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ON THE MEROMORPHIC NON-INTEGRABILITY OF SOME N-BODY PROBLEMS

Abstract. We present a proof of the meromorphic non–integrability of the planar N-Body Problem for some special cases. A simpler proof is added to those already existing for the Three-Body Problem with arbitrary masses. The N-Body Problem with equal masses is also proven non-integrable. Furthermore, a new general result on additional integrals is obtained which, applied to these specific cases, proves the non-existence of an additional integral for the general Three-Body Problem, and provides for an upper bound on the amount of additional integrals for the equal-mass Problem for $N = 4, 5, 6$. These results appear to qualify differential Galois theory, and especially a new incipient theory stemming from it, as an amenable setting for the detection of obstructions to Hamiltonian integrability.

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Dedicated to professor Carles Simó on his sixtieth birthday

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1. **Introduction.** Arguably the cornerstone of Celestial Mechanics, the N-Body Problem has long been seen in Astrophysics and Applied Mathematics as an epitome of chaotic behavior. The search for a global solution to the problem was first glanced upon in the 1880s by K. T. W. Weierstrass who, with the aid of G. Mittag-Leffler and under the auspices of King Oscar of Sweden, favoured the announcement of a prize in Acta Mathematica (volume 7, 1885/86) for finding the solution as a uniformly convergent series. The difficulty of finding such a series, let alone a convergent one, is inferred from the revised draft of H. Poincaré's attempt which, although thwarted, won the prize and is nowadays considered landmark in the theory of Dynamical Systems; that alone attests the complexity of it all. The problem as stated in the terms of the prize was finally solved (except for the case of zero angular momentum) by K. Sundman in [**?**] though, unfortunately, the series he found was far too slowly convergent and thus of no practical use – not even for numerical computations. Q. D. Wang obtained a similar general solution for the N-body problem, but the problems arising from slow convergence were present in his infinite sum, too, and

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the question of singularities was completely left off: [?], [?]. See [?] for details on the subject's evolution from Weierstrass and Poincaré's "brilliant failure" onward.

It may still be argued that there is no debate on the Problem's "solvability", in view of those results by Sundman and Wang. But a solution in the form of a slowly-converging series not only has virtually no numerical utility: it does not predict the existence of periodic orbits, unbounded motion, or collision of two or more bodies either, in turn yielding further open problems which could only be settled with more information than is provided by an infinite series, such as stability, central configurations, variational problems, properties of the eight solution, existence of choreographies, Levi's problem, constancy of moment of inertia, Saari's conjecture, etc. And, although an adequate set of conserved quantities could help solving these problems, finding it stands as an obstacle on its own since only a comparatively small set of such (so-called *classical*) first integrals is known, and any other algebraic first integral, in the case of $N = 3$ bodies, would necessarily be algebraically dependent with the classical ones in virtue of Bruns' theorem (Theorem 2.5) – a result which has recently been generalized by E. Julliard (Theorem 2.6) to arbitrary N.

Hence, the Problem's history of parallel attempts both at looking for new first integrals and proving it analytically or meromorphically non-integrable should not come up as a surprise. Even less surprising is the partial success of the latter, especially in recent times thanks to two parallel lines of study with more than a trait in common: the line of study initiated by S. L. Ziglin ([?]) and the one begotten by the present paper's first author and J.-P. Ramis: see [?] and [?]. Ziglin's theory relied strictly on the monodromy generators of the variational equations around a given particular solution, whereas the latter theory, used in the present work, uses linear algebraic groups containing the aforementioned monodromies and is naturally immersed in the Galois theory of linear differential equations, which we assume the reader is already familiar with – otherwise, see [?] or [?, Chapter 2] for the minimum necessary concepts.

Using a consequence of this new theory as applied to the factorization of linear operators, D. Boucher and J.-A. Weil ([?], [?]) proved the meromorphic non-integrability of the Three-Body Problem. On the other hand, using the Ziglin approach, A. V. Tsygvintsev ([?], [?], [?], [?], [?]) proved the meromorphic non-integrability of the Three-Body Problem and ultimately settled the non-existence of a single meromorphic first integral; *he established both things for all except three special cases (see Remark 4.1)*. It is finally worth noting that Ziglin ([?, Sections 3.1 and 3.2]) managed to settle strong conditions on the integrability of the Three-Body Problem and the equal-mass N-Body Problem.

Our work is aimed at reobtaining in simpler ways, strengthening and generalizing the results mentioned in the previous paragraph using the aforementioned theory started in [?] as applied to Hamiltonians of a specific kind: to wit, those which are classical with an integer degree homogeneous potential. Although conjectures and open problems will still prevail (see Section 5), the proofs given here are significantly shorter thanks to a significant step forward made in [?, Theorem 3]. Furthermore, using this same Theorem a new necessary condition is established in Section 2.3 of this paper on the existence of a single additional integral for any classical conservative system – a condition in turn allowing us to discard the existence of an additional integral for the Three-Body Problem with arbitrary positive masses, and of a certain amount of additional first integrals for the N-Body Problem with equal masses if $N = 4, 5, 6$.

Regarding notation and basic conventions, all vectors will be denoted in boldface and their norms will be written in ordinary face. All norms will be assumed *Euclidean* by default, for it is through these that the N-Body Problem finds its simplest known formulation. For every vector whose entries are likely to be broken down in separate vectors of lesser size, at most two different boldface types will be used, albeit with the same letter: for any $n, m \in \mathbb{N}$, a vector in $\mathbb{C}^{nm}$ will be written with italic boldface, $\boldsymbol{q}$ (its norm being q) if the n consecutive m-vectors making up for its entries are also being considered; in such case, these latter will be written in regular boldface, $\mathbf{q}_1, \dots, \mathbf{q}_n \in \mathbb{C}^m$, their norms written as $q_1, \dots, q_n$, respectively. If further hierarchy is needed, we will maintain either italic or regular boldface. Vectors will be freely written in concatenation, e.g. $\boldsymbol{z}^T = \left(\boldsymbol{q}^T, \boldsymbol{p}^T\right) = \left(\mathbf{q}_1^T, \dots, \mathbf{q}_n^T, \mathbf{p}_1^T, \dots, \mathbf{p}_n^T\right)^T$, but we will avoid the ^{T} superindex unless we have to make specific reference to scalar products, e.g. in Rayleigh quotients. Boldface as described in all of the above considerations will be applied exclusively to constant vectors and vector functions of *one* variable, e.g. $\boldsymbol{q} = \boldsymbol{q}(t)$, whereas vector functions with more than one argument, e.g. $f = f(t, \boldsymbol{q})$, will be written in regular face. Since there will only be one independent variable t properly regarded as time, an overdot will stand for $\frac{d}{dt}$ all through the text and $^{(k)}$ will stand for $\frac{d^k}{dt^k}$, $k \geq 4$, whereas $'$ will usually imply derivation with respect to phase variables of Hamiltonian systems. It is worth noting this time variable t will be complex by default all through the text. Γ will often stand for Riemann surfaces, and $\mathbb{P}^1$ will always stand for the (complex) projective line. Defining the Kronecker delta $\delta_{i,j}$ as usual, $\left\{\boldsymbol{e}_{n,k} = \left(\delta_{i,k}\right)_{i=1,\dots,n}^T\right\}$ will be the canonical basis for $\mathbb{R}^n$. Zero vectors and zero and identity matrices will be written with their dimension as a subindex whenever deemed necessary, e.g. $\mathbf{0}_n \in \mathbb{C}^n$ or $0_{n\times n}, \mathrm{Id}_n \in \mathcal{M}_n(\mathbb{C})$. $|\cdot|$ will denote absolute value or modulus indistinctively. $\sqrt{-1} =$ will always be denoted in Roman, non-italic font. The consideration of points in the plane as either complex numbers or real 2-vectors will also be tacit depending on the context. The determination for complex square roots will be that given by the analytic continuation of the *positive* real square root, i.e. $\sqrt{z} := \sqrt{r}e^{\frac{\theta}{2}}$ whenever $z = re^{\theta}$ and $\theta \in [0, 2\pi]$.

2. **Linear and Hamiltonian integrability.**

2.1. **Differential Galois Theory.**

See [?] or [?] for more information. Given a **linear** differential system (whether or not autonomous), with coefficients in a differential field (K, ∂) whose field of constants $\mathbb{C}$ is algebraically closed (e.g. $\left(\mathbb{C}(t), \frac{d}{dt}\right)$),

$$\partial \boldsymbol{y} = A(t)\, \boldsymbol{y}, \tag{1}$$

differential Galois theory assures the existence of

- a differential field $L \supset K$, unique up to K-isomorphism, containing all entries of a fundamental matrix $\Psi = [\boldsymbol{\psi}_1, \dots, \boldsymbol{\psi}_n]$ of (1);
- an algebraic group (see Appendix A) G linked to $K \subset L$ (the **differential Galois group** of (1)), such that G acts over the $\mathbb{C}$-vector space $\langle \boldsymbol{\psi}_1, \dots, \boldsymbol{\psi}_n \rangle$ of solutions of (1) as a linear transformation group over $\mathbb{C}$, and the **monodromy group** of (1) is contained in G.

(1) is called *integrable* if its general solution can be written as a finite sequence of quadratures, exponentials, and algebraic functions (and any of their inverses). In the Galoisian setting, assertion "(1) is integrable" is equivalent to the following: the identity component G^0 *of the differential Galois group G of (1) is* ***solvable***.

2.2. **A general non-integrability theorem.** *Heuristics of all non-integrability results considered here* are firmly rooted in the following general principle: if we assume any system

$$\dot{\boldsymbol{y}} = X(\boldsymbol{y}) \tag{2}$$

"integrable" in some reasonable sense, then the corresponding variational equations along any integral curve Γ of (2) must be also integrable (in the sense of linear Galois differential theory). Any attempt at ad-hoc formulations of this heuristic principle for a specific system (2) has an asset and a drawback. As seen above, there *is* a definition of integrability for linear systems (and thus, for the variational system): that the identity component of its Galois group be *solvable*. But still, in order to transform this principle into a true conjecture it is necessary to clarify a notion of "integrability " for (2). *Everything is considered in the complex analytical setting from now on.*

There is a specific notion of integrability for *Hamiltonian* systems, namely in the sense of Liouville-Arnold, for which the aforementioned general principle does have an implementation:

Theorem 2.1. ***(J. Morales-Ruiz & J.-P. Ramis, 2001)*** *Let H be an n-degree-of-freedom Hamiltonian having n independent first integrals in pairwise involution, defined on a neighborhood of an integral curve Γ. Then, the identity component of the Galois group of the variational equations of H along Γ is a* ***commutative*** *group.*

See [**?**, Corollary 8] or [**?**, Theorem 4.1] for a precise statement and a proof.

Remark 1. *An essential tool in the proof, which does not require the dynamical system to be Hamiltonian, is the following (*[**?**, Lemma 9]*, see also* [**?**, Lemma 4.6]*).* Let f be a meromorphic first integral of any autonomous dynamical system (2). Then, the Galois group of the variational system has a non-trivial rational invariant

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2.3. **A special case: homogeneous potentials.**

2.3.1. *Prior results.* This Subsection is nothing but a reenaction of [**?**, §5.1.2], Section 7 in the second issue of [**?**] (pp. 97–111 of the same volume) and [**?**, §1–3]. Assume X_H is given by a **classical** n-degree-of-freedom Hamiltonian,

$$H(\boldsymbol{q},\boldsymbol{p}) = T + V = \frac{1}{2}\boldsymbol{p}^T\boldsymbol{p} + V(\boldsymbol{q}), \tag{3}$$

$V(\boldsymbol{q})$ **being homogeneous of degree** $k \in \mathbb{Z}$. Hamiltonians such as these are by no means generical. The fact V is homogeneous implies the observance of the principle of mechanical similarity ([**?**]): the orbits on any integral manifold can be rescaled to one of a finite set of such manifolds (typically corresponding to energy values $-1, 0, 1$), i.e. freedom of choice of the energy constant is only countered by discrete gaps in the dynamics generated by V; indeed, transformation $\boldsymbol{q} \mapsto \alpha^2\boldsymbol{q}$, $\boldsymbol{p} \mapsto \alpha^k\boldsymbol{p}$, with possible change in time $t \mapsto it$, yields the new energy $\tilde{H} = (\pm)\,\alpha^{2k}H$ for any given α. In order to see further uses of this fact, as well as generalizations to not necessarily finite values of the energy, see [**?**], [**?**] and [**?**].

X_H defined as above, every vector function $\widehat{\boldsymbol{z}}(t) = \left(\phi(t)\,\boldsymbol{c}, \dot{\phi}(t)\,\boldsymbol{c}\right)$, such that $\ddot{\phi} + \phi^{k-1} = 0$ and $\boldsymbol{c} \in \mathbb{C}^n$ satisfies $\boldsymbol{c} = V'(\boldsymbol{c})$, is a solution of Hamilton's equations for H, as may be easily proven using the fact that the n entries in vector $V'(\boldsymbol{q})$ are homogeneous polynomials of degree $k-1$. Such a vector $\boldsymbol{c}$ is usually called a **homothetic point** of potential V. Other references call this vector a *Darboux point* as well.

Writing infinitesimal variations on the canonical variables as $\delta\boldsymbol{q}=\tilde{\boldsymbol{\xi}}$ and $\delta\boldsymbol{p}=\tilde{\boldsymbol{\eta}}$, the equations satisfied by these are

$$\frac{d}{dt}\tilde{\boldsymbol{\xi}}=\tilde{\boldsymbol{\eta}},\quad \frac{d}{dt}\tilde{\boldsymbol{\eta}}=-\phi\left(t\right)^{k-2}V''\left(\boldsymbol{c}\right)\tilde{\boldsymbol{\xi}},$$

or equivalently $\frac{d^2}{dt^2}\tilde{\boldsymbol{\xi}}=-\phi\left(t\right)^{k-2}V''\left(\boldsymbol{c}\right)\tilde{\boldsymbol{\xi}}$. Assume $V''\left(\boldsymbol{c}\right)$ is diagonalizable; this is the case, for instance, if $\boldsymbol{c}\in\mathbb{R}^n$. Then, any transformation $\tilde{\boldsymbol{\xi}}=U\boldsymbol{\xi}$, $\tilde{\boldsymbol{\eta}}=U\boldsymbol{\eta}$ with an adequate $U\in\mathrm{GL}_n\left(\mathbb{C}\right)$ transforms the system, written as

$$\frac{d}{dt}\boldsymbol{\xi}=\boldsymbol{\eta},\quad \frac{d}{dt}\boldsymbol{\eta}=-\phi\left(t\right)^{k-2}\left[U^{-1}V''\left(\boldsymbol{c}\right)U\right]\boldsymbol{\xi},$$

into

$$\frac{d^2}{dt^2}\boldsymbol{\xi}=-\phi\left(t\right)^{k-2}\begin{pmatrix}\lambda_1&&&\\&\lambda_2&&\\&&\ddots&\\&&&\lambda_n\end{pmatrix}\boldsymbol{\xi},$$

where $\{\lambda_1,\dots,\lambda_n\}=\mathrm{Spec}\,V''\left(\boldsymbol{c}\right)$.

In other words, along $\widehat{\boldsymbol{z}}$, variational equations may be split into a direct sum $\bigoplus_{i=1}^n\mathrm{VE}_i$ of n uncoupled equations, each of the form

$$\frac{d^2\xi_i}{dt^2}+\lambda_i\left[\phi\left(t\right)\right]^{k-2}\xi_i=0,\qquad i=1,\dots,n, \tag{4}$$

Furthermore,

$$V''\left(\boldsymbol{c}\right)\boldsymbol{c}=\left(k-1\right)\boldsymbol{c}, \tag{5}$$

is easily established as a special case of Euler's Theorem; thus, we may set $\lambda_1=k-1$; the corresponding variational equation, VE_1, is trivially integrable. The remaining $n-1$ eigenvalues $\lambda_2,\dots,\lambda_n$ may be enough to determine the non-integrability de X_H in this special case of [**?**, Corollary 8]; indeed, (4) following [**?**], the finite branched covering map $\overline{\Gamma}\to\mathbb{P}^1$ is considered, given by $t\mapsto x:=\phi\left(t\right)^k$, where $\overline{\Gamma}$ is the compact hyperelliptic Riemann surface of the hyperelliptic curve $w^2=\frac{2}{k}\left(1-\phi^k\right)$ (see [**?**, §4.1.1)], [**?**, §4.1]). With this covering in consideration, (4) are finally written as a system of *hypergeometric differential equations* ([**?**], [**?**]) in the new independent variable x, each of them of the form:

$$x\left(1-x\right)\frac{d^2\xi_i}{dx^2}+\left(\frac{k-1}{k}-\frac{3k-2}{2k}x\right)\frac{d\xi_i}{dx}+\frac{\lambda_i}{2k}\xi_i=0. \tag{6}$$

These equations are usually called the *algebraic variational equations* ($\mathrm{AVE}=\bigoplus_{i=1}^n\mathrm{AVE}_i$). Kimura's table ([**?**]), in turn owing to Schwarz's ([**?**]), provides a concise list of those cases in which hypergeometric equations are integrable by quadratures, *i.e. in which the Galois group of* (6) *has a solvable identity component.* Both tables were based on properties of the monodromy group ([**?**]). Adapting both tables to the new hypothesis, namely that the Galois group of each of the variational equations must have a *commutative* identity component, yields the following fundamental result:

Theorem 2.2. ([**?**, Theorem 3] (see also [**?**, Theorem 5.1])) *Assume X_H, given by* (3)*, is completely integrable with meromorphic first integrals; let $\boldsymbol{c}\in\mathbb{C}^n$ a solution* *to $V'(\boldsymbol{c})=\boldsymbol{c}$ and assume $V''(\boldsymbol{c})$ is diagonalizable; then, if $\lambda_1,\ldots,\lambda_n$ are the eigenvalues of $V''(\boldsymbol{c})$ and we define $\lambda_1=k-1$, each pair (k,λ_i), $i=2,\ldots,n$ matches one of the following items (p being an arbitrary integer):*

TABLE 1

	k	λ		k	λ
1	k	$p+p(p-1)\frac{k}{2}$	**10**	-3	$\frac{25}{24}-\frac{1}{24}\left(\frac{12}{5}+6p\right)^2$
2	2	*arbitrary* $z\in\mathbb{C}$	**11**	3	$-\frac{1}{24}+\frac{1}{24}(2+6p)^2$
3	-2	*arbitrary* $z\in\mathbb{C}$	**12**	3	$-\frac{1}{24}+\frac{1}{24}\left(\frac{3}{2}+6p\right)^2$
4	-5	$\frac{49}{40}-\frac{1}{40}\left(\frac{10}{3}+10p\right)^2$	**13**	3	$-\frac{1}{24}+\frac{1}{24}\left(\frac{6}{5}+6p\right)^2$
5	-5	$\frac{49}{40}-\frac{1}{40}(4+10p)^2$	**14**	3	$-\frac{1}{24}+\frac{1}{24}\left(\frac{12}{5}+6p\right)^2$
6	-4	$\frac{9}{8}-\frac{1}{8}\left(\frac{4}{3}+4p\right)^2$	**15**	4	$-\frac{1}{8}+\frac{1}{8}\left(\frac{4}{3}+4p\right)^2$
7	-3	$\frac{25}{24}-\frac{1}{24}(2+6p)^2$	**16**	5	$-\frac{9}{40}+\frac{1}{40}\left(\frac{10}{3}+10p\right)^2$
8	-3	$\frac{25}{24}-\frac{1}{24}\left(\frac{3}{2}+6p\right)^2$	**17**	5	$-\frac{9}{40}+\frac{1}{40}(4+10p)^2$
9	-3	$\frac{25}{24}-\frac{1}{24}\left(\frac{6}{5}+6p\right)^2$	**18**	k	$\frac{1}{2}\left(\frac{k-1}{k}+p(p+1)k\right)$

(7)

Remarks 2.1.

1. Theorem 2.2 strengthens what was done by H. Yoshida for $n=2$ from reference [**?**] onward; indeed, his result, which is not generalizable to $n>2$ in a simple, straightforward manner, pivoted on the use of Ziglin's Theorem in which, as said in [**?**, Remark 2.3.2(2)], complete integrability may only be assumed if $n=2$. Hence, Yoshida's line of study only allowed one non-trivial integer λ_2; besides, it ended up in a wider set of non-integrability regions for λ_2, each with a non-zero Lebesgue measure. Since Yoshida's result is a corollary to Theorem 2.2 for $n=2$ ([**?**, p. 6], see also [**?**, p. 105]), and since the latter works for arbitrary $n\geq 2$ and restricts the non-integrability regions much further (namely, to discrete sets rather than infinite unions of intervals), Table 1 appears, in expectation for advances concerning the higher variational equations (see [**?**, Subsection 5.3.1]), as the strongest current tool for testing the non-integrability of Hamiltonians of the form (3) from the Galoisian viewpoint.
2. It is not difficult to see that, for any given $i=2,\ldots,n$, if λ_i does not appear in Table (7), then the Galois group G_i of equation (4) is precisely $\mathrm{SL}_2(\mathbb{C})$; indeed, the fact λ_i falls out of the Table guarantees the non-solvability of the identity component $\widehat{G}_i^0$ of the Galois group $\widehat{G}_i$ of the hypergeometric equation (6). It now only takes recalling the result [**?**, Theorem 5] (see also [**?**, Theorem 2.5]), according to which the identity component of the Galois group remains invariant under finite branched coverings. Since $t\mapsto\phi(t)^k$ is precisely one such covering, G_i^0 is non-commutative. The fact $G_i\subset\mathrm{SL}_2(\mathbb{C})$ (due to the absence of $\frac{d\xi_i}{dt}$ in (4), see e.g. [**?**, §2.2]) obviously implies $G_i^0\subset\mathrm{SL}_2(\mathbb{C})$ and the fact G_i^0 is not solvable renders $G_i^0=G_i=\mathrm{SL}_2(\mathbb{C})$ in virtue of the classification of subgroups of $\mathrm{SL}_2(\mathbb{C})$ given in [**?**, Proposition 2.2] and the analysis done thereof in the last paragraph of [**?**, §2.1].

2.3.2. *Existence of an additional integral.* If X_H has p first integrals $f_1 = H, \ldots, f_p$ in pairwise involution and independent over a neighborhood of the integral curve $\overline{\Gamma}$ defined by $\phi(t)\,\boldsymbol{c}$, the *normal variational equations* ([?, §4.3], see also [?, §4.1.3]) are equal to $n-p$ of the initial variational equations; reordering indexes if needed, let us write them as $\mathrm{VE}_{p+1}, \ldots, \mathrm{VE}_n$ with corresponding differential Galois groups $G_{p+1}, \ldots, G_n$ and let us write the eigenvalues corresponding to $\mathrm{VE}_{p+1}, \ldots, \mathrm{VE}_n$ (each of them of the form (4)) as $\lambda_{n-p+1} = k-1, \ldots, \lambda_n$ and assume they are all in Table (7). In virtue of what was stated in [?, Remark 2.2.12], the differential Galois group $G_{\mathrm{NVE}} = \mathrm{Gal}\left(\bigoplus_{i=1}^{n-p} \mathrm{VE}_i\right)$ of the normal variational equations satisfies $G_{\mathrm{NVE}} \subset G_1 \times \cdots \times G_{n-p}$ and, defining $\pi_1, \ldots, \pi_{n-p}$ as the usual projections of $G_1 \times \cdots \times G_{n-p}$, $\pi_i\left(G_{\mathrm{NVE}}\right) \simeq G_i$ for $i = 1, \ldots, n-p$. In an similar manner, applying the covering $t \mapsto \phi^k$ to each one of the normal variational equations $\mathrm{VE}_1, \ldots, \mathrm{VE}_{n-p}$ we obtain the *algebraic normal variational equations*, $\mathrm{ANVE} = \bigoplus_{i=1}^{n-p} \mathrm{AVE}_i$.

Recently A. J. Maciejewski, M. Przybylska and H. Yoshida proved the following:

Theorem 2.3. ([?, Theorem 1.2]) *Let X_H be a Hamiltonian field given by* (3)*. If there is at least an additional single first integral f independent with $\{f_1, \ldots, f_p\}$ on a neighborhood of $\overline{\Gamma}$ (but may be dependent on $\overline{\Gamma}$), then we have one of the following two situations:*

1. *At least one of the eigenvalues $\lambda_1, \ldots, \lambda_{n-p}$ belongs to Table* 1*.*
2. *There are $1 \le i < j \le n-p$ such that*

$$\sqrt{(k-2)^2 + 8k\lambda_i} - \sqrt{(k-2)^2 + 8k\lambda_j} \in 2k\mathbb{Z}. \quad \square$$

This theorem was in turn based on a essential result by E. Kolchin about algebraic dependence ([?], see also [?, Theorem A.2]). We will actually perform a step further and, as a by-product, obtain an alternative proof for Theorem 2.3 without resorting to Kolchin's result.

Theorem 2.4. *Let X_H be a Hamiltonian field given by* (3)*. If there is (at least) an additional single first integral f independent with $\{f_1, \ldots, f_p\}$ on a neighborhood of $\overline{\Gamma}$, then we have one of the following two situations:*

1. *At least one of the eigenvalues $\lambda_1, \ldots, \lambda_{n-p}$ belongs to Table* 1*.*
2. *There exist $1 \le i < j \le n-p$ such that*

$$\sqrt{(k-2)^2 + 8k\lambda_i} - \sqrt{(k-2)^2 + 8k\lambda_j} \in 2k\mathbb{Z}. \tag{8}$$

Moreover if we divide the set of eigenvalues $\{\lambda_1, \ldots, \lambda_{n-p}\}$ in equivalence classes, $\Lambda_1 = \{\lambda_{1,1}, \ldots, \lambda_{1,k_1}\}, \ldots \Lambda_r = \{\lambda_{r,1}, \ldots, \lambda_{r,k_r}\}, \{\lambda_{K+1}\}, \ldots, \{\lambda_{n-p}\}$, with respect to the relation defined by (8) *with $k_1, k_2, \ldots, k_r$ all greater than* 1 *(by reordering the eigenvalues we can assume this) and $K := \sum_{i=1}^{r} k_i$. Then X_H can have at most $2K - 3r$ additional meromorphic first integrals.*

Proof. In the proof we will use three distinct kinds of normal variational equations, two of which have already been introduced above:

- $\mathrm{NVE} = \bigoplus_{i=1}^{n-p} \mathrm{VE}_i$ corresponding to equations (4) for $i = 1, \ldots, n-p$ and having Galois group G_{NVE};
- $\mathrm{ANVE} = \bigoplus_{i=1}^{n-p} \mathrm{AVE}_i$ corresponding to equations (6) for $i = 1, \ldots, n-p$ with Galois group G_{ANVE};

- the *invariant algebraic normal equations* obtained from (6) for $i = 1, \ldots, n-p$ by means of the classical transformation

$$\xi_i = \eta_i \exp\left(\frac{1}{2}\int \frac{\frac{k-1}{k} - \frac{3k-2}{2k}x}{x(1-x)}dx\right),$$

  aimed at vanishing the coefficients in $\frac{d\eta_i}{dx}$ in the resulting $n-p$ second-order equations in η_i; the Galois group of such a system of equations will be written as H. For each $i = 1, \ldots, n-p$, let H_i be the Galois group of the corresponding equation in η_i; it is immediate that $H \subset H_1 \times \cdots \times H_{n-p}$ and that $\pi_i(H) \simeq H_i$ for $i = 1, \ldots, n-p$, as was the case for G_{NVE}.

In virtue of Remark 2.1(2) and the fact that only algebraic functions are introduced by the changes $\xi_i \mapsto \eta_i$, *the identity component of all three groups is one and the same.* Hence, in virtue of what was said at the end of the proof of Theorem A.10, G_{NVE} is q-Ziglin for some q if and only H is. Furthermore, G_{NVE} is contained in $\mathrm{SL}_2(\mathbb{C})^{n-p}$; such is the case for H, as well.

Assume none of $\lambda_1, \ldots, \lambda_{n-p}$ belongs to Table (7); then, in virtue of Remark 2.1(2), we have $G_i \simeq \mathrm{SL}_2(\mathbb{C})$ for all $i = 1, \ldots, n-p$. If there is an additional first integral f which is independent with the set $\{f_1, \ldots, f_p\}$, then by Ziglin's Lemma ([**?**, Lemma 4.3], [**?**, Lemma 6], [**?**, Remark 2.3.2(2)]) the normal variational equations must have a non-trivial rational first integral $\hat{f}$ with coefficients in $\mathcal{M}\left(\overline{\Gamma}\right)$ and thus, in virtue of the fundamental lemma [**?**, Lemma 9], see also [**?**, Lemma 4.6], G_{NVE} must have a non-trivial rational invariant, i.e. G_{NVE} is at least 1-Ziglin. Inclusion $G_{\mathrm{NVE}} \subset G_1 \times \cdots \times G_{n-p}$, isomorphisms $G_i \simeq \mathrm{SL}_2(\mathbb{C}), i = 1, \ldots n-p$ and [**?**, Remark 2.2.12] yield a faithful representation of G_{NVE} in $\mathrm{SL}_2(\mathbb{C})^{n-p}$ such that $\pi_i(G_{\mathrm{NVE}}) \simeq \mathrm{SL}_2(\mathbb{C})$ for each $i = 1, \ldots, n-p$. Thus, we are in the situation of Appendix A, G_{NVE} being at least 1-Ziglin. Since $2K - 3r$ is zero if and only if $r = 0$, and by Theorems A.10 and A.8, we know that the structure of G_{NVE} is as in Theorem A.8 with $r \geq 1$ and $m = n - p - K$. That is, elements of G_{NVE} are expressible as

$$\mathrm{diag}\left([A_1]_{k_1}^{\boldsymbol{\chi}_1}, \ldots, [A_r]_{k_r}^{\boldsymbol{\chi}_r}, A_{r+1}, \ldots, A_{n-p-K}\right), \quad A_1, \ldots, A_{n-p-K} \in \mathrm{SL}_2(\mathbb{C}). \quad (9)$$

All of the above assertions concerning G_{NVE} are true, *mutatis mutandis*, for $H \subset H_1 \times \cdots \times H_{n-p}$. Therefore, we may also apply Theorems A.10 and A.8 and conclude that the elements of H are of the form (9) as well. The remainder of the proof will be done exclusively using H. Let us fix $j \in \{1, \ldots, r\}$.

In each diagonal block $[A_j]_{k_j}^{\boldsymbol{\chi}_j}$ (denoted accordingly as in (45) and Theorem A.8), we have a pairwise relation between the 2×2 submatrices: for each $i_1, i_2 = 0, \ldots, k_j - 1$, $\chi_{j,i_1} A_j^{(i_1)}$, $\chi_{j,i_2} A_j^{(i_2)}$ are such that $A_j^{(i_1)}$, $A_j^{(i_2)}$ are equivalent (in the sense of the representation theory, see Subsection A.3.2), with the conditions $\chi_{j,0} := 1$ and $A_j^{(0)} = A_j$. As we are going to see, this relation corresponds exactly to equation (8), as was also shown in [**?**] in order to prove Theorem 2.3. We know $\chi_{j,i_1}, \chi_{j,i_2} \in \{\pm 1\}$, hence the 2×2 matrices $\chi_{j,i_1} A_j^{(i_1)}, \chi_{j,i_2} A_j^{(i_2)}$ have the same eigenvalues up to a sign.

We recall that each matrix $\chi_{j,i} A_j^{(i)}$ corresponds to a faithful representation of the Galois group of one of the equations in $\eta_1, \ldots, \eta_{n-p}$, which we may denote as H_i without loss of generality. For i_1, i_2 as above, it is a well-known fact that the monodromy groups around $x = \infty$ of the corresponding two equations belong to the

Galois groups H_{i_1}, H_{i_2}, respectively; hence, the local monodromy matrices M_{i_1}, M_{i_2} of these two equations around $x=\infty$ are precisely equal to two matrices of the form $\chi_{j,i_1}A_j^{(i_1)}, \chi_{j,i_2}A_j^{(i_2)}$ introduced in the above paragraph.

The differences of exponents at infinity of these local monodromies M_{i_1} and M_{i_2} are given by

$$\frac{\sqrt{(k-2)^2+8k\lambda_{i_1}}}{2k}, \qquad \frac{\sqrt{(k-2)^2+8k\lambda_{i_2}}}{2k},$$

respectively (see [?, §5.1.2]). It is now a simple exercise to verify, by means of the computations in the proof of Theorem 2.3, that the identity between the eigenvalues of M_{i_1} and M_{i_2} up to a sign implies the relation (8) (see reference [?], and especially §6 therein, for details). In particular, to each block $[A_j]_{k_j}^{\boldsymbol{\chi}_j}$ in the structure Theorem A.8 corresponds one of the equivalence classes $\Lambda_1,\ldots,\Lambda_r$.

The fact that we can have at most $2K-3r$ meromorphic first integrals follows from Ziglin's lemma and the fundamental Lemma in [?, Lemma 9] (see also [?, Lemma 4.6]), since if X_H has q additional meromorphic first integrals then the Galois group G_{NVE} must be q-Ziglin; the result now follows from Theorem A.10. □

Remark 2. In the hypotheses of Theorem (namely, right before items 1 and 2), the presumed additional single first integral f independent with $\{f_1,\ldots,f_p\}$ on a neighborhood of $\overline{\Gamma}$ may still be dependent therewith on $\overline{\Gamma}$.

2.4. The N-Body Problem.

2.4.1. *Definitions.* Let $d,N\geq 2$ be two integers. The **(General d-dimensional) N-Body Problem** is the model describing the motion of N mutually interacting point-masses in an Euclidean d-space led solely by their mutual gravitational attraction. It is determined by the initial-value problem given by the $2N$ initial conditions $\mathbf{x}_1(t_0),\ldots,\mathbf{x}_N(t_0)\in\mathbb{R}^d$ and $\dot{\mathbf{x}}_1(t_0)\ldots,\dot{\mathbf{x}}_N(t_0)\in\mathbb{R}^d$, such that $\mathbf{x}_j(t_0)\neq\mathbf{x}_k(t_0)$ if $j\neq k$, and the system of Nd scalar second-order differential equations

$$m_i\ddot{\mathbf{x}}_i=-G\sum_{k\neq i}^{N}\frac{m_im_k}{\|\mathbf{x}_i-\mathbf{x}_k\|^3}(\mathbf{x}_i-\mathbf{x}_k), \qquad i=1,\ldots,N, \tag{10}$$

where, for each $i=1,\ldots,N$, $\mathbf{x}_i\in\mathbb{R}^d$ is a d-dimensional vector function of the time variable t describing the position of a body and m_i is the mass of the body with position $\boldsymbol{q}_i$. G, the gravitational constant, may and will be set equal to one from now on by an appropriate choice of units.

Hamiltonian formulation ensues in a most natural way; defining

$$M=\mathrm{diag}(m_1,\ldots,m_1,\cdots,m_N,\ldots,m_N)\in\mathcal{M}_{Nd}(\mathbb{R}),$$

and assembling the coordinates of our phase space among the Nd-dimensional vectors

$$\boldsymbol{x}(t)=(\mathbf{x}_i(t))_{i=1,\ldots,N}, \qquad \boldsymbol{y}(t)=(\mathbf{y}_i(t))_{i=1,\ldots,N}:=(m_i\dot{\mathbf{x}}_i(t))_{i=1,\ldots,N}$$

of **positions** and **momenta**, respectively, the equations of motion may now be expressed as

$$\dot{\boldsymbol{x}}=M^{-1}\boldsymbol{y}, \qquad \dot{\boldsymbol{y}}=-\nabla U_{N,d}(\boldsymbol{x}), \tag{11}$$

where $U_{N,d}(\boldsymbol{x}):=-\sum_{1\leq i<k\leq N}\frac{m_im_k}{\|\mathbf{x}_i-\mathbf{x}_k\|}$ is the **potential function** of the gravitational system. System (11) is the set of Hamilton's equations linked to the Hamiltonian

$$H_{N,d}(\boldsymbol{x},\boldsymbol{y}):=\frac{1}{2}\boldsymbol{y}^TM^{-1}\boldsymbol{y}+U_{N,d}(\boldsymbol{x}). \tag{12}$$

Most of the bibliography on the subject deals with either the **planar** ($d = 2$) or **spatial** ($d = 3$) N-Body Problem since raising the dimension of the ambient space deprives the problem of most of its physical significance; it must be said, nevertheless, that further research has been attempted assuming d is an arbitrary integer – needless to say, the reader can already infer that such an assumption is by no means a symptom of confidence in our knowledge of the planar and spatial problems, as may be ascertained in the following Sections of this paper.

2.4.2. *Known first integrals.* Transformations of the form $\mathbf{x} \mapsto T_{Q,\mathbf{v},\mathbf{w},t}(\mathbf{x}) := Q\mathbf{x} + \mathbf{v} + t\mathbf{w}$, formed by a rotation $Q \in \mathrm{O}_{dN}(\mathbb{R})$ and a translation linear with respect to time, are easily proven to be symmetries of (10). $\mathbf{v}$ represents constant translation, and $t\mathbf{w}$ represents the change to a moving frame which moves with a constant velocity $\mathbf{w}$. Since symmetries come paired with first integrals (see [?]), the first step is looking for conserved quantities linked to symmetries as basic as $T_{Q,\mathbf{v},\mathbf{w},t}$. The vector $\mathbf{c}_G(t) := \frac{1}{m}\sum_{i=1}^N m_i \mathbf{x}_i(t)$, where $m = \sum_{i=1}^N m_i$, is the **center of mass** of the configuration $\boldsymbol{x}(t)$. It corresponds to a configuration whose movement is rectilinear and uniform: $\ddot{\mathbf{c}}_G = \frac{1}{m}\sum_{i=1}^N m_i \ddot{\mathbf{x}}_i = 0$, due to the symmetry of the expression in the second addition. Thus,

$$\mathbf{c}_G(t) = \mathbf{c}_1 t + \mathbf{c}_2, \qquad \mathbf{c}_i \in \mathbb{R}^d. \tag{13}$$

In particular $\mathbf{I}_L := m\mathbf{c}_1 = \sum_{i=1}^N m_i \dot{\mathbf{x}}_i$, usually called the **linear momentum**, is a vector of conserved quantities of the system; the ones associated to translation, that is. The conserved quantities linked to rotation all lie in the **angular momentum** $\mathbf{I}_A = (I_{A,k,l})_{1 \le k < l \le d} \in \mathbb{R}^{d(d-1)/2}$,

$$I_{A,k,l} = \sum_{i=1}^N x_{d(i-1)+k}\dot{x}_{d(i-1)+l} - x_{d(i-1)+l}\dot{x}_{d(i-1)+k}, \qquad 1 \le k < l \le d,$$

obviously summing up to a single scalar quantity if $d = 2$: $I_A := \sum_{i=1}^N m_i \mathbf{x}_i \wedge \dot{\mathbf{x}}_i$. In view of (13), $\mathbf{c}_G$ can always be assumed fixed at the origin since $T_{\mathrm{Id},-\mathbf{c}_1 t,-\mathbf{c}_2,t}$ is a symmetry for (10); except for Definition 2.7, we will assume $\mathbf{c}_G = \mathbf{0}$ from now on.

Let us define the scalar product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle := (M\boldsymbol{x})^T \boldsymbol{y}$ in $\mathbb{R}^{Nd}$. The **moment of inertia** for a given solution $\boldsymbol{x}(t)$ of (10) is defined as $I(\boldsymbol{x}) := \langle \boldsymbol{x}, \boldsymbol{x} \rangle$. This is *not* a first integral of the problem but will be useful in the next Subsection.

All in all, the N-body problem has $\frac{1}{2}(d+2)(d+1)$ (so-called *classical*) first integrals (see [?]):

1. $2d$ for the invariance of the linear momentum $\mathbf{I}_L$, i.e. for the uniform linear motion of the center of mass;
2. $d(d-1)/2$ for the invariance of the angular momentum $\mathbf{I}_A$;
3. one for the invariance of the Hamiltonian $H_{N,d}$.

That makes 6 for the planar problem and 10 for the spatial problem. Bruns' theorem, given in 1887, asserts these are the only first integrals algebraic with respect to phase variables for the Three-Body Problem:

Theorem 2.5 (Bruns' Theorem, [?]). *Every first integral of the spatial Three-Body Problem which is algebraic with respect to positions, momenta and time is an algebraic function of the classical ten first integrals.*

An attempt at extending this result was done by P. Painlevé, namely at proving that any integral depending algebraically on the moments $\mathbf{y}_1, \dots, \mathbf{y}_N$, regardless of how it depends on the positions $\mathbf{x}_1, \dots, \mathbf{x}_N$, is a function of the classical integrals.

The proof of this assertion, written in [?], is wrong, though; see also [?]. The best generalization of Theorem 2.5 known to date is the following:

Theorem 2.6 (Julliard's Theorem, [?]). *In the d-dimensional N-body problem with $1 \leq d \leq N$, every first integral which is algebraic with respect to positions, momenta and time is an algebraic function of the classical $\frac{1}{2}(d+2)(d+1)$ integrals.*

Our obvious aim, both in the present paper and in the future, is to take the thesis in Theorem 2.6 to its most extreme generalization.

2.4.3. *Central configurations of the N-body problem.* Despite the general lack of faith in finding *simple* closed-form solutions for the N-body problem ([?]), there are special solutions whose orbits allow for a complete qualitative study without having to resort only to the infinite series given in [?], [?] and [?]. Such solutions, called **homographic**, are those preserving the initial figure formed by the bodies, except for homothecies and rotations:

Definition 2.7. *A solution $\boldsymbol{x}(t)$ of the N-body problem is called* ***homographic*** *if there are functions $r : J \subset \mathbb{R} \to \mathbb{R}$ and $\Phi : J \subset \mathbb{R} \to \mathrm{SO}_d(\mathbb{R})$ defined on an open interval $J \subset \mathbb{R}$, such that*

$$\mathbf{x}_i(t) - \mathbf{c}_G(t) = r(t)\,\Phi(t)\,(\mathbf{x}_i(t_0) - \mathbf{c}_G(t_0)),$$

Using the homogeneity of $U_{N,d}(\boldsymbol{x})$ and $I(\boldsymbol{x})$ of degree -1 and 2, respectively, the Euler relation for homogeneous functions and the method of Lagrange multipliers, it may be easily proven that initial conditions $\boldsymbol{x}$ of homographic solutions satisfy system

$$U'_{N_d}(\boldsymbol{x}) = \lambda M \boldsymbol{x}, \tag{14}$$

where $\lambda > 0$; actually $\lambda = U_{N,d}(\boldsymbol{x})/I(\boldsymbol{x})$. If the bodies are released with zero initial velocity, these initial conditions give rise to simple, explicit **homothetical** solutions of the N-Body Problem (i.e. solutions showing homothetical collapse to the origin).

Definition 2.8. *An initial configuration $\boldsymbol{x}(t_0)$ of a homographic solution (i.e. a solution to* (14)*) will be called a* ***central configuration****.*

Remark 3. λ may be set equal to one; indeed, the -2-homogeneity of $U'_{N,d}$ assures us $U'_{N,d}(\lambda^\alpha \boldsymbol{x}) = \lambda^{-2\alpha} U'_{N,d}(\boldsymbol{x})$; thus, assuming $U'_{N,d}(\boldsymbol{x}) = \lambda M \boldsymbol{x}$, defining $\tilde{\boldsymbol{x}} = \lambda \boldsymbol{x}$ and asking for $U'_{N,d}(\tilde{\boldsymbol{x}}) = M\tilde{\boldsymbol{x}}$ to hold, we obtain $\alpha = -1$.

The above remark implies that the set of solutions to (14) is independent of the value of λ and thus has the same cardinal as the set of solutions to $U'(\boldsymbol{x}) = -\lambda^* M \boldsymbol{x}$ for any other $\lambda^* > 0$. Measuring such a cardinal is a fundamental problem in Celestial Mechanics; in order for this problem to make sense, the usual procedure is studying the quotient modulo symmetries of rotation $\mathrm{O}_d(\mathbb{R})$, translation $(\mathbb{R}^d)$ and homothecy $(\mathbb{R} \setminus 0)$, i.e. counting classes of central configurations modulo these symmetries. For planar central configurations, the set of mutual distances between the bodies may occasionally prove an adequate coordinate system for this quotient space, albeit a rather redundant one since its cardinality is equal to $\binom{N}{2}$ and a set of merely $2N-4$ coordinates suffices in the planar case. See [?].

Examples 2.1.

1. Regardless of m_1, m_2, m_3, there exists a central configuration of the Three-Body Problem, called a **Lagrange (triangular) configuration**, consisting of an equilateral triangle whose vertexes are the point-masses (see [?] or Remark 5 and Section 4.1 below).
2. Generalizing Example 1 above, the regular d-simplex is a central configuration of the d-dimensional Problem for any $d \geq 2$ and $N = d + 1$ (see [?]): for instance, Lagrange's triangular configuration if $d = 2$ or a regular tetrahedron if $d = 3$ ([?]).
3. Again regardless of m_1, m_2, m_3, each ordering of three bodies arranged on a straight line forms a central configuration, called an **Euler (collinear) configuration** (see [?]).
4. Yet again we may generalize Example 3: for each $N \geq 3$ and each set of positive values $m_1, \ldots, m_N$, N bodies with masses $m_1, \ldots, m_N$ arranged in a straight line lead to $N!/2$ central configurations – one for each ordering of the point-masses; we call these the **Moulton (or Euler-Moulton) configurations** (see [?]).
5. Whenever the masses are equal, regular N-polygons with the point-masses at the vertexes are central configurations, see [?], [?], [?], [?] or Remark 5 and Lemma 4.1. Conversely, for $N > 3$, regular polygons are central configurations if and only if the masses are equal (again [?], [?], [?] or [?]).
6. Whenever N of the masses are equal and an additional mass is allowed into the system, regular N-polygons with the bodies of equal masses at the vertexes and the body corresponding to the isolate mass m_{N+1} placed at the center of the polygon (i.e. the center of mass) are central configurations, see Remark 5 and Lemma 5.2.
7. Depending on N and on the specific masses, other special configurations may be proven to exist. See for instance [?] and [?] for the so-called *pyramidal configurations*, and [?] and [?] for some insight and new results on the case $N = 4$.

Remark 4. Inasmuch as in Examples 1, 2, 5 and 6, the exact coordinates of the solution in Example 3 may be found explicitly, albeit in a less straightforward way: indeed, for an adequate mutual-distance quotient parameter ρ, the so-called **Euler quintic** holds along any collinear three-body solution:

$$\begin{aligned}(m_2 + m_3) + (2m_2 + 3m_3)\,\rho + (3m_3 + m_2)\,\rho^2 - (3m_1 + m_2)\,\rho^3 \\ - (3m_1 + 2m_2)\,\rho^4 - (m_1 + m_2)\,\rho^5 \quad = \quad 0 \qquad (15)\end{aligned}$$

Equation (15) may be solved explicitly by transforming P to Bring reduced form $P_B(\rho) = \rho^5 - \rho - \beta$ by means of three Tschirnhaus transformations and expressing the roots of $P_B(\rho)$ in terms of generalized hypergeometric functions ${}_4F_3$, although such calculus is not necessary for our study and will be skipped; see [?].

For more information on central configurations, see [?].

There are some facts proving the importance of research in central configurations for the N-body problem:

1. Besides the orbits of the two-body problem, the only known explicit solutions for the N-body problem are homographic orbits, i.e. those having as an initial condition a central configuration.

2. Thanks to Sundman ([**?**]), we know all orbits beginning or ending at a total collision are asymptotic to a homothetic movement, i.e. the configuration formed by the bodies tends to a central configuration.

3. All changes in the topology of the integral varieties $V_{H,\mathbf{I}_A}$ corresponding to the energy H and the angular momentum $\mathbf{I}_A$ are due to central configurations ([**?**], [**?**], [**?**], [**?**]). However, the concise description of these varieties with prescribed values of $H, \mathbf{I}_A$ is not even concluded for $N = 3$ ([**?**, §2], [**?**]).

4. The sixth problem proposed by S. Smale in [**?**] is whether or not, given $m_1, \ldots, m_N$, the number of classes of central configurations is finite. His program pivoted precisely on the topology of the $V_{H,\mathbf{I}_A}$ so as to pursue topological stability; namely pivoting on the impossibility of transition between connected components. This is useful if $N = 3$, since there exist ranges for which $V_{H,\mathbf{I}_A}$ has some connected component projecting on a bounded set of the $\boldsymbol{x}$-space. For $N \geq 4$, however, there is always only one connected component, and it has unbounded $\boldsymbol{x}$-projection: see [**?**, §2] and, especially, [**?**].

3. **Preliminaries.**

3.1. **Statement of the main results.**

Symplectic change $\boldsymbol{x} = M^{-1/2}\boldsymbol{q}$, $\boldsymbol{y} = M^{1/2}\boldsymbol{p}$ renders $H_{N,d}$ a classical Hamiltonian $\mathcal{H}_{N,d} = \frac{1}{2}p^2 + V_{N,d}(\boldsymbol{q})$ with a potential which is homogeneous of degree -1:

$$V_{N,d}(\boldsymbol{q}) := -\sum_{1\leq i<j\leq N} \frac{(m_i m_j)^{3/2}}{\left\| \sqrt{m_j}\mathbf{q}_i - \sqrt{m_i}\mathbf{q}_j \right\|}. \tag{16}$$

In virtue of Theorem 2.2, performing the following two steps would prove $\mathcal{H}_{N,d}$ not meromorphically integrable:

Step I either explicitly finding or proving the existence of an adequate constant vector $\boldsymbol{c} \in \mathbb{C}^{2N}$ such that

$$V'_{N,d}(\boldsymbol{c}) = \boldsymbol{c}; \tag{17}$$

Assume $V''_{N,d}(\boldsymbol{c})$ is diagonalizable.

Step II proving that at least one of the eigenvalues of $V''_{N,d}(\boldsymbol{c})$ does not belong to the set given by items **1** and **18** in Table (7), which happens to be a set of integers:

$$S := \left\{ -\frac{p(p-3)}{2} : p \in \mathbb{Z} \right\} = \left\{ -\frac{(p+2)(p-1)}{2} : p \in \mathbb{Z} \right\} \subset \mathbb{Z}, \tag{18}$$

whose symmetry allows for the assumption $p > 1$; the size of the consecutive gaps in this discrete set is strictly increasing, as is seen in its first elements: $\{1, 0, -2, -5, -9, -14, -20, -27, -35, \ldots\}$.

In virtue of Theorem 2.4, isolating an adequate set of eigenvalues and performing the following third step would be enough to set a very precise upper bound on the amount of additional meromorphic integrals:

Step III proving that, except for a set $\tilde{S}$ of notable eigenvalues corresponding to the set of classical first integrals, there is no other eigenvalue of $V''_{N,d}(\mathbf{c})$ in S.

This is exactly what will be done for the equal-mass 4, 5, 6-Body Problem (item 1 in Theorem 3.2 below).

And in virtue of either [**?**, Theorem 1.2] or Theorem 2.4, the following fourth step would be enough to discard the existence of even a *single* additional meromorphic integral; in other words, we would prove a generalized version of Theorems 2.5 and 2.6:

Step IV performing Step III and proving that, except for said notable set $\tilde{S}$, Spec $\left(V''_{N,d}(\boldsymbol{c})\right) \setminus \tilde{S}$ consists exclusively of eigenvalues *not* satisfying relation (8) pairwise.

As asserted in Theorem 3.1 below, *this last step has been attained for* $N = 3$; see Subsection 4.1 for a proof.

Remark 5. *Solving* (17) *for the general case appears as anything but trivial.* In virtue of Remark 3, real vector solutions to $V'_{N,d}(\boldsymbol{c}) = \boldsymbol{c}$ correspond exactly to homothetical central configurations, since $M^{1/2}V'_{N,d}(\boldsymbol{q}) = U'_{N,d}\left(M^{-1/2}\boldsymbol{q}\right)$ and thus $U'_{N,d}(\boldsymbol{x}) = M\boldsymbol{x}$ (for $\boldsymbol{x} = M^{-1/2}\boldsymbol{q}$) is equivalent to

$$V'_{N,d}(\boldsymbol{q}) = M^{-1/2}MM^{-1/2}\boldsymbol{q} = \boldsymbol{q}.$$

Were solving (17) a straightforward task, so would be computing central configurations; in view of the egregious amount of research involving or needed for the latter, even in special cases, e.g. the lines of study hinted at in [?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?], [?], or [?], such a premise is arguable at best.

We are proving the following two main results:

Theorem 3.1. *For every* $d \geq 2$, *there is no additional meromorphic first integral for* $X_{\mathcal{H}_{3,d}}$ *with arbitrary positive masses which is independent with the classical first integrals.*

Theorem 3.2. *Let* $X_{\tilde{\mathcal{H}}_{N,d}}$ *stand for any d-dimensional* equal-mass *N-Body Problem:*

1. *For the* planar *Problem* $X_{\tilde{\mathcal{H}}_{N,2}}$, *the number of additional meromorphic first integrals is no larger than:*
   - ***a)*** *one if* $N = 4$;
   - ***b)*** *three if* $N = 5, 6$.

   In particular, the Problem is not meromorphically integrable in the sense of Liouville for all three values of N.
2. *For* $N \geq 3$ *and* $d \geq 2$, $X_{\tilde{\mathcal{H}}_{N,d}}$ *is not meromorphically integrable in the sense of Liouville.*

Consider any triangular homographic solution (Example 2.1(1)) corresponding to energy level zero; such a solution is usually called the *parabolic* Lagrangian solution since the orbit of each of the point-masses is precisely a parabola. By means of Ziglin's Theorem, A. V. Tsygvintsev not only proved there is no complete set of meromorphic first integrals for the *planar* Three-Body Problem in a neighborhood of a parabolic Lagrangian solution; he further transited from this non-integrability proof to one of the absence of a *single* additional integral, except for the three special cases shown in (31) below. See [?, Theorems 2 and 4]; see also [?, Theorem 1.1 and Corollary 1.2], [?, Theorems 6.1 and 6.3], [?, Theorem 1.1], [?, Theorem 4.1]. In [?, Section 3.1], S. L. Ziglin himself established a non-integrability proof provided (m_1, m_2, m_3) belongs to the intersection of some neighborhood of $\{m_1 = m_2\} \cup \{m_1 = m_3\} \cup \{m_2 = m_3\}$ in $\mathbb{R}^3_+$ with the set of deleted lines $\bigcup_{k \neq i} \{m_k/m_i \neq 11/12, 1/4, 1/24\}$; this he did exploiting the proximity of the particular solutions with respect to a certain collinear configuration. Although by no means proven valid for a wide set of values of the masses, Ziglin's result had the advantage of considering general dimension d for the point masses. D. Boucher and J.-A. Weil also proved the planar Three-Body Problem non-integrable in [?, Theorem 9] (see also [?, Theorem 2] and [?, Theorem 3]) by using a criterion of their own (e.g. [?, Theorem 2], [?, Theorem 8], [?, Criterion 1]) devised from Theorem 2.1, and consisting on the detection of logarithms in the factorization of a certain reduced variational system; the particular solution along which variational equations were reduced and factorized was a Lagrange zero-energy solution, just as in the results by Tsygvintsev. As for the equal-mass N-Body Problem, in [?, Section 3.2] Ziglin allowed one of the masses, say m_N, to be different from the others and made attempts at the very same thesis we use here: to wit, that the trace of the Hessian matrix for $V''_{N,d}(\boldsymbol{c})$ is not contained in $\mathbb{Z}$ for some solution $\boldsymbol{c}$ of (17). The main result in [?, Section 3.2] was the existence of at most finitely many values m_N for which the Problem is integrable, although none of these values was actually given.

Theorem 3.1 completes the aforementioned results by Tsygvintsev by discarding the three special cases remaining therein. Furthermore, the proof given here is shorter thanks to Theorems 2.2 and 2.4. Theorem 3.1 also completes what was done by S. L. Ziglin in [?, Section 3.1] and complements the non-integrability result by D. Boucher and J.-A. Weil by extending it to arbitrary dimension, besides being a consistent generalization of Bruns' Theorem 2.5 and the case $N = 3$ of Julliard's Theorem 2.6. Theorem 3.2, on the other hand, completes the results in [?, Section 3.2], though the tools used here hardly qualify as a theoretical step forward since, as said above, the author of the latter reference shared our aim. A comment will be made in Section 5.3.2 concerning the hypotheses in [?, Section 3.2].

Remark 6. We must observe that Hamiltonian $\mathcal{H}_{N,d}$ is *not meromorphic*. However, any first integral of $X_{\mathcal{H}_{N,d}}$ (e.g. $\mathcal{H}_{N,d}$ itself), when restricted to a domain of each determination of $\mathcal{H}_{N,d}$, is meromorphic and thus amenable to the whole theory explained so far; see, for instance, [?, pp. 156-157] for more details as applied to a different homogeneous potential.

3.2. **Setup for the proof.**

3.2.1. *Known eigenvalues.* Let us find the exceptional set $\tilde{S}$ hinted at in Steps III and IV, which consists of $p = d + n + 1$ eigenvalues, all belonging to $\{-2, 0, 1\}$. For the sake of a more comfortable notation, we will denote them from subindex 1 onward, say $\{\lambda_1, \ldots, \lambda_{d+n+1}\}$, as opposed to the notation used in Subsection 2.3.2. d of them, for instance $\lambda_2, \ldots, \lambda_{d+1}$, appear for any solution of Hamilton's equations, and the remaining ones appear specifically for solutions of the form $\phi\boldsymbol{c}$ with $\ddot{\phi} + \phi^{-2} = 0$ and $V'_{N,d}(\boldsymbol{c}) = \boldsymbol{c}$.

Lemma 3.3. *Let $\boldsymbol{q}(t) = (\mathbf{q}_1(t), \ldots, \mathbf{q}_N(t))$ be a solution of the N-Body Problem. Then, d of the eigenvalues of $V''_{N,d}(\boldsymbol{q})$ are identically zero.*

Proof. This results from the invariance of the linear momentum $\mathbf{I}_L$ (Subsection 2.4.2), which after symplectic change $\mathbf{x}_i = \frac{1}{\sqrt{m_i}}\mathbf{q}_i$ and $\mathbf{y}_i = \sqrt{m_i}\mathbf{p}_i$ becomes $\sum_{i=1}^N \sqrt{m_i}\ddot{\mathbf{q}}_i = \mathbf{0}$. Since $\ddot{\mathbf{q}}_i = \dot{\mathbf{p}}_i = -\frac{\partial V_{N,d}}{\partial \mathbf{q}_i}$ for $i = 1, \ldots, N$, we obtain

$$\sum_{i=1}^N \sqrt{m_i}\frac{\partial V_{N,d}}{\partial q_{d(i-1)+k}} = 0, \qquad k = 1, \ldots, d,$$

and derivating these equations with respect to $\boldsymbol{q}$ we obtain d distinct relations of linear dependence between the columns of the Hessian,

$$\sum_{i=1}^{N} \sqrt{m_i} \frac{\partial^2 V_{N,d}}{\partial q_{d(i-1)+k} \partial q_j} = 0, \qquad j = 1, \ldots, 2N, \quad k = 1, \ldots, d,$$

rendering $\left\{ \sum_{i=1}^{N} \sqrt{m_i} \boldsymbol{e}_{dN,d(i-1)+j} : j = 1, \ldots, d \right\}$ an independent eigensystem for the eigenvalue 0; that alone allows us to write $\lambda_2 = \lambda_3 = \cdots = \lambda_{d+1} = 0$. □

Let $\boldsymbol{q} = \phi(t)\, \boldsymbol{c}$ as above in the next two Lemmae. The first of them takes no other effort in proving than referring the reader back to the consequence (5) of Euler's Theorem while setting $k = -1$:

Lemma 3.4. *We may write* $\lambda_1 = -2$. □

Lemma 3.5. $1 \le n \le \binom{d}{2}$ *of the eigenvalues, say* $\lambda_{d+2}, \ldots, \lambda_{d+n+1}$*, are equal to* 1.

Proof. This is a consequence of the invariance of the angular momentum; derivating $\mathbf{I}_A$ once after expressing it in coordinates $\boldsymbol{q}$, $\boldsymbol{p}$, we obtain

$$0 = \sum_{i=1}^{N} q_{d(i-1)+k} \ddot{q}_{d(i-1)+l} - q_{d(i-1)+l} \ddot{q}_{d(i-1)+k}, \qquad 1 \le k < l \le d,$$

and thus

$$0 = \sum_{i=1}^{N} q_{d(i-1)+k} \frac{\partial V_N}{\partial q_{d(i-1)+l}} - q_{d(i-1)+l} \frac{\partial V_N}{\partial q_{d(i-1)+k}}, \qquad 1 \le k < l \le d,$$

which derivated with respect to $\boldsymbol{q}$ yields

$$\begin{aligned} 0 \;&=\; \sum_{i=1}^{N} \left(\delta_{d(i-1)+k,j} \frac{\partial V_{N,d}}{\partial q_{d(i-1)+l}} - \delta_{d(i-1)+l,j} \frac{\partial V_{N,d}}{\partial q_{d(i-1)+k}} \right) \\ &\quad + \sum_{i=1}^{N} \left(q_{d(i-1)+k} \frac{\partial^2 V_{N,d}}{\partial q_{d(i-1)+l} \partial q_j} - q_{d(i-1)+l} \frac{\partial^2 V_{N,d}}{\partial q_{d(i-1)+k} \partial q_j} \right), \\ 1 \;&\le\; k < l \le d, \qquad j = 1, \ldots, dN; \end{aligned}$$

thus, assuming $\boldsymbol{q} = \phi(t)\, \boldsymbol{c}$ as above we have

$$\begin{aligned} 0 \;&=\; \sum_{i=1}^{N} \phi^{-2} \left(\delta_{d(i-1)+k,j} c_{d(i-1)+l} - \delta_{d(i-1)+l,j} c_{d(i-1)+k} \right) \\ &\quad + \sum_{i=1}^{N} \phi^{-2} \left(c_{d(i-1)+k} \frac{\partial^2 V_N}{\partial q_{d(i-1)+l} \partial q_j} (\boldsymbol{c}) - c_{d(i-1)+l} \frac{\partial^2 V_N}{\partial q_{d(i-1)+k} \partial q_j} (\boldsymbol{c}) \right), \\ j \;&=\; 1, \ldots, dN, \qquad 1 \le k < l \le d, \end{aligned}$$

which means $\sum_{i=1}^{N} \boldsymbol{k}_{i,k,l}$ is an eigenvector of $V''_{N,d}(\boldsymbol{c})$ of eigenvalue 1, where $\boldsymbol{k}_{i,k,l} = -c_{d(i-1)+l} \boldsymbol{e}_{dN,d(i-1)+k} + c_{d(i-1)+k} \boldsymbol{e}_{dN,d(i-1)+l}$, for each $1 \le k < l \le d$. $\binom{d}{2}$ is clearly an upper bound for the dimension of vector space $\left\langle \sum_{i=1}^{N} \boldsymbol{k}_{i,k,l} : 1 \le k < l \le d \right\rangle$. □

Corollary 3.6. *Assume* $\boldsymbol{q} = \phi(t) \left(\mathbf{c}_1^T, \ldots, \mathbf{c}_N^T \right)^T$*, where*

$$\mathbf{c}_i = \left(c_{d(i-1)+1}, c_{d(i-1)+2}, 0, \ldots, 0 \right)^T, \qquad i = 1, \ldots, N,$$

and there are at least two $\mathbf{c}_{i_1}, \mathbf{c}_{i_2}$ *such that* $c_{d(i_j-1)+1}c_{d(i_j-1)+2} \neq 0,\ j = 1,2$ *and*

$$\frac{c_{d(i_1-1)+1}}{c_{d(i_1-1)+2}} \neq \frac{c_{d(i_2-1)+1}}{c_{d(i_2-1)+2}}.$$

Then, there are at least $n = 2d-3$ *eigenvalues equal to one.*

Proof. Let $\tilde{\mathbf{c}} = \left(\tilde{\mathbf{c}}_1^T, \ldots, \tilde{\mathbf{c}}_N^T\right)^T$ be the vector formed by shifting the first two entries in each $\mathbf{c}_i$ and multiplying the first of them by -1:

$$\tilde{\mathbf{c}}_i = \left(-c_{d(i-1)+2}, c_{d(i-1)+1}, 0, \ldots, 0\right)^T, \qquad i = 1, \ldots, N.$$

According to the previous Lemma, $\tilde{\boldsymbol{c}} \in \ker\left(V''_{N,d}(\boldsymbol{c}) - \mathrm{Id}_{dN}\right)$. The same Lemma asserts that the set $W \cup \tilde{W} := \{\boldsymbol{v}_k : 3 \leq k \leq d\} \cup \{\tilde{\boldsymbol{v}}_k : 3 \leq k \leq d\}$, where each of its elements is defined as

$$\boldsymbol{v}_k := \left(c_{d(i-1)+1}\mathbf{e}_{d,k}\right)_{i=1,\ldots,N}, \quad \tilde{\boldsymbol{v}}_k := \left(c_{d(i-1)+2}\mathbf{e}_{d,k}\right)_{i=1,\ldots,N}, \qquad k = 3, \ldots, d,$$

is also set of eigenvectors of $V''_{N,d}(\boldsymbol{c})$ for eigenvalue 1, all of them independent with $\tilde{\boldsymbol{c}}$ by hypothesis $c_{d(i_1-1)+1}c_{d(i_1-1)+2} \neq 0$. The dimension of the space spanned by W (resp. $\tilde{W}$) is $d-2$, and any relation of linear independence of a vector of $\boldsymbol{v}_k \in W$ with one vector in $\tilde{\boldsymbol{v}}_l \in \tilde{W}$ would necessarily imply $k = l$; in particular, we would have

$$\frac{c_{d(i_1-1)+1}}{c_{d(i_1-1)+2}} = \frac{c_{d(i_2-1)+1}}{c_{d(i_2-1)+2}},$$

which contradicts our hypothesis. Hence, $\dim W \oplus \tilde{W} = 2d-4$ and adjoining $\tilde{\boldsymbol{c}}$ to $W \cup \tilde{W}$ yields $2d-3$ independent eigenvectors for $V''_{N,d}(\boldsymbol{c})$. □

3.2.2. *Notation for the planar case.* Defining $\mathbf{q}_i = (q_{2i-1}, q_{2i})$ for $i = 1, \ldots, N$ and $\boldsymbol{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_N)$, we have

$$\frac{\partial V_{N,2}}{\partial \mathbf{q}_i} = \sum_{k=1,k\neq i}^{n} \sqrt{m_k}\,(m_i m_k)^{3/2} D_{i,k}^{-3}\mathbf{D}_{i,k}, \quad i = 1, \ldots, N, \tag{19}$$

where $\mathbf{D}_{i,j} = \left(d_{2i-1,2j-1}, d_{2i,2j}\right)^T := \sqrt{m_j}\mathbf{q}_i - \sqrt{m_i}\mathbf{q}_j$ for each $i,j = 1, \ldots, N$, and we obtain the block expression for the Hessian matrix: $V''_{N,2}(\boldsymbol{q}) = \left(\tilde{U}_{i,j}\right)_{i,j=1,\ldots,N}$, defining

$$\tilde{U}_{i,j} := \begin{cases} -\sqrt{m_i m_j}U_{i,j}, & i \neq j, \\ \sum_{k\neq i} m_k U_{i,k}, & i = j \end{cases} \tag{20}$$

where

$$U_{i,j} = U_{j,i} = \begin{cases} 0_{2\times 2}, & i = j, \\ (m_i m_j)^{3/2}\left(d_{2i-1,2j-1}^2 + d_{2i,2j}^2\right)^{-5/2} S_{i,j}, & i < j, \end{cases} \tag{21}$$

and

$$S_{i,j} = S_{j,i} := \begin{pmatrix} d_{2i,2j}^2 - 2d_{2i-1,2j-1}^2 & -3d_{2i-1,2j-1}d_{2i,2j} \\ -3d_{2i-1,2j-1}d_{2i,2j} & d_{2i-1,2j-1}^2 - 2d_{2i,2j}^2 \end{pmatrix}, \qquad i \neq j. \tag{22}$$

3.2.3. *Reduction to the planar case.* We are now justifying our future trend to restrict ourselves to $d=2$. All there is to prove is that, assuming $\boldsymbol{c}$ is embedded in a particular way into a wider ambient space, the only changes in Spec $\left(V''_{N,d}\right)$ are possibly the multiplicity of its existing elements, and possibly the addition of new ones:

Lemma 3.7. *For any given $d \geq 2$, let*

$$\boldsymbol{c} : (\mathbf{c}_1, \dots, \mathbf{c}_N) \in \mathbb{C}^{2d}, \qquad \mathbf{c}_i : (u_{i,1}, u_{i,2}), \qquad i = 1, \dots, N,$$

be a solution to $V'_{N,2}(\boldsymbol{c}) = \boldsymbol{c}$, and

$$\widetilde{\boldsymbol{c}} : (\tilde{\mathbf{c}}_1, \dots, \tilde{\mathbf{c}}_N) \in \mathbb{C}^{Nd}, \qquad \tilde{\mathbf{c}}_i : (u_{i,1}, u_{i,2}, 0, \dots, 0), \qquad i = 1, \dots, N.$$

Then, $V'_{N,d}(\widetilde{\boldsymbol{c}}) = \widetilde{\boldsymbol{c}}$ and Spec $\left(V''_{N,2}(\boldsymbol{c})\right) \subset$ Spec $\left(V''_{N,d}(\boldsymbol{c})\right)$.

Proof. $V'_{N,d}(\widetilde{\boldsymbol{c}}) = \widetilde{\boldsymbol{c}}$ is immediate since

$$\left.\frac{\partial V_{N,d}}{\partial \mathbf{q}_i}\right|_{\mathbf{q}_i = \tilde{\mathbf{c}}_i} = \left.\left(\begin{array}{c} \sum_{k=1,k\neq i}^{n} \sqrt{m_k}\,(m_i m_k)^{3/2} D_{i,k}^{-3} \mathbf{D}_{i,k} \\ \mathbf{0}_{d-2} \end{array} \right)\right|_{\mathbf{q}_i = \mathbf{c}_i} = \left.\left(\begin{array}{c} \frac{\partial V_{N,2}}{\partial \mathbf{q}_i} \\ \mathbf{0}_{d-2} \end{array} \right)\right|_{\mathbf{q}_i = \mathbf{c}_i}.$$

$V''_{N,d}(\widetilde{\boldsymbol{c}})$ takes the following form: $V''_{N,d}(\widetilde{\boldsymbol{c}}) = \left(\tilde{U}_{d,i,j}\right)_{i,j=1,\dots,N}$, where

$$\tilde{U}_{d,i,j} := \left\{ \begin{array}{ll} -\sqrt{m_i m_j}\, U_{d,i,j}, & i \neq j, \\ \sum_{k\neq i} m_k U_{d,i,k}, & i = j \end{array} \right. \tag{23}$$

and the block structure of these matrices will be

$$U_{d,i,j} = \left(\begin{array}{cc} U_{i,j} & \mathbf{0}_{d-2}^T \\ \mathbf{0}_{d-2} & \alpha_{i,j} \mathrm{Id}_{d-2} \end{array} \right), \qquad i,j = 1, \dots, N,$$

where $U_{i,j}$ is defined as in (21) and

$$\alpha_{i,j} = \alpha_{j,i} = \left\{ \begin{array}{ll} 0, & i = j, \\ (m_i m_j)^{3/2} D_{i,j}^{-3}, & i \neq j = 1, \dots, N. \end{array} \right.$$

Thus, if

$$\boldsymbol{v} : (\mathbf{v}_1, \dots, \mathbf{v}_N) \in \mathbb{C}^{2d}, \qquad \mathbf{v}_i : (v_{i,1}, v_{i,2}), \qquad i = 1, \dots, N,$$

is an eigenvector of $V''_{N,2}(\boldsymbol{c})$, then

$$\widetilde{\boldsymbol{v}} : (\tilde{\mathbf{v}}_1, \dots, \tilde{\mathbf{v}}_N) \in \mathbb{C}^{Nd}, \qquad \tilde{\mathbf{v}}_i : (v_{i,1}, v_{i,2}, 0, \dots, 0), \qquad i = 1, \dots, N,$$

is an eigenvector of $V''_{N,d}(\tilde{\boldsymbol{c}})$ for the same eigenvalue. □

We will define $V_N := V_{N,2}$ from now on, and save for indication of the contrary (e.g. for Section 4.1), we will assume we are dealing exclusively with the *planar* case.

4. **Proofs of Theorems 3.1 and 3.2.**

4.1. **Proof of Theorem 3.1.** Step I in Section 3.1 is computing a solution $\boldsymbol{c}$ of (17) for $N=3$. Let us define $m = m_1 + m_2 + m_3$ (which may be always set to 1 by the reader if even simpler calculations are sought all through this section) and $D = m_1 m_2 + m_2 m_3 + m_3 m_1$, and consider vectors of the form $\boldsymbol{c} = m^{-2/3} M^{1/2} \hat{\boldsymbol{c}}$, where $M = (m_i \mathrm{Id}_d)_{i=1,\dots,N}$ as in Subsection 2.4.1 and

$$\hat{\boldsymbol{c}} = \begin{pmatrix} a_2 m_2 + a_3 m_3 \\ b_2 m_2 + b_3 m_3 \\ a_3 m_3 - a_2 (m_1 + m_3) \\ b_3 m_3 - b_2 (m_1 + m_3) \\ a_2 m_2 - a_3 (m_1 + m_2) \\ b_2 m_2 - b_3 (m_1 + m_2) \end{pmatrix} \tag{24}$$

and a_2, a_3, b_2, b_3 are solutions to

$$\left(a_2^2 + b_2^2\right)^{3/2} = \left(a_3^2 + b_3^2\right)^{3/2} = \left[(a_2 - a_3)^2 + (b_2 - b_3)^2\right]^{3/2} = 1.$$

See Subsection 5.2.2 for an explanation of such an assumption. An example of such a vector $\hat{\boldsymbol{c}}$ is

$$\hat{\boldsymbol{c}} = \begin{pmatrix} (m_2 + 2m_3)\,\alpha \\ m_2 \beta \\ -(m_1 - m_3)\,\alpha \\ -(m_1 + m_3)\,\beta \\ -(2m_1 + m_2)\,\alpha \\ m_2 \beta \end{pmatrix}, \tag{25}$$

where $\alpha^2 + \beta^2 = 1$ and $\alpha^3 = 1/8$. The possible choices of α and β add up to two such vectors as (25), and thus two solutions $\boldsymbol{c} = m^{-2/3} M^{1/2} \hat{\boldsymbol{c}}$ and $\boldsymbol{c}^* = m^{-2/3} M^{1/2} \hat{\boldsymbol{c}}^*$ for (17): those corresponding to $\alpha = 1/2$ and $\alpha^* = \frac{-1+\mathrm{i}\sqrt{3}}{4}$, respectively; keeping with what was said in Section 1, square roots are taken in their principal determination. A simple, if tedious computation proves $\boldsymbol{c}$ and $\boldsymbol{c}^*$ solutions to (17), indeed. $\boldsymbol{c}$ *yields an explicit parametrization for the (homothetical) Lagrange triangular solution (Example 2.1(1)).*

The rest of the proof is based on performing both Steps II and III in Section 3.1 at a time. The eigenvalues of $V_3''(\boldsymbol{c})$ are $\{-2, 0, 0, 1, \lambda_+, \lambda_-\}$, where

$$\lambda_\pm := -\frac{1}{2} \pm \frac{3\sqrt{m_1^2 + m_2^2 + m_3^2 - m_1 m_2 - m_1 m_3 - m_2 m_3}}{2(m_1 + m_2 + m_3)}.$$

As said in Theorem 2.4, the existence of a single additional meromorphic integral for $X_{\mathcal{H}_3}$ implies either $\lambda_+^* \in S$ or $\lambda_-^* \in S$, where $S = \left\{-\frac{1}{2} p (p-3) : p > 1\right\}$, which means (defining $R := \sqrt{m^2 - 3D}$) that $\pm 3R \in \left\{\left(p^2 - 3p - 1\right) m : p > 1\right\}$ and therefore

$$-27 (m_1 m_2 + m_1 m_3 + m_2 m_3) \in \left\{m^2 (p-1)(p-2)(p-4)(p+1) : p > 1\right\}, \tag{26}$$

impossible if $p \in \{2, 4\}$ or $p > 4$ since it would have a strictly negative number equaling a non-negative one. For $p = 3$ (26) becomes $8m^2 = 27D$, that is,

$$\frac{m_1 m_2 + m_1 m_3 + m_2 m_3}{(m_1 + m_2 + m_3)^2} = \frac{8}{27}. \tag{27}$$

Thus, we could at this point assure the absence of an additional meromorphic integral *except when* (27) *holds.*

The eigenvalues of $V_3''(\boldsymbol{c}^*)$ are $\{-2,0,0,1,\lambda_+^*,\lambda_-^*\}$, where $\lambda_\pm^* = -\frac{1}{2} \pm \frac{3\sqrt{A}}{2\sqrt{2}m}$, and $A = 2m_1^2 + 2m_2^2 + 2m_3^2 - 5m_1m_2 - 5m_2m_3 + 7m_1m_3 - \mathrm{i}\sqrt{3}(m_1m_2 + m_2m_3 - 5m_1m_3)$. See Appendix B for details. Again, the thesis in Theorem 2.4 amounts to either $\lambda_+^* \in S$ or $\lambda_-^* \in S$, which here becomes $\pm 3\sqrt{A} = (p^2 - 3p - 1)\sqrt{2}m$, and thus

$$A - 2m^2 \in \left\{ \frac{2}{9}(p-1)(p-2)(p-4)(p+1)m^2 : p > 1 \right\};$$

a necessary condition for this to hold with real masses is the vanishing of the imaginary term in A

$$-\mathrm{i}\sqrt{3}\,(m_1m_2 + m_2m_3 - 5m_1m_3) = 0, \tag{28}$$

implying $m_1m_2 + m_2m_3 = 5m_1m_3$. Thus,

$$-378m_1m_3 = 2(p-1)(p-2)(p-4)(p+1)m^2, \tag{29}$$

for some $p > 1$. We discard $p = 2, 4$ in (29) assuming the strict positiveness of m_1 and m_3. The only integer $p > 1$ for which the right side can be negative is: 3, implying $-378m_1m_3 = -16(m_1 + m_2 + m_3)^2$. These two constraints arising from (28) and (29),

$$5m_1m_3 = m_1m_2 + m_2m_3, \qquad \frac{189}{8}m_1m_3 = (m_1 + m_2 + m_3)^2, \tag{30}$$

cannot hold at the same time as condition (27). Indeed, the former two substituted into the latter would yield $\frac{(5m_1m_3 + m_1m_3)}{\frac{189}{8}m_1m_3} = \frac{8}{27}$, i.e. $\frac{16}{63} = \frac{8}{27}$ which is obviously absurd. Thus, either (27) holds or both equations in (30) hold. In particular, term A in $\lambda_\pm^* = -\frac{1}{2} \pm \frac{3\sqrt{A}}{2\sqrt{2}m}$ does not vanish if (27) holds, which implies $\lambda_-^* \neq \lambda_+^*$ and thus $V_3''(\boldsymbol{c}^*)$ has a *diagonal* Jordan canonical form; indeed, the Jordan blocks for eigenvalues $0, -2, 1$ are already diagonal since the eigenvectors provided by the proofs Lemmae 3.3 and 3.4 and Corollary 3.6 are eigenvectors here as well. In other words, in spite of being complex, the second vector $\boldsymbol{c}^*$ does not prevent the symmetrical matrix from being diagonalizable, and thus amenable to the application of Theorem 2.4.

Let us now prove that V_3 does not satisfy the remaining hypothesis in said Theorem. The difference in (8), $E(\lambda_i, \lambda_j) = \left(\sqrt{9 - 8\lambda_j} - \sqrt{9 - 8\lambda_i}\right)/2$, will be studied both for $\operatorname{Spec}(V_3''(\boldsymbol{c}^*))$ and $\operatorname{Spec}(V_3''(\boldsymbol{c}))$. Let

$$a := \left(m_1^2 + m_2^2 + m_3^2 - m_1m_2 - m_1m_3 - m_2m_3\right)^{1/2}(m_1 + m_2 + m_3)^{-1} \geq 0.$$

The only case worth considering for the real eigenvalues is

$$E(\lambda_+, \lambda_-) = \frac{\sqrt{13 + 12a} - \sqrt{13 - 12a}}{2},$$

which is real only if $a \in \left[0, \frac{13}{12}\right]$. In this interval, moreover, the only possible integer values of $E(\lambda_+, \lambda_-)$ are $0, 1, 2$. Note that $a = \sqrt{1 - 3Q}$, where $Q = D/m^2 = (m_1m_2 + m_1m_3 + m_2m_3)(m_1 + m_2 + m_3)^{-2}$. The solution to $\sqrt{1 - 3Q} = n$ for $n = 0, 1, 2$ is, respectively, $Q = 1/3, 0, -1$, among which the only possible value for Q is $1/3$. Hence, $E(\lambda_+, \lambda_-)$ can only be real if $a = 0$, i.e. $Q = 1/3$.

Now consider the *complex* eigenvalues $\lambda_\pm^* = -\frac{1}{2} \pm \frac{3\sqrt{a^*}}{2}$ of $V_3''(\boldsymbol{c}^*)$. Since

$$E\left(\lambda_+^*, \lambda_-^*\right) = \sqrt{13}\left(\frac{\sqrt{1 + \frac{12}{13}a^*} - \sqrt{1 - \frac{12}{13}a^*}}{2}\right),$$

it is enough to prove that $(a^*)^2$ is always complex, non-real whenever $Q = 1/3$. Indeed, if $z = z_1 + z_2\mathrm{i}$ with $z_1 z_2 \neq 0$, then $\sqrt{1+z} - \sqrt{1-z}$ is always complex: $\left(\sqrt{1+z} - \sqrt{1-z}\right)^2 = 2 - 2\sqrt{1-z^2}$ and since z^2 is non-real, so is $2 - 2\sqrt{1-z^2}$.

In order to prove $a^*, (a^*)^2 \in \mathbb{R} \setminus \mathbb{C}$, we will see that the imaginary term inside the square root, $-5m_1m_3 + m_2m_1 + m_2m_3$, is always nonzero if $Q = \frac{1}{3}$. Indeed, otherwise $\frac{5m_1m_3+m_1m_3}{(m_1+m_2+m_3)^2} = \frac{1}{3}$, i.e. $16m_1m_3 - m_1^2 - 2m_2m_1 - m_2^2 - 2m_2m_3 - m_3^2 = 0$; from $5m_1m_3 = m_2m_1 + m_2m_3$, we also deduce $m_2 = \frac{5m_1m_3}{m_1+m_3}$ and therefore

$$16m_1m_3 - m_1^2 - 2m_2m_1 - m_2^2 - 2m_2m_3 - m_3^2 = \frac{4m_1^3m_3 - 15m_1^2m_3^2 + 4m_1m_3^3 - m_1^4 - m_3^4}{(m_1+m_3)^2} = 0,$$

and the only values of m_3 allowing this are

$$\frac{(2+3\mathrm{i}) \pm (1+2\mathrm{i})\sqrt{3}}{2} m_1 \qquad \frac{(2-3\mathrm{i}) \pm (1-2\mathrm{i})\sqrt{3}}{2} m_1,$$

which are obviously not positive real numbers. The lack of an additional meromorphic first integral for arbitrary $m_1, m_2, m_3 > 0$ is thus proven in the planar case.

Furthermore, for the general case $d \geq 3$, we may embed $\boldsymbol{c}$ and $\boldsymbol{c}^*$ into vectors $\tilde{\mathbf{c}}, \tilde{\mathbf{c}}^* \in \mathbb{C}^{3d}$ as in Lemma 3.7. In virtue of Lemmae 3.3 and 3.4 and Corollary 3.6, we have $d + 1 + 2d - 3 = 3d - 2$ eigenvalues (that is, all of them but two) belonging to $\{-2, 0, 1\}$ and due to the classical first integrals; the remaining two eigenvalues of $V_{3,d}''(\boldsymbol{c})$ (resp. $V_{3,d}''(\boldsymbol{c}^*)$) are $\lambda_\pm$ (resp. $\lambda_\pm^*$) due to Lemma 3.7. □

Remarks 4.1.

1. It is worth noting that the only cases forcing us to resort to a second solution to (17) are precisely two of the three cases exceptional to A. V. Tsygvintsev's proof ([**?**]):
$$\frac{D}{m^2} \in \left\{\frac{1}{3}, \frac{2^3}{3^3}, \frac{2}{3^2}\right\}. \tag{31}$$
2. Yet another valid (and even shorter) proof would be feasible were more knowledge available concerning the collinear solution; see Section 5.2, and especially (39), for details.
3. A proof could be attempted at by using Bring forms as in Remark 4, although the amount of calculations involving generalized hypergeometric functions ${}_4F_3$ appears to be rather cumbersome. We are therefore avoiding this for the sake of simplicity.

4.2. **Proof of Theorem 3.2.** In this specific case, since every choice of mass units amounts to a symplectic change in the extended phase space, we may set $m_1 = \cdots = m_N = 1$. Expressions (19) and (20) may be found explicitly in terms of trigonometric functions if we choose the polygonal configuration (Example 2.1 (5)) as a solution to (17). Define

$$s_k := \sin\frac{\pi k}{N}, \quad c_k := \cos\frac{\pi k}{N}, \qquad k \in \mathbb{N},$$

and $\zeta = e^{\frac{2\pi}{N}} = c_2 + s_2$.

Lemma 4.1. *Vector* $\boldsymbol{c}_P = (\mathbf{c}_1, \ldots, \mathbf{c}_N)$ *defined by* $\mathbf{c}_j = \beta_N^{1/3}(c_{2j}, s_{2j})$, *where* $\beta_N = \frac{1}{4}\sum_{k=1}^{N-1} \csc\left(\frac{\pi k}{N}\right)$, *is a solution for* $V_N'(\boldsymbol{q}) = \boldsymbol{q}$.

Proof. Indeed, assume $\mathbf{c}_j = A\left(\cos\frac{2\pi j}{N}, \sin\frac{2\pi j}{N}\right)$ for some $A>0$. We have

$$\frac{\partial V_N}{\partial \mathbf{q}_j}\left(\boldsymbol{c}_P\right) = \frac{1}{4A^2}\begin{pmatrix} \sum_{k=1}^{N-1}\frac{\cos\frac{2\pi j}{N}}{\sin\frac{\pi}{N}k} \\ \sum_{k=1}^{N-1}\frac{\sin\frac{2\pi j}{N}}{\sin\frac{\pi}{N}k}\end{pmatrix}$$

due to the fact that

$$\sum_{k=1,k\neq j}^{N}\frac{\zeta^j-\zeta^k}{\left|\zeta^j-\zeta^k\right|^3} = \zeta^j\sum_{k=1}^{N-1}\frac{1-(c_{2k}+\mathrm{i}s_{2k})}{\left|1-\zeta^k\right|^3},$$

and, since the imaginary part of this sum satisfies:

$$\sum_{k=1}^{N-1}\frac{s_{2k}}{\left|1-\zeta^k\right|^3} = \sum_{k=1}^{N-1}\frac{2s_kc_k}{8c_k^3} = \frac{1}{4}\sum_{k=1}^{N-1}\frac{c_k}{s_k^2} = 0,$$

we finally obtain $\zeta^j\sum_{k=1}^{N-1}\frac{1-(c_{2k}+\mathrm{i}s_{2k})}{\left|1-\zeta^k\right|^3} = \frac{1}{4}\zeta^j\sum_{k=1}^{N-1}s_k^{-1}$. Now $V'(\boldsymbol{c}_P)=\boldsymbol{c}_P$ if and only if $\sum_{k=1}^{N-1}\frac{1}{4A^2s_k} = A$. The latter holds for $A=\beta_N^{1/3}$. □

Let us see how this specific vector simplifies V_N''. Keeping expression (20) in consideration we have $d_{2i-1,2j-1}+d_{2i,2j} = \beta_N^{1/3}\left(\zeta^i-\zeta^j\right)$ which implies

$$S_{i,j} = 2\left(\beta_N^{1/3}s_{i-j}\right)^2\begin{pmatrix} 3c_{2(i+j)}-1 & 3s_{2(i+j)} \\ 3s_{2(i+j)} & -3c_{2(i+j)}-1\end{pmatrix},$$

for each $1\leq i,j\leq N$, and thus

$$\begin{aligned} U_{i,i} &= 0_{2\times 2}, \qquad i=1,\ldots,N,\\ U_{i,j} &= U_{j,i} = \left(2\beta_N^{1/3}s_{i-j}\right)^{-5}S_{i,j}\\ &= \frac{\left|s_{i-j}\right|^{-3}}{16\beta_N}\begin{pmatrix} 3c_{2(i+j)}-1 & 3s_{2(i+j)} \\ 3s_{2(i+j)} & -3c_{2(i+j)}-1\end{pmatrix}, \qquad i\neq j,\end{aligned}$$

from which defining

$$\begin{aligned} \tilde{U}_{i,i} &= \sum_{j\neq i}\frac{\left|s_{i-j}\right|^{-3}}{16\beta_N}\begin{pmatrix} 3c_{2(i+j)}-1 & 3s_{2(i+j)} \\ 3s_{2(i+j)} & -3c_{2(i+j)}-1\end{pmatrix},\\ \tilde{U}_{i,j} &= \frac{\left|s_{i-j}\right|^{-3}}{16\beta_N}\begin{pmatrix} 1-3c_{2(i+j)} & -3s_{2(i+j)} \\ -3s_{2(i+j)} & 3c_{2(i+j)}+1\end{pmatrix}, \qquad i\neq j,\end{aligned}$$

we have $V_N''\left(\boldsymbol{c}_P\right) = \left(\tilde{U}_{i,j}\right)_{i,j=1,\ldots,N}$.

Lemma 4.2. *The trace for $V_N''\left(\boldsymbol{c}_P\right)$ is equal to $-(N/8)\left(\alpha_N/\beta_N\right)$, where $\alpha_N = \sum_{k=1}^{N-1}\csc^3\left(\frac{\pi k}{N}\right)$ and β_N is defined as in Lemma 4.1.*

Proof. In virtue of the above simplifications for (20), $\operatorname{tr}\left(V_N''\left(\boldsymbol{c}_P\right)\right)$ is equal to

$$\mu_N := -\frac{2}{\beta_N}\sum_{1\leq k_1<k_2\leq N}\left|\zeta^{2k_1}-\zeta^{2k_2}\right|^{-3}.$$

We have $-\frac{\mu_N}{4}\sum_{k=1}^{N-1}\csc\left(\frac{\pi k}{N}\right) = \sum_{1\leq k_1<k_2\leq N}2\left|\zeta^{2k_1}-\zeta^{2k_2}\right|^{-3}$; on the other hand, the symmetry of a regular polygon assures

$$\sum_{1\leq k_1<k_2\leq N}2\left|2s_{k_2-k_1}\right|^{-3} = N\sum_{k=1}^{N-1}\left(2s_k\right)^{-3};$$

thus, $2\mu_N \sum_{k=1}^{N-1} \csc\left(\frac{\pi k}{N}\right) = -N \sum_{k=1}^{N-1} \csc^3\left(\frac{\pi k}{N}\right)$. □

4.2.1. *Case 1:* $N = 3,4,5,6$. We can afford a result stronger than non-integrability for these values without using Lemma 4.2, in view of Theorem 2.4. We just have to prove the following:

Lemma 4.3. $V_N''(\boldsymbol{c}_P)$, $N = 3,4,5,6$, *has only four eigenvalues in* S: $\lambda_1 = -2, \lambda_2 = \lambda_3 = 0, \lambda_4 = 1$. *Furthermore, the sets of equivalence classes given by relation* $E(\lambda_i, \lambda_j) \in \mathbb{Z}$ *in* (8) *with cardinality greater than one are (assuming* $j > 4$*):*

1. *a double eigenvalue for* $N = 3,4$*;*
2. *three double eigenvalues for* $N = 5,6$.

Proof. The eigenvalues of $V_3''(\boldsymbol{c}_P)$ are $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and $\lambda_{5,6} = -1/2$. Those of $V_4''(\boldsymbol{c}_P)$ are $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and $\lambda_5 = \frac{2(5-3\sqrt{2})}{7}, \lambda_{6,7} = \frac{2(\sqrt{2}-4)}{7}, \lambda_8 = \frac{6\sqrt{2}-17}{7}$. The corresponding relations are

$$
\begin{aligned}
E\left(\frac{2(5-3\sqrt{2})}{7}, \frac{2(\sqrt{2}-4)}{7}\right) &= -\frac{1}{14}\sqrt{-119+336\sqrt{2}} + \frac{1}{14}\sqrt{889-112\sqrt{2}},\\
E\left(\frac{6\sqrt{2}-17}{7}, \frac{2(\sqrt{2}-4)}{7}\right) &= -\frac{1}{14}\sqrt{1393-336\sqrt{2}} + \frac{1}{14}\sqrt{889-112\sqrt{2}},\\
E\left(\frac{6\sqrt{2}-17}{7}, \frac{2(5-3\sqrt{2})}{7}\right) &= -\frac{1}{14}\sqrt{1393-336\sqrt{2}} + \frac{1}{14}\sqrt{-119+336\sqrt{2}}.
\end{aligned}
$$

$V_5''(\boldsymbol{c}_P)$ has six different non-trivial double eigenvalues:

$$\lambda_{5,6,7,8} = \frac{\sqrt{5}-5 \pm \sqrt{518-222\sqrt{5}}}{4}, \; \lambda_{9,10} = \frac{\sqrt{5}-4}{2}.$$

Relations are

$$
\begin{aligned}
E(\lambda_{5,6}, \lambda_{7,8}) &= \frac{\sqrt{19-2\sqrt{5}+6\sqrt{37}-2\sqrt{37}\sqrt{5}} - \sqrt{19-2\sqrt{5}-6\sqrt{37}+2\sqrt{37}\sqrt{5}}}{2},\\
E(\lambda_{5,6}, \lambda_{9,10}) &= \frac{\sqrt{25-4\sqrt{5}} - \sqrt{19-2\sqrt{5}-6\sqrt{37}+2\sqrt{185}}}{2},\\
E(\lambda_{7,8}, \lambda_{9,10}) &= \frac{\sqrt{25-4\sqrt{5}} - \sqrt{19-2\sqrt{5}+6\sqrt{37}-2\sqrt{185}}}{2}.
\end{aligned}
$$

The eight non-trivial eigenvalues for $V_6''(\boldsymbol{c}_P)$ are

$$
\begin{aligned}
\lambda_5 &= \frac{4(29\sqrt{3}-94)}{59}, & \lambda_{6,7} &= \frac{34\sqrt{3}-\sqrt{133465-59584\sqrt{3}}-157}{118},\\
\lambda_{8,9} &= \frac{2(7\sqrt{3}-41)}{59}, & \lambda_{10,11} &= \frac{34\sqrt{3}+\sqrt{133465-59584\sqrt{3}}-157}{118},\\
\lambda_{12} &= \frac{4(53-22\sqrt{3})}{59}.
\end{aligned}
$$

The relations are

$$\begin{aligned}
E(\lambda_5,\lambda_{6,7}) &= \frac{\sqrt{s_1+236s_2}-s_3}{118}, & E(\lambda_5,\lambda_{8,9}) &= \frac{s_4-s_3}{118},\\
E(\lambda_5,\lambda_{10,11}) &= \frac{\sqrt{s_1-236s_2}-s_3}{118}, & E(\lambda_5,\lambda_{12}) &= \frac{s_5-s_3}{118},\\
E(\lambda_{8,9},\lambda_{10,11}) &= \frac{\sqrt{s_1-236s_2}-s_4}{118}, & E(\lambda_{8.9},\lambda_{12}) &= \frac{s_5-s_4}{118},\\
E(\lambda_{6,7},\lambda_{8,9}) &= \frac{s_4-\sqrt{s_1+236s_2}}{118}, & E(\lambda_{6,7},\lambda_{12}) &= \frac{s_5-\sqrt{s_1+236s_2}}{118},\\
E(\lambda_{10,11},\lambda_{12}) &= \frac{s_5-\sqrt{s_1-236s_2}}{118}, & &\\
E(\lambda_{6,7},\lambda_{10,11}) &= \frac{\sqrt{s_1-236s_2}-\sqrt{s_1+236s_2}}{118}, & &
\end{aligned}$$

with $s_1 = 68381 - 8024\sqrt{3}$, $s_2 = \sqrt{133465 - 59584\sqrt{3}}$, $s_3 = \sqrt{208801 - 54752\sqrt{3}}$, $s_4 = \sqrt{70033 - 6608\sqrt{3}}$, $s_5 = \sqrt{-68735 + 41536\sqrt{3}}$. □

Let us now determine an upper bound for the amount of meromorphic first integrals for the equal–mass Problem. We will reorder non-trivial eigenvalues according to their multiplicity as in Theorem 2.4. Let Γ be the integral curve given by the solution $\boldsymbol{z} = \phi \boldsymbol{c}_P$ of $X_{\mathcal{H}_{N,2}}$ and $G_{\mathrm{NVE}} = \mathrm{Gal}\left(\mathrm{NVE}_\Gamma\right)$ as in the proof of Theorem 2.4.

1. For $N = 3$, we have $\Lambda_1 = \{\lambda_{5,6}\} = \{1/2\}$ and the structure of the representation given in Section A.3.4, modulo equivalence, is
$$\begin{pmatrix} A_1 & \\ & \chi A_1 \end{pmatrix}, \qquad A_1 \in \mathrm{SL}_2(\mathbb{C}),$$
where either $\chi = 1$ (the connected case) or $\chi = -1$. Hence, G_{NVE} has a polynomial invariant: $J_1 = \det(\mathbf{v}_1, \mathbf{v}_2)$, in turn allowing the existence of an additional meromorphic first integral around Γ. That possibility, however, is ruled out by the complex solution given in the proof of Theorem 3.1.
2. For $N = 4$, we have $\Lambda_1 = \{\lambda_{6,7}\}$ and two simple eigenvalues: $\{\lambda_5\}$ and $\{\lambda_8\}$; reordering the blocks in the representation in the same manner, we obtain:
$$\begin{pmatrix} \boxed{\begin{matrix} A_1 & \\ & \chi A_1 \end{matrix}} & & \\ & A_2 & \\ & & A_3 \end{pmatrix}, \qquad A_1, A_2, A_3 \in \mathrm{SL}_2(\mathbb{C}).$$
Hence the action of the Galois group,
$$(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) \mapsto (A_1\mathbf{v}_1, \chi A_1 \mathbf{v}_2, A_2\mathbf{v}_3, A_3\mathbf{v}_4), \qquad \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathbb{C}^2,$$
has a single polynomial invariant: $J_1 = \det(\mathbf{v}_1, \mathbf{v}_2)$. In other words: there may be at most one additional meromorphic first integral defined on a neighborhood of Γ.

3. For $N=5$, we have $\Lambda_1=\{\lambda_{5,6}\}$, $\Lambda_2=\{\lambda_{7,8}\}$, $\Lambda_3=\{\lambda_{9,10}\}$, rendering G_{NVE} a group whose representation, according to Theorem A.8, may adopt the form:

$$\left(\begin{array}{cccccc} A_1 & & & & & \\ & \chi_1 A_1 & & & & \\ & & A_2 & & & \\ & & & \chi_2 A_2 & & \\ & & & & A_3 & \\ & & & & & \chi_3 A_3 \end{array}\right),$$

where $A_1,A_2,A_3\in\mathrm{SL}_2(\mathbb{C})$ and $\chi_1,\chi_2,\chi_3\in\{1,-1\}$. The action of G_{NVE} has three invariants:

$$J_1=\det(\mathbf{v}_1,\mathbf{v}_2), \qquad J_2=\det(\mathbf{v}_3,\mathbf{v}_4), \qquad J_3=\det(\mathbf{v}_5,\mathbf{v}_6).$$

Obviously, $\{J_1,J_2\}=\{J_2,J_3\}=\{J_1,J_3\}=0$, $\{\cdot,\cdot\}$ being the *Poisson bracket*; see [?, §3.4] for more details. Hence, there may be at most three additional meromorphic integrals for $X_{\tilde{\mathcal{H}}_{N,2}}$ in a neighborhood of Γ.

4. For $N=6$, we have $\Lambda_1=\{\lambda_{6,7}\}$, $\Lambda_2=\{\lambda_{8,9}\}$, $\Lambda_3=\{\lambda_{10,11}\}$, and two simple eigenvalues $\{\lambda_5\}$ and $\{\lambda_{12}\}$, rendering G_{NVE} a group whose representation according Subsection A.3.4 may adopt the form:

$$\left(\begin{array}{cccccccc} A_1 & & & & & & & \\ & \chi_1 A_1 & & & & & & \\ & & A_2 & & & & & \\ & & & \chi_2 A_2 & & & & \\ & & & & A_3 & & & \\ & & & & & \chi_3 A_3 & & \\ & & & & & & A_4 & \\ & & & & & & & A_5 \end{array}\right),$$

for $A_1,A_2,A_3,A_4,A_5\in\mathrm{SL}_2(\mathbb{C})$ and $\chi_i^2=1$. The scenario is the same as for $N=5$: three invariants in pairwise involution – hence, at most three additional meromorphic integrals.

Thus follows item 1 in Theorem 3.2. □

Remark 7. The above pattern appears to persist for higher values of N, although a rigorous proof is still unfinished. See Conjecture 7 for a precise statement.

4.2.2. *Case 2:* $N=7,8,9$. Proceeding from Lemma 4.2, it is straightforward to see the traces for $V_N''(\boldsymbol{c})$ for these three values of N are non-integers since

$$\begin{aligned} \mu_7 &= -\frac{\sqrt{413+56\sqrt{7}\cos\left(\frac{1}{3}\arctan 3\sqrt{3}\right)}}{2\cos\left(\frac{1}{6}\arctan\frac{3\sqrt{3}}{13}\right)} \in(-12,-11), \\ \mu_8 &= \frac{4\left(-2633+766\sqrt{2}+4\sqrt{118010-68287\sqrt{2}}\right)}{241} \in(-17,-16), \\ \mu_9 &= -\frac{9}{2}\,\frac{\frac{8\sqrt{3}}{9}+\csc^3\frac{\pi}{9}+\csc^3\frac{2\pi}{9}+\csc^3\frac{4\pi}{9}}{\frac{2\sqrt{3}}{3}+\csc\frac{\pi}{9}+\csc\frac{2\pi}{9}+\csc\frac{4\pi}{9}} \in(-22,-21). \end{aligned}$$

4.2.3. *Case 3: $N \geq 10$. We will prove $V_N''(\boldsymbol{c}_P)$ has at least an eigenvalue greater than* 1. We know the following holds ([?]),

$$\csc x = \frac{1}{x} + f(x) := \frac{1}{x} + \sum_{k\geq 1} \frac{(-1)^{k-1} 2 \left(2^{2k-1}-1\right) B_{2k} x^{2k-1}}{(2k)!}, \tag{32}$$

f being analytical for $|x| < \pi$ (which obviously holds if $x = \frac{\pi j}{N}$, $j = 1, \ldots, N-1$) and $B_k, k \geq 1$, being the Bernoulli numbers ([?, Chapter 23], [?, §3.3]).

Lemma 4.4. *For each $N \geq 10$, $S_N := 2\sum_{j=1}^{N-1} \left(\csc^2 \frac{j\pi}{N} - 5\right) \csc \frac{j\pi}{N} > 0$.*

Proof. Recall the *Euler-MacLaurin summation formula* ([?, §3.3]): for any $f \in \mathbb{C}^{2s+2}([a,b])$ and $n \in \mathbb{N}$, and defining $h = \frac{b-a}{n}$, the following holds,

$$\sum_{j=0}^{n} f(a+jh) = \frac{\int_a^b f}{h} + \frac{f(a)+f(b)}{2} + \sum_{r=1}^{s} h^{2r-1} B_{2r} \frac{f^{(2r-1)}(b) - f^{(2r-1)}(a)}{(2r)!} + R_s,$$

where $R_s = nh^{2s+2}\frac{B_{2s+2}}{(2s+2)!} f^{(2s+2)}(\alpha)$ for some $\alpha \in (a, a+nh)$. Substituting in $a = h = \pi/N$, $n = N-2$, $b = a + hn = \frac{\pi(N-1)}{N}$, $f(x) = 2\left(\csc^2 x - 5\right)\csc x$ and $s = 2$, we obtain

$$\begin{aligned}
\frac{\int_a^b f(x)\,dx}{h} &= \frac{2N}{\pi}\left(\cot\frac{\pi}{N}\csc\frac{\pi}{N} + 9\ln\left(\tan\frac{\pi}{2N}\right)\right), \\
\frac{f(a)+f(b)}{2} &= 2\left(\csc^2\frac{\pi}{N} - 5\right)\csc\frac{\pi}{N}, \\
hB_2\frac{f'(b)-f'(a)}{2} &= \frac{\pi\cot\frac{\pi}{N}\csc\frac{\pi}{N}\left(3\csc^2\frac{\pi}{N}-5\right)}{3N}, \\
h^3B_4\frac{f'''(b)-f'''(a)}{4!} &= -\frac{\pi^3\csc^6\frac{\pi}{N}\left(742\cos\frac{\pi}{N} + 213\cos\frac{3\pi}{N} + 5\cos\frac{5\pi}{N}\right)}{2880N^3} \\
&> -\frac{\pi^3(742+213+5)\csc^6\frac{\pi}{N}}{2880N^3} = -\frac{\pi^3\csc^6\frac{\pi}{N}}{3N^3},
\end{aligned}$$

and

$$R_2(\alpha) = \frac{\csc^9(\alpha)(N-2)\pi^6 P(\alpha)}{1935360N^6},$$

where $P(x) := 1110231 + 1256972\cos 2x + 206756\cos 4x + 6516\cos 6x + 5\cos 8x$;. In previous formulae, we have used $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$ and several trigonometric identities in order to express the different terms in a suitable way for what follows.

The remainder of the proof is a shorter version of the original one, for whose development we are indebted to C. Simó. Introducing variable $w = \cos 2x$, we may write the function defined by the first three terms in $P(x)$ as

$$\widehat{P}(w) := 903475 + 1256972w + 413512w^2.$$

Then, for each $w \in [-1,1]$, one has $\widehat{P}'(w) > 0$; hence, for $x \in (0,\pi)$ we obtain $P(x) \geq \widehat{P}(-1) - 6516 - 5 > 0$ and therefore $R_2(\alpha) > 0$, which leads to the following:

$$\begin{aligned} S_N &= \frac{\int_a^b f}{h} + \frac{f(a)+f(b)}{2} + \sum_{r=1}^{2} h^{2r-1} B_{2r} \frac{f^{(2r-1)}(b) - f^{(2r-1)}(a)}{(2r)!} + R_2(\alpha) \\ &> \frac{\int_a^b f(x)\,dx}{h} + \frac{f(a)+f(b)}{2} + \sum_{r=1}^{2} h^{2r-1} B_{2r} \frac{f^{(2r-1)}(b) - f^{(2r-1)}(a)}{(2r)!} \\ &> \frac{2N\left(\cot\frac{\pi}{N}\csc\frac{\pi}{N} + 9\ln\left(\tan\frac{\pi}{2N}\right)\right)}{\pi} + 2\left(\csc^2\frac{\pi}{N} - 5\right)\csc\frac{\pi}{N} \\ &\quad + \frac{\pi\cot\frac{\pi}{N}\csc\frac{\pi}{N}\left(3\csc^2\frac{\pi}{N} - 5\right)}{3N} - \frac{\pi^3\csc^6\frac{\pi}{N}}{3N^3}. \end{aligned}$$

There is a number of possible ways of proving this latter lower bound strictly positive. For instance, since, for $N \geq 10$, $\cot\frac{\pi}{N} > 3$, we have

$$\begin{aligned} S_N &> \frac{2N}{\pi}\left(\cot\frac{\pi}{N}\csc\frac{\pi}{N} + 9\ln\left(\tan\frac{\pi}{2N}\right)\right) + 2\left(\csc^2\frac{\pi}{N} - 5\right)\csc\frac{\pi}{N} \\ &+ \frac{\pi}{N}\csc\frac{\pi}{N}\left(3\csc^2\frac{\pi}{N} - 5\right) - \frac{\pi^3\csc^6\frac{\pi}{N}}{3N^3} \\ &=: \sigma_N. \end{aligned}$$

The first term in that sum is exactly $\frac{2N}{\pi}F\left(\tan\frac{\pi}{2N}\right)$, where

$$F : (0,\infty) \to \mathbb{R}, \qquad F(z) := \frac{z^{-2} - z^2}{4} + 9\ln z,$$

is strictly decreasing in $\left(0, \sqrt{5}-2\right)$. Since $\tan\frac{\pi}{2N} < \sqrt{5}-2$ for all $N \geq 10$, we have

$$F\left(\tan\frac{\pi}{2N}\right) \geq F\left(\tan\frac{\pi}{20}\right) > -\frac{20}{3},$$

and thus,

$$\begin{aligned} \sigma_N &> \frac{2N}{\pi}\left(-\frac{20}{3}\right) + 2\left(\csc^2\frac{\pi}{N} - 5\right)\csc\frac{\pi}{N} + \frac{\pi}{N}\csc\frac{\pi}{N}\left(3\csc^2\frac{\pi}{N} - 5\right) - \frac{\pi^3\csc^6\frac{\pi}{N}}{3N^3} \\ &> \frac{\csc\frac{\pi}{N}}{3N^3} G_N\left(\csc\frac{\pi}{N}\right), \end{aligned}$$

where $G_N(x) := -\pi^3 x^5 + 3N^2(2N+3\pi)x^2 - N^2(55N+15\pi)$ and we have used $\csc(x) > \frac{1}{x}$ for all $x \in (0,\pi)$ (see (32)) and thus $-\frac{40N}{3\pi} > -\frac{40}{3}\csc\left(\frac{\pi}{N}\right)$ for all $N \geq 2$. It is immediate that $G_N'(x) > 0$ if

$$x \in \left(0, \frac{N}{\pi}\left(\frac{12 + 18\frac{\pi}{N^2}}{5}\right)^{1/3}\right) \supset \left(0, \frac{N}{\pi}\frac{4}{3}\right).$$

For all $N \geq 3$, the latter interval contains $\left[\frac{N}{\pi}, \csc\frac{\pi}{N}\right]$, thus allowing us to lower-bound $G_N\left(\csc\frac{\pi}{N}\right)$ by

$$G_N\left(\frac{N}{\pi}\right) = \frac{N^5}{\pi^2}\left(-1 + 6 + \frac{9\pi}{N} - \frac{55\pi^2}{N^2} - \frac{15\pi^3}{N^4}\right) > 0, \qquad N \geq 10.$$

In this way we obtain $S_N > \sigma_N > \frac{\csc\left(\frac{\pi}{N}\right)}{3N^3}G\left(\csc\frac{\pi}{N}\right) > 0, \qquad N \geq 10.$ □

Lemma 4.5. *For $N \geq 10$, $V_N''(\boldsymbol{c}_P)$ has at least one eigenvalue greater than 1.*

Proof. Indeed, let $A = (a_{i,j})_{i,j=1,\dots,2N} = V_N''(\boldsymbol{c}_P)$. The Rayleigh quotient for vector $\boldsymbol{v} = \boldsymbol{e}_{2N,2N-1} = (0,0,\cdots,0,1,0)^T$ is

$$\frac{\boldsymbol{v}^T A \boldsymbol{v}}{\boldsymbol{v}^T \boldsymbol{v}} = \frac{\mathbf{v}_N^T \tilde{U}_{N,N} \mathbf{v}_N}{\mathbf{v}_N^T \mathbf{v}_N} = a_{2N-1,2N-1} = \frac{\sum_{j=1}^{N-1} \left(\csc^3 j\frac{\pi}{N}\right)\left(3\cos 2j\frac{\pi}{N} - 1\right)}{4\sum_{j=1}^{N-1} \csc j\frac{\pi}{N}},$$

and it will be strictly greater than 1 if and only if

$$\sum_{j=1}^{N-1}\left(3\cos\frac{2j\pi}{N} - 1\right)\csc^3\frac{j\pi}{N} - 4\sum_{j=1}^{N-1}\csc\frac{j\pi}{N} = \sum_{j=1}^{N-1} 2\left(\csc^2\frac{j\pi}{N} - 5\right)\csc\frac{j\pi}{N} > 0,$$

which we already know holds for $N \geq 10$ by Lemma 4.4. Elementary Linear Algebra then yields the existence of at least one eigenvalue $\tilde{\lambda} > 1$ for $V_N''(\boldsymbol{c}_P)$. □

Since $\max S = 1 < \tilde{\lambda}$, $\tilde{\lambda} \notin S$ and this ends the proof for Theorem 3.2, item 2. □

4.3. **Proof isolate: $N = 2^m$ equal masses.** For the sake of a (modest) diversification, and in order to show yet another way of confronting issues of non-integrability with arithmetical tools, we include this alternative proof of a weaker version of Theorem 3.2, item 2: namely, the case $N = 2^m$ with $m \geq 2$.

We know we can reorder the eigenvalues so as to obtain $\lambda_1 = k - 1 = -2$, $\lambda_2 = \lambda_3 = 0$ and $\lambda_4 = 1$. These four eigenvalues belong to S. If all of $\lambda_5, \dots, \lambda_{2N}$ did too, their sum

$$\operatorname{tr}\left(V_N''(\boldsymbol{c}_P)\right) = -1 + \lambda_5 + \cdots + \lambda_{2N} = -N\left(\frac{\sum_{k=1}^{N-1}\frac{1}{\sin^3\left(\frac{\pi k}{N}\right)}}{2\sum_{k=1}^{N-1}\frac{1}{\sin\left(\frac{\pi k}{N}\right)}}\right), \tag{33}$$

would be an integer number μ_N such that $-\infty < \mu_N \leq 2N - 5$ since the only positive term in S is 1.

Proving the trace of $V_N''(\boldsymbol{c})$, i.e. the sum of its eigenvalues, a non-integer will be enough to settle the rest of Corollary 4.9; in view of (33), such a condition is immediate if we prove that any relation of the form

$$n_1\sum_{k=1}^{N-1}\csc\frac{\pi}{N}k + n_2\sum_{k=1}^{N-1}\csc^3\frac{\pi}{N}k = 0, \tag{34}$$

where $n_1, n_2 \in \mathbb{Z}$, implies $n_1 = n_2 = 0$.

As in the previous Subsection, let $\zeta = \cos\frac{\pi}{N} + \mathrm{i}\sin\frac{\pi}{N}$ be a primitive $2N^{\text{th}}$ root of unity. Then, $\sin\frac{\pi k}{N} = \frac{1}{2\mathrm{i}}\left(\zeta^k - \zeta^{-k}\right)$ for each k, and thus

$$\sum_{k=1}^{N-1}\csc\frac{\pi}{N}k = 2\mathrm{i}\sum_{k=1}^{N-1}\frac{1}{\zeta^k - \zeta^{-k}}, \qquad \sum_{k=0}^{N-1}\csc^3\frac{\pi}{n}k = -8\mathrm{i}\sum_{k=1}^{N-1}\left(\frac{1}{\zeta^k - \zeta^{-k}}\right)^3.$$

Any relation of the form (34) would thus yield

$$\sum_{k=1}^{N-1}\frac{1}{\zeta^k - \zeta^{-k}} - \alpha\sum_{k=1}^{N-1}\left(\frac{1}{\zeta^k - \zeta^{-k}}\right)^3 = 0,$$

for some $\alpha \in \mathbb{Q}$. Singling out summands with index $N/2$ yields

$$2\sum_{k=1}^{N-1}\frac{1}{\zeta^k - \zeta^{-k}} + \frac{1}{\zeta^{N/2} - \zeta^{-N/2}} = \alpha\left[2\sum_{k=1}^{N-1}\frac{1}{\left(\zeta^k - \zeta^{-k}\right)^3} + \frac{1}{\left(\zeta^{N/2} - \zeta^{-N/2}\right)^3}\right] = 0,$$

which, since $\zeta^{N/2} = \mathrm{i}$, and thus $\zeta^{-N/2} = -\mathrm{i}$, becomes

$$2\sum_{k=1}^{N/2-1} \frac{1}{\zeta^k - \zeta^{-k}} - \frac{\mathrm{i}}{2} = \alpha \left(2\sum_{k=1}^{N/2-1} \frac{1}{\left(\zeta^k - \zeta^{-k}\right)^3} + \frac{\mathrm{i}}{8} \right) \tag{35}$$

for some $\alpha \in \mathbb{Q}$. The next lemmae are aimed at proving that such an equation as (35) is unfeasible for the only possible value of α, which will be found to be -4.

Remark 8. We recall that since $\dim_{\mathbb{Q}} \mathbb{Q}(\zeta) = N = 2^m$, the set of roots of unity $\left\{1, \zeta, \ldots, \zeta^{N-1}\right\}$ is rationally independent. So is, thus, any set $\left\{\zeta^{kj} : 1 \leq j \leq M\right\}$ of cardinality $M \leq N-1$, where $k > 0$ is an arbitrary integer.

We may find two possible expressions of $\frac{1}{\zeta^k - \zeta^{-k}}$ depending on the parity of k:

Lemma 4.6. *Let $k = 1, \ldots, N/2$. Then,*

$$\frac{1}{\zeta^k - \zeta^{-k}} = \begin{cases} -\frac{1}{2}\sum_{j=1}^{N/2} \zeta^{(N-2j+1)k}, & k \text{ odd}, \\ -\frac{1}{2}\sum_{j=1}^{2^{m-n-1}} \zeta^{\left(2^{m-n}-2j+1\right)k}, & k = 2^n q, \ q \text{ odd}. \end{cases}$$

Proof. In general, if $u = \zeta^k$ and $\frac{1}{u-u^{-1}} = -\frac{1}{2}\left(u + u^3 + u^5 + \cdots + u^{r-3} + u^{r-1}\right)$ for some $r \leq N$,

$$\begin{aligned} -2 &= \left(u - u^{-1}\right)\left(u + u^3 + u^5 + \cdots + u^{r-3} + u^{r-1}\right) \\ &= u^2 + u^4 + \cdots + u^{r-2} + u^r - \left(1 + u^2 + u^4 + \cdots + u^{r-2}\right) \\ &= u^r - 1, \end{aligned}$$

meaning $\zeta^{kr} = -1$, i.e. $kr = N(2p+1)$ for some $p \in \mathbb{N}$.

1. If k is odd, the facts $kr = N(2p+1)$ and $N = 2^m$ imply $k \mid 2p+1$ and thus $r = \tilde{q}2^m$ for some odd $\tilde{q}$; the minimum value of r satisfying this is $r = 2^m = N$, and indeed $\frac{1}{\zeta^k - \zeta^{-k}} = -\frac{1}{2}\left(\zeta^k + \zeta^{3k} + \cdots + \zeta^{(N-1)k}\right)$ as may be checked multiplying both sides by $\zeta^k - \zeta^{-k}$.
2. For even k we have $k = 2^n s < N/2 = 2^{m-1}$ for some odd integer s, implying $n < m-1$; furthermore, $kr = N(2p+1)$ implies $sr = (1+2p)\,2^{m-n}$; since s is odd, $s \mid 2p+1$ and thus $r = \tilde{q}2^{m-n}$ for some odd $\tilde{q}$; the minimal such r is $r = 2^{m-n}$, and again a simple check indeed assures $\frac{1}{\zeta^k - \zeta^{-k}} = -\frac{1}{2}\left(\zeta^k + \zeta^{3k} + \cdots + \zeta^{\left(2^{m-n}-1\right)k}\right)$.

□

Let $P(\zeta)$ (resp. $Q(\zeta)$) be the polynomial expression of $\sum_{k=1}^{N/2-1} \frac{1}{\zeta^k - \zeta^{-k}}$ (resp. $\sum_{k=1}^{N/2-1} \left(\frac{1}{\zeta^k - \zeta^{-k}}\right)^3$) of degree smaller than or equal to $N-1$, attained by reduction via $\zeta^N = -1$. This means (35) may be written as $2P(\zeta) - \frac{\mathrm{i}}{2} = \alpha\left(2Q(\zeta) + \frac{\mathrm{i}}{8}\right)$; let us write $P(\zeta) = \sum_{k=0}^{N-1} a_k \zeta^k$ and $Q(\zeta) = \sum_{k=0}^{N-1} b_k \zeta^k$. We are now going to discard cross-contributions to two particular powers of ζ in these polynomials:

Lemma 4.7. *Let $\tilde{k} \in \{1, \ldots, N-1\}$. Then,*

1. *if $\tilde{k} = N/2$, $a_{\tilde{k}} = b_{\tilde{k}} = 0$. In particular, $\alpha = -4$.*
2. *If $\tilde{k} = 2^{m-2}$, the only summand $\frac{1}{\zeta^k - \zeta^{-k}}$ in $P(\zeta)$ (resp. $\left(\frac{1}{\zeta^k - \zeta^{-k}}\right)^3$ in $Q(\zeta)$) whose polynomial in powers of ζ contains a non-zero coefficient of $\zeta^{\tilde{k}}$, is precisely $\frac{1}{\zeta^{\tilde{k}} - \zeta^{-\tilde{k}}}$ (resp. $\left(\frac{1}{\zeta^{\tilde{k}} - \zeta^{-\tilde{k}}}\right)^3$).*

Proof.

1. Let $j \in \{1,\dots,N-1\}$. We may assume $j < N/2$ due to (35) and in view of Lemma 4.6 there is an even $r_j \in \{2^{m-n}, N\}$ such that

$$\begin{cases} \left(\zeta^j - \zeta^{-j}\right)^{-1} = -\frac{1}{2}\left(\zeta^j + \zeta^{3j} + \cdots + \zeta^{(r_j-1)j}\right), \\ \left(\zeta^j - \zeta^{-j}\right)^{-3} = -\frac{1}{8}\left(\zeta^j + \zeta^{3j} + \cdots + \zeta^{(r_j-1)j}\right)^3; \end{cases} \tag{36}$$

a) if j is odd, $\zeta^j + \zeta^{3j} + \cdots + \zeta^{(r_j-1)j}$ includes exclusively odd powers of ζ, i.e. $a_{N/2} = 0$; this is also the case with $\left(\zeta^j + \zeta^{3j} + \cdots + \zeta^{(N-1)j}\right)^3$, since it is a polynomial containing powers of the form $\zeta^{(q_1+q_2+q_3)j}$ where $q_1+q_2+q_3 > 1$ is an odd positive integer. Thus, in particular $b_{N/2} = 0$.

b) If j is even, say $j = 2^n q$ with q odd (which implies $n < m-1$), $\zeta^j + \zeta^{3j} + \cdots + \zeta^{\left(2^{m-n}-1\right)j}$ consists of powers of the form $\zeta^{\tilde{q}2^n}$ with $\tilde{q}$ odd. These even exponents $\tilde{q}2^n$ are different (mod $2N$) from $2^{m-1} = N/2$. Indeed, any relation of the form $2^n \cdot \tilde{q} = 2^{m-1} + p2^{m+1}$ for some integer p would imply $\tilde{q} = 2^{m-n-1} + p2^{m-n+1}$, impossible since $2^{m-n-1} + p2^{m-n+1}$ is even. Meanwhile, the exponents in $\left(\zeta^j + \zeta^{3j} + \cdots + \zeta^{\left(2^{m-n}-1\right)j}\right)^3$ are again of the form $(q_1+q_2+q_3)\,j$ as in a), and thus a particular case of the form $\tilde{q}2^n$ just studied, which implies $[(q_1+q_2+q_3)\,j]_{2N} \neq [N/2]_{2N}$ and thus $b_{N/2} = 0$.

Thus, for each j neither of the sum expressions in (36) contains $\zeta^{N/2}$, implying $a_{N/2} = b_{N/2} = 0$, and since $\mathrm{i} = \zeta^{N/2}$ and $\left\{\zeta^{kj} : 1 \le j \le N-1\right\}$ is an independent set (Remark 8), the only contribution to i in each side of (35) is precisely the one we singled out of each sum in that equation, i.e. $-\frac{\mathrm{i}}{2} = \alpha\frac{\mathrm{i}}{8}$, meaning $\alpha = -4$.

2. For the same reasons as in item 1, we may restrict to $j \in \{1,\dots,N/2-1\}$.

a) If j is odd, as seen in 1.a) above both $\zeta^j + \zeta^{3j} + \cdots + \zeta^{(2^m-1)j}$ and $\left(\zeta^j + \zeta^{3j} + \cdots + \zeta^{(2^m-1)j}\right)^3$ are a sum of odd powers of ζ, none of them congruent to the *even* number $\tilde{k} = 2^{m-2}$ (mod $2N$).

b) If $j < 2^{m-1} = N/2$ is even and $j \neq 2^{m-2}$, writing $j = 2^n \cdot q$ for some n and some odd q, implies $n < m-2$ (since $n = m-2$ would imply $q = 1$ and thus $j = 2^{m-2}$) and $\zeta^j + \zeta^{3j} + \cdots + \zeta^{\left(2^{m-n}-1\right)j}$, has exponents different modulo N from 2^{m-2} as is proven by the exact same reasoning as in item 1.b) while since every expression of the form $2^{m-n-2} + Q2^{m-n}$, $Q \in \mathbb{Z}$, is even if $n < m-2$. Same applies thus to $\left(\zeta^j + \zeta^{3j} + \cdots + \zeta^{\left(2^{m-n}-1\right)j}\right)^3$, as in item 1 *mutatis mutandis.*

□

We finally obtain the result which is central to this Subsection:

Theorem 4.8. *For any $N \in \mathbb{N}$ of the form $N = 2^m$, $m \ge 2$, $\sum_{k=1}^{N-1} \csc\frac{\pi}{N}k$ and $\sum_{k=1}^{N-1} \csc^3\frac{\pi}{N}k$ are $\mathbb{Q}$-independent, i.e., any equation of the form (34), where $n_1, n_2 \in \mathbb{Z}$, implies $n_1 = n_2 = 0$.*

Proof. As said before, any relation of the form (34) may be written in the form (35) for some $\alpha \in \mathbb{Q}$. In virtue of item 1 in Lemma 4.7, $\alpha = -4$, and (35) thus provides for

$$2\sum_{k=1}^{N/2-1}\frac{1}{\zeta^k-\zeta^{-k}}-\frac{\mathrm{i}}{2}=-4\left(2\sum_{k=1}^{N/2-1}\frac{1}{\left(\zeta^k-\zeta^{-k}\right)^3}+\frac{\mathrm{i}}{8}\right),$$

i.e. for $2\sum_{k=0}^{N-1}a_k\zeta^k-\frac{\mathrm{i}}{2}=-4\left(2\sum_{k=0}^{N-1}b_k\zeta^k+\frac{\mathrm{i}}{8}\right)$ (according to the notation introduced immediately prior to Lemma 4.7), which in view of Remark 8 implies $a_k=-4b_k$ for $k=1,\dots N-1$. However, let us express $a_{\tilde{k}}=\alpha b_{\tilde{k}}$ for $\tilde{k}=2^{m-2}=N/4$; this we can do since, in virtue of Lemma 4.7 (item 2), we just have to compare the coefficients in $\zeta^{\tilde{k}}$ of $\frac{1}{\zeta^{\tilde{k}}-\zeta^{-\tilde{k}}}$ and $\left(\frac{1}{\zeta^{\tilde{k}}-\zeta^{-\tilde{k}}}\right)^3$. Since $\zeta^{4\tilde{k}}=\zeta^N=-1$, we have $\zeta^{6\tilde{k}}=-\mathrm{i}=\zeta^{-2\tilde{k}}$, meaning

$$\frac{1}{\zeta^{\tilde{k}}-\zeta^{-\tilde{k}}}=-\frac{1}{2}\left(\zeta^{\tilde{k}}+\zeta^{3\tilde{k}}\right),\qquad\left(\frac{1}{\zeta^{\tilde{k}}-\zeta^{-\tilde{k}}}\right)^3=\frac{1}{4}\left(\zeta^{\tilde{k}}+\zeta^{3\tilde{k}}\right),$$

which would imply $-\zeta^{\tilde{k}}/2=\alpha\zeta^{\tilde{k}}/4$, i.e. $\alpha=-2$, an absurd since we know $\alpha=-4$. $\square$

Hence, the trace of $V_3''\left(\boldsymbol{c}_P\right)$, written in (33), is irrational, and thus a non–integer; in virtue of Theorem 2.2 and Lemma 3.7, we conclude the following:

Corollary 4.9. *The d-dimensional N-Body Problem with N equal masses is meromorphically non-integrable for $N=2^m$ with $m\geq 2$.* $\square$

5. **Conclusions and work in progress.**

5.1. **Overview.** With the aid of a special case of the Morales–Ramis Theorem we have established a necessary condition on the existence of a single additional first integral for Hamiltonian systems with a homogeneous potential. Using this condition we have generalized Theorems 2.5 and 2.6 for $N=3$ with arbitrary masses, and (partially) for $N=3,4,5,6$ with equal masses. Finally, we have proven the non-integrability of the N-Body Problem for $N\geq 7$ equal masses.

Proving non–integrability for the given instances of the N-Body Problem required nothing but the exploration of the eigenvalues of a given matrix, with the advantage of knowing four of them explicitly: $-2,0,0,1$. Thus, whether it be for generalizations of Bruns' Theorem or just for proofs of non-integrability, not all variational equations were needed but those *not* corresponding to these four eigenvalues – this is exactly what transpires from the reduction of variational systems and the introduction of normal variational equations in Section 2.3.2.

The main goal of the present paper was presenting a number of (old and new) possible ways of proving Hamiltonian non-integrability, rather than exhausting all possible open problems that might appear. Our immediate goal at this point is proving one of the following:

Conjecture 1 (Non-integrability of the N-Body Problem)**.** *Regardless of the value of the masses $m_1,\dots,m_N>0$, the d-dimensional N-Body Problem has no set of dN meromorphic first integrals independent and in pairwise involution.*

Conjecture 2. *Except for an identifiable, zero-measure family $\mathfrak{M}\in\mathbb{R}_+^N$ of mass vectors $(m_1,\dots,m_N)$, the d-dimensional N-Body Problem has no meromorphic first integral independent and in involution with the classical ones.*

The latter, which in some sense may be seen as a generalization of Bruns' Theorem 2.5, obviously implies the former whenever $(m_1, \dots, m_N) \notin \mathfrak{M}$, although the difference in complexity between both can only be a source of speculation at this point. Besides, proving any of these will definitely call for a further extension of our present knowledge regarding central configurations and Galois differential theory.

5.2. **Perspectives on Conjectures 1 and 2.**

5.2.1. *The N-body problem with arbitrary masses.* Numerical exploration does suggest special values of the masses for which at least one of the eigenvalues of V_N'' may belong to Table (7). Refining of these values has been done in order to obtain generalizations of relation (31) – to no avail. Thus, most of what follows for arbitrary masses would be more likely applied to Conjecture 1 than to Conjecture 2.

Let $\boldsymbol{c}_L = (\mathbf{c}_1, \dots, \mathbf{c}_N) \in \mathbb{R}^{Nd}$ be the collinear solution defined in Section 2.4.3. We assume

$$\mathbf{c}_i : \left(\sqrt{m_1} c_i, 0, \dots, 0\right), \qquad i = 1, \dots, N, \tag{37}$$

are, respectively, the coordinates of the bodies of masses $m_1, \dots, m_N$. Tracing the steps in Moulton's existence and unicity proof it is easy to prove there exists such a solution as (37). The very particular form of $\boldsymbol{c}_L$ allows for a more specific version of Lemma 3.7. $V_N''(\boldsymbol{c}_L) = (V_{i,j})_{i,j=1,\dots,N}$, where for each $i, j = 1, \dots, N$ we have

$$\begin{aligned} V_{i,i} &= \left(\sum_{k\neq i,k=1}^{N} \frac{m_k}{|c_i - c_k|^3}\right) A, \qquad 1 \le i \le N, \\ V_{i,j} &= V_{j,i} = -\frac{\sqrt{m_i}\sqrt{m_j}}{|c_i - c_j|^3} A, \qquad 1 \le i < j \le N, \end{aligned}$$

where $A = \begin{pmatrix} -2 & \mathbf{0}^T \\ \mathbf{0} & \mathrm{Id}_{d-1} \end{pmatrix}$. The following appears to be a direct consequence of this:

Conjecture 3. *The following holds:*

$$\mathrm{Spec}\left(V_{N,d}''(\boldsymbol{c}_L)\right) = \{\mu_1, \dots, \mu_N, -2\mu_1, \dots, -2\mu_N\},$$

where $\mu_i \ge 0$ and $-2\mu_i$ has multiplicity $d-1$ for every $i = 1, \dots, N$.

Hence, we may cling to the planar collinear solution

$$\boldsymbol{c}_L : \left(\sqrt{m_1} c_1, 0, \sqrt{m_2} c_2, 0, \sqrt{m_3} c_3, 0, \dots, \sqrt{m_N} c_N, 0\right).$$

The main line of study pivots around a property which seems true for all values numerically tested:

Conjecture 4. *There is at least an $i = 1, \dots, N$ such that $\sum_{k\neq i,k=1}^{N} \frac{m_k}{|c_k - c_i|^3} > 1$.*

The known result closest resembling our goal is apparently what was done for $m_1 = \cdots = m_N = m$ in [**?**], although deviating one, two or more of the masses away from the common value m has consequences still unknown to us. Anyway, proving Conjecture 4 proves Conjecture 1. Indeed, we have

$$V_N''(\boldsymbol{c}_L) = \mathrm{diag}\left\{ \begin{pmatrix} -2\sum_{k\neq i,k=1}^{N} \frac{m_k}{|c_k - c_i|^3} & 0 \\ 0 & \sum_{k\neq i,k=1}^{N} \frac{m_k}{|c_k - c_i|^3} \end{pmatrix} : 1 \le i \le N \right\} + B_N,$$

B_N being null along its three main diagonals; hence, inasmuch as was done in Subsection 4.2.3, we could now proceed to search for vectors yielding a Rayleigh quotient greater than 1. One such vector is $\boldsymbol{w}_i := \boldsymbol{e}_{2N,2i}$ (i as in Conjecture 4), since the following holds: $\frac{\boldsymbol{w}_i^T A \boldsymbol{w}_i}{\boldsymbol{w}_i^T \boldsymbol{w}_i} = \sum_{k\neq i, k=1}^{N} \frac{m_k}{|c_i - c_k|^3} > 1$; this proves the existence of an eigenvalue strictly greater than one, and thus not belonging to $S = \left\{-\frac{1}{2}p(p-3) : p > 1\right\}$.

The second line of study, using Conjecture 3, would be based on the following:

Lemma 5.1. *Assume all of the eigenvalues of* $V_N''(\boldsymbol{c}_L)$ *belong in Table* (7)*. Then, they all belong to* $\tilde{S} = \{-2, 0, 1\}$.

Proof. For any $\lambda = -\frac{1}{2}p(p-3) \in S$, assume $\lambda = -2\mu$ for some other $\mu \in \tilde{S}$. Then defining $\mu = -\frac{1}{2}q(q-3)$, we would have

$$-\frac{1}{2}p(p-3) = q(q-3), \tag{38}$$

implying $p = p_{\pm} = \frac{3}{2} \pm \frac{1}{2}\sqrt{\Delta}$, where $\Delta = -8q^2 + 24q + 9$. $\Delta \geq 0$ only holds for $q \in \left[3\left(2-\sqrt{6}\right)/4, 3\left(2+\sqrt{6}\right)/4\right] \subset (-1, 4)$, and for $q = 0, 1, 2, 3$ the corresponding values of $p_{\pm}$ are easily proven to yield either -2 or 0 for both sides of (38). □

Hence, if we prove the following we are done with Conjecture 1:

Conjecture 5. *There is at least one eigenvalue of* $V_N''(\boldsymbol{c}_L)$ *not in* $\{-2, 0, 1\}$.

Numerical evidence of this is overwhelming.

5.2.2. *Other possibilities.* Since only four of the eigenvalues are known for sure and little is known about central configurations, most of the remaining possible methods of proving Conjectures 1 and 2 are likely to be dead-end sidings, at least if we are expecting simple proofs for these conjectures.

1. Matrix deflation is already useless for $N = 3$ in the Euler collinear case $\boldsymbol{c}_L$ and arguably remains so for higher N: if we choose for instance null-vectors

$$v_1 : \left(\sqrt{m_1}, 0, \sqrt{m_2}, 0, \sqrt{m_3}, 0\right), \qquad v_2 : \left(\sqrt{m_1}, 0, \sqrt{m_2}, 0, \sqrt{m_3}\right),$$

for the corresponding 6×6 and 5×5 matrices to be deflated with, respectively, it is easy to see that Spec $V_N''(\boldsymbol{c}_L) = \{-2, 0, 0, 1, \lambda, -2\lambda\}$, where

$$\lambda = -1 + \frac{m_1 + m_2}{|c_1 - c_2|^3} + \frac{m_1 + m_3}{|c_1 - c_3|^3} + \frac{m_2 + m_3}{|c_2 - c_3|^3}. \tag{39}$$

Proving that one or both of λ and -2λ lies outside $\tilde{S}$ is as open a problem as the one posed in Conjecture 4 and requires more knowledge on the collinear solution than we currently have.

2. Another apparent dead end is the use of a more general family of solutions than the one appearing in Section 4.1. It may be shown that a solution for $V_N'(\boldsymbol{c}) = \boldsymbol{c}$ is $\widehat{\boldsymbol{c}} = \left(\sum_{k=1}^{N} m_k\right)^{-2/3} \boldsymbol{c}$, where

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \sqrt{m_1} \sum_{k\neq 1} \begin{pmatrix} a_k m_k \\ b_k m_k \end{pmatrix},$$

$$\begin{pmatrix} c_{2i-1} \\ c_{2i} \end{pmatrix} = \sqrt{m_i}\left[\sum_{k\neq i, k\geq 2} m_k \begin{pmatrix} a_k \\ b_k \end{pmatrix} - \left(\sum_{k\neq i, k\geq 2} m_k\right)\begin{pmatrix} a_i \\ b_i \end{pmatrix}\right], \quad i \geq 2,$$

and $a_2, \dots, a_N, b_2, \dots, b_N$ are solutions to

$$\left(a_i^2 + b_i^2\right)^{3/2} = 1, \quad \left((a_i - a_j)^2 + (b_i - b_j)^2\right)^{3/2} = 1, \qquad i \neq j = 2, \dots, N.$$

A special case for $N = 3$ is the solution (24) used Section 4.1. The problem, though, is assuring the existence of such a set $\{a_2, \dots, a_N, b_2, \dots, b_N\} \subset \mathbb{C}$ when $N \geq 4$. Another problem is determining how many solutions of (17) *do not* match pattern $\widehat{\boldsymbol{c}}$; in particular, determining whether or not (24) and collinear solutions are the only possible complex solutions of (17) for $N = 3$.

3. A formula of the sorts of

$$f(A) = \frac{1}{2\pi \mathrm{i}} \int_{\partial\Omega} \left(A - z\mathrm{Id}_{2N}\right)^{-1} f(z)\, dz, \tag{40}$$

where $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is any given analytical function with a matrix counterpart $f(A) := \sum_{k=0}^{\infty} a_k A^k$ and $\mathrm{Spec}\, A \subset \Omega$, is hardly of any use here no matter how simple f is, since everything basically boils down to observing obstructions to an equality such as (40) on the complementary of a *discrete* set and this is arguably the opposite of the way a proper proof works, especially considering our scarce knowledge of the Hessian matrix A. This is especially evident when trying to compute, for instance, the *matrix sine* $f(A) = \sin(\pi A) := \frac{1}{2\mathrm{i}}\left[\exp(\mathrm{i}\pi A) - \exp(-\mathrm{i}\pi A)\right]$, the matrix exponential $\exp : \mathcal{M}_{2N\times 2N}(\mathbb{C}) \to \mathcal{M}_{2N\times 2N}(\mathbb{C})$ being defined as usual. Proving $\sin(\pi A)$ has not a single zero (resp. at least a non-zero) eigenvalue would establish Conjecture 2 (resp. 1), but finding plausible properties (or patterns, for that matter) for the infinite series involved requires a knowledge on A which we currently don't have, not even for the relatively sparse form $A = V_N''(\boldsymbol{c}_L)$ it has in the collinear case.

4. Geršgorin and Bauer-Fike bounds ([**?**, §6.9]) are probably just as useless here since numerical evidence yields non-void pairwise intersection of nearly all of the disks containing the eigenvalues for a widespread set of values of the masses.

5. Finally, and in spite of some distant similarities, the reduction of $V_N''(\boldsymbol{c})$ to a Toeplitz matrix ([**?**], [**?**]) seems difficult to perform, even for solutions such as those given by the polygonal and collinear configurations. Hence, none of the well-known results of detection of extreme eigenvalues for such matrices is likely to hold here, at least not regardless of N and $\boldsymbol{c}$.

5.3. **Candidates for a partial result.**

5.3.1. *The N-body problem with equal masses.* We already generalized Bruns' Theorem for this special case with $N \leq 6$, and proved non-integrability for $N \geq 7$. Let $\boldsymbol{c}_P$ be the polygonal solution (Example 2.1(5) and Section 4.2). Numerical evidence supports the following fact for all $N \geq 3$: $\mathrm{Spec}\, V_N''(\boldsymbol{c}_P) = \tilde{S} \cup \{\mu_1, \dots, \mu_n\}$, where $\tilde{S} = \{-2, 0, 1\}$ (-2 and 1 simple, 0 double) and $\mu_1 \leq \cdots \leq \mu_n$, where:

1. if N is even, μ_1 and μ_n are simple, and the remaining $\mu_2, \dots, \mu_{n-1}$ are double eigenvalues;

2. if N is odd, all of $\mu_1, \dots, \mu_n$ are double eigenvalues;

and, most importantly:

Conjecture 6. *There is not a single element in $\{\mu_1, \dots, \mu_n\}$ belonging to $\tilde{S}$.*

A partial result, weaker than the above Conjecture 2, would be given by both the multiplicity of the eigenvalues just hinted at and a generalization of Theorem 3.2:

Conjecture 7. *$\tilde{\mathcal{H}}_{N,2}$ has at most $2\left[\frac{N-1}{2}\right]-1$ first integrals, independent both pairwise and with respect to the classical ones.*

We may also hint at the following generalization of Theorem 4.8, although the result it implies (namely, that the Problem with equal masses is not integrable) has been already obtained by other means in Theorem 3.2, item 2:

Conjecture 8. *For* any *$N \in \mathbb{N}$, $N \geq 7$, $\sum_{k=1}^{N-1} \csc \frac{\pi}{N} k$ and $\sum_{k=1}^{N-1} \csc^3 \frac{\pi}{N} k$ are $\mathbb{Q}$-independent.*

5.3.2. *The $N+1$-body problem with N equal masses.* Assume $m_1 = \cdots = m_N = 1$ and $m_{N+1} > 0$ is the additional mass. The next two Lemmae are as immediate to prove as Lemmae 4.1 and 4.2:

Lemma 5.2. *The vector $\boldsymbol{c}_C = \tilde{\beta}_N^{1/3}\left(\mathbf{c}_1, \ldots, \mathbf{c}_N, \mathbf{c}_{N+1}\right)$, defined by*

$$\mathbf{c}_j = \left(c_{2j-1}, c_{2j}\right) = \begin{cases} \left(\cos \frac{2\pi j}{N}, \sin \frac{2\pi j}{N}\right), & j < N+1, \\ (0,0), & j = N+1, \end{cases} \tag{41}$$

where $\tilde{\beta}_N := m_{N+1} + \frac{1}{4}\sum_{k=1}^{N-1} \csc\left(\frac{\pi k}{N}\right)$, is a solution for $V_{N+1}'(\boldsymbol{c}) = \boldsymbol{c}$. □

Lemma 5.3. *The trace for $V_{N+1}''(\boldsymbol{c}_C)$ is equal to*

$$\tilde{\mu}_N := -\frac{N}{2}\frac{\sum_{k=1}^{N-1} \csc^3\left(\frac{\pi k}{N}\right) + 8\left(m_{N+1}+1\right)}{\sum_{k=1}^{N-1} \csc\left(\frac{\pi k}{N}\right) + 4m_{N+1}}. \quad \square$$

Observation of Lemma 4.4 for $N \geq 10$ and a direct check for $N < 0$ assure the following fact: $\sum_{k=1}^{N-1} \csc^3\left(\frac{\pi k}{N}\right) + 8 > 2\sum_{k=1}^{N-1} \csc\left(\frac{\pi k}{N}\right)$ for all N; hence, we have

$$\sum_{k=1}^{N-1} \csc^3\left(\frac{\pi k}{N}\right) + 8\left(m_{N+1}+1\right) > 2\sum_{k=1}^{N-1} \csc\left(\frac{\pi k}{N}\right) + 8m_{N+1},$$

and thus $\frac{\sum_{k=1}^{N-1} \csc^3\left(\frac{\pi k}{N}\right)+8(m_{N+1}+1)}{2\sum_{k=1}^{N-1} \csc\left(\frac{\pi k}{N}\right)+8m_{N+1}} > 1$; hence, as was already stated in reference [?, Section 3.2]:

Corollary 5.4. *Given N, $\operatorname{tr} V_{N+1}''(\boldsymbol{c}_C)$ is a non-integer for all but a finite number of values of $m_{N+1} > 0$. The cardinality of this exceptional set depends on N.* □

Let $\boldsymbol{c}_C$ be as in Lemma 5.2. Numerics seem to corroborate the following assertions:

Conjecture 9. *$V_{N+1}''(\boldsymbol{c}_C)$ has at least an eigenvalue $\lambda > 1$.*

Conjecture 10. *$V_{N+1}''(\boldsymbol{c}_C)$ has all of its eigenvalues out of S, except for -2 and 1 (simple) and 0 (double).*

Proving these would settle the matter for Conjectures 1 and 2, respectively on $\mathcal{H}_{N+1}$ with arbitrary $m_{N+1} > 0$ and $m_1 = \cdots = m_N$.

5.3.3. *The Spatial Four-Body Problem.* Let $\boldsymbol{c}_T = (\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4) \in \mathbb{R}^{12}$ be a vector such that $V''_{4,3}(\boldsymbol{c}_T) = \boldsymbol{c}_T$ and $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4$ are the vertexes of a regular tetrahedron. Such a vector exists in virtue of Remark 5 and what was said in Example 2.1(2), and in turn yields a homographic solution for the three-dimensional Four-Body Problem. The following appears to hold:

Conjecture 11. *The eigenvalues of* $V''_{4,3}(\boldsymbol{c}_T)$ *are*

$$\lambda_1 = -2, \quad \lambda_2 = \lambda_3 = \lambda_4 = 0, \quad \lambda_5 = \lambda_6 = \lambda_7 = 1, \quad \lambda_8, \ldots, \lambda_{12},$$

at least one of $\lambda_8, \ldots, \lambda_{12}$ *being a non-integer.*

A stretch may be attempted by asking for Conjecture 2 to hold, at least for a generic family of masses m_1, m_2, m_3, m_4. $\boldsymbol{c}_T$, as is the case for the triangular solution used in Subsection 4.1, is fairly easy to compute; the main drawback here is computing the eigenvalues of $V''_{4,3}(\boldsymbol{c}_T)$.

6. Hamiltonians with a homogeneous potential.

6.1. **Higher variational equations.** All of what follows is the product of a personal communication from J.-P. Ramis during a short-term stay in Toulouse in 2005 as well as a couple of conversations with J.-P. Ramis and J.-A. Weil in Luminy and Barcelona in 2006.

The first variational equations along solutions of the form $\phi(t)\,\boldsymbol{c}$ such that (17) holds are expressible in terms of hypergeometric functions, as was seen in Subsection 2.3. A first step should be done forward into expressing higher-order variational equations along those solutions in terms of generalized hypergeometric functions; the most general instance of such functions for which a significant amount of study has been done is the *Meijer G-function* ([**?**, §5.3]),

$$G^{m,n}_{p,q}\left(x \left| \begin{array}{c} a_1 \cdots a_p \\ b_1 \cdots b_q \end{array} \right.\right) := \frac{1}{2\pi \mathrm{i}} \int \frac{\prod_{j=1}^{m} \Gamma(\beta_j - \tau) \prod_{j=1}^{m} \Gamma(1 - \alpha_j + \tau)\, x^{\tau}}{\prod_{j=n+1}^{p} \Gamma(\alpha_j - \tau) \prod_{j=m+1}^{q} \Gamma(1 - \beta_j + \tau)} d\tau \quad (42)$$

where $m, n, p, q \in \mathbb{N}$. The change $t \mapsto x$ will probably involve a branched covering much in the way explained in Subsection 2.3. Hence, the study of monodromy and Galois groups done by Yoshida, Morales-Ruiz and Ramis is here substituted in by the computation of those groups for differential equations with functions of the form (42). Since higher variational equations are solvable by quadratures along any known integral curve (using variation of constants), the corresponding linear differential operators given by (42) are reducible; this places us in the least studied case, since most of the bibliography concerning a Galoisian approach to generalized hypergeometric functions corresponds to the irreducible case (e.g. [**?**], [**?**]). The most reliable sources concerning this are probably [**?**], [**?**], [**?**] and [**?**], in which relevant information has been collected on the Galois group G of these operators: for instance, that G is the semi-direct product of a reductive group (computable in terms of the first variational equations), and its unipotent radical; furthermore, a thorough study has been made of this unipotent radical in the first three references, for instance concerning its usual commutativity. However, it is still not clear whether or not this information (especially the non-trivial direct product structure) is useful for our purposes here. And even if it were, and the aforesaid direct product were to yield families of masses $m_1, \ldots, m_N$ for which the identity component of G is non-commutative, the task would still remain to find such families – a rather involved task ahead of us, considering we have not one but N parameters to work with.

Appendix A. Useful results from Algebraic Geometry.

See [?], [?], [?], [?], [?], [?] or [?] for technical details and further information.

A.1. Preliminaries.

From now on, each group G will have its unit element written as e_G, subindex $_G$ being dropped for the most part. It is straightforward to establish that the kernel of any group homomorphism, as well as the image of a normal subgroup under an *epimorphism* is always a normal subgroup of the source group. A sequence of subgroups

$$G = G_0 \supset G_1 \supset \cdots \supset G_m, \tag{43}$$

for any given $m \in \mathbb{N}$, is called a **tower** of subgroups. Tower (43) is called **normal** if G_{i+1} is a normal subgroup of G_i for each $i = 0, \dots, m-1$. A group G is called **solvable** if there is at least one $m \in \mathbb{N}$ such that G has a normal tower (43) in which $G_m = \{e_G\}$. It is a known fact that given a normal subgroup $H \subset G$ then G is solvable if and only if H and G/H are solvable; in particular, $f : H \to H' = f(H)$ given, $\ker f$ is a solvable normal subgroup and thus $H/\ker f \simeq H'$ is solvable as well, meaning: *solvability is preserved under group epimorphisms.*

Given a finite-dimensional vector space V over an algebraically closed field K, let S be a finitely-generated K-algebra of K-valued functions on V. Two such algebras are:

1. the K-algebra $K[V]$ of polynomial functions on V, i.e. functions of the form $f = P \circ \varphi : V \to K$, $P : K^n \to K$ being a polynomial, $P \in K[x_1, \dots, x_n]$, and φ being an isomorphism between V and K^n;
2. and the quotient field of $K[V]$, i.e. the K-algebra $K(V)$ of **rational** functions defined on V, i.e. functions of the form $f = F \circ \varphi : V \to K$, $F : K^n \to K$ being a quotient of polynomials, $P(x_1, \dots, x_n) / Q(x_1, \dots, x_n)$ with $P, Q \in K[x_1, \dots, x_n]$, and again φ being an isomorphism between V and K^n.

If $S = K[V]$ it may be easily proven (e.g. [**?**, Proposition 5.2 (Chapter 10)]) that the sets $\mathbb{Z}(I)$ of zeros of ideals $I \in S$ are *affine varieties* over K ([**?**, §1.1]) and thus closed sets of a certain topology called the *Zariski topology* ([**?**, §1.2]). For the remainder of this Section, any reference to topology will be henceforth set exclusively in either the Zariski topology or the one therefrom induced on subsets or cartesian products.

We recall a topological space X is **irreducible** if two non-empty open subsets of X have a non-empty intersection. In the next results, as said in the previous paragraph, subsets $X \subset V$ will be systematically endowed with the subspace topology induced by the Zariski topology of V. It is easy to establish that V is irreducible ([**?**, Corollary 1.3.8]) and thus:

Lemma A.1. *Any non-empty open set $A \subset V$ is dense in V.* □

A.2. Linear algebraic groups and Lie algebras.

A.2.1. *Linear algebraic groups.* Recall an **algebraic group** over K as being an affine algebraic variety over K endowed with a group structure such, that the two maps $\mu : G \times G \to G$, $\iota : G \to G$ defined by $\mu(x, y) = xy$ and $\iota(x) = x^{-1}$ are morphisms of varieties. In particular, a special type of algebraic group is a **linear algebraic group** which is defined as a Zariski closed subgroup of some $\mathrm{GL}(V)$, V being finite-dimensional K-vector space as above. We also recall ([**?**, §7.4]) a **morphism of algebraic groups** as being a group homomorphism $\phi : G \to G'$ which is also a morphism of varieties; whenever $G' = \mathrm{GL}_n(K)$ we say morphism ϕ is a **(rational) representation**; in light of this, it is usually advisable to view $\mathrm{GL}(V)$ as an algebraic group all its own, specifying its Zariski topology in an unambiguous way by any arbitrary choice of basis for $V \simeq K^n$ since any such choice in K^n corresponds to an inner automorphism $x \mapsto yxy^{-1}$ in $\mathrm{GL}_n(K)$. Since the product topology in $G_1 \times \cdots \times G_n$ is precisely the initial topology with respect to projection maps $\pi_i : G \to G_i$ defined by $\pi_i(g_1, \ldots, g_n) := g_i$, each of these projections will be continuous with respect to the Zariski topology in G. In particular, if $G_1, \ldots, G_n$ are algebraic groups, then for any connected subgroup $H \subset G_1 \times \cdots \times G_n$ each image $\pi_i(H)$, $i = 1, \ldots, n$, is a connected subgroup of G_i with respect to the Zariski topology in G_i.

A representation is called **faithful** if it is injective. Given any representation $\phi : G \to \mathrm{GL}(V)$ of an algebraic group G, the operation

$$G \times V, \qquad (x, \boldsymbol{v}) \mapsto x \cdot \boldsymbol{v} := \phi(x)\, \boldsymbol{v},$$

is clearly a group action of G on V. In this case V is usually called a (rational) G-**module**. For any algebraic group G acting over V, we call $G\boldsymbol{v} = O(\boldsymbol{v}) = \{g \cdot \boldsymbol{v} : g \in G\}$ the G-**orbit** of $\boldsymbol{v} \in E$. G-module V is called **faithful** if $(x, \boldsymbol{v}) \mapsto x \cdot \boldsymbol{v}$ is faithful as a group action, i.e. if ϕ is a faithful representation. Module V is called **irreducible** if it has exactly two submodules: $\{\mathbf{0}\}$ and V itself. More generally, a finite-dimensional G-module V is **completely reducible** if for every submodule $V_1 \subset V$ there is another submodule $V_2 \subset V$ such that $V = V_1 \oplus V_2$ or, equivalently, if V is the direct sum of some of its irreducible submodules.

Given an algebraic group G, the **identity component** G^0 of G is the unique (topologically) irreducible component containing e_G. Any algebraic group has a unique largest normal solvable subgroup, which is automatically closed ([**?**, Corollary 7.4 and Lemma 17.3(c)]). Its identity component is thus the largest connected normal solvable subgroup of G; it is called the **radical** of G and denoted $R(G)$. The subgroup of $R(G)$ consisting of all its **unipotent** elements (i.e., those elements expressible as the sum of the identity and a nilpotent element) is normal in G; it is called the **unipotent radical** ([**?**, §19.5]) of G, denoted as $R_u(G)$, and may be characterized as the largest closed, connected, normal subgroup formed by unipotent elements of G. If $R(G)$ is trivial and $G \neq \{e\}$ is connected, G is called **semisimple**; this is the case, for instance, for $\mathrm{SL}_n(K)$ ([**?**, §19.5]). If G is semisimple, then every G-module V is completely reducible. G is furthermore called **simple** if it has no closed connected normal subgroups other than itself and $\{e\}$; $\mathrm{SL}_n(K)$ is again a valid example ([**?**, §27.5]).

A.2.2. *Lie algebras.* Everything defined and asserted in this Subsection is found and verified in detail in [**?**, Chapter 1, from §3 onward], [**?**, Chapters 9 and 10], [**?**, Chapters 2, 3 and 4] or [**?**, Chapters 1 and 3].

A **Lie algebra** over K is a particular kind of algebra over a field; it is defined as a K-vector space $\mathfrak{a}$ together with a bilinear binary operation $[\cdot, \cdot] : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}$, called the **Lie bracket**, such that $[\boldsymbol{x}, \boldsymbol{x}] = \mathbf{0}$ for all $\boldsymbol{x} \in \mathfrak{a}$ and the **Jacobi identity** holds:

$$[\boldsymbol{x}, [\boldsymbol{y}, \boldsymbol{z}]] + [\boldsymbol{y}, [\boldsymbol{z}, \boldsymbol{x}]] + [\boldsymbol{z}, [\boldsymbol{x}, \boldsymbol{y}]], \qquad \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathfrak{a}.$$

Lie subalgebras will be accordingly defined as subspaces of a Lie algebra which are closed under the Lie bracket. An **ideal** of the Lie algebra $\mathfrak{a}$ is a subspace $\mathfrak{h}$ of $\mathfrak{a}$ such that $[\boldsymbol{a}, \boldsymbol{x}] \in \mathfrak{h}$ for all $\boldsymbol{a} \in \mathfrak{g}$ and $\boldsymbol{x} \in \mathfrak{h}$. All ideals are trivially subalgebras, although the converse is not always true.

The **commutator series** of a Lie algebra $\mathfrak{a}$, sometimes also called the **derived series**, is the sequence of subalgebras recursively defined by $\mathfrak{a}^{k+1} := \left[\mathfrak{a}^k, \mathfrak{a}^k\right]$, $k \geq 0$, with $\mathfrak{a}^0 := \mathfrak{a}$. A Lie algebra $\mathfrak{a}$ is **solvable** if its Lie algebra commutator series $\left\{\mathfrak{a}^k\right\}_k$ vanishes for some k. $\mathfrak{a}$ is **simple** if it is not abelian and has no nonzero proper ideals; it is straightforward to prove that *solvable* implies *not simple* for any Lie algebra. A Lie algebra is **semisimple** if it is a direct sum of simple Lie algebras.

Let G be an algebraic group over $\mathbb{C}$; since, being an affine variety, it may be endowed with the usual complex topology as well as with the Zariski topology, it is actually a **Lie group** ([**?**, §1 (Chapter 1)]), i.e. a group which is also a differential manifold, such that the group operations are compatible with the differential structure. To every Lie group G we can associate a Lie algebra (whose indication in blackletter, $\mathfrak{g}$, is usually the only change in notation), in a way completely summarizing the *local* structure of the group; the underlying vector space of $\mathfrak{g}$ is the tangent space of G at the e_G, and we can heuristically characterize all elements of the Lie algebra as elements of G which are "infinitesimally close" to e_G. We will usually call $\mathfrak{g}$ the **Lie algebra** of G, writing it alternatively as $\mathrm{Lie}\,(G)$. See [**?**, Chapter 1] for concise definitions and properties. It is also reasonably immediate to prove that the Lie algebra of a semisimple algebraic group is semisimple itself.

We have the following result (see also [**?**, Proposition 2.2]):

Lemma A.2. $\mathfrak{sl}_2(\mathbb{C})$*, i.e. the Lie algebra of* $\mathrm{SL}_2(\mathbb{C})$*, has no simple subalgebras other than itself.*

Proof. Indeed, the dimension of $\mathfrak{sl}_2(\mathbb{C})$ is three, and thus any proper subalgebra of $\mathfrak{sl}_2(\mathbb{C})$ should be of dimension smaller than or equal to two; all such subalgebras are solvable ([**?**, §2.1]), thus not simple. □

A.3. Rational invariants.

A.3.1. *Introduction.* See [**?**, §2] for more details on the definitions and concepts introduced in the following paragraph.

Let $G \subset \mathrm{GL}\,(V)$ be a linear algebraic group. We may define, as is done in [**?**, §4.2], the action of G on $\mathbb{C}[V]$ or $\mathbb{C}(V)$:

$$g \cdot f := f \circ g^{-1},\ g \in G, f \in \mathbb{C}(V).$$

We define by $\mathbb{C}[V]^G$ (resp. $\mathbb{C}(V)^G$) the $\mathbb{C}$-algebra of G-invariant elements of $\mathbb{C}[V]$ (resp. $\mathbb{C}(V)$); hence the denomination **rational invariant** for any $f \in \mathbb{C}(V)^G$. We may furthermore assume G is connected, since G has an invariant if, and only if, G^0 has an invariant; this fact, which is a consequence of the finite index of G^0 in G, may be found proven in the first Lemma of [**?**, Chapter 1]; see also [**?**]. Let G be an algebraic group, V be a G-module and L^G the field of rational invariants of V as a G-module. We say G is r-**Ziglin** if $\deg\operatorname{tr}\left(L^G\right) = r$.

Lemma A.3. *Let* $\mathfrak{g}$ *be a simple Lie subalgebra of* $\bigoplus_{i=1}^n \mathfrak{sl}_2(\mathbb{C}) = \mathrm{Lie}\left(\mathrm{SL}_2(\mathbb{C})^n\right)$*. Then* $\mathfrak{g} \simeq \mathfrak{sl}_2(\mathbb{C})$*.*

Proof. For each $i = 1, \dots, n$ let $\left.\pi_i\right|_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{sl}_2(\mathbb{C}),\ (\boldsymbol{x}_1, ..., \boldsymbol{x}_n) \mapsto \boldsymbol{x}_i$, be the restriction of the canonical projection $\pi_i : \bigoplus_{i=1}^n \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{sl}_2(\mathbb{C})$ to $\mathfrak{g}$. There is at least one i such that $\left.\pi_i\right|_{\mathfrak{g}}(\mathfrak{g}) \neq \{\mathbf{0}\}$, since each element $\boldsymbol{x} = (\boldsymbol{x}_1, ..., \boldsymbol{x}_n) \in \mathfrak{g}$ is precisely equal to $(\pi_1(\boldsymbol{x}), ..., \pi_n(\boldsymbol{x}))$, and were $\left.\pi_i\right|_{\mathfrak{g}} \equiv \{\mathbf{0}\}$, $i = 1, ..., n$, we would then have $\mathfrak{g} = \{\mathbf{0}\}$. Thus, there is at least one i for which $\left.\pi_i\right|_{\mathfrak{g}}$ has a non-trivial image $\pi_i(\mathfrak{g}) \neq \{\mathbf{0}\}$, itself a subalgebra of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ which admits no simple subalgebras other than itself, as said in Lemma A.2; this latter fact implies $\pi_i(\mathfrak{g}) = \mathfrak{sl}_2(\mathbb{C}) \simeq \mathfrak{g}/\ker \pi_i|_{\mathfrak{g}}$. But $\mathfrak{g}$ is simple as well, and thus the ideal $\ker \pi_i|_{\mathfrak{g}}$ must be either $\{\mathbf{0}\}$ or $\mathfrak{g}$. It is clear that $\ker \pi_i|_{\mathfrak{g}} = \{\mathbf{0}\}$, since $\ker \pi_i|_{\mathfrak{g}} = \mathfrak{g}$ would imply $\mathfrak{sl}_2(\mathbb{C}) \simeq \mathfrak{g}/\ker \pi_i|_{\mathfrak{g}} = \{\mathbf{0}\}$ which is obviously absurd. □

A.3.2. *Basic premises.* We will now establish the hypotheses for the rest of the Section.

First of all, we will adopt the following notation for each $A \in \mathrm{SL}_2(\mathbb{C})$:

$$[A]_k := \left(A^{(0)}, A^{(1)}, \dots, A^{(k-1)}\right) = \begin{pmatrix} A^{(0)} & 0 & \cdots & 0 \\ 0 & A^{(1)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A^{(k-1)} \end{pmatrix}, \tag{44}$$

where $A^{(0)} := A, A^{(1)}, \dots, A^{(k-1)}$ are bound to be equivalent to A in the sense of equivalence of representations: there are $B^{(1)}, \dots, B^{(k-1)} \in \mathrm{GL}_2(\mathbb{C})$ such that $A^{(i)} = B^{(i)} A^{(i)} \left[B^{(i)}\right]^{-1}$ regardless of the choice of $A^{(i)}$.

Let $k \in \mathbb{N}$ and $\boldsymbol{\chi} = (\chi_1, \dots, \chi_{k-1}) \in \{\pm 1\}^{k-1}$ be any vector such that either $\chi_i = 1$ for $i = 1, \dots, k-1$ or both 1 and -1 appear as entries in $\boldsymbol{\chi}$. We denote

$$[A]_k^{\chi} := \left(A^{(0)}, \chi_1 A^{(1)}, \dots, \chi_{k-1} A^{(k-1)}\right), \tag{45}$$

again assuming $A^{(0)} := A, A^{(1)}, \dots, A^{(k-1)}$ to be equivalent to A in the sense of representation equivalence as was defined above.

All through subsections A.3.3 and A.3.4, we will assume the following: G will be an algebraic group and V a G-module such that G is faithfully represented as a subgroup of $\mathrm{SL}_2(\mathbb{C})^n$,

$$\rho : G \to \mathrm{SL}_2(\mathbb{C})^n .$$

We will assume $\pi_i(G) = \mathrm{SL}_2(\mathbb{C})$ for $i = 1, \dots, n$,

$$\pi_i : \mathrm{SL}_2(\mathbb{C})^n \to \mathrm{SL}_2(\mathbb{C}), \qquad (A_1, \dots, A_n) \mapsto A_i,$$

being the i-th projection for each $i = 1, \dots, n$.

A.3.3. *Case* 1. *G is connected.*

Lemma A.4. *Let G be an algebraic group satisfying the hypotheses in A.3.2, with the additional property of being connected. Then,*

1. *G is semisimple;*
2. *$G \simeq \mathrm{SL}_2(\mathbb{C})^m$ for some $m \le n$.*
3. *$\rho(G) = \left([A_1]_{k_1}, \dots, [A_m]_{k_m}\right)$, where $k_1 + \cdots + k_m = n$.*

Proof.

1. The hypotheses imply V is a completely reducible G-module. In order to further prove G semisimple, let us assume the contrary, i.e. that $R(G) \neq \{e\}$; then not every $\pi_i(R(G))$ would be nontrivial since ρ is injective and thus so is $\rho|_{R(G)}$, i.e. $R(G)$ is represented faithfully as a subgroup of $\mathrm{SL}_2(\mathbb{C})^n$: $R(G) \hookrightarrow \pi_1(R(G)) \times \cdots \times \pi_n(R(G)) \subset \mathrm{SL}_2(\mathbb{C})^n$. But this is absurd since $\pi_i(R(G))$ is trivial, $i = 1, \dots, n$; indeed, each $\pi_i(R(G)) \subset \mathrm{SL}_2(\mathbb{C})$ is a normal, connected, solvable subgroup of a simple algebraic group since π_i is a group epimorphism and $\mathrm{SL}_2(\mathbb{C})$ is simple. Thus, $\pi_i(R(G)) = \{\mathrm{Id}_2\}$ for each $i = 1, \dots, n$ implying $R(G) = \{e\}$, i.e. G is a *semisimple* algebraic group.

2. In virtue of the preceding item, $\mathfrak{g} := \mathrm{Lie}\,(G)$ is a semisimple Lie algebra. Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$ be its decomposition in simple algebras. From Lemma A.3, we know $\mathfrak{g}_i \simeq \mathfrak{sl}_2\,(\mathbb{C})\,,\ i = 1, \ldots, m$, and thus $\mathfrak{g} \simeq \bigoplus_{i=1}^m \mathfrak{sl}_2\,(\mathbb{C})$. Since $\rho(G) \subset \mathrm{SL}_2\,(\mathbb{C})^n$ and ρ is faithful, the rest follows from the standard theory of representations of semisimple groups (see [?, Chapter XI]); indeed, if $G = G_1 \times \cdots \times G_m$ is the decomposition in simple algebraic groups, with $\mathrm{Lie}\,(G_i) = \mathfrak{g}_i = \mathfrak{sl}_2\,(\mathbb{C})$, $i = 1, ..., m$, out of the two possibilities for each G_i, $\mathrm{SL}_2\,(\mathbb{C})$ or $\mathrm{PSL}_2\,(\mathbb{C})$ (see [?, §32.4]), only $\mathrm{SL}_2\,(\mathbb{C})$ is possible, since $\rho(G) \subset \mathrm{SL}_2\,(\mathbb{C})^n$.

3. Every representation of the semisimple group $G \simeq \mathrm{SL}_2\,(\mathbb{C})^m$ is a direct sum of completely irreducible representations ([?, Prop. 1.8]).

□

A.3.4. *Case 2. G is not connected.* Let us now assume G fulfills all of the hypotheses in the above Lemma, *save for connectivity.* From this point onward we will write $\rho\,(g)$ and g indistinctively for elements of G. With the notation $\sigma_g\,(h) := ghg^{-1}$, let $\mathrm{Int}\left(G^0\right)$ be the group of *internal automorphisms* of $G^0 \simeq \mathrm{SL}_2\,(\mathbb{C})^m$:

$$\mathrm{Int}\left(G^0\right) := \{\sigma_g : G^0 \to G^0 : g \in G^0\}.$$

Given $g \in G \subset \mathrm{SL}_2\,(\mathbb{C})^n$, the automorphism σ_g of G^0 is in fact internal, i.e., it belongs to $\mathrm{Int}\left(G^0\right)$, since any automorphism of the form

$$\sigma : \mathrm{SL}_2\,(\mathbb{C})^m \to \mathrm{SL}_2\,(\mathbb{C})^m\,, \qquad \sigma\,(A_1, \ldots, A_m) = (\sigma_1 A_1, ..., \sigma_m A_m)\,,$$

(σ_i being the restriction of σ to the i-th component) which preserves the order of the $\mathrm{SL}_2\,(\mathbb{C})$-blocks of the representation, is given by automorphisms

$$\sigma_i : \mathrm{SL}_2\,(\mathbb{C}) \to \mathrm{SL}_2\,(\mathbb{C})\,.$$

But it is well-known that the group of automorphisms of $\mathrm{SL}_2\,(\mathbb{C})$ is $\mathrm{Int}\,(\mathrm{SL}_2\,(\mathbb{C}))$. Another somewhat abstract argument is that the group $G^0 \simeq \mathrm{SL}_2\,(\mathbb{C})^m$ is a semisimple, simply connected algebraic group and for such groups $\mathrm{Aut}\left(G^0\right)$ splits as a product of the finite group Γ of symmetries of the Dynkin diagram (which in $\mathrm{Aut}\left(G^0\right)$ is given as a permutation of elements $g = (A_1, ..., A_m)$ of G^0) and of $\mathrm{Int}\left(G^0\right)$,

$$\mathrm{Aut}\left(G^0\right) \simeq \Gamma \cdot \mathrm{Int}\left(G^0\right),$$

and here the action of $G \subset \mathrm{SL}_2\,(\mathbb{C})^n$ over G^0 preserves the order in $(A_1, ..., A_m)$ (see [?, §27.4], [?, p. 35, 203]). Now we can define an epimorphism

$$\Psi : G \to \mathrm{Int}\left(G^0\right), \qquad \Psi\,(g) : G^0 \to G^0, \quad \Psi\,(g) = \sigma_g.$$

As usual we denote

$$Z_G\,(H) = \{g \in G : gh = hg,\ h \in H\}\,, \qquad Z\,(H) = \{g \in H : gh = hg,\ h \in H\}\,,$$

the *centralizer* (in G) and *center* of a subgroup $H \subset G$, respectively. It is clear that $\mathrm{Int}\left(G^0\right) = G^0/Z\left(G^0\right)$ and that $\ker\,(\Psi)$ is given by all the elements in G which commute with the elements of G^0, i.e., by $Z_G\left(G^0\right)$. In virtue of Schur's Lemma [?] (applied to any of the irreducible representations of G in $\mathrm{SL}_2\,(\mathbb{C})^n$), we may assert that

$$Z_G(G^0) = Z(G) = G \cap \{(\chi_1 \mathrm{Id}_2, ..., \chi_n \mathrm{Id}_2) \,:\, \chi_i \in \{\pm 1\}\},$$

Id_2 denoting the identity matrix in $\mathrm{SL}_2\,(\mathbb{C})$. Indeed, by Schur's Lemma the center of $\mathrm{SL}_2\,(\mathbb{C})$ is given by the scalar matrices (i.e. Id_2 multiplied by a constant), and the scalar matrices inside $\mathrm{SL}_2\,(\mathbb{C})$ are $\pm\mathrm{Id}_2$ and it is clear that $Z(G) \subset Z_G(G^0)$, the equality also following from Schur's Lemma. Thus, applying the isomorphism theorem we conclude:

Lemma A.5. $G/Z(G) \simeq G^0/Z(G^0)$.□

Corollary A.6. $G = Z(G) \cdot G^0$

Proof. $Z(G) \cdot G^0/Z(G) \simeq G^0/Z(G) \cap G^0 \simeq G^0/Z(G^0) \simeq G/Z(G^0)$. Hence

$$\mathrm{Id}_{2n} \simeq G/Z(G)/Z \cdot G^0/Z(G) \simeq G/Z(G^0).$$

□

Corollary A.7. $G = Z(G)/Z(G^0) \times G^0$.

Proof. We prove $G/G^0 \simeq Z(G)/Z(G^0)$:

$$G/G^0 \simeq Z(G) \cdot G^0/G^0 \simeq Z(G)/G^0 \cap Z(G) \simeq Z(G)/Z(G^0).$$

□

We remark that it is possible to immerse $Z(G)/Z(G^0)$ in G.

Defining $E_{k_i} := \{(\mathrm{Id}_2, \chi_1 \mathrm{Id}_2, \dots, \chi_{k_i-1}\mathrm{Id}_2) : \chi_i \in \{\pm 1\}\}$, a straightforward argument yields

$$Z(G)/Z(G^0) = G \cap \prod_{i=1}^{m} E_{k_i} = \{(B_1, \dots, B_m) : B_i \in E_{k_i}\},$$

and thus $G = Z(G)/Z(G^0) \times G^0$; hence, we have proven:

Theorem A.8. *Let G be an algebraic group satisfying the hypotheses in Subsection A.3.2. Then, there exist integers $k_1, \dots, k_m$ such that $k_1 + \dots + k_m = n$, $k_i > 1$ for $i = 1, \dots, r$ and $k_{r+1} = \dots = k_m = 1$, and there exist r sets $X_1, \dots, X_r$ of vectors $\boldsymbol{\chi}$ satisfying the hypotheses in Subsection A.3.2, for which each element of G is expressible, using* (45), *as* $\mathrm{diag}\left([A_1]_{k_1}^{\boldsymbol{\chi}_1}, \dots, [A_r]_{k_r}^{\boldsymbol{\chi}_r}, A_{r+1}, \dots, A_m\right)$ *where $\boldsymbol{\chi}_i \in X_i$ for $i = 1, \dots, r$ and $A_i \in \mathrm{SL}_2(\mathbb{C})$ for $i = 1, \dots, m$.* □

Remarks A.1.

1. The thesis of the above Theorem implies that all representations of G are equivalent to one for which each element of the group is expressed as:

$$\begin{pmatrix}
\boxed{\begin{array}{cccc} A_1 & & & \\ & \chi_{1,1}A_1 & & \\ & & \ddots & \\ & & & \chi_{1,k_1-1}A_1 \end{array}} & & & & & \\
& \ddots & & & & \\
& & \boxed{\begin{array}{cccc} A_r & & & \\ & \chi_{r,1}A_r & & \\ & & \ddots & \\ & & & \chi_{r,k_r-1}A_r \end{array}} & & & \\
& & & A_{r+1} & & \\
& & & & \ddots & \\
& & & & & A_m
\end{pmatrix}$$

for $A_1, \dots, A_m \in \mathrm{SL}_2(\mathbb{C})$ and some set $\{\chi_{1,1}, \dots, \chi_{m,k_m-1}\} \subset \{\pm 1\}$.

2. For this particular structure of the group G, Kolchin's Theorem [**?**, p. 1152-1153] on algebraic dependence in the context of Picard-Vessiot theory appears as a corollary of the above Theorem A.8. We do not extend on this further, though.

3. Case $r = 0$ is possible in the above theorem, it corresponds to $G = SL(2, \mathbf{C})^n$.

A.3.5. *Invariants of* G. We are now going to analyze the rational invariants of G. We first need to recall the two classical theorems concerning the invariants of $\mathrm{SL}_2(\mathbb{C})$. Consider the faithful representation of $\mathrm{SL}_2(\mathbb{C})$ on $\mathrm{SL}_2(\mathbb{C})^k$ defined by

$$\mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_2(\mathbb{C})^k, \qquad A \mapsto (A, \dots, A).$$

k is assumed to be greater than one. The action of $\mathrm{SL}_2(\mathbb{C})$ on $V = \mathbb{C}^{2k}$ is given by $(\mathbf{v}_1, \dots, \mathbf{v}_k) \mapsto (A\mathbf{v}_1, \dots, A\mathbf{v}_k)$; in canonical coordinates, we adopt the notation $\mathbf{v}_i : (x_i, y_i)^T$. A set of generators of the algebra R^G (where $R = \mathbb{C}[V]$) of polynomial invariants of this representation of $\mathrm{SL}_2(\mathbb{C})$ is formed by $J_{i,j} := \det(\mathbf{v}_i, \mathbf{v}_j) = x_i y_j - x_j y_i$, $i < j$. This is exactly the first theorem of invariants of unimodular groups applied to $\mathrm{SL}_2(\mathbb{C})$ ([**?**, p. 30]).

Lemma A.9. *Let* $L = \mathbb{C}(V)$ *be the field of rational functions over* V. *Then,* $\deg \operatorname{tr}\left(L^{\mathrm{SL}_2(\mathbb{C})}\right) = 2k - 3$.

Proof. $\deg \operatorname{tr}\left(L^{\mathrm{SL}_2(\mathbb{C})}\right) = \dim V - \Delta$, where Δ is the maximal dimension of orbits of the G-module V (see [**?**, Theorem 2.10]). $\Delta = 3$: indeed, the orbit along a given $\boldsymbol{v} = (\mathbf{v}_1, \dots, \mathbf{v}_k) \in V$ is

$$\mathcal{O}_{\boldsymbol{v}} = \left\{ A \begin{pmatrix} x_1 & x_2 & \cdots & x_k \\ y_1 & y_2 & \cdots & y_k \end{pmatrix} =: AC : A \in \mathrm{SL}_2(\mathbb{C}) \right\},$$

and $\dim(\mathcal{O}_{\boldsymbol{v}}) = \dim(\mathrm{SL}_2(\mathbb{C})) - \Delta_{\boldsymbol{v}}$, where $\Delta_{\boldsymbol{v}}$ is the dimension of the isotropy group in $\boldsymbol{v}$. We may now choose C such that $\operatorname{rank}(C) = 2$ (for instance taking $\mathbf{v}_1, \mathbf{v}_2$ linearly independent); in that case, if a matrix A were such that $AC = C$, that would imply that $\mathbf{v}_1, \mathbf{v}_2$ be a basis de $\ker(A - \mathrm{Id}_2)$; since

$$\dim(\ker(A - \mathrm{Id}_2)) = 2 - \operatorname{rank}(A - \mathrm{Id}_2),$$

this implies $A = \mathrm{Id}_2$; therefore, $\Delta_{\boldsymbol{v}} = 0$ for this choice of $\boldsymbol{v}$. Hence, $\dim(\mathcal{O}_{\boldsymbol{v}}) = 3$. □

Remarks A.2.

1. An alternative proof can be given using the dimension of the associated Grassmannian.

2. Another proof could be done using that $\deg \operatorname{tr}\left(R^{\mathrm{SL}_2(\mathbb{C})}\right) = 2k - 3$ and that $Q\left(R^{\mathrm{SL}_2(\mathbb{C})}\right) = L^G$ (due to the fact that G is semisimple).

3. Lemma A.9 is obviously also true for the equivalent representation $A \mapsto [A]_k$ given in (44).

Let G be a group satisfying the hypotheses in Subsection A.3.2.

Assume G is connected; Lemma A.4 assures us that

$$G = \left\{ \left([A_1]_{k_1}, \dots, [A_m]_{k_r}\right) : A_1, \dots, A_r \in \mathrm{SL}_2(\mathbb{C}) \right\},$$

for some $r \in \mathbb{N} \cup \{0\}$. Our G-module is now $V = \mathbb{C}^{2n}$. We define $R = C[V]$. Assume $k_i > 1$, $i = 1, \dots, r$ without loss of generality, otherwise defining r as maximal with this property. Then, G is $(\sum_{i=1}^{r} 2k_i - 3r)$-Ziglin due to Lemma A.9.

Let G be any group, whether or not connected, albeit still under the hypotheses of Theorem A.8. G^0 is s-Ziglin if and only if G is (see for instance [**?**, Prop. 2.9]). We finally obtain, by this and the previous paragraph:

Theorem A.10. *G is $\left(\sum_{i=1}^{r} 2k_i - 3r\right)$-Ziglin.* □

Appendix B. **Computations for Theorem 3.1.** We have, using the notation in Subsection 3.2.2,

$$\begin{aligned}
\mathbf{D}_{1,2} &= \begin{pmatrix} d_{1,3} \\ d_{2,4} \end{pmatrix} := \sqrt{m_2}\mathbf{q}_1 - \sqrt{m_1}\mathbf{q}_2 = \sqrt{m_1 m_2} m^{1/3} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \\
\mathbf{D}_{1,3} &= \begin{pmatrix} d_{1,5} \\ d_{2,6} \end{pmatrix} := \sqrt{m_3}\mathbf{q}_1 - \sqrt{m_1}\mathbf{q}_3 = \sqrt{m_1 m_3} m^{1/3} \begin{pmatrix} 2\alpha \\ 0 \end{pmatrix}, \\
\mathbf{D}_{2,3} &= \begin{pmatrix} d_{3,5} \\ d_{4,6} \end{pmatrix} := \sqrt{m_3}\mathbf{q}_2 - \sqrt{m_2}\mathbf{q}_3 = \sqrt{m_2 m_3} m^{1/3} \begin{pmatrix} \alpha \\ -\beta \end{pmatrix},
\end{aligned}$$

and thus, using the fact that $\alpha^2 + \beta^2 = 1$,

$$\begin{aligned}
\tilde{D}_{1,2} &= \sqrt{d_{1,3}^2 + d_{2,4}^2} = \sqrt{\left(\alpha^2 + \beta^2\right) m_1 m_2 m^{2/3}} = \sqrt{m_1 m_2 m^{2/3}}, \\
\tilde{D}_{1,3} &= \sqrt{d_{1,5}^2 + d_{2,6}^2} = 2\sqrt{\alpha^2 m_1 m_3 m^{2/3}}, \\
\tilde{D}_{2,3} &= \sqrt{d_{3,5}^2 + d_{4,6}^2} = \sqrt{\left(\alpha^2 + \beta^2\right) m_2 m_3 m^{2/3}} = \sqrt{m_2 m_3 m^{2/3}};
\end{aligned}$$

take into consideration $\tilde{D}_{1,2}, \tilde{D}_{1,3}, \tilde{D}_{2,3}$ need not be Euclidean norms (hence the unusual notation, as opposed to the one introduced in Section 1), though this will be the case if the terms inside the parentheses are real. Furthermore, we will at this point assume that either $\alpha \in (0, \infty)$ or $\alpha = re^{\theta \mathrm{i}}$, with $\theta \in [0, \pi)$, as is the case in the proof of Theorem 3.1: $\alpha = \frac{1}{2}, \frac{-1+\sqrt{3}\mathrm{i}}{4}$. In both cases, we have $\sqrt{\alpha^2} = \alpha$ according to our positive determination of the square root.

We know, using the notation in Subsection 3.2.2, that

$$V_3''(q) = \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{1,2} & A_{2,2} & A_{2,3} \\ A_{1,3} & A_{2,3} & A_{3,3} \end{pmatrix},$$

where

$$
\begin{aligned}
A_{1,1} &= m_1^{\frac{3}{2}}\begin{pmatrix} \frac{\left(\tilde{D}_{1,2}^2-3d_{1,3}^2\right)m_2^{\frac{5}{2}}}{\tilde{D}_{1,2}^5}+\frac{\left(\tilde{D}_{1,3}^2-3d_{1,5}^2\right)m_3^{\frac{5}{2}}}{\tilde{D}_{1,3}^5} & -\frac{3d_{1,3}d_{2,4}m_2^{\frac{5}{2}}}{\tilde{D}_{1,2}^5}-\frac{3d_{1,5}d_{2,6}m_3^{\frac{5}{2}}}{\tilde{D}_{1,3}^5} \\ -\frac{3d_{1,3}d_{2,4}m_2^{\frac{5}{2}}}{\tilde{D}_{1,2}^5}-\frac{3d_{1,5}d_{2,6}m_3^{\frac{5}{2}}}{\tilde{D}_{1,3}^5} & \frac{\left(\tilde{D}_{1,2}^2-3d_{2,4}^2\right)m_2^{\frac{5}{2}}}{\tilde{D}_{1,2}^5}+\frac{\left(\tilde{D}_{1,3}^2-3d_{2,6}^2\right)m_3^{\frac{5}{2}}}{\tilde{D}_{1,3}^5} \end{pmatrix}, \\
A_{1,2} &= \frac{m_1^2m_2^2}{\tilde{D}_{1,2}^5}\begin{pmatrix} 3d_{1,3}^2-\tilde{D}_{1,2}^2 & 3d_{1,3}d_{2,4} \\ 3d_{1,3}d_{2,4} & 3d_{2,4}^2-\tilde{D}_{1,2}^2 \end{pmatrix}, \\
A_{1,3} &= \frac{m_1^2m_3^2}{\tilde{D}_{1,3}^5}\begin{pmatrix} 3d_{1,5}^2-\tilde{D}_{1,3}^2 & 3d_{1,5}d_{2,6} \\ 3d_{1,5}d_{2,6} & 3d_{2,6}^2-\tilde{D}_{1,3}^2 \end{pmatrix}, \\
A_{2,2} &= m_2^{\frac{3}{2}}\begin{pmatrix} \frac{\left(\tilde{D}_{1,2}^2-3d_{1,3}^2\right)m_1^{\frac{5}{2}}}{\tilde{D}_{1,2}^5}+\frac{\left(\tilde{D}_{2,3}^2-3d_{3,5}^2\right)m_3^{\frac{5}{2}}}{\tilde{D}_{2,3}^5} & -\frac{3d_{1,3}d_{2,4}m_1^{\frac{5}{2}}}{\tilde{D}_{1,2}^5}-\frac{3d_{3,5}d_{4,6}m_3^{\frac{5}{2}}}{\tilde{D}_{2,3}^5} \\ -\frac{3d_{1,3}d_{2,4}m_1^{\frac{5}{2}}}{\tilde{D}_{1,2}^5}-\frac{3d_{3,5}d_{4,6}m_3^{\frac{5}{2}}}{\tilde{D}_{2,3}^5} & \frac{\left(\tilde{D}_{1,2}^2-3d_{2,4}^2\right)m_1^{\frac{5}{2}}}{\tilde{D}_{1,2}^5}+\frac{\left(\tilde{D}_{2,3}^2-3d_{4,6}^2\right)m_3^{\frac{5}{2}}}{\tilde{D}_{2,3}^5} \end{pmatrix}, \\
A_{2,3} &= \frac{m_2^2m_3^2}{\tilde{D}_{2,3}^5}\begin{pmatrix} 3d_{3,5}^2-\tilde{D}_{2,3}^2 & 3d_{3,5}d_{4,6} \\ 3d_{3,5}d_{4,6} & 3d_{4,6}^2-\tilde{D}_{2,3}^2 \end{pmatrix}, \\
A_{3,3} &= m_3^{\frac{3}{2}}\begin{pmatrix} \frac{\left(\tilde{D}_{1,3}^2-3d_{1,5}^2\right)m_1^{\frac{5}{2}}}{\tilde{D}_{1,3}^5}+\frac{\left(\tilde{D}_{2,3}^2-3d_{3,5}^2\right)m_2^{\frac{5}{2}}}{\tilde{D}_{2,3}^5} & -\frac{3d_{1,5}d_{2,6}m_1^{\frac{5}{2}}}{\tilde{D}_{1,3}^5}-\frac{3d_{3,5}d_{4,6}m_2^{\frac{5}{2}}}{\tilde{D}_{2,3}^5} \\ -\frac{3d_{1,5}d_{2,6}m_1^{\frac{5}{2}}}{\tilde{D}_{1,3}^5}-\frac{3d_{3,5}d_{4,6}m_2^{\frac{5}{2}}}{\tilde{D}_{2,3}^5} & \frac{\left(\tilde{D}_{1,3}^2-3d_{2,6}^2\right)m_1^{\frac{5}{2}}}{\tilde{D}_{1,3}^5}+\frac{\left(\tilde{D}_{2,3}^2-3d_{4,6}^2\right)m_2^{\frac{5}{2}}}{\tilde{D}_{2,3}^5} \end{pmatrix}.
\end{aligned}
$$

In this case, thus, we have

$$
\begin{aligned}
A_{1,1} &= \frac{1}{m}\begin{pmatrix} \frac{4\left(1-3\alpha^2\right)m_2-m_3\alpha^{-3}}{4} & -3\alpha\beta m_2 \\ -3\alpha\beta m_2 & \frac{8\left(1-3\beta^2\right)m_2+m_3\alpha^{-3}}{8} \end{pmatrix}, \\
A_{1,2} &= \frac{\sqrt{m_1}\sqrt{m_2}}{m}\begin{pmatrix} 3\alpha^2-1 & 3\alpha\beta \\ 3\alpha\beta & 3\beta^2-1 \end{pmatrix}, \\
A_{1,3} &= \frac{\sqrt{m_1}\sqrt{m_3}}{m}\begin{pmatrix} \alpha^{-3}/4 & 0 \\ 0 & -\alpha^{-3}/8 \end{pmatrix}, \\
A_{2,2} &= \frac{1}{m}\begin{pmatrix} \left(1-3\alpha^2\right)(m_1+m_3) & 3\alpha\beta\,(m_3-m_1) \\ 3\alpha\beta\,(m_3-m_1) & \left(1-3\beta^2\right)(m_1+m_3) \end{pmatrix}, \\
A_{2,3} &= \frac{\sqrt{m_2}\sqrt{m_3}}{m}\begin{pmatrix} -1+3\alpha^2 & -3\alpha\beta \\ -3\alpha\beta & -1+3\beta^2 \end{pmatrix}, \\
A_{3,3} &= \frac{1}{m}\begin{pmatrix} \frac{4\left(1-3\alpha^2\right)m_2-m_1\alpha^{-3}}{4} & 3\alpha\beta m_2 \\ 3\alpha\beta m_2 & \frac{8\left(1-3\beta^2\right)m_2+\alpha^{-3}m_1}{8} \end{pmatrix},
\end{aligned}
$$

which under the assumption $\alpha^3 = 1/8$ become

$$
\begin{aligned}
A_{1,1} &= \frac{1}{m}\begin{pmatrix} m_2\left(1-3\alpha^2\right)-2m_3 & -3\alpha\beta m_2 \\ -3\alpha\beta m_2 & m_2\left(1-3\beta^2\right)+m_3 \end{pmatrix},\\
A_{1,2} &= \frac{\sqrt{m_1m_2}}{m}\begin{pmatrix} 3\alpha^2-1 & 3\alpha\beta \\ 3\alpha\beta & 3\beta^2-1 \end{pmatrix},\\
A_{1,3} &= \frac{\sqrt{m_1m_3}}{m}\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix},\\
A_{2,2} &= \frac{1}{m}\begin{pmatrix} \left(1-3\alpha^2\right)(m_1+m_3) & 3\alpha\beta\,(m_3-m_1) \\ 3\alpha\beta\,(m_3-m_1) & \left(1-3\beta^2\right)(m_1+m_3) \end{pmatrix},\\
A_{2,3} &= \frac{\sqrt{m_2m_3}}{m}\begin{pmatrix} 3\alpha^2-1 & -3\alpha\beta \\ -3\alpha\beta & 3\beta^2-1 \end{pmatrix},\\
A_{3,3} &= \frac{1}{m}\begin{pmatrix} -2m_1+\left(1-3\alpha^2\right)m_2 & 3\alpha\beta m_2 \\ 3\alpha\beta m_2 & m_1+\left(1-3\beta^2\right)m_2 \end{pmatrix}
\end{aligned}
$$

The characteristic polynomial for $V_3''(c)$ is $P(x) = x^2(x-1)\frac{Q(x)}{m^2}$, where
$Q(x) = p_1p_3m_1^2+p_1^2(x-1)m_2^2+p_1p_3m_3^2+p_1p_2m_1m_2+2(2+x)p_4m_1m_3+p_1p_2m_2m_3$,
and

$$
\begin{aligned}
p_1(x) &= x+3\left(\alpha^2+\beta^2\right)-1,\\
p_2(x) &= 3\alpha^2(x-1)+3\beta^2(x+2)+(x-1)(2x+1),\\
p_3(x) &= (x+2)(x-1),\\
p_4(x) &= (x-1)\left(x+3\beta^2-1\right)+3\alpha^2\left(x+6\beta^2-1\right);
\end{aligned}
$$

substituting in $\alpha^2+\beta^2=1$ once again, we obtain

$$
\begin{aligned}
p_1(x) &= x+2,\\
p_2(x) &= 2x^2+2x+6\beta^2-3\alpha^2-1,\\
p_4(x) &= x^2+x+18\alpha^2\beta^2-2,
\end{aligned}
$$

and thus

$$
P(x) = x^2(x-1)(x+2)\frac{Q(x)}{m^2},
$$

$Q(x) := p_3m_1^2+p_1(x-1)m_2^2+p_3m_3^2+p_2m_1m_2+p_2m_2m_3+2p_4m_1m_3$ having six roots: $-2,0,0,1,\lambda_+,\lambda_-$ where $\lambda_\pm = -\frac{1}{2}\pm\frac{\sqrt{3}\sqrt{\rho}}{2m}$ and

$$
\begin{aligned}
\rho &= 3\left(m_1^2+m_2^2+m_3^2\right)+2\left(1+2\alpha^2-4\beta^2\right)m_2m_3\\
&+ 2m_1\left[m_2\left(1+2\alpha^2-4\beta^2\right)+2m_3\left(1-8\alpha^2\beta^2\right)\right],
\end{aligned}
$$

which assuming once again that $\beta^2 = 1-\alpha^2$ becomes

$$
\rho = 3\left(m_1^2+m_2^2+m_3^2+2\left(2\alpha^2-1\right)\left(m_2m_3+m_1m_2\right)+m_3+8\alpha^2\left(\alpha^2-1\right)m_3\right),
$$

and assuming $\alpha^4 = \alpha/8$ we finally obtain

$$
\rho = 3\left(m_1^2+m_2^2+m_3^2+2\left(2\alpha^2-1\right)\left(m_2m_3+m_1m_2\right)-2\left(8\alpha^2-\alpha-1\right)m_1m_3\right).
$$

For $\alpha = 1/2$, we obtain $\rho = 3\left(m_1^2+m_2^2+m_3^2-m_2m_1-m_2m_3-m_1m_3\right)$, as was the case for the *real* eigenvalues $\lambda_\pm$ in Subsection 4.1, whereas, defining

$$
\begin{aligned}
B_1 &= 2m_1^2+2m_2^2+2m_3^2-5m_1m_2-5m_2m_3+7m_1m_3,\\
B_2 &= \sqrt{3}\left(m_1m_2+m_2m_3-5m_1m_3\right),
\end{aligned}
$$

for $\alpha = \frac{-1+\sqrt{3}\mathrm{i}}{4}$ we have the discriminant $\rho = \frac{3(B_1 - \mathrm{i}B_2)}{2}$, precisely the one appearing in the *complex* eigenvalues $\lambda_{\pm}^{*}$ in Subsection 4.1.

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