ldots, \phi_n)$, with $m$ exponential potentials

$$V(\phi) = -\sum_{J=1}^{m} V_J e^{-\varepsilon_J \phi}$$  \hspace{1cm} (7.1)$$

where $\varepsilon_J = (\varepsilon_{J1}, \ldots, \varepsilon_{Jn})$. We recover the new ekpyrotic model, described above, as a special case of two orthogonal vectors $\varepsilon_1 \cdot \varepsilon_2 = 0$, but in the following we will consider the more general case of non-orthogonal or “tilted” potentials, such that $\varepsilon_I \cdot \varepsilon_J \neq 0$ [24]. We restrict our discussion to $V_J > 0$ so that every term in Eq. (7.1) is negative and $V < 0$. The case of positive potentials, $V > 0$, was discussed previously in the context of assisted inflation [135, 136]. We will assume that the $m$ different vectors, $\varepsilon_J$, are linearly independent. Hence our analysis is also restricted to $m \leq n$ and we assume that the fields are not trapped, so that there always exists a regime with finite energy density in which $V_J e^{-\varepsilon_J \phi} \to 0$ for any given potential.

We note that we could choose to work with fields $\varphi_J \propto \varepsilon_J \phi$ aligned with the potentials in Eq. (7.1) but then these fields would have a non-diagonal metric in field space, i.e., be non-orthogonal for $\varepsilon_I \cdot \varepsilon_J \neq 0$.

The evolution equation for the canonical fields is given by

$$\ddot{\phi} + 3H \dot{\phi} + \sum_J \varepsilon_J V_J e^{-\varepsilon_J \phi} = 0$$  \hspace{1cm} (7.2)$$

where the Friedmann equation for $H \equiv \dot{a}/a$ is

$$H^2 = \frac{1}{3} \left( \frac{1}{2} \dot{\phi}^2 - \sum_J V_J e^{-\varepsilon_J \phi} \right)$$  \hspace{1cm} (7.3)$$

We have set $8\pi G = 1$ and dots correspond to derivatives with respect to cosmic time.

7.1.1 Dynamics and Stability

Firstly let’s look at the stability of this type of system. To do so let’s follow [27, 137] and define

$$x_i = \frac{\dot{\phi}_i}{\sqrt{6H}}$$  \hspace{1cm} (7.4)$$

$$y_J = \frac{\sqrt{V_J e^{-\varepsilon_J \phi}}}{\sqrt{3H}}.$$  \hspace{1cm} (7.5)$$
Using Eqs (7.2) and (7.3) one finds

\[
\frac{dx_i}{dN} = -3x_i \left(1 - \sum_k x_k^2\right) - \sqrt{\frac{3}{2}} \sum_J c_{ji} y_j^2
\]  
(7.6)

\[
\frac{dy_J}{dN} = y_J \sum_i x_i \left(3x_i - \sqrt{\frac{3}{2}} c_{ji}\right)
\]  
(7.7)

We can then study fixed points corresponding to scaling solutions [136, 137, 138, 139]. A fixed point is called to a particular family of solutions \((x_i, y_J)\) that satisfy \(dx_i/dN = dy_J/dN = 0\). In cosmology, we call a scaling solution to a solution of the system where ratio between the energy density of a component of the universe and the universe energy density is always constant. In other words, the energy density of such a component of the universe is linearly dependent to \(H^2\). We find the following fixed points:

1. Zero-potential fixed point

These points are characterised by \(y_J = 0, \forall J\) and \(x_\\cdot x = X^2 = \sum_i x_i^2 = 1\). This kinetic energy-dominated collapse with \(a \propto (-t)^{1/3}\) is unstable whenever there exists at least one potential with \(c_J^2 > 6\).

2. Single-potential fixed point

In the case where only one potential is non-zero, i.e., \(y_K \neq 0\) while \(y_J = 0, \forall J \neq K\), we then have a fixed point

\[
y_K = \sqrt{\frac{c_{K}^2}{6} - 1}, \quad x_i = \frac{c_{K}i}{\sqrt{6}};
\]  
(7.8)

where \(c_{K}^2 = \sum_i c_{Ki}^2\) and we require \(c_{K}^2 > 6\). This corresponds to a power law solution of the scale factor with \(a \propto (-t)^p\), where \(p = 2/c_{K}^2\). This collapse is stable with respect to the zero-potential solution.

3. Double-potential fixed point

In the case where two potentials are non-zero, \(y_K \neq 0\) and \(y_L \neq 0\), we have a fixed point where

\[
x_i = \frac{c_i}{\sqrt{6}},
\]  
(7.9)

and

\[
y_K = \sqrt{\frac{c_{L}^2 - c_{L} c_{K} c_{K}}{|c_{L} - c_{K}|^2} \left(\frac{c_{K}^2}{6} - 1\right)};
\]  
(7.10)

with

\[
\xi = \frac{(c_{K}^2 - c_{L} c_{K}) c_{K} + (c_{L}^2 - c_{L} c_{K}) c_{K}}{|c_{L} - c_{K}|^2}.
\]  
(7.11)
This corresponds to a power-law solution of the scale factor with \( p = 2/c^2 \) where 
\[
 c^2 = c_K^2 c_L^2 - (\xi_K \cdot \xi_L)^2.
\] For this solution to exist we require \( c^2 > 6 \) and \( c_K \cdot c_L < \min\{c_K^2, c_L^2\} \), and in this case this double-potential collapse is always stable with respect to kinetic-dominated collapse but unstable with respect to single-potential collapse.

4. Multiple-potential fixed points

The scaling solution for two potentials can be generalised to the case of multiple tilted potentials. We again find 
\[
 x_i = c_i / \sqrt{6}
\] where we have
\[
 c^2 = \sum_{I,J} (M^{-1})_{IJ} c_J
\] where we define the matrix \( M_{IJ} \equiv \xi_I \cdot \xi_J \) and \( (M^{-1})_{IJ} \) is its inverse. Hence we have a power-law solution with \( p = 2/c^2 \) where \( c^2 = \sum_{I,J} (M^{-1})_{IJ} c_J \). A system with many exponential potentials can have many different fixed points. \( n \) fields with \( m \leq n \) potentials of the form given in Eq. (7.1) with independent \( \xi_J \) will have \( 2^m - 1 \) different fixed points with at least one non-zero potential. For instance, if we have 3 potentials there will be one scaling solution with 3 non-zero potentials \( (y_K \neq 0) \), three scaling solutions with 2 non-zero potentials, and three fixed points each with a single non-zero potential. In each case we can use the general result for the multiple-potential fixed point given in Eqs. (7.12) and (7.13) where the sums are to be taken over the non-zero potentials. This reduces to Eq.(7.11) for two tilted potentials, or 
\[
 c^{-2} = \sum_I c_I^{-2}
\] for multiple orthogonal potentials.

7.2 A working example

Let’s consider a new ekpyrotic scenario with non-orthogonal potentials. From now on we will discuss the case when we have two potentials and two fields. Without further loss of generality we set
\[
 c_1 = c_1(1,0)
\]
\[ c_2 = c_2(\sin \theta, \cos \theta). \] (7.15)

We recover the case of orthogonal potentials in the limit \( \sin \theta \to 0 \). The rotation to adiabatic and isocurvature fields in field space is given by

\[
\sigma = \frac{(c_2 \cos \theta) \phi_1 + (c_1 - c_2 \sin \theta) \phi_2}{\Delta},
\]

\[
\chi = \frac{(c_1 - c_2 \sin \theta) \phi_1 - (c_2 \cos \theta) \phi_2}{\Delta},
\]

where \( \Delta^2 \equiv c_1^2 - 2c_1c_2 \sin \theta + c_2^2 \). The potential in terms of the adiabatic field \( \sigma \) and the isocurvature field \( \chi \) is \( V(\sigma, \chi) = -e^{-\sigma}U(\chi) \) where

\[ c^2 = \frac{c_1^2c_2^2 \cos^2 \theta}{c_1^2 + c_2^2 - 2c_1c_2 \sin \theta}, \]

and

\[
U(\chi) = V_1 \exp \left[ -\frac{(c_1 - c_2 \sin \theta)c}{c_1 \cos \theta} \chi \right] + V_2 \exp \left[ \frac{(c_2 - c_1 \sin \theta)c}{c_1 \cos \theta} \chi \right].
\]

which reduces to (2.80) when \( \theta = 0 \). We note that \( V_1, V_2 > 0 \) so \( U(\chi) \) is bounded from below and has a minimum at \( \chi = \chi_0 \) for \( c_1c_2 \sin \theta < \min\{c_1^2, c_2^2\} \). Thus there is an ekpyrotic, power-law solution with \( \chi = \chi_0 \) and \( V \propto e^{-\sigma} \). Around the minimum we can expand \( U(\chi) \) up to fourth order as

\[
U(\chi) \simeq U_0 \left( 1 + \frac{\mu_\chi^2}{2} (\chi - \chi_0)^2 + \frac{\mu_\chi^2 \hat{c}_3}{6} (\chi - \chi_0)^3 + \frac{\mu_\chi^2 \hat{c}_4^2}{24} (\chi - \chi_0)^4 + \ldots \right)
\]

where

\[
\mu_\chi^2 = \frac{(c_1^2 - c_1c_2 \sin \theta) (c_2^2 - c_1c_2 \sin \theta)}{c_1^2 - 2c_1c_2 \sin \theta + c_2^2} = c^2 - c_1c_2 \sin \theta > 0,
\]

and

\[
\hat{c}_3 = \mu_\chi \sqrt{1 + \sin^2 \theta} \frac{c_1^2 - c_2^2}{c_1^4 c_2^2 - (c_1^2 + c_2^2) c_1c_2 \sin \theta},
\]

\[
\hat{c}_4 = \mu_\chi^4 \left( \frac{(c_1^2 - c_1c_2 \sin \theta)^3 + (c_2^2 - c_1c_2 \sin \theta)^3}{(1 + \sin^2 \theta) c_1^4 c_2^2 - (c_1^2 + c_2^2) c_1c_2 \sin \theta} \right)^{\frac{2}{3}}.
\]

Note that in the new ekpyrotic \( \theta = 0 \). These expressions recover \( \mu_\chi = c, \hat{c}_3 = \hat{c} \) and

\[
\hat{c}_4 = c^4 \left( \frac{c_1^2}{c_4^2} + \frac{c_2^2}{c_4^2} \right),
\]

(7.24)
for $\theta = 0$. In the case of the new ekpyrotic the isocurvature and the adiabatic fields have the same mass. This allows several cancelations that won’t happen in the tilted case.

The adiabatic field $\sigma$ has the same background evolution as described in Eq. (2.70). Effectively the tilting of the potentials changes the effective mass of the isocurvature field with respect to the adiabatic field. Quantum fluctuations in the adiabatic field, $\sigma$, lead to a steep blue spectrum of density perturbations, $n_\zeta = 3$ in the fast-roll limit[20]. A nearly scale-invariant spectrum of perturbations can instead originate from isocurvature fluctuations in the isocurvature field, $\chi$.

### 7.2.1 Linear Perturbations

As described in Chapter 3, linear perturbations in the isocurvature field obey

$$\ddot{\delta \chi} + 3H \dot{\delta \chi} + \left(\frac{k^2}{a^2} - \frac{2\mu^2 \chi (c^2 - 6)}{c^4 t^2}\right) \delta \chi = 0. \tag{7.25}$$

It is convenient to work in conformal time defined by $dt = ad\tau$. Using the Bunch-Davies vacuum state to normalise the amplitude of quantum fluctuations at early times ($k\tau \to -\infty$) one finds

$$a\delta \chi(\tau) = \frac{\sqrt{\pi}}{2} e^{-i\tau (\nu + \frac{1}{2})} (-\tau)^{1/2} H^{(2)}_{\nu}(-k\tau) \tag{7.26}$$

where $H^{(2)}_{\nu}(-k\tau)$ is a Hankel function of the second kind and

$$\nu^2 = \frac{9}{4} - \frac{2c^4 - 2(3 + \mu^2 \chi) c^2 + 12\mu^2 \chi}{(c^2 - 2)^2}. \tag{7.27}$$

The scale dependence of the power spectrum is given by $n_\chi - 1 \equiv d \ln P_\chi / d \ln k = 3 - 2\nu$. To obtain an exactly scale-invariant spectrum we require $\nu = 3/2$ and hence

$$c_1 c_2 \sin \theta = -\frac{3c^2}{c^2 - 6}. \tag{7.28}$$

Note that, as expected, a scale-invariant spectrum requires that the two-potential collapse is an unstable point in the phase space.

In the fast-roll limit $c^2 \gg 1$ we have

$$\nu^2 \simeq \frac{1}{4} + \frac{2(3 + \mu^2 \chi) c^2 - 12\mu^2 \chi}{c^4}. \tag{7.29}$$
We will focus on the fast roll, $c^2 \gg 1$, and small angle case, $\theta \ll 1$. In this case

$$n_\chi - 1 \simeq \frac{4}{c^2} \left( 1 + \frac{c_1 c_2}{3} \theta \right). \quad (7.30)$$

So we find a nearly scale-invariant red spectrum if $\theta \lesssim -3/c_1 c_2$. The relative tilt of the potentials in field space, $\theta$, alters the effective mass of the isocurvature field enabling us to obtain a slightly red spectrum in contrast to the case of orthogonal potentials. Tilting the potentials down does not introduce any running of the power spectrum, $\alpha_\zeta = 0$.

### 7.2.2 Non-linear Perturbations

Another important constraint comes from the observed Gaussian distribution of primordial perturbations. Due to the steep exponential potentials, the isocurvature field fluctuations do not remain Gaussian on super-Hubble scales. The resulting non-Gaussianity of the subsequent density perturbations depends on the conversion process. In the simplest case the $\chi$ field perturbations evolve away from the two-potential fixed point leading to a tachyonic transition to the single-potential solution.

The calculation of the non-linear parameters is analogous to the one described in subsection 3.7. Here we will extend the calculation to third order. If we include cubic and quartic self interactions into the evolution equation of $\chi$ on large scales we have

$$\ddot{\chi} + 3H \dot{\chi} + m^2_\chi \chi = -m^2_\chi \left( \frac{\dot{c}_3}{2} \chi^2 + \frac{c_3^2}{6} \chi^3 + \ldots \right), \quad (7.31)$$

where

$$m^2_\chi = -\frac{2\mu^2(1 - 3p)}{c^2 t^2}. \quad (7.32)$$

As before, the $\chi$-field won’t be Gaussian through the collapse. We therefore expand it into a linear part, $\chi_L$, and a perturbative expansion about $\chi_L$ as

$$\chi = \chi_L + \frac{1}{2} C_\chi \chi_L^2 + \frac{1}{6} D_\chi \chi_L^3. \quad (7.33)$$

Using the ansatz $\chi_L \propto t^{-\beta}$ we find for the growing mode

$$\beta = -\frac{1}{2} \left[ 1 - \frac{6}{c^2} - \sqrt{\left( 1 - \frac{6}{c^2} \right)^2 + \frac{8\mu^2}{c^2} \left( 1 - \frac{6}{c^2} \right)} \right]. \quad (7.34)$$

If we compare with Eq. (3.92), tilting the potential only alters the last term inside the square root. We are interested in the fast-roll limit and in the small angle approximation,
then,
\[ \beta \simeq 1 - \frac{2}{c^2} \left( 1 + \frac{c_1 c_2}{3} \sin \theta \right) . \] (7.35)

Using the perturbative expansion Eq. (7.33) we find that the coefficients \( C \) and \( D \) are related with the coefficients in the potential expansion (Eq. (7.20)) via
\[
C = \frac{\mu_H^2 (1 - 3p)/c^2}{2\beta^2 + (1 - 3p)(\beta - \mu_H^2/c^2)} c_3 , \tag{7.36}
\]
\[
D = \frac{2\mu_H^2 (1 - 3p)/c^2}{9\beta^2 + (1 - 3p)(3\beta - 2\mu_H^2/c^2)} \left( 3C \hat{c}_3 + \hat{c}_4^2 \right) . \tag{7.37}
\]

As in subsection 3.7 we use a Gaussian parameter \( \alpha \), that characterises \( \chi_L \) trajectories, to be able to use the \( \delta N \) formalism. We have already seen that \( \alpha = \chi_L H^{-\beta} \), but we need to write it in terms of \( \chi \) since we want to know how \( \alpha \) relates with the integrated expansion on the transition hyper-surfaces \( \chi_T \). Then if we invert Eq. 7.33 we get:
\[
\alpha = \frac{\chi}{H^\beta} \left( 1 - \frac{1}{2} C \chi - \frac{1}{6} D \chi^2 \right) . \tag{7.38}
\]

The transition hyper-surface, \( H = H_T \), corresponds to the transition from a two-field scaling solution to a single field scaling solution. In this example, the integrated expansion is given by
\[
N = -\frac{2}{c_1^2} \ln |H_T| + \text{constant} , \tag{7.39}
\]
with
\[
\tilde{c}_1 = \frac{c_1 c_2 \cos \theta}{c_2 - c_1 \sin \theta} . \tag{7.40}
\]

Then using the \( \delta N \) expansion in term of \( \alpha \)
\[
\delta N = \frac{2}{c_1^2 \beta \alpha} - \frac{1}{c_1^2 \beta} \left( \frac{\delta \alpha}{\alpha} \right)^2 + \frac{2}{3c_1^2 \beta} \left( \frac{\delta \alpha}{\alpha} \right)^3 , \tag{7.41}
\]
we find
\[
N_\alpha = \frac{2}{c_1^2 \beta \alpha} , \tag{7.42}
\]
\[
N_{\alpha\alpha} = -\frac{2}{c_1^2 \beta \alpha^2} , \tag{7.43}
\]
\[
N_{\alpha\alpha\alpha} = \frac{4}{c_1^2 \beta \alpha^3} . \tag{7.44}
\]

Hence, we can determine the the three non-Gaussian parameters
\[
f_{NL} = -\frac{5 \beta}{12} \tilde{c}_1^2 , \tag{7.45}
\]
\[ \tau_{NL} = \frac{c_1^4 \beta^2}{4}, \quad (7.46) \]

\[ g_{NL} = \frac{25}{108} c_1^4 \beta^2. \quad (7.47) \]

Note that, in the non-tilted case these results agree with what is found in the literature [44, 140].

In the fast roll limit and small angle we have

\[ f_{NL} = -\frac{5}{12} \left[ 1 - \frac{2}{c^2} \left( 1 + \frac{c_1 c_2}{3} \theta \right) \right] \left[ \frac{c_1 c_2}{c_2 - c_1 \theta} \right]^2. \quad (7.48) \]

To lowest order \( f_{NL} = -(5/12)c_1^2 < -(5/12)c^2 \). Although \( c \) is no longer uniquely determined by the tilt of the power spectrum, it is nonetheless required to be large in the fast-roll limit. On the other hand, if we require \( f_{NL} > -10 \) this demands \( c^2 < 24 \).

**Scale-invariance of non-linear parameters**

There are other quantities of interest to study the non-linearities of a particular model. We can study the scale dependence of the non-linear parameter. The scale dependence of \( f_{NL} \) is defined by \([100]\)

\[ n_{f_{NL}} = d \ln f_{NL} \frac{d \ln k}{d \ln k}. \quad (7.49) \]

It is easy to check from Eq. (7.45) that the ekpyrotic collapse won’t source any scale dependence, i.e., \( n_{f_{NL}} = 0 \). Similarly, the scale dependence of the second non-linear parameters is defined by

\[ n_{\tau_{NL}} = d \ln \tau_{NL} d \ln k, \quad (7.50) \]

\[ n_{g_{NL}} = d \ln g_{NL} d \ln k. \quad (7.51) \]

Again, we see from Eqs. (7.46) and (7.47) that there is no scale dependence of \( \tau_{NL} \) and \( g_{NL} \).

### 7.3 Issues with initial conditions

Having successfully obtained a red spectrum of density perturbations by tilting the potentials, we have also created a new problem of initial conditions for the scenario as we will now demonstrate.

Although a classical solution can spend an arbitrarily long period of time close to the unstable double-potential fixed point, quantum fluctuations in the isocurvature field
inevitably lead to a tachyonic transition within a finite time. The transition occurs when 
\( \mu_\chi^2 \langle \delta \chi^2 \rangle \sim 1 \), where
\[
\langle \delta \chi^2 \rangle = \int \mathcal{P}_\chi(k) d\ln k ,
\] (7.52)
where the integral is over all wave numbers for which we can treat the fluctuations as 
effectively classical, usually taken to be all super-Hubble scales. For a blue spectrum 
of perturbations, the integral is dominated by the shortest wavelengths, i.e., the Hubble 
scale, and we have \( \langle \delta \chi^2 \rangle \simeq (c^4/4) (H/2\pi)^2 \). Thus, taking \( \mu_\chi^2 \simeq c^2 \), the transition must 
occur when \( |H| \lesssim c^{-3} \), but the double-potential phase begin far in the past.

For a red spectrum of fluctuations in the isocurvature field, the variance of the field is 
dominated by the longest wavelengths. If the new ekpyrotic phase started far in the past, 
then the variance of the field on the largest scales would be infinitely large leading to a 
contradiction as we require the field to be close to the double-potential fixed point. Thus, 
the new ekpyrotic phase must have started a finite time in the past and we have
\[
\mu_\chi^2 \langle \delta \chi^2 \rangle \simeq \left( \frac{c^6}{4} \right) \left( \frac{H}{2\pi} \right)^2 \left( \frac{k_\ast}{k_i} \right)^{1-n} .
\] (7.53)
where \( k_\ast \) is the comoving Hubble scale and \( k_i \) is the initial comoving Hubble scale at the 
beginning of the new ekpyrotic phase. Assuming we have a slightly red spectrum with 
\( 1 - n \simeq 0.01 \) and requiring a new ekpyrotic phase which lasts at least 10 e-folds, i.e., 
\( k_\ast/k_i > e^{10} \), we require that the transition completes when \( |H| < (1 - n)^{1/2}c^{-3} \).

We require a phase preceding the new ekpyrotic phase which sets the classical back-
ground field sufficiently close to the fixed point, and ensures that the isocurvature field 
has a sufficiently small variance on large scales at the start of the new ekpyrotic phase. 
There are several possibilities, one being that the isocurvature field has a mass parameter 
that changes during the evolution, inserting \( \mu_\chi^2(\sigma) \) in Eq.(7.20). This could both stabilise 
\( \chi = \chi_0 \) at early times and offers another way produce a red spectrum at late times [141]. 
However such a potential cannot be realised within the context of simple exponential po-
tentials (7.1), and lies outside the class of scale-invariant potentials with scaling solutions 
[138]. We expect that a time-dependent \( \mu_\chi^2(\sigma) \) would lead to a running of the tilt, \( n_c(k) \) 
and a scale-dependent non-Gaussianity, \( f_{NL}(k) \).

Finally, we note that in principle we might disregard the ensemble average for \( \langle \chi^2 \rangle \) on 
large scales and assume that simply by chance quantum fluctuations away from \( \chi = \chi_0 \) 
in our local patch are unusually small. This is unlikely a priori but one might appeal to some anthropic argument that only these regions are capable of giving rise to observers 
[142].
7.3.1 Kinetic Dominated Phase

Within our simple model (7.1), one can address the issues of initial conditions with a preceding phase described by a multiple-potential scaling solution, that itself would be unstable with respect to the two-potential solution. But the spectrum of the isocurvature perturbations requires a careful analysis of the three (or more) potential system, and could be highly model-dependent. An alternative preceding fixed point already present in our two field model is the kinetic fixed point with vanishing potential energy. This is an unstable fixed point but it does describe the generic behaviour of the system as \( t \to -\infty \).

The kinetic-dominated fixed point where the potentials are negligible is in fact the basis of the pre-big bang models proposed by Gasperini and Veneziano [130]. It is well-known that a kinetic-dominated collapse leads to a steep blue spectrum of perturbations for any massless fields. Thus the isocurvature field naturally has negligible perturbations on large scales. On the other hand, a priori there seems no particular reason why the classical background trajectories should approach close to the new ekpyrotic (double-potential) solution which is a saddle point in the phase-space [27] rather than proceeding directly to the old ekpyrotic (single-potential) solutions which are the stable late-time attractors.

7.3.2 Hybrid-Ekpyrotic contraction

A way to stabilise the \( \chi \)-field at early times is to have a positive mass of the isocurvature field prior to the collapse. Therefore, we require an isocurvature mass that evolves, i.e., that becomes negative as \( \sigma \)-field evolves down the potential. For that we need to break the scale invariance of the exponential potential. Now we expect scale dependencies in the observables.

Let’s consider the potential

\[
V(\sigma, \chi) = -U_0 e^{-c(\chi)\sigma} \left( 1 + \frac{\mu_2}{2} \chi^2 + \frac{\mu_3^2}{6} \chi^3 \right),
\]

(7.54)

where to lowest order we take

\[
c(\chi) = c_0 + \frac{\lambda}{2} \chi^2 + \ldots.
\]

(7.55)

Note that this change in the rotated potential is a phenomenological approach to study such hybrid-like collapses. The fields are normalised with respect to the Planck mass \( m_{Pl} \). One should bear in mind that this potential solves classically the initial condition problem, nonetheless it may not be stable once we treat it quantum mechanically.
The potential has the same functional shape for \( \sigma \) for fixed \( \chi \) but with \( c \) varying slightly with \( \chi \). On the other hand, \( c(\chi) \) introduces mass corrections to the isocurvature field which alter its dynamics considerably. The first derivative of the potential with respect to the isocurvature field is

\[
V_\chi = -U_0 e^{-(c_0 + \frac{1}{2} \chi^2) \sigma} \left[ \mu_\chi^2 \chi + \frac{\mu_\chi^2 \hat{c}_3}{2} \chi^2 - \lambda \sigma \chi \left(1 + \frac{\mu_\chi^2}{2} \chi^2 + \frac{\mu_\chi^2 \hat{c}_3}{6} \chi^3\right) \right],
\]

which vanishes on the ridge \( \chi = 0 \). The second derivative of the potential with respect to the isocurvature field is given by

\[
V_{\chi\chi} = -U_0 e^{-(c_0 + \frac{1}{2} \chi^2) \sigma} \times \left[ \mu_\chi^2 (1 + \hat{c}_3 \chi) - \lambda \sigma \left(1 + \frac{5}{2} \mu_\chi^2 \chi^2 + \frac{7}{6} \mu_\chi^2 \hat{c}_3 \chi^3\right) + \lambda^2 \chi^2 \sigma^2 \left(1 + \frac{\mu_\chi^2}{2} \chi^2 + \frac{\mu_\chi^2 \hat{c}_3}{6} \chi^3\right) \right].
\]

For \( \sigma > \mu_\chi^2 / \lambda > 0 \) in the ridge we have \( m_\chi^2 > 0 \). Hence, at early times, a \( \chi = 0 \) would remain identically zero since the field would be stabilised at this value. This justifies taking \( \chi = 0 \) for the background evolution of the two fields. Then, the background evolution of the adiabatic field remains unchanged and obeys to Eq. (2.83). Furthermore, all of the background dynamics stay unchanged with respect to the case \( c = c_0 \).

We will define the mass of the isocurvature field as

\[
m_\chi^2 \equiv V_{\chi\chi}(\sigma, 0) = -U_0 e^{-c_0 \sigma} \left( \mu_\chi^2 - \lambda \sigma \right).
\]

If we substitute Eq. (2.83) into the previous equation we get

\[
m_\chi^2 = -\frac{m_0^2}{t^2} \left[ 1 - K \ln \left(\frac{-t}{t_0}\right) \right],
\]

where

\[
m_0^2 = \frac{2 \mu_\chi^2 (1 - 3p)}{c_0^2},
\]

\[
K = \frac{2 \lambda}{c_0 \mu_\chi^2},
\]

\[
t_0 = \sqrt{\frac{2(1 - 3p)}{c_0^2 U_0}}.
\]

For \( t < -t_0 \exp(1/K) \) the mass is positive and the isocurvature field is stabilised at \( \chi = 0 \). In principle ekpyrosis could be happening in this regime already. The second regime
happens when $t > -t_0 \exp(-1/K)$. In this case the dominant term is the logarithmic
dependence of the mass. In the fast roll limit the mass becomes

$$m^2_\chi \simeq \frac{m_0^2 K}{t^2} \ln \left( -\frac{t}{t_0} \right).$$  \hspace{1cm} (7.64)

Note that $m^2_\chi < 0$ since $t \ll -t_0$. The logarithmic time dependence spoils the previous
observed scale invariance. In the intermediate regime $\exp(-1/K) < -t/t_0 < \exp(1/K)$
we can treat the logarithm component as a perturbative correction. One can say that we
have $\Delta N = p/K$ e-folds in the perturbative regime. If we require at least 10 e-folds of
scale invariant power-spectrum then $p \ll 1$ gives $K \ll 10^{-1}$. In fact this just restates
that the correction $\lambda$ is sub-leading with respect to $\mu^2_\chi$.

The logarithmic correction can be treated a scale dependent quantity rather than time
dependent. Noting that $k = aH$ on finds that

$$\ln \left( -\frac{t}{t_0} \right) = \frac{c^2}{2 - c^2} \ln \frac{k}{k_0},$$ \hspace{1cm} (7.65)

where $k_0$ is a pivot scale.

In the perturbation evolution equation (7.25) one only need to substitute $\mu^2_\chi$ by

$$\mu^2_\chi \left( 1 + K \frac{c^2}{c^2 - 2} \ln \frac{k}{k_0} \right).$$ \hspace{1cm} (7.66)

To lowest order the solution of Eq. (7.25) is the same as given in Eq. (7.26) but with the
Hankel function ended given by

$$\nu^2 = \frac{9}{4} - \frac{2}{(c^2 - 2)^2} \left( c^4 - 3c^2 - \mu^2_\chi \left( 1 + K \frac{c^2}{c^2 - 2} \ln \frac{k}{k_0} \right)(c^2 - 6) \right).$$ \hspace{1cm} (7.67)

Assuming that $\lambda \ll c_0$, i.e. $K \ll 1$, then we consider the logarithm part as a running of
the parameter $\mu^2_\chi$. Doing so we don’t alter the amplitude but we introduce a running of the
power. To compute the spectral index we need to know the ratio $\mu^2_\chi/c^2$. For simplicity
and to study how this hybrid-like potential alters the observables in the new ekpyrotic
scenario we fix the ratio to one, i.e., $\mu^2_\chi = c^2$. Then the spectral index measure at the
pivot scale $k_0$, in the fast roll limit, remains unchanged,

$$\left. (n_{\delta\chi} - 1) \right|_{k = k_0} = \frac{4}{c^2}.$$ \hspace{1cm} (7.68)
but we introduce running of the power spectrum

\[ \alpha_{\delta x}|_{k=k_0} = - \frac{2K}{3} \left(1 - \frac{6}{c^2}\right) . \] (7.69)

Although we have a running, we expect it to be very small.

The expression (3.106) for \( f_{\text{NL}} \) will remain unchanged. We just need to change the expression for \( \beta \) to

\[ \beta = -\frac{1}{2} \left(1 - \frac{6}{c^2} - \sqrt{\left(1 - \frac{6}{c^2}\right)^2 + 8 \left(1 + K \frac{c^2}{c^2 - 2 \ln \frac{k}{k_0}}\right) \left(1 - \frac{6}{c^2}\right)}\right) . \] (7.70)

Note that \( \beta \) is now scale dependent but to lowest order we have

\[ \beta \simeq 1 - \frac{2}{c^2} + \frac{2}{3} K \ln \frac{k}{k_0} . \] (7.71)

Assuming that there is a one potential scaling solution with mass \( c_2^2 \), then the first non-Gaussian parameter is given by

\[ f_{\text{NL}} = -\frac{5\beta(k)}{12} c_1^2 , \] (7.72)

where \( c_1 \) is defined by \( 1/c_1^2 = 1/c^2 - 1/c_2^2 \). We find the scale dependence of \( f_{\text{NL}} \) to be

\[ n_{f_{\text{NL}}}|_{k=k_0} = \frac{2}{3} K . \] (7.73)

Although we find a scale dependence of \( f_{\text{NL}} \), \( f_{\text{NL}} \) should be nearly scale invariant since we expect \( K \) to be a very small parameter.

### 7.4 Summary

In this chapter [143] we considered a simple model of cosmological collapse driven by canonical fields with exponential potentials. We generalised the two-field ekpyrotic collapse to consider non-orthogonal or tilted potentials and give the general condition for isocurvature field fluctuations to have a scale-invariant spectrum in this model. In particular we have shown that an ekpyrotic collapse driven by two scalar fields with non-orthogonal potentials can give a scale-invariant or slightly red tilted spectrum of perturbations. This is in contrast to the original ekpyrotic collapse with a single field [132]
which produces a steep blue spectrum [20], or new ekpyrotic collapse with two orthogonal potentials [21, 22, 23] which yields an almost scale-invariant, but slightly blue spectrum of perturbations. To obtain a slightly red spectrum we fine-tune the tilt such that the angle $\theta \sim 0.01$ in the fast-roll limit. However, a red spectrum of fluctuations implies that the two-field ekpyrotic phase must have a finite duration and requires a preceding phase which sets the initial conditions for what otherwise appears to be a fine-tuned trajectory in the phase space. We presented two phenomenological ways of fixing this issue and its structure predictions. We also studied the predicted non-Gaussianities from the tilted ekpyrotic model, the predicted $f_{\text{NL}}$ is still negative and large in disagreement with observations.
Chapter 8

Conclusion

The origin of structure in the universe is a subject of major investigation nowadays. It is within that quest that the research presented in this thesis was done. We reviewed the basics of cosmology and models of the early universe. We then reviewed perturbation theory to have the tools to compute primordial perturbations from different models of the early universe. We computed several observational quantities from different inflationary models. Although the inflationary paradigm is the simplest and most self-consistent explanation of the early universe, we also studied perturbations from alternative proposals.

8.1 Extending $\delta N$ and the physics at the end of hybrid inflation

The challenge for the standard $\delta N$ approach in hybrid inflation is that the homogeneous classical background solution, with smooth fields on scales much larger than the Hubble-horizon, fails to provide a good description of the dynamics. A classical solution which starts precisely in the false vacuum state stays there and inflation never ends. Thus, even a tiny perturbation away from this background solution results in a huge apparent change in the local expansion, $N$. In fact the false vacuum is destabilised mainly due to fluctuations of the waterfall field on scales close to the Hubble-horizon at the end of inflation.

The coupling between the slow-rolling inflaton and the waterfall field that leads to the tachyonic instability also makes the waterfall field massive before the transition. Thus quantum fluctuations of the waterfall field are not amplified during inflation and remain in a quantum state even on super-Hubble scales until the critical value of the inflaton is reached. Once the tachyonic instability is triggered, there is an explosive growth of long-wavelength modes, but they retain the steep blue spectrum on super-Hubble scales. On the other hand much smaller scales, well inside the Hubble horizon at the transition are stabilised by spatial gradients and remain in their vacuum state. The resulting power
spectrum for the waterfall field thus peaks on scales around the Hubble scale at the transition, as shown in Figure 4.3, and these modes play an essential role in the dynamics that must be included when calculating the effect of large scale fluctuations in the waterfall field.

Thus, we identify the primordial curvature perturbation due to perturbations in the waterfall field, $\delta \chi_L$ on some large scale $L$, with the perturbation in the average expansion, $\langle N_f \rangle$, including fluctuations in the waterfall field on Hubble scales at the transition. We adopt a Gaussian distribution for the field whose average value in a region of size $L$ is $\delta \chi_L$, but whose variance is given by $\sigma^2 = P_u(k_*)/a^2$, where $P_u(k_*)$ is given by Eq. (4.29). We find two main results:

- from the symmetry of the potential (4.1) under $\chi \rightarrow -\chi$ we see that the primordial curvature perturbation is independent of $\delta \chi_L$ at first order, and the curvature perturbation is second-order in $\delta \chi_L$.

- the spectrum of the primordial curvature perturbation due to fluctuations in the waterfall field on large scales is suppressed by a factor of order $(k_*/L)^3$ which for cosmological scales is likely to be of order $10^{-54}$.

While we have considered the specific hybrid potential (4.1) and presented numerical solutions for specific parameter choices we believe the general conclusions will hold for all hybrid models in which the waterfall field is massive during slow-roll inflation and for which the end of inflation occurs due to a rapid tachyonic instability.

It is not surprising that the effect of small scale fluctuations must be taken into account when estimating the primordial curvature perturbation when the large-scale field is very close to zero. This is the case, for example, in the curvaton model where the curvature perturbation, $\zeta \sim \delta \sigma/\sigma$, could become very large when the background field, $\sigma$, is close to zero in some regions of the universe [144]. The apparent singular behaviour of the curvature perturbation in this case is regularised by smaller scale fluctuations in the curvaton field [82].

In the analysis in chapter 4 we have considered only the early stages of the tachyonic instability. Eventually inflation ends and the coupled fields oscillate about the true vacuum $\phi = 0$ and $\chi^2 = 2M^2/\sqrt{\Lambda}$. Numerical simulations are required to study resonant particle production and other non-perturbative effects in this regime which goes beyond the scope of this thesis. Resonance is often most efficient in long-wavelength modes, but fields which are massive throughout slow-roll inflation necessarily give rise to a steep blue spectrum of perturbations and hence the large-scale power is suppressed with respect to smaller scales [145, 146, 147].
8.2 Further possible constraints in the curvaton model

In the simplest curvaton model we neglected inflaton contributions to the power spectrum. For the quadratic potential for the curvaton, bounds on the tensor-to-scalar ratio place an upper bound on the dimensionless decay rate, ruling out large regions of parameter space that would yield a large primordial non-Gaussianity in the distribution of scalar perturbations. Simultaneous measurement of both the non-linearity parameter, $f_{NL}$, and the tensor-to-scalar ratio, $r_T$, can determine both the expectation value of the field during inflation, $\chi_*$, and the dimensionless decay rate, $\Gamma_\chi/m_\chi$.

In the conventional inflaton scenario for the origin of structure we have three free parameters: the inflation scale $H_*$ and two slow-roll parameters, $\epsilon_*$ and $\eta_{\phi \phi}$. These can be determined by the power of the primordial scalar perturbations, $P_\zeta$, the tensor perturbations, $P_T$, and the spectral index of the scalar spectrum, $n_\zeta$. The spectral index of the tensor spectrum, if measurable, would give a valuable consistency check [110]. Another important consistency condition for canonical, slow-roll inflation is that the primordial density perturbations should be Gaussian and the non-linearity parameter, $f_{NL}$, should be much less that unity [89].

In the curvaton scenario with a simple quadratic potential we have 5 free parameters: the inflation scale $H_*$, the expectation value of the curvaton during inflation $\chi_*$, the decay rate of the curvaton relative to its mass, $\Gamma_\chi/m_\chi$, and the slow roll parameters $\epsilon_*$ and $\eta_\chi = m_\chi^2/3H_*^2$. For a curvaton, we find that $H_*$, $\chi_*$ and $\Gamma_\chi/m_\chi$ are determined by the primordial scalar perturbations, $P_\zeta$, the tensor perturbations, $P_T$, and the non-linearity parameter, $f_{NL}$, but the mass and decay rate of the curvaton are not separately determined. The two slow-roll parameters $\epsilon_*$ and $\eta_\chi$ are then determined by the two spectral indices $n_\zeta$ and $n_T$.

Another possible observable in the curvaton model is the scale dependence of the non-linearity parameter, defined as [99]

$$n_{f_{NL}} \equiv \frac{d \ln |f_{NL}|}{d \ln k}.$$  \hspace{1cm} (8.1)

In the curvaton scenario, neglecting any possible contributions to the power spectrum from inflaton perturbations, the scale dependence of $f_{NL}$ is given by a simple expression [98, 101]

$$n_{f_{NL}} = \eta_3 \frac{5}{m_{Pl} g' 4 R_\chi f_{NL}},$$  \hspace{1cm} (8.2)

where we define $\eta_3 \equiv m_{Pl}^3 V'''/V$. This can be rewritten in terms of observable quantities and $\eta_3$

$$n_{f_{NL}} = \eta_3 \frac{5}{12 \sqrt{2} f_{NL}}.$$  \hspace{1cm} (8.3)
Thus it offers the possibility of testing the curvaton self interactions. Future observations may be able to detect $|f_{NL} n_{f_{NL}}| > 5$ [148], corresponding to $|\eta_3|\sqrt{r_T} > 17$. For the quadratic potential we have the consistency condition $n_{f_{NL}} = 0$.

Deviations from a quadratic potential introduce at least one further model parameter, $f$, corresponding to the mass scale associated with the non-linear corrections. This leads to a degeneracy in model parameters consistent with the five observables $\mathcal{P}_\zeta, \mathcal{P}_T, f_{NL}, n_\zeta$ and $n_T$, but this can be broken by a measurement of $n_{f_{NL}}$.

In the case of a cosine-type curvaton potential the self interaction corrections became important near the top of the potential, i.e., when $\chi_s \sim \pi f$ [98] and the tensor-to-scalar ratio no longer places an upper bound on $\Gamma_\chi/m_\chi$. As for a quadratic curvaton, we still find $f_{NL} > -5/4$ and hence any large non-Gaussianity, $|f_{NL}| \gg 1$, has positive $f_{NL}$. But for $\chi_s \sim f$ we have $\eta_3 \sim -(m_{Pl}/f^3) < 0$, and if $f$ is well below the Planck scale there could be strong scale dependence.

In the case of a hyperbolic-type potential $f_{NL}$ can become large and negative, for $\chi_s \sim f$. However the tensor-to-scalar ratio again plays an important role, in this case placing a lower bound on $f_{NL}$, e.g., $f_{NL} > -100$ for $r_T < 0.1$ when $f = 10^{16}$ GeV. In this regime we find $\eta_3 \sim (m_{Pl}/f^3) > 0$, which can be large, leading to strong scale dependence for $f < m_{Pl}$, with $n_{f_{NL}} < 0$ for $f_{NL} < 0$.

Running of either the scalar tilt, $\alpha_\zeta$, or the non-linearity, $\alpha_{f_{NL}}$ [98], yields additional information about the higher derivatives of the potential, and in particular curvaton-inflaton interactions which we have assumed are negligible in our analysis.

Significant non-Gaussianity in the primordial perturbations opens up the possibility to extract information from the higher-order correlations in the scalar spectrum, such as the trispectrum [40, 82, 84, 87, 149]

$$ T_\zeta(k_1, k_2, k_3, k_4) = \frac{54}{25} g_{NL} [P_\zeta(k_2) P_\zeta(k_3) P_\zeta(k_4) + 3 \text{ perms}] + \frac{36}{25} f_{NL}^2 [P_\zeta(k_{13}) P_\zeta(k_3) P_\zeta(k_4) + 11 \text{ perms}] . \tag{8.4} $$

which are sensitive to higher-order derivatives of the expansion history with respect to the curvaton field value during inflation through $g_{NL} = (25/54) N''/N'^3$. Differentiating Eq. (5.30) we obtain

$$ g_{NL} = \frac{25}{24} \left[ \frac{R_{\chi}''}{R_{\chi}^3} g^2 + 2 \frac{R_{\chi}'}{R_{\chi}^3} \left( \frac{g'^2 g''}{g'^3} - \frac{g''}{g'} \right) + \frac{1}{R_{\chi}^2} \left( \frac{g'^2 g''}{g'^3} - \frac{3gg''}{g'^2} + 2 \right) \right] . \tag{8.5} $$

$g_{NL}$ and its scale dependence $n_{g_{NL}}$ [100, 101], thus provide additional observable parameters which then offer consistency conditions for generalised curvaton models such as
the cosine or hyperbolic potentials. In practice we require more accurate numerical sim-
ulations than those used in chapter 5 to reliably determine the required higher-derivatives
with respect to the initial field value across the range of model parameters used here.

8.3 Testing the mixed curvaton-inflaton model with non-
linear perturbations

In chapter 6 we generalize the work presented in chapter 5 to include the possible effects
of inflaton perturbations.

To differentiate between different scenarios for the origin of non-Gaussianity we
should examine further the statistics of the primordial density field. For example, in
the absence of curvaton self-interactions but including inflaton perturbations, the scale-
dependence of the non-linearity parameter (8.1) is given by [99, 100, 101, 124]

\[ n_{f_{\text{NL}}} = 2(1 - w_\chi)(n_\chi - n_\phi), \]

(8.6)

where \( w_\chi \) is the weight of the curvaton contribution to the power spectrum and \( n_\phi \) and
\( n_\chi \) are the tilts of the inflaton and curvaton power spectra, respectively. Note that if the
curvaton dominates both the power spectrum and the higher-order correlators, \( w_\chi \approx 1 \),
then \( f_{\text{NL}} \) is independent of scale. If the power spectrum is dominated by inflaton per-
turbations (\( w_\chi \ll 1 \)), such that \( n_\zeta = n_\phi \), then the bispectrum and higher-order corre-
lators are still dominated by the curvaton perturbations. Hence we generally expect a
scale-dependence of the non-linearity parameters \( f_{\text{NL}} \) and \( g_{\text{NL}} \) since they determine the
higher-order correlators relative to the power spectrum. The higher-order correlators and
the power spectrum inherit different scale-dependence from the curvaton and inflaton
perturbations respectively. In terms of slow-roll parameters

\[ n_{f_{\text{NL}}} \approx 4(1 - w_\chi)(2\epsilon_* + \eta_{\chi\chi} - \eta_{\phi\phi}), \]

(8.7)

which is small if we assume slow-roll for both the curvaton and inflaton. However in this
case only the inflaton tilt is constrained by current observations of the power spectrum
and if the curvaton scale-dependence is large then \( n_{f_{\text{NL}}} \) could be large.

The primordial trispectrum also gives important clues about the origin of non-linearity.
Figure 8.1 shows the trispectrum parameters \( g_{\text{NL}} \) and \( \tau_{\text{NL}} \) as a function of curvaton
parameters for a quadratic curvaton potential. In the absence of self-interactions non-Gaussianity only becomes large when $R_\chi \ll 1$ and in this limit we have

$$g_{NL} \simeq -\frac{10w_\chi}{3} f_{NL}, \quad \tau_{NL} = \frac{36}{25w_\chi} f_{NL}^2. \quad (8.8)$$

Even allowing for a mixed inflaton plus curvaton model with $0 \leq w_\chi \leq 1$, we can eliminate $w_\chi$ to obtain a consistency relation between the bispectrum and trispectrum parameters in this case:

$$g_{NL}\tau_{NL} \simeq -\frac{24}{5} f_{NL}^3. \quad (8.9)$$

Note that from (8.8) we can deduce that $\tau_{NL} > 0.1 g_{NL}^2$ for a quadratic curvaton potential. If both $g_{NL}$ and $\tau_{NL}$ are large then the curvaton potential must include self-interaction terms [84]. Such self-interactions can give rise to large scale-dependence of $f_{NL}$ and $g_{NL}$ even in the curvaton-dominated limit, $w_\chi \simeq 1$ [98, 99, 100, 101, 150].

Unlike single-field models of inflation, the predictions of the curvaton model are dependent on the initial value of the curvaton field during inflation. Although it may not be possible to identify a unique initial value for the curvaton, we may be able to specify the expected probability distribution for different models.

If we take a stochastic approach for the distribution of the curvaton VEV [151] then for a quadratic curvaton potential, and assuming that inflation lasted long enough (and

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure8.1.png}
\caption{Contour lines for the trispectrum non-linear parameters $g_{NL}$ (thin black lines) and $\tau_{NL}$ (thick green lines) as a function of the curvaton parameters, $\chi_*$ and $m_\chi/\Gamma$, for the quadratic curvaton potential, Eq. (6.64), with a fixed value of the inflation slow-roll parameter, $\epsilon_*=0.02$.}
\end{figure}
assuming a light curvaton, \( m_\chi^2 < H^2 \), we find a Gaussian distribution with variance (see Appendix A)

\[
\langle \chi_*^2 \rangle = \frac{3}{8\pi^2} \frac{H_*^4}{m_\chi^2} .
\]

Huang [91] has argued that in this case, a detection of primordial non-Gaussianity then places a lower bound on the tensor-scalar ratio. On the other hand for the self-interacting potential considered here there is an additional scale, \( f \), in addition to the effective mass, \( m_\chi \). For a cosine type potential one obtains an almost uniform distribution for \( 0 \leq \chi_* \leq \pi f \) assuming inflation lasts long enough and is at a high-enough energy scale, while for a hyperbolic potential which becomes steep for \( \chi_* \gg f \) we expect values with \( \chi_* \gtrsim f \) to be suppressed. It would be interesting to interpret observations from future observations of primordial non-Gaussianity and/or tensor-scalar ratio in terms of curvaton model parameters, incorporating a prior probability distribution for the curvaton VEV to marginalise over at least one unknown model parameter.

8.4 Issues with pre-Big Bang models

The two-potential ekpyrotic solution presented in chapter 7 is an unstable saddle point in the phase-space and a red tilted spectrum of tachyonic field fluctuations can therefore only exist over a finite range of scales. Thus the two-potential solution can only exist for a finite time. This is possible for a particular class of solutions in the phase space which must evolve from a kinetic-dominated initial state to approach sufficiently close to the two-potential saddle point. The late-time attractor in the phase space is a single-potential-dominated collapse, i.e., the old ekpyrotic collapse [132].

If the tachyonic transition from two-potential to single-potential collapse occurs then this naturally converts the isocurvature field fluctuations into density perturbations. However this potentially leads to \( f_{\text{NL}} < -(5/12)c^2 \), a large and negative non-Gaussianity parameter, in the fast-roll limit, \( c^2 \gg 1 \), in contradiction to the observations.

By studying a simple two-field system we have a well-defined model within which we can calculate the quantum field perturbations about classical trajectories during a cosmological collapse. However it leaves unanswered the question of whether the required tilted potentials can be realised within a string theory setting, as originally envisaged in the ekpyrotic scenario [132], or how the initial state evolves sufficiently close to an unstable saddle point in the phase space. This would require a preceding phase [141], such as is envisaged within the cyclic scenario [152]. Here we presented two phenomenological possibilities to address this issue.

In all these scenarios we still need to understand whether, and if so how, the universe emerges from a collapse phase to the standard expanding hot big bang. For such transition
to happen we require \( \dot{H} \) to change sign. From equations (2.3) and (2.6) we find, ignoring curvature,

\[
\dot{H} = -\frac{1}{2}(\rho + p). \tag{8.11}
\]

A possibility is to have a singular bounce where the Hubble rate changes sign rapidly, as originally proposed by a collision of orbifold branes in M-theory [132]. So far such brane collision has only been studied successfully semi-classically [153] but not fully quantum mechanically. On the other hand, a non-singular bounce can be obtained with a violation of the null energy condition, \( \rho + p > 0 \). Models with equation of state \( w < -1 \) are usually associated with instabilities, the most common are fields with negative kinetic energy, the so called ghosts. Ghost condensate models [154] can violate the null energy condition in a stable and controlled way although is not yet clear if such models can arise from string theory. More details on the transition between a collapsing to an expanding universe can be found in [155] and in section V of [156].

### 8.5 Future observational prospects

Future experiments, like the Planck satellite, will either give more accurate measurements or stringent bounds of cosmological parameters. In this thesis we mainly focussed on determining the amplitude of the power spectrum of curvature perturbations \( P_\zeta \), its tilt \( n_\zeta \) and running \( \alpha_\zeta \), the non-Gaussian parameters \( f_{\text{NL}} \), \( g_{\text{NL}} \) and \( \tau_{\text{NL}} \), and the tensor-to-scalar ratio \( r_T \).

The amplitude of the power spectrum is already very well measured [15]. Planck will just improve this measurement [157]. On the other hand, for Planck, \( \Delta n_\zeta = 4.0 \times 10^{-3} \) [157]. Taking the most likely value from WMAP7 [15], \( n_\zeta = 0.968 \), then Planck will be capable of excluding a blue tilt and the Harrison Zel’dovich spectrum at several sigma. This completely excludes observationally the simplest new ekpyrotic model, although it would not excluded new ekpyrotic with non-orthogonal potentials. The running of the power spectrum will be better measured as well. Forecast for how well the running will be measured by Planck gives \( \Delta \alpha_\zeta = 5.8 \times 10^{-3} \) [157]. For single field slow-roll inflation \( \alpha_\zeta \sim (n_\zeta - 1)^2 \) which is at the verge of detectability by Planck. If a sizable \( \alpha_\zeta \) is detected then it would indicate deviations from the simplest model. It would indicate that one of the slow-roll parameters is bigger than expected or the second slow-roll parameters are sizable. In the case of the curvaton model it would indicate the presence of curvaton self-interactions.

The detection of gravitational waves is usually stated as the smoking gun for inflation since all inflationary models predict them at some level, in opposition to ekpyrotic contraction. The forecast for Planck give an upper 95\% confidence limit for the error on
the tensor-to-scalar ratio, $\Delta r_T < 0.03[157]$. If we consider $\eta_{\phi\phi} < \mathcal{O}(\epsilon_*)$ then for single field inflation models $r_T \simeq 8(1 - n_\zeta)/3 \simeq 0.09$. Hence we would expect a detection of gravitational waves by Planck from these models. On the other hand if tensor modes are not detected we can still explain such bound within the inflationary paradigm since a low tensor-to-scalar ratio happens naturally when other degrees of freedom are present during inflation, like the curvaton.

In the upcoming Planck satellite there is the possibility of detecting local $f_{NL}$. The current forecasts give $\Delta f_{NL, Local} = 8$. [158]. WMAP7 gives $-10 < f_{NL, Local} < 74 [15]$, if we assume the central value of the bound, $f_{NL, Local} = 32$, then Planck would give a definite detection. If so this rules out single field slow-roll inflation. On the other hand a non-detection of $f_{NL, Local}$ favours single field slow-roll inflation. Current bounds on $f_{NL, Local}$ already exclude some conversion mechanisms in the new ekpyrotic scenario as seen in 8.4. From [158] we have that the detectability of $\tau_{NL}$ is $\Delta \tau_{NL} = 1550$ which is already able to distinguish between multiple fields models and single field model of inflation (depending on the value of $f_{NL, Local}$). For $g_{NL}$, Planck will not give much information since $\Delta g_{NL} = 1.3 \times 10^5 [158]$. In the future we will have better observations of the 2- and 3-point function of the curvature perturbation. This will allow to distinguish between the simplest single field slow roll inflation and others. Similarly, better bounds on the tensor modes will give a better idea of the inflationary model of the early universe.
Appendix A

Stochastic Approach to fields in de Sitter space

In this appendix we will derive the variance of a massive scalar field in de Sitter space-time. We will follow [119]. We wish to study fields in de-Sitter in order to apply such results during slow-roll inflation. Nonetheless, if it lasts long enough one can approximate calculations during inflation by de-Sitter results.

The de Sitter metric is given by

$$ds^2 = -dt^2 + e^{2Ht}d\mathbf{x}^2 = a^2(\tau) \left(-d\tau^2 + d\mathbf{x}^2\right),$$  \hspace{1cm} (A.1)

where we take \(a(t = 0) = 1\). From the definition of conformal time Eq. (2.45) we have

$$a(\tau) = -\frac{1}{H\tau}. \hspace{1cm} (A.2)$$

Let us consider a massive canonical field. Its Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{2} \partial_\mu \phi \partial^{\mu} \phi + V(\phi), \hspace{1cm} (A.3)$$

where the field mass is given by

$$m^2 \equiv \frac{\partial^2 V(\phi)}{\partial^2 \phi}. \hspace{1cm} (A.4)$$

The evolution equation for the field in de-Sitter space (A.1) is

$$\phi''(\tau, \mathbf{x}) - 2\frac{1}{\tau} \phi'(\tau, \mathbf{x}) - \nabla^2 \phi(\tau, \mathbf{x}) + a^2(\tau) \frac{\partial V(\phi)}{\partial \phi} = 0, \hspace{1cm} (A.5)$$

where primes denote derivatives with respect to conformal time \(\tau\).
For now let us just consider a massless free field, i.e., we neglect the last term in Eq. (A.5). Doing the transformation \( u(\tau, x) = a(\tau)\phi(\tau, x) = -\frac{\phi(\tau, x)}{H\tau} \), then Eq. (A.5) becomes
\[
u''(\tau, x) - \left(\frac{2}{\tau^2} + \nabla^2\right) u(\tau, x) = 0. \tag{A.6}
\]
Before proceeding let’s quantize the massless scalar field. The massless scalar field operator \( u(\tau, x) \) can be expanded as
\[
u(\tau, x) = \int \frac{d^3k}{(2\pi)^3} \left( a_k u_k(\tau) e^{i\vec{k}.\vec{x}} + a_k^\dagger u_k^*(\tau) e^{-i\vec{k}.\vec{x}}\right), \tag{A.7}
\]
where \( a_k \) and \( a_k^\dagger \) are the creation and annihilation operators. We can easily see that both mode functions will obey to the wave equation
\[
u_k''(\tau) + \left(k^2 - \frac{2}{\tau^2}\right) u_k(\tau) = 0, \tag{A.8}
\]
which has the general solution
\[
u_k(\tau) = \frac{\sqrt{\pi}}{2} \sqrt{-\tau} \left( C_1(k) H^{(1)}_{3/2}(-k\tau) + C_2(k) H^{(2)}_{3/2}(-k\tau) \right). \tag{A.9}
\]
The functions \( H^{(1),(2)}_{3/2} \) are the Hankel functions of index \( 3/2 \). We just want to consider solutions for \( a(\tau) > 0 \), therefore \( \tau < 0 \). The normalization is picked such that the mode functions obey the Wronskian equal to \(-i\), i.e.,
\[
u_k(\tau) u_k^*(\tau) - u_k(\tau) u_k^*(\tau) = -i. \tag{A.10}
\]
We want to recover the Minkowski quantization in the high frequency limit, \( k \to \infty \), i.e, \( u_k(\tau) \to \frac{e^{i\tau\kappa}}{\sqrt{2k}} \), for \( \tau < 0 \). This condition is satisfied for \( C_1(k) = 0 \) and \( C_2(k) = -1 \). Then
\[
u_k(\tau) = \frac{1}{\sqrt{2k}} \left( 1 - \frac{i}{k\tau} \right) e^{-ik\tau}, \tag{A.11}
\]
or
\[
\phi_k(\tau) = \frac{-H\tau}{\sqrt{2k}} \left( 1 - \frac{i}{k\tau} \right) e^{-ik\tau}. \tag{A.12}
\]
The variance of the \( \phi \) field is given by
\[
\langle \phi^2 \rangle = \frac{1}{(2\pi)^3} \int d^3k \phi_k \phi_k^* = \frac{1}{(2\pi)^3} \int d^3k \frac{(H\tau)^2}{2k} \left( 1 + \frac{\phi}{(k\tau)^2} \right). \tag{A.13}
\]
The first term is the usual contribution from vacuum fluctuations in Minkowski space. Its contribution can be eliminated using renormalization. The second term comes from the
fact that the field lives in a de Sitter space-time. In this set up inflation started at $t = 0$.
We are only interested in modes that left the horizon between the beginning of inflation
($k = H$ since $a = 1$) and some later time where the modes at horizon crossing have
$k = aH$. Then
\begin{align}
\langle \phi^2 \rangle &= \frac{H^2}{4\pi^2} \int_H^{aH} \frac{dk}{k}, \\
&= \frac{H^3 t}{4\pi^2}.
\end{align}
As $t \to \infty$, $\langle \phi^2 \rangle$ diverges. One would naively expect such a result. Since the field
is massless all field values have the same likelihood. As the field randomly “walks”
through its field values with time the uncertainty of finding it at a particular value grows.

During inflation modes with $k > aH$ initially, leave the Hubble scale $k < aH$.
Outside the horizon the amplitude of the modes freezes and the modes start obeying the
spatially homogeneous equation of motion. One can look at the classical field evolution
as a diffusion problem of modes, from $k > aH$ to $k < aH$. A convenient way to describe
the system at any point is to consider the probability distribution function of field values
$P(\phi, t)$. This obeys the diffusion equation
\begin{equation}
\frac{\partial P(\phi, t)}{\partial t} = \frac{\partial}{\partial \phi} \left( D(\phi) \frac{\partial P(\phi, t)}{\partial \phi} \right), \tag{A.16}
\end{equation}
where $D(P, \phi)$ is the diffusion coefficient. For a free field we can take $D = \text{const}$
since there’s no reason for some field values to be more likely than others. If we remind
ourselves that
\begin{equation}
\langle \phi^2 \rangle = \int \phi^2 P(\phi, t) d\phi, \tag{A.17}
\end{equation}
then differentiating with respect to time and using Eqs. (A.15) and (A.16) one concludes
that
\begin{equation}
D = \frac{H^3}{4\pi^2}. \tag{A.18}
\end{equation}
If the initial conditions are such that $P(\phi, 0) = \delta(\phi)$ then the probability distribution
function becomes a Gaussian with with variance $\sigma^2 = H^3 t / 4\pi^2$. Then
\begin{equation}
P(\phi, t) = \sqrt{\frac{2\pi}{H^3 t}} e^{-\frac{2\sigma^2 \phi^2}{H^3 t}}. \tag{A.19}
\end{equation}
Let’s now consider a massive field. As for the massless field one can look at the
probability distribution function of the classical (background) field. The more general
form of the diffusion equation is
\[ \frac{\partial P(\phi, t)}{\partial t} = D \frac{\partial^2 P(\phi, t)}{\partial \phi^2} + b \frac{\partial}{\partial \phi} \left( P(\phi, t) \frac{dV(\phi)}{d\phi} \right). \] (A.20)

The diffusion coefficient \( D \) measures the rate at which quantum fluctuations are transferred from modes with \( k > aH \) to \( k < aH \). We are only interested in light fields, \( m^2 \ll H^2 \). In this regime the mass term is negligible around \( k = aH \) so we don’t expect \( D \) to be altered. The mobility coefficient \( b \) is defined by
\[ \dot{\phi} = -b \frac{dV(\phi)}{d\phi}. \] (A.21)

If we take the slow-roll approximation then \( b \simeq 1/3H \). Then the evolution equation of the probability distribution function of a light field in a de-Sitter space is
\[ \frac{\partial P}{\partial t} = \frac{\partial^2}{\partial \phi^2} \left( \frac{H^3 P}{8\pi^2} \right) + \frac{\partial}{\partial \phi} \left( \frac{P}{3H} \frac{dV}{d\phi} \right). \] (A.22)

For a quadratic theory \( V(\phi) = m^2 \phi^2/2 \) we can still assume that \( P(\phi, t) \) is a Gaussian. Then, the variance follows the evolution equation
\[ \dot{\sigma}^2 + \frac{2m^2}{3H} \sigma^2 - \frac{H^3}{4\pi^2} = 0. \] (A.23)

The general solution is going to be given by the sum of a homogeneous part \( \sigma_H^2 \) and a particular solution \( \sigma_P^2 \), i.e.,
\[ \sigma^2 = \sigma_H^2 + \sigma_P^2. \] (A.24)

The solution for the homogeneous part of the differential equation is
\[ \sigma_H^2 = Ae^{-\frac{2m^2}{3H}t}. \] (A.25)

It is easy to see that the particular equation is going to be a constant, i.e.,
\[ \sigma_P^2 = 3H^4/(8\pi^2 m^2). \] (A.26)

Initially the field is at a fixed value, i.e., \( \sigma^2(t = 0) = 0 \), then \( A = -\sigma_P^2 \). Hence the variance of the field is given by [119]
\[ \sigma^2 = \frac{3H^4}{8\pi^2 m^2} \left[ 1 - e^{-\frac{2m^2}{3H}t} \right]. \] (A.27)
If we take the limit $t \to \infty$ (same as assuming that inflation lasted long enough) we obtain the stochastic result for the variance of a light field

$$\langle \phi^2 \rangle = \frac{3H^4}{8\pi^2m^2}.$$ (A.28)
References


[17] W. G. Unruh,


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