Intrinsic correlations of galaxy sizes and luminosities in weak lensing

Ph. D. Thesis

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Declaration

The work presented in this thesis was carried out at the Institute of Cosmology and Gravitation, University of Portsmouth, United Kingdom under the supervision of Prof. Robert Crittenden.

I hereby declare that except where specific reference is made to the works of others the contents of this dissertation are original. Whilst registered as a candidate for the above degree, I have not been registered for any other research award. The results and conclusions embodied in this thesis are the work of the named candidate and have not been submitted for any other academic award.

The work in this thesis is my own unless otherwise stated.


This thesis contains approximately 33000 words.

Sandro Ciarlariello
March 2016
To Adelina Fierro and Mario Ciarlariello,
the best people I have ever met
Abstract

Correlations of the observed sizes and luminosities of galaxies can be used to estimate the magnification that arises through weak gravitational lensing. However, the intrinsic properties of galaxies can be similarly correlated through local physical effects, and these present a possible contamination to the weak lensing estimation.

In this thesis we model these intrinsic correlations using the halo model, assuming that both the sizes and luminosities of galaxies reflect the mass in the associated halo, assuming the observed galaxy properties correlate closely with the mass of the haloes and sub-haloes. Larger and more luminous galaxies live in more massive haloes, and even if the sub-halo population is largely independent of the halo mass, the sizes and luminosities of the largest sub-haloes will still be limited by the total halo mass. We use this simple model to predict what would be observed for a magnification estimator based on galaxy sizes and magnitudes, and how the intrinsic signal correlates with the true lensing convergence. Additionally, we include in our analysis the effects of cuts in the sample and we model the size-magnitude distribution and study how the cuts in the survey affect the intrinsic mean size and magnitude as well as the inferred convergence power spectrum when using galaxy sizes and magnitudes.

Studying these correlations is important both to improve our understanding of galaxy properties and because they are a potential systematic for weak lensing size magnification measurements. Our model assumes that the density field drives these intrinsic correlations and we also model the distribution of satellite galaxies. We calculate the possible contamination to measurements of lensing convergence power spectrum from galaxy sizes and luminosities, and show that the cross-correlation of intrinsic properties with convergence is potentially an important systematic. We also explore how these intrinsic correlations may affect surveys with different redshift depth. We find that, in this simple approach, intrinsic size and luminosities correlations cannot be neglected in order to estimate lensing convergence power spectrum for constraining cosmological parameters.
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Introduction

*Cosmologists are often in error, but never in doubt.*

Lev Landau

Cosmology can be recognised as the most ancient science ever studied on our planet. Indeed, its history is strongly tied to human history. It would be quite hard to say when the first human being started to think about where all the universe came from, but since then, huge steps have been made in understanding how our universe works. It has obviously been a long process spanning several thousand of years for several reasons. First of all, the development of science in general has had to wait until Galileo Galilei (1564-1640) when the scientific method has been established after many years of controversy. Secondly, we needed the appropriate theoretical framework to be developed and the suitable instruments to reach the desired accuracy and precision in cosmology. While the latter is strongly dependent on the technological capability we have available at a given historical time, the former has been continuously developed over the last centuries after Galilei’s work.

Although the first theory of gravity has been proposed by Sir Isaac Newton in his *Principia* published in 1687, we could state that modern cosmology was born when Albert Einstein (1879-1955) published his theory of gravity called General Relativity in 1915 [4] upon which the equations able to predict the evolution of our universe are based. A few years later, in 1929, Edwin Hubble came up with a landmark observation regarding the expansion of the universe [5] by means of Cepheids-calibrated measurements of galaxy distances.

Since then, we can recognise, among the other important discoveries, at least four milestone observations which contributed to the development of cosmology. Moving in chronological order, we start to mention the existence of a dark matter component in the universe as suggested in 1937 by Fritz Zwicky [6] who noticed that the luminous matter cannot account for the velocity dispersion of galaxies in clusters. The presence of dark matter has been confirmed in 1970 by Vera Rubin [7] who studied the rotation curve of spiral galaxies and later on by other independent probes such as the ones discussed below.
The second milestone represents presumably the most important discovery in cosmology so far and it has been the detection of the Cosmic Microwave Background (CMB) in 1964 by Arno Penzias and Robert Wilson [8]: this has brought a strong evidence in favour of the Big Bang model which tells that the early universe was quite hot and dense. The third milestone was in 1990 regards the discovery of small anisotropies (of order of $10^{-5}$) in the CMB temperature fluctuations angular power spectrum by the NASA satellite COBE (COsmic Background Explorer, [9]): these anisotropies are thought to be the seed of the perturbation which led to the formation of the structures we can see today in the universe. The measurements of CMB anisotropies has been further studied with increased precision also by another NASA mission called WMAP (Wilkinson Microwave Anisotropy Probe, [10]) as well as many ground and balloon based observations and more recently by the ESA satellite Planck [11].

The fourth milestone discovery to mention has been made in 1998 by Riess et al. (1998) [12], Schmidt et al. (1998) [13], Perlmutter et al. (1999) [14] regarding the evidence for an accelerated expansion of the universe and, particularly, for the presence in the universe of a component called dark energy which drives this acceleration. This detection has been achievable by means of the supernovae Type Ia which helps cosmologists to extend to higher redshift the analysis made by Hubble in 1929.

It is also remarkable to mention that, apart from the dark matter observation, each of the other three major discoveries mentioned above has been worth a Nobel Prize for Physics.

Alongside these milestones, other important observations are the measurement of the abundances of the light elements produced right after the Big Bang and important cosmological probes such as the spatial distribution of galaxies and weak gravitational lensing.

Last but not least, we must mention the possibly most important element of this list which is the enormous amount of data available nowadays that makes testing the cosmological model against observations uniquely scientifically powerful. Putting things together, we can see how, in the last century, we obtained a picture in which the universe, started being hot and dense, expands and it is filled mainly of dark energy and dark matter while the ordinary baryonic matter represents only around 5% and the cosmic structures we observe today are the results of the evolution of tiny perturbations originated in the early universe.

Cosmologists have planned for the next years several remarkable projects such as Euclid1, the Large Synoptic Survey Telescope (LSST2) and the Square Kilometer

1http://www.euclid-ec.org/
2http://www.lsst.org/lsst/
Array (SKA$^3$) which will increase further the quantity of data to study in order
to try to understand the nature of dark energy and dark matter, and the nature
and evolution of the cosmological perturbations by combining several cosmological
probes. In this respect, that this is the most exciting time of the human history to
be a cosmologist.

In this common effort of looking for even more precision we cannot obviously for-
get to be accurate. Accounting for systematics which can undermine both precision
and accuracy is a great challenge for these future experiments and it is fundamental
in order to correctly interpret data.

In this thesis we discuss in particular this problem in the context of weak gravi-
tational lensing.

Weak lensing has become a powerful tool to constrain the cosmological model
(see [3] for a review). The coherent distortion of galaxy images due to the gravita-
tional field of the large scale structures has been measured as correlation of galaxy
shapes, the so-called cosmic shear, and enhancement of number density of a back-
ground population (e.g. quasars, [15]) around foreground galaxies, an effect called
magnification bias. In order to obtain precise lensing measurements it is essential to
carefully model how our instruments acquire galaxy image data; on the other hand,
galaxy properties may be intrinsically correlated and this can introduce spurious
correlations which can bias the estimation of cosmological parameters from weak
lensing data resulting in a lack of accuracy.

Intrinsic correlations of galaxy ellipticities are called intrinsic alignments (IAs)
and have extensively been studied in literature both theoretically, developing mod-
els, as well as by means of N-body simulations, and observationally through direct
measurements (for a review see [16]).

Size and magnitude information comes for free from a weak lensing survey. In
order to use that information to probe the cosmological model then we need to
understand the potential impact of analogous correlations of sizes and magnitudes.
Recently, some authors [17, 18] pointed out the importance of adding size informa-
tion to ellipticities measurements in order to help to obtain better constraints on
cosmological parameters. In this thesis we discuss how these intrinsic correlations
can impact weak lensing analyses, focusing in particular on intrinsic correlations of
galaxy sizes and magnitudes.

In order to give to the reader the right framework in which this work is embedded
in the general field of cosmology, we start to discuss, in Chapter 1, the cosmological
background introducing the basic equations describing the evolution of the universe
assuming General Relativity and the fundamental observations supporting the cur-
rent cosmological model and, after that, we review the theory of the cosmological

$^3$https://www.skatelescope.org/
perturbations providing the statistical tools and physical motivations in the context of the current scenario including the theory of inflation.

In Chapter 2 we review the basics of weak gravitational lensing discussing theoretical and observational aspects as well as focusing on the impact of systematics.

We define the backbone of our models for intrinsic size and luminosity correlations in Chapter 3 where we discuss the essential elements of this formalism based on the halo model.

In Chapter 4 we discuss how it is possible to model correlations of intrinsic galaxy sizes and luminosities on realistic surveys, including modelling the effects of limiting cuts affecting cosmological data, before eventually concluding in Chapter 5.
Chapter 1

Cosmological background

Never underestimate the joy people derive from hearing something they already know.

Enrico Fermi

Cosmology is the application of the scientific method to the study of the universe as a whole.

In this chapter we review the essential elements of cosmology starting from the basic principles which form the basis of this science to the most important observational evidences supporting the current cosmological model which we refer as the ΛCDM, model indicating that an expanding universe containing cold dark matter in addition to ordinary baryonic matter and a cosmological constant acting as dark energy driving the acceleration expansion of the universe, and relativistic species such as radiation (CMB photons) and neutrinos. Furthermore, we review the formation of primordial fluctuations in the density field which the cosmic structures we observe today are from.

We refer to Dodelson’s book [19] and Mukhanov’s book [20] for the reader interested in further details about the topics discussed in this chapter.

1.1 Homogeneous and isotropic universe

An important issue in cosmology is given by the fact that this discipline has a severe restriction: there is only one universe accessible to our observations. Thus, we need to set up the necessary assumptions in order to use statistical tools. Basically, this means that in principle we cannot make any experiment in cosmology because we cannot make any statistical studies among many universes.

Nevertheless, it is possible to overcome this problem, introducing a fundamental assumption known as the Cosmological Principle, which states that on sufficiently
large scales (larger than around 100 Mpc$^3$) an observer co-moving with the matter distribution views the universe to be homogeneous and isotropic. By means of this principle, instead of averaging over an ensemble of different universes we can think that large enough regions are fair representations of the global properties of the universe and then we can perform an average over these regions.

Because of that, it is then extremely important to discuss to what extent the universe can be considered homogeneous and isotropic. The main problem is that we cannot observe homogeneity neither in the galaxy distribution nor in the CMB. This is for the simple reason that we observe past light cone rather than spatial surfaces. The good news is that we can test isotropy of observations but in order to link isotropy to homogeneity we must assume the Copernican Principle, which states that our position as observer is not a special one. Furthermore, our universe is not exactly isotropic but there are small anisotropies we can measure in the CMB temperature fluctuations (see Section 1.3.2 for more details). However, it is possible to show that if all the observers measure an almost isotropic CMB then, assuming the Copernican Principle, the universe is almost homogeneous and isotropic. This is exactly what the Cosmological Principle states.

In particular, regarding homogeneity, even though we cannot directly probe it, we can test for departures from it by means of observations. This, of course, would be an indirect evidence for homogeneity and there are several probes which can be used for this purpose. For the reader interested in more details, we refer to Maartens (2011) [21] where these arguments are carefully discussed.

Hence, assuming the Copernican Principle and given the almost-isotropic CMB and no signals for violation of homogeneity, we can safely assume the validity of the Cosmological Principle throughout this thesis. Of course, we do not observe an universe which is homogeneous and isotropic around us on small scales. Without anisotropies, the cosmic structures we observe today could not exist; therefore we should keep in mind that when referring to homogeneity and isotropy, we actually mean the background dynamical evolution of the universe (we will discuss this in Section 1.4).

1.2 Theoretical framework

Assuming the Cosmological Principle, we have a model to describe geometry of the space-time. Homogeneity and isotropy are related to invariant properties of basic geometric transformations of coordinates, in particular translations and rotations. This leads to Friedmann-Leimatre-Robertson-Walker (FLRW) metric described below. Furthermore, we need a set of equations of motion in a given space-time and

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$^1$1 Mpc = $3.0856 \times 10^{24}$ cm
also a field equation is required to determine the metric. In order to describe these two aspects of the cosmological model we use the formalism and the equations of Einstein’s General Relativity.

1.2.1 General Relativity as theory of gravity

The basic concepts of General Relativity required in cosmology are the metric, the geodesic and Einstein equations. Given any differentiable manifold we can always build locally a map between that manifold and the Minkowskian space $M^n$ in such a way that we can define a coordinate system. In particular, here we are interested in 4-dimensional space.

A metric transforms the coordinates of the chosen reference frame into space-time distances. Commonly it is indicated with the notation $g_{\mu\nu}$ and the relation between coordinates $x^\mu$ and the element of distance $ds$ is:

$$ds^2 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} dx^\mu dx^\nu ,$$

where the tensor indices $\mu$ and $\nu$ range from 0 to 3, with 0 indicating the time coordinate and the last three spatial coordinates. Also, we use the convention for which repeated indices are summed over and therefore in the following we will not write down the summation sign explicitly. The assumption of the Cosmological Principle allows us to use a particular metric describing an isotropic and homogeneous space-time, the FLRW metric:

$$ds^2 = c^2 dt^2 - a^2(t) \left[ d\chi^2 + f_K(\chi)^2 (d\theta^2 + \sin^2(\theta) d\phi^2) \right],$$

where we used the $(+,−,−,−)$ as convention for the signature of the metric, $c = 299 792 458 \text{ m s}^{-1}$ [22] is the value of the speed of light in vacuum, $t$ is the time coordinate, $\chi$ the radial co-moving coordinate, $\theta$ and $\phi$ are the azimuthal and polar angle of spherical coordinates, the function $a(t)$ is the dimensionless scale factor normalised to unity at the present epoch, $K$, which has the dimension of the inverse square of a length, is the Gaussian curvature of the space at the present time, and the transverse distance function $f_K(\chi)$ depends on $K$ as follows:

$$f_K(\chi) = \begin{cases} \sqrt{K} \sin(\sqrt{K} \chi) & K > 0 \\ \chi & K = 0 \\ \sqrt{-K} \sinh(\sqrt{-K} \chi) & K < 0 . \end{cases}$$

Note that in the rest-frame of a comoving observer, i.e. at rest with respect to the spatial coordinate system, we have $ds^2 = c^2 dt^2$ and for this reason $ds$ is also called
proper time. Given a time $t$, the physical distance $r(t)$ is linked to the comoving radial distance through:

$$r(t) = a(t) \chi.$$  \hfill (1.4)

Therefore, at any time $t$ we have the objects in the universe at certain coordinates and what we need then to describe the evolution of their reciprocal distances and velocities is just the scale factor. By means of simple transformation of coordinates $d\tau = dt/a(t)$ we can re-write the metric in 1.2 as follows (see [20]):

$$ds^2 = a^2(\tau)[c^2d\tau^2 - d\chi^2 - f_K(\chi)^2(d\theta^2 + \sin^2(\theta)d\phi^2)].$$  \hfill (1.5)

This transformation is called a conformal transformation and the coordinate $\tau$ is the conformal time; this form of the metric is quite useful in order to describe horizons and singularities.

A geodesic is the generalisation of the idea of straight line for an arbitrary space-time. The concept is quite simple: given a path of the space-time, it is possible to parametrise it as $x^\mu(u)$ where $u$ is a parameter which monotonically increases along the path called an affine parameter. A vector tangent to the path is then defined as $dx^\mu/du$.

We now need to spend few words on the concept of covariant derivative. This is just the geometrical tool which allows us to use differential operators when the curvature of the space-time is taken into account. Since the derivative is a differentiation of tensors, we need a tool to parallel transport tensor appropriately along the path. It is clear that meaningful parallel transports can only be local: parallel transports of tensors on large scales end up with different results depending on the chosen path.

This can be seen when we think about curved space we know very well: our planet surface. In this case we have a positive curvature and if we start with a vector tangent to Earth’s surface at some point and we transport it tangentially along the surface on large distances we will obtain a vector pointing in a different direction. Keeping this in mind we can see the power of a covariant derivative. When the covariant derivative of a tensor vanishes that means that tensor is constant over that path of the space-time in the sense of being parallel transported. Then we can define a geodesic as a path for which the covariant derivative of a tangent vector vanishes.

The covariant derivative along a particular direction $\nabla_\beta$ of a vector $V^\alpha$ is defined as follows:

$$\nabla_\beta V^\alpha = \frac{dV^\alpha}{dx^\beta} + \Gamma^\alpha_{\mu\lambda} V^\lambda,$$  \hfill (1.6)

where the symbols $\Gamma^\mu_{\alpha\beta}$ are the Christoffel symbols and account for the fact that the
metric can itself depend on the coordinates:

\[ \Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left( \frac{\partial g_{\alpha\nu}}{\partial x^\beta} + \frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu} \right). \]  

(1.7)

Note that in this thesis we will often use also a different notation for the covariant derivative using a semi-colon, such that \( V^\alpha_{;\beta} = \nabla_\beta V^\alpha \). Given the covariant derivative we can also define the Riemann tensor which is a more fundamental quantity to describe the geometry of the space-time. Imagine we want to transport a vector \( V^\nu \) around a closed path; if the path is a geodesic then the vector is parallel transported and will come back to its starting point exactly as it started as the covariant derivative vanishes. Otherwise, it will come back different. This difference is encoded in the definition of Riemann tensor:

\[ V^\nu_{;\rho\sigma} - V^\nu_{;\sigma\rho} = V^\beta R^\beta_{\nu\sigma\rho}. \]  

(1.8)

Therefore the Riemann tensor tells us how much a tensor changes relative to what it would have been if it had been parallel transported and it is directly related to the curvature of the space-time.

We now define the following operator:

\[ \frac{D}{du} = \frac{dx^\beta}{du} \nabla_\beta, \]  

(1.9)

and therefore, from the definition of geodesic given above we have:

\[ \frac{D}{du} \frac{dx^\mu}{du} = 0, \]  

(1.10)

which is called geodesic equation. This can be re-written also as:

\[ \frac{d^2 x^\mu}{du^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{du} \frac{dx^\beta}{du} = 0, \]  

(1.11)

where we can also see why the parameter \( u \) is called an affine parameter as a new parametrisation \( u \rightarrow \bar{u} \) preserve the form of the geodesic equation if, and only if, it is an affine transformation \( \bar{u} = bu + d \) (where \( b \) and \( d \) are constants).

So far we just introduced some elements of geometry. General Relativity comes into play effectively when we consider that a gravitational field can be actually described locally by a metric. Also, the universe can be empty or be full of different types of matter. How these components influence the metric, hence the gravitational field, and vice-versa is described by Einstein equations:

\[ G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} + g_{\mu\nu} \Lambda. \]  

(1.12)
In the equation above the tensor $G_{\mu\nu}$ is called the Einstein tensor and it satisfies $G_{\mu\nu} = 0$.

$R_{\mu\nu}$ is the Ricci tensor that depends on the derivatives of the metric and can be written in terms of the Christoffel symbols:

$$R_{\mu\nu} = \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\alpha,\nu} + \Gamma^\alpha_{\alpha\beta} \Gamma^\beta_{\mu\nu},$$  \hspace{1cm} (1.13)

where commas indicate derivatives with respect to $x$; the Ricci tensor is itself a contraction of the Riemann tensor with the metric:

$$R_{\mu\nu} = g^{\gamma\delta} R_{\mu\nu\gamma\delta}. \hspace{1cm} (1.14)$$

With $R \equiv g^{\mu\nu} R_{\mu\nu}$ we indicate the Ricci scalar; the Newtonian gravitational constant is indicated by $G = 6.6738(8) \times 10^{-11} \text{m}^3\text{kg}^{-1}\text{s}^{-2}$ [22]; $T_{\mu\nu}$ is the energy-momentum tensor containing information about the various components filling the universe defined as follows:

$$T^\mu_\nu = \partial_\nu q \frac{\partial L}{\partial (\partial_\mu q)} - g^{\mu\omega} L,$$ \hspace{1cm} (1.15)

where $L(q, \partial q/\partial x_\nu)$ is the Lagrangian density, a function characterising the system which depends on the variables $q$ determining the degrees of freedom of the system. There is also a conservation law involving the energy-momentum tensor: $T^\mu_\nu = 0$.

It is straightforward to see that $[G_{\mu\nu} - (8\pi G/c^4) T_{\mu\nu}]_{;\nu} = 0$. This means that Einstein equations can include a constant term (multiplied by the metric because $g_{\mu\nu} = 0$). For this reason we have $\Lambda$ in eq. (1.12); as we will see later, it is a constant of integration which can be interpreted as a cosmological constant driving the accelerated expansion of the universe.

In addition to the Cosmological Principle we will assume that the contents of the universe can be described as barotropic perfect fluid. This means that the fluid, being forced to move in the pressure ($p$), density ($\rho$) and temperature ($T$) space on a manifold described by the perfect gas law $pV = n k_B T$, can only change its state through transformations such as $p = p(\rho)$, where $n$ is the number density, and $k_B = 1.3806488(13) \times 10^{-23} \text{JK}^{-1}$ is the Boltzmann constant [22].

In this case the energy-momentum tensor is:

$$T_{\mu\nu} = (p + \rho c^2) v_\nu v_\mu - g_{\mu\nu} p \hspace{1cm} (1.16)$$

where $v^\nu = dx^\nu/ds$ is the fluid four-velocity, $\rho$ is the fluid density and the pressure $p$ is given by the equation of state:

$$p_w = w p_w c^2, \hspace{1cm} (1.17)$$
where $\rho = \sum_i \rho_{w,i}$ and $p = \sum_i p_{w,i}$. For instance, the equation of state parameter is $w = 0$ for non-relativistic matter, $w = 1/3$ for relativistic species such as radiation and neutrinos, $w = -1$ for dark energy such as a cosmological constant.

From the conservation law for the energy-momentum tensor $T_{\mu\nu} = 0$ and by means of eq. (1.17), we obtain:

$$\dot{\rho}_w + 3H(1 + w)\rho_w = 0,$$

where we indicate with $\rho_w$ the density of each component present in the universe.

It directly follows that:

$$\rho_w(t) = \rho_{w,0}a(t)^{-3(1+w)}.$$  

In particular we have:

$$\begin{cases} 
\rho_m(t) = \rho_{m,0}a(t)^{-3} & \text{matter}, \\
\rho_r(t) = \rho_{r,0}a(t)^{-4} & \text{radiation}, \\
\rho_\Lambda(t) = \rho_{\Lambda,0} & \text{cosmological constant}.
\end{cases}$$  

To summarise, Einstein equations essentially link the geometry of the space-time to the kind of content present in the universe. This means the curvature is determined by the density field and, on the other hand, that the motion of the particles is determined by the curvature itself.

Given these three concepts we can now write down the equations for the evolution of the universe and for the cosmological distances.

### 1.2.2 Redshift and distances in cosmology

#### Cosmological redshift

We can find the geodesic equation for a massless particle as a photon. In this case we have that the proper time vanishes, $ds^2 = 0$ and then we use an affine parameter to define our quantities. In order to do that we start by defining the momentum four-vector $p^\mu = dx^\mu/du$. The time component is the energy of the particle and the scalar product of the momentum four-vector of a massless particle vanishes: $g_{\mu\nu}p^\mu p^\nu = 0$. By means of this and using the FLRW metric, the zero-th component of the geodesic equation is:

$$\frac{dE}{dt} + \frac{\dot{a}}{a}E = 0,$$

where we simply indicated with $E = dp^0/du$ the energy, which is the first component of the momentum four-vector.
From 1.21 we obtain the following solution for a photon:

\[ E = \frac{2\pi \hbar c}{\lambda} \propto a^{-1} \]  

(1.22)

where \( \hbar = 1.054571726(47) \times 10^{-34} \) Js is the reduced Planck constant\(^2\) [22] and \( \lambda \) is the photon wavelength. The fact that the wavelength is proportional to the scale factor can be also seen directly from the FLRW metric in eq. (1.2) when considering, for simplicity, only radial propagation and that for photons \( ds^2 = 0 \).

This change of the photon wavelength happens during light propagation, between the emission and the observation of the event. The amount of change in photon wavelength due to the geometry of the space-time is called cosmological redshift and indicated with \( z \) and defined as follows:

\[ \frac{\lambda_{\text{obs}}}{\lambda_{\text{em}}} = \frac{a(t_{\text{obs}})}{a(t_{\text{em}})} = 1 + z. \]  

(1.23)

For our purposes in this thesis we define the present time \( t_0 \) as the time of observation and normalise to unity the scale factor that is \( a(t_0) = 1 \). Then the equation above is simply \( a = (1 + z)^{-1} \).

**The Hubble Expansion**

It is straightforward to see from the FLRW metric that given eq. (1.4) and taking its derivative we have:

\[ \frac{dr}{dt} = \frac{da}{dt} \chi = H(t)r(t) \]  

(1.24)

where from eq. (1.2) we have

\[ \chi(t) = c \int_0^t \frac{dt'}{a(t')} \]  

(1.25)

and we have introduced the Hubble parameter \( H(t) \):

\[ H(t) = \frac{\dot{a}}{a} \]  

(1.26)

where we indicate the derivative with respect the time coordinate by a dot. The Cosmological Principle then essentially tells us that there is no privileged point in a homogeneous and isotropic universe: all the observers see the universe expanding at the same rate given by \( H(t) \).

The Hubble parameter measured at the current time of observation \( t_0 \) is called Hubble constant and is indicated by \( H_0 \equiv H(t_0) \). Normally, since the Hubble constant appears almost everywhere in cosmological definitions, it is common to

\[^{2}\text{1eV} = 1.602176565(35) \times 10^{-19} \text{ J}\]
give uncertainties in this parameter as follows:

\[ H_0 = 100 \, h \, \text{km s}^{-1}\text{Mpc}, \tag{1.27} \]

where the parameter \( h \) indicates our level of knowledge about the Hubble constant.

At low redshift, the Hubble constant enters in the relation between redshift of a galaxy and its distance. This relation can be found given eq. (1.23) and by making a Taylor series expansion:

\[ z = \frac{a(t_{\text{obs}})}{a(t_{\text{em}})} - 1 \simeq \frac{a(t_{\text{obs}})}{a(t_{\text{obs}})[1 + (t_{\text{em}} - t_{\text{obs}})H_0]} - 1 \simeq (t_{\text{obs}} - t_{\text{em}})H_0, \tag{1.28} \]

and defining the distance \( D \) as \( D = c(t_{\text{obs}} - t_{\text{em}}) \):

\[ v_r = cz \simeq DH_0, \tag{1.29} \]

where \( v_r \) is called recession velocity of the galaxy.

We also must recall that the Hubble Law is valid on large scales. When observing structures on small scales, the FLRW metric is not valid any longer as a description of the universe. In this case the change in redshift is dominated by the peculiar velocity caused by inhomogeneities on small scales.

The relation given in eq. (1.29) is called Hubble Law and, as already mentioned, it was observed for the first time by Hubble in 1929 [5], although Hubble inferred a quite large value because of errors in distance estimates.

By means of the Hubble constant it is possible to define the Hubble time \( t_H \) that would be the age of the universe if all the galaxies were receding at same velocity they are today:

\[ t_H = \frac{1}{H_0} = 9.78 \times 10^9 \, h^{-1} \text{yr} = 3.09 \times 10^{17} \, h^{-1} \text{s}. \tag{1.30} \]

Of course, it is clear that the Hubble time cannot be the actual age of the universe because the receding velocities depend on the density content of the universe.

Given the Hubble time, we can define the Hubble distance \( D_H \) as the distance a photon would travel in that time:

\[ D_H = ct_H = \frac{c}{H_0} = 3000 \, h^{-1} \text{Mpc} = 9.26 \times 10^{25} \, h^{-1} \text{m}. \tag{1.31} \]

Given eq. (1.29), it seems we could obtain, in an expanding universe, recession velocities that exceed the value of the speed of light in vacuum, for a galaxy located at a large physical distance far from us, but this actually leads to two quite common misconceptions.
The first is the concept of superluminal expansion of the universe; this is a misconception because given the definition of parallel transport (see Section 1.2.1) we cannot define a relative velocity between that galaxy and us, as this sort of definition of velocity makes sense only locally. Of course, for measurements of $H_0$ and distances, eq. (1.29) gives us the recession velocities of galaxies but there is nothing in contradiction with General Relativity in obtaining velocities larger than the speed of light for the simple reason that they cannot be interpreted as relative velocities. To say it in other words, in the current concordance cosmological model, all galaxies with redshift greater than $z \sim 1.46$ are receding faster than the speed of light for several cosmological models. This can been seen combining eq. (1.24) and eq. (1.25), as shown in Figure 1 of Davis & Lineweaver (2000) [23] and in Figure 2 of Davis & Lineweaver (2013) [24]. However, this is not in contradiction at all with Relativity because the motion does not happen in any observer’s inertial frame. The second misconception is in considering the Hubble distance as a special distance. Were it as such, then we should observe an infinite redshift for galaxies at that distance. We further discuss this in section 1.2.2 where the concept of horizon is introduced.

Furthermore, on small scales the universe is obviously not homogeneous and isotropic. This simply means we can no longer describe the gravitational field by means of a FLRW metric. Therefore, the fact the we must use a different metric on small scales means that galaxies do not follow the Hubble Law on those scales. This can result in observing some galaxies approaching to us rather than run away. We refer the reader interested in the discussion of the misconception regarding recession velocities to Davis & Lineweaver (2013) [24] and reference in there; regarding the meaning of the concept of a local counterpart of the expanding universe we refer to Francis et al. (2013) [25] and reference in there.

**Distances in cosmology**

Perhaps obtaining distances of galaxies is the most important topic of cosmology. This is because once one has very good distance measurements, then it would be possible to test the cosmological model directly using the geometry of the universe. There are several definitions of distance depending on which observable is used.

We first consider a light beam moving radially along the line of sight. For such a beam we have the comoving radial distance from eq. (1.2):

$$\chi(z) = c \int_0^z \frac{dz'}{H(z')}.$$  \hspace{1cm} (1.32)

The comoving transverse distance is given by the transverse distance function $f_k(\chi)$ of the FLRW metric in eq. (1.3). The comoving transverse distance is nec-
essary in order to define the angular diameter distance $d_A$ which is the ratio of the physical size $\Delta R$ of an object to its angular size $\Delta \theta$:

$$d_A(z) = \frac{\Delta R}{\Delta \theta} = \frac{f_K[\chi(z)]}{1 + z}. \tag{1.33}$$

Another important quantity is the luminosity distance, defined as the square root of the ratio of the total luminosity $L$ to the total flux $F$ of an object:

$$d_L(z) = \sqrt{\frac{L}{4\pi F}} = (1 + z)f_K[\chi(z)] = (1 + z)^2d_A(z). \tag{1.34}$$

Angular diameter distances and luminosity distances are fundamental in several fields such as weak lensing and supernovae, and errors in calibrating distances can lead to wrong estimations of the cosmological parameters. We will return on these distances in Chapter 2.

**Horizons**

The principle of causality states that there is no observer who can detect the effect of a certain phenomenon before the cause generating it. If it were possible to send information faster than the speed of light, this principle would be violated. As we do not observe violations of causality we can say that no particle can travel faster than light. Given this and the fact that the speed of light as a finite value, the concept of horizon naturally arises.

If the universe has a finite age, then light can travel only a finite distance in that time and, consequently, the volume we can obtain information from is limited. The boundary of this volume is called the particle horizon. Therefore, in comoving units, by means of eq. (1.5):

$$\chi_p(\tau) = \tau - \tau_1 = \int_{\tau_1}^\tau \frac{c \, dt'}{a(t')} , \tag{1.35}$$

where $\tau_1$ and $t_1$ refer to the initial time for the universe. Hence, at a generic time $\tau_1$ information about events located at $\chi > \chi_p$ are not accessible to the observer sitting in $\chi = 0$.

There is another kind of horizon called event horizon. This is defined as the region where the signals sent at a certain time $\tau$ will never be received by an observer outside this region. Therefore, the event horizon is defined as follows:

$$\chi_e(\tau) = \tau_{\text{max}} - \tau = \int_{t_1}^{t_{\text{max}}} \frac{c \, dt'}{a(t')} , \tag{1.36}$$

where now $\tau_{\text{max}}$ and $t_{\text{max}}$ is the final time for the universe. For instance, if the
universe will expand forever, then $t_{\text{max}} = +\infty$.

The event horizons play an important role in understanding the phenomenological theory of inflation and the behaviour of an universe dominated by a cosmological constant.

### 1.2.3 Friedmann equations

Alexander Friedmann (1888-1925) derived for the first time the equations for the dynamical evolution of the universe in 1924 ([26]); by inserting the components of the FLRW metric of eq. (1.2) in the Einstein equations given in eq. (1.12), and assuming the energy-momentum tensor for a barotropic perfect fluid from eq. (1.16), we obtain:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{Kc^2}{a^2},$$  \hspace{1cm} (1.37)

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3\frac{p}{c^2}\right),$$  \hspace{1cm} (1.38)

where the dots indicate time derivatives.

It is possible to re-arrange the equations above by means of some definitions. The critical density at given epoch of the universe is defined as follows:

$$\rho_{\text{cr}}(t) = \frac{3H^2(t)}{8\pi G},$$  \hspace{1cm} (1.39)

and, at the present day, we have $\rho_{\text{cr},0} = 2.775\,366\,27 \times 10^{11}\,M_\odot\,\text{Mpc}^{-3}$ [22] where the solar mass is $M_\odot = 1.988\,5(2) \times 10^{30}\,\text{kg}$ [22]. The total density parameter $\Omega$ for an universe filled with different components $\rho_{w,i}$ can be written as:

$$\Omega(t) = \sum_i \frac{\rho_{w,i}(t)}{\rho_c(t)} = \sum_i \Omega_{w,i}(t).$$  \hspace{1cm} (1.40)

The critical density is called as such because we can re-write eq. (1.37) such that:

$$\frac{Kc^2}{a^2(t)} = H^2(t)(\Omega(t) - 1).$$  \hspace{1cm} (1.41)

If we have, for example, $\Omega > 1$ at a certain time $t$ then this is true for any time. The same reasoning is valid for other values assumed by $\Omega$. The critical density is clearly that value of the density which is crucial for determining the curvature of the universe. In particular, if the density of the universe is exactly equal to $\rho_{\text{cr}}$ the universe is flat ($K = 0$). Given these definitions, we can rewrite the first Friedmann
equation as a function of redshift:
\[ E(z) \equiv \frac{H(z)}{H_0} = \sqrt{\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_k(1+z)^2 + \Omega_{\Lambda,0}}. \]

where we introduced the expansion rate \( E(z) \) and \( \Omega_{r,0}, \Omega_{m,0}, \Omega_{\Lambda,0} \) indicate respectively the density parameters for radiation, matter and cosmological constant at the present time, and \( \Omega_k = 1 - \Omega \). In particular \( \Omega_{m,0} = \Omega_{b,0} + \Omega_{c,0} \) where \( \Omega_{b,0} \) is the baryon density parameter and \( \Omega_{\text{cdm},0} \) is the cold dark matter density parameter. Also, from eq. (1.42) clearly follows that \( \rho_{cr}(z) = \rho_{cr,0}E(z) \).

Through this new form of the first Friedmann equation we can see some interesting results. Dividing eq. (1.19) by \( \rho_{cr} \) and using eq. (1.42), we find the following equation for each component:
\[ \Omega(z) = \frac{\Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{\Lambda,0}}{E^2(z)}. \]

From the equation above is clear that:
\[ \lim_{z \to +\infty} \Omega(z) = 1, \]
that is, the geometry of the early universe is flat.

### 1.2.4 The Big Bang and the ΛCDM model

So far, we have described the equations governing the evolution of the universe, including information about what it contains. However this is not the end of the story, since there are other processes involving the various components which fill the universe. The current accepted model supported by observations is called ΛCDM model.

This is basically a model in which the dynamics and geometry of the universe are described, respectively, by FLRW and General Relativity and the universe is filled with components such as the relic microwave radiation of the CMB and matter \( \Omega_m \) in the form of ordinary baryonic matter \( \Omega_b \) and non-relativistic (cold) collisionless dark matter \( \Omega_{\text{cdm}} \), and an unknown component called dark energy in the form of cosmological constant \( \Omega_{\Lambda} \). The latest constraints on the ΛCDM model have been provided by the ESA Planck satellite [11], which reported the following values for
the cosmological parameters with 68% confidence limits:

\[
\begin{align*}
h &= 0.6751 \pm 0.0064 \\
\Omega_{b,0} h^2 &= 0.02226 \pm 0.00016 \\
\Omega_{cdm,0} h^2 &= 0.1193 \pm 0.0014 \\
\Omega_{m,0} &= 0.3121 \pm 0.0087 \\
\Omega_{\Lambda,0} &= 0.6879 \pm 0.0087.
\end{align*}
\]

(1.45)

The Big Bang is a singularity of the space-time from which the universe began to expand. A common misconception is that the Big Bang is an explosion into space; actually, an explosion requires some space that it can propagate into; more correctly, we should refer to the Big Bang as the theory that describes the expansion of the space. It is possible to show a condition for the existence of the Big Bang singularity by means of Friedmann equations. In eq. (1.38) if we have the following condition on the scale factor satisfied:

\[
\rho + 3 \frac{p}{c^2} > 0,
\]

then \( \dot{a}(t) < 0 \) and since \( a(t) \) is a positive function, that means there is a time \( t = t_{BB} \) when \( a(t_{BB}) = 0 \). This holds when the dominant component in the universe has \( w > -1/3 \), as can be seen from eq. (1.46). If a component with \( w < -1/3 \) is dominant then \( \ddot{a} > 0 \), the universe is accelerating; for instance, this is the case of a universe dominated by a cosmological constant, which has \( w = -1 \). In general, any component with \( w < -1/3 \) is called dark energy. Also, from eq. (1.43), we can see that the early universe \( (z \to +\infty) \) can be thought as being dominated by radiation, a component with an equation of state parameter \( (w = 1/3) \), which allows a Big Bang singularity. Because of the different scaling of the various density parameters, it is straightforward to see that after a radiation-dominated era, matter domination follows, eventually leading to the current era of dark energy domination in which we observe an accelerated expansion of the universe. The intermediate period of matter domination, however, does not necessarily occur; this depends on the value of the various density parameters. In our universe, the radiation-matter equality happened at \( z_{eq} = 3380 \), while the matter-dark energy equivalence was at \( z_{\Lambda} = 1.26 \).

We can now sketch a brief history of our universe given the considerations above. We do not have any theory for the first \( 10^{-43} \) s. This time is called Planck time and below this interval of time General Relativity cannot be applied and scientists are working hard in order to find a theory that is suitable on these temporal scales. Right after the Big Bang singularity, the universe experienced a period of accelerating expansion called cosmic inflation, which is thought to have occurred during the first \( 10^{-32} \) s. Currently, evidence supports an inflation driven by a single scalar field called
the inflaton. Additionally, during inflation the seeds of the density perturbations we can observe today were formed. We will see how this happened in Section 1.4.2.

At the end of this accelerated expansion, the inflaton decays into particles of the standard model of particle physics during a phase called reheating. As we will see later in Section 1.4.2, inflation is a phenomenological theory introduced in order to solve some problems occurring in the $\Lambda$CDM framework and there are still several points we do not fully understand about it.

During the first minutes, the first light elements, such as Helium-4 and trace amounts of Lithium, were produced in Big Bang Nucleosynthesis. Matter starts to be the dominant component only after around 70,000 years after the Big Bang.

When the age of the universe was around 300,000 years the cosmic microwave background formed (see more in Section 1.3.2) and with protons to form the first atoms of hydrogen, the universe finally became transparent setting photons free to stream and electrons combined (a process usually called recombination by astronomers). Since then, baryons started to fall into dark matter potential wells because of gravitational collapse in order to form the first structures of the universe such stars and galaxies. During an epoch called reionisation occurred after the recombination and the first billion year of the universe, ultraviolet light coming from these first hot stars ionised the atoms of hydrogen.

After that, more galaxies and several generation of stars formed and even planets such as our own. Finally, the cosmological constant become the dominant component of the universe around 5 billion years after the Big Bang, leading to a phase of accelerated expansion of the universe because of the equation of state of this component. Current data set the age of the universe to be around 13.7 billion years.

1.3 Main probes of the cosmological model

Important evidences for the Big Bang theory and the $\Lambda$CDM model are given by the observation of the abundances of light elements produced in the early universe during the Big Bang Nucleosynthesis (BBN), the Hubble diagram, the Cosmic Microwave Background (CMB) and dark matter measurements. Although these are not directly part of the work done in this thesis, for the sake of completeness, we briefly review these evidences in the next sections.

1.3.1 Big Bang Nucleosynthesis

Soon after the Big Bang, the universe was extremely hot and there was no chance to obtain any bound atomic nucleus because of the high-energy photons able to destroy.
them. When the temperature dropped below the typical binding energy necessary to keep together protons and neutrons, the so-called Big Bang Nucleosynthesis (BBN) took place. This happened in the first 20 minutes after the Big Bang. Following nuclear and atomic physics, it is possible to predict the abundances of these light elements. By means of these predictions it is possible to constrain some cosmological parameters such as the baryon density $\Omega_b$.

The abundances of deuterium (D), helium-3 ($^3\text{He}$), helium-4 ($^4\text{He}$) and lithium-7 ($^7\text{Li}$) have been widely used to test the Big Bang paradigm. In particular, the primordial abundance of $^4\text{He}$, according to the prediction of BBN, is around 25% [27] (this value depends on the baryon density and the effective number of neutrinos); for instance, a measurements of the primordial $^4\text{He}$-abundance can be done using observations of low metallicity H II regions in dwarf galaxies, e.g. see discussion in [28]. Clearly, a inferred value lower than 25% in stars would be against the Big Bang paradigm but, so far, there is no evidence against the Big Bang nucleosynthesis. We refer the reader interested in this topic to Burles et al. (1999) [29], and Cyburt et al. (2015) [30] where the information provided in this section comes from.

Lithium can be used as well to test the Big Bang theory and recently [31] there has been a further confirmation of the baryon to photon ratio $\eta_b$, in combination with CMB measurement from Planck [11].

The abundances of light elements are naturally explained in the context of an expanding universe which was hot and dense in its early stage and hence we have a quite strong observational evidence that theories other than the Big Bang framework are not able to fully explain.

1.3.2 Cosmic Microwave Background

At redshift $z \simeq 1100$, when the universe was around 300 000 years old, photons started to travel freely in the universe. Before that time, the universe was filled with the atomic nuclei formed during the BBN and photons were coupled with free electrons through electromagnetic interactions and this photon-baryon fluid was in thermal equilibrium. In this situation of equilibrium, photons are expected to have a black-body spectrum of the form:

$$I(\nu) = \frac{4\pi \hbar \nu^3 / c^2}{\exp[2\pi \hbar \nu / k_B T] - 1}$$ (1.47)

where $I(\nu)$ is the specific intensity of a gas with a Bose-Einstein statistics with vanishing chemical potential\(^3\); current constraints on the chemical potential are $\mu / T < 10^{-9}$ (from [32], where a dimensionless chemical potential is used). Also,

\(^3\)The chemical potential $\mu$ is related to the conservation of the number density of particles. A vanishing chemical potential means that the number of photons is conserved.
Cosmological Background

Figure 1.1 – Planck map from [11] showing CMB temperature anisotropies.

in this case we have that the density of photons scales with the temperature as $\rho_{\text{rad}} \propto T^4$; since $\rho_{\text{rad}} \propto (1 + z)^4$ then follows that:

$$T(z) = T_0 \left(1 + z\right), \quad (1.48)$$

where we normalised to the current observed mean CMB black-body temperature $T_0 = 2.72548 \pm 0.00057$ K [33].

While the universe continued to expand, the rate of interaction between photons and electrons dropped and around $z \simeq 1100$ the two components decoupled. At this point electrons combined with the nuclei forming the first atoms. The photons freely streaming out from this process formed the Cosmic Microwave Background. Here there is an important consideration to be made. Before decoupling the photon-baryon fluid was in thermal equilibrium. Afterwards this is no longer the case and therefore the concept of temperature of the photons, in principle, should not be valid. However, although it is an out-of-equilibrium situation, the expansion of the universe does not change the blackbody spectrum of the photons; this is due to the fact that the denominator in eq. (1.47) is proportional to the ration $\nu/T$ which has a constant scaling in redshift. Hence, we can continue to consider the effective temperature of the photons as the blackbody spectrum is expected to be observed now.

The black-body spectrum of the CMB as well as the existence of temperature fluctuations of order of $10^{-5}$ have been confirmed by the COBE satellite [9]. Recently the Planck satellite has measured with great precision these fluctuations and put tight constraints on the cosmological parameters (see Fig. 1.1).

The CMB anisotropies contain a lot of information about the universe. This can
be extracted by means of the analysis of the temperature fluctuations carried out through the study of their angular power spectrum (we will describe the general statistical tools used in cosmology such as the power spectrum in Section 1.4). Current data from Planck [11] fit extremely well the $\Lambda$CDM model (see Fig. 1.2).

The various peaks and troughs present in Fig. 1.2 are the imprint of the acoustic oscillations in the plasma before recombination. At that time, dark matter has already decoupled from the photon-baryon plasma and statistically distributed according to the density perturbation power spectrum originated at the time of inflation (as we will see in Section 1.4.2).

Imagine an overdensity: matter is attracted towards it because of gravity and, at the same time, the radiation pressure counter-balances the action of gravitational collapse producing the so-called acoustic oscillations. The largest scale a sound wave could propagate from inflation was the sound horizon; that means it is more likely to find to overdensities separated by that distance (analogously for underdensities, of course), which is the reason for the strength of the first peak around $\ell \sim 300$ which roughly corresponds to an angular scale $\theta \sim 0.5$ degree ($\theta \sim \pi/\ell$). The other peaks refer to the same mechanism and, in particular, on small scales (large $\ell$’s in Fig. 1.2) we cannot see any oscillations because on those scales the photons could free-stream erasing any perturbation.

The acoustic oscillations are not only visible in the CMB power spectrum. Indeed, it is possible find similar feature in the distribution of the matter we observe today as an excess of correlations. The oscillation we observe in the matter power
Figure 1.3 – The Hubble diagram from the Joint Light-curve Analysis (JLA) collaboration. Plot from [35]. On the vertical axis in the plot above is the distance modulus, which is the difference between the apparent and the absolute magnitude (parametrised as shown for the calibration procedure). The plot below shows the residual with respect to the $\Lambda$CDM prediction.

spectrum are called Baryonic Acoustic Oscillations (BAOs). The BAO feature is usually referred as a standard ruler because the sound horizon can be calculated as a function of the cosmic epoch; then, measuring the angular scale where the correlation excess is observed, we can infer the angular diameter distance hence directly probe the geometry of the universe assuming an angular diameter distance given by the FLRW metric (e.g. [34]).

Given this wealth of information, it is not surprising that the CMB is often considered the most important probe cosmologists have to constrain the cosmological model.

1.3.3 The Hubble diagram

The Hubble diagram is the relation between distance (or some function of the distance as we will see soon) and redshift. As already mentioned, the first attempt to do that was made by Edwin Hubble [5]. In order to obtain the Hubble diagram, two measurements are necessary: redshifts and distances. Redshift measurements can be mainly done in two ways: through spectroscopy, or through photometry. Spectroscopic redshift are more accurate, though more time-expensive as they require the spectrum of a galaxy. Photometric redshift are faster to obtain because they just need to measure the flux in different wavebands using different filters but this
kind of redshift are subject to larger error. However, improvements are expected to be reached with the next generation of surveys.

Measuring distances of galaxies is potentially the most powerful tool cosmologists have to constrain the cosmological models because it would be possible to directly measure the geometry and the dynamics of the universe. However, it represents a very challenging task due to the fact that we need to calibrate very accurately local measurements and, at the same time, to find a class of object whose intrinsic properties are very well-known. We could use galaxies where either their intrinsic luminosity (standard candles) or their intrinsic size (standard rulers) can be standardised. An example of standard ruler is given by the Baryonic Acoustic Oscillations (BAOs) discussed in the previous section.

Standard candles, based on the knowledge of the galaxy intrinsic luminosities have been more widely used so far in cosmology to infer distances. For instance, Hubble in his work used Cepheids, which are periodic pulsating stars with a well-known relation between luminosity and the pulsation interval. Observing the light curve it is possible to infer the pulsation interval, then the luminosity and the distance, once the flux is measured, according to eq. (1.34). Although Cepheids represent the baseline for distance measurements, in order to obtain the distance of far-away galaxies we need to use Supernovae Type Ia as standard candles. This kind of supernovae is a binary star system in which at least one of the two components is a white dwarf, and also double-degenerate models, where both the components are white dwarfs, can occur.

A white dwarf is the almost final stage of low-mass star (below $8 \, M_\odot$) in which nuclear reactions in its core do not occur and the stability is given by the pressure of a degenerate gas of fermions (mainly electrons). Because of this configuration, the mass of a white dwarf cannot exceed a certain maximum value; Chandrasekhar in 1931 [36] was the first to calculated this limit for an ideal white dwarf. The value currently used for this maximum value is around $M_{\text{chandra}} \sim 1.38 \, M_\odot$ (e.g. [37]). Therefore, in binary system, a white dwarf starts accumulating material from its companion and when it approaches a mass around $M \sim M_{\text{chandra}}$ it explodes in a Supernova Type Ia.

Because of the typical mass generating this kind of supernovae, it has been possible standardise and use them as standard candles. Extraordinary results have been obtained when the Hubble diagram has been extended to high-redshift by using Supernovae Type Ia: it has been found that the universe is undergoing a stage of accelerated expansion [12, 13, 14].

The most recent Hubble diagram has been compiled by Betoule et al. (2014) [35]; it is in Fig. 1.3 and further confirmed the discovery of an accelerating expansion of the universe supporting the evidence in favour of a cosmological constant acting as
dark energy driving this acceleration.

1.3.4 Dark matter

The discovery of the accelerated expansion of the universe has been awarded a well-deserved Nobel prize in 2011, as it represents a breakthrough in cosmology. However, there is another outstanding discovery, the existence of dark matter in the universe, that was made during last century but that unfortunately it has not been recognised in the same way yet.

The first hint for the presence of some matter we are not able to observe was considered by Fritz Zwicky in 1937 [6] who noticed that the velocities of galaxies in the Coma cluster could not be accounted by the luminous matter in the cluster and Zwicky proposed to assume the existence of a new kind of collisionless matter which interacts with ordinary baryonic matter only gravitationally. Furthermore, Zwicky suggested that the distortions produced by gravitational lensing could be used to infer the existence of the dark matter.

The first confirmation of the existence of such dark matter came from Vera C. Rubin and W. Kent Ford in 1970 [7] and in the following years, who measured the rotation curve of the spiral galaxy M31 and found that on large distances far from the centre the radial velocities do not follow a keplerian curve as planets in our solar system do, but instead maintain a constant value (see Fig. 1.4).

Other important evidence of the existence of the dark matter come from the CMB [11], gravitational lensing in galaxy clusters (e.g. the Bullet Cluster, [38], and galaxy clustering and baryonic acoustic oscillations (e.g SDSS data, [39], [34]).

Current cosmological probes support the existence of a cold dark matter. This means the dark matter was non-relativistic during the first stages of the history of the universe, decoupling from the baryon-photon fluid. Because of this, dark matter structures could have grown and, after recombination, baryons collapsed in these structures forming galaxies, moving from small structures to large structures through merging. This framework is usually referred as the hierarchical formation of structures.

Current efforts in research are towards a detection of dark matter particle. Possible candidates are, for example, represented by Weakly Interacting Massive Particles (WIMPs) predicted by supersymmetry theories (e.g. see some detection attempts in [40] and [41]) and sterile neutrinos [42]. Both these ideas would represent possible exciting extensions of the standard model of particle physics and would lead to new understanding of how the universe works, were they detected.
1.4 Formation of structures

So far, we have discussed the properties of a homogeneous and isotropic universe, at least on large scales. However, the universe we observe now is not homogeneous and isotropic on small scales; additionally, we have seen we have from some evidence, such as the CMB data, that the universe contains small anisotropies already at early times. Therefore, it is natural to ask how these small fluctuations arise and evolve in the history of the universe.

In this section we will briefly review the mechanism of cosmic inflation which naturally allows a framework for the origin of primordial fluctuations as well as helping to solve some generic problems associated with the Big Bang model. Then we show how these perturbations evolve in the linear regime.

Fluctuations in the density field contain a wealth of information. Correlations of these fluctuations tell us a lot about the cosmological model and in this section we also review the statistical tools we need to study these correlations.

1.4.1 Statistics of random fields

A random field $G(x)$ is a stochastic process, that is, it is a collection of random variables characterised by a probability density function (PDF) $p[G(x)]$ which describes the relative likelihood for a given realisation of the field $\hat{G}(x)$. In particular, the expectation value of any function of the random field is just an integration over all the possible realisation of the field (at each space point) weighted by the PDF:

$$\langle F[G(x)] \rangle = \int D[\hat{G}(x)] p[\hat{G}(x)] F[\hat{G}(x)] ,$$

(1.49)
Common functions calculated to describe a random field are the $n$-point correlation functions $F_n[G(x)] = \langle G(x_1) \ldots G(x_n) \rangle$. Since we deal with a homogeneous and isotropic universe, we recall the properties of a homogenous and isotropic random field. A random field can be homogeneous if the PDF cannot statistically be distinguished after an arbitrary translation vector: $p[G(x_1) \ldots G(x_n)] = p[G(x_1 + r) \ldots G(x_n + r)]$. It is called isotropic if it has the same statistic properties as the random field after an arbitrary rotation. For instance, in this case, a 2-point correlation function (that we indicate by $\xi$) depends only on the absolute value of the difference vector between two points:

$$
\xi(r) = \langle G(x)G(x + r) \rangle, \quad (1.50)
$$

where $r = |r|$. The Fourier transform of the 2-point correlation function is called power spectrum $P(k)$:

$$
\langle G(k)G(k') \rangle = 2\pi^3 \delta_D(k + k') P(k) \quad (1.51)
$$

where $k = 2\pi/r$ is the modulus of the wave-vector associated with a particular scale and $G(k)$ represents the Fourier transform of the random field:

$$
G(k) = \frac{1}{(2\pi)^3} \int d^3r \exp(-i k \cdot r)G(r), \quad (1.52)
$$

and

$$
G(k) = |G(k)| \exp(i\theta(k)) = \text{Re}[G(k)] + i \cdot \text{Im}[G(k)]. \quad (1.53)
$$

The 2-point correlation function and the power spectrum are related each other as follows when dealing with a homogeneous and isotropic field:

$$
\xi(r) = \frac{1}{2\pi^2} \int_0^{+\infty} dk k^2 j_0(kr) P(k), \quad (1.54)
$$

where $j_0(y) = \sin(y)/y$ is the zero-th order spherical Bessel function. In particular, the variance of the field $\sigma^2$ is defined as follows:

$$
\sigma^2 = \xi(0) = \frac{V}{2\pi^2} \int_0^{+\infty} dk k^2 P(k), \quad (1.55)
$$

where $V$ is a normalising volume. It is also useful to introduce a dimensionless definition of the power spectrum:

$$
\Delta^2(k) = \frac{V k^3}{2\pi^2} P(k), \quad (1.56)
$$
Gaussian random fields are characterised by the property that the probability distribution of any linear combination of the random field is Gaussian. A Gaussian random field has a probability distribution as follows:

\[ p_{Gauss}[G] = (\det K)^{-1/2} \exp \left( -\frac{1}{2} \int d^3x_1 d^3x_2 G(x_1) K^{-1}(x_1, x_2) G(x_2) \right), \]  

(1.57)

where the \( K(x_1, x_2) \) represents an invertible, symmetric operator and \( \xi(x_1, x_2) = K(x_1, x_2) \). Additionally, for a Gaussian random field we have that the real and imaginary parts of \( G(k) \), as given in eq. (1.53), are mutually independent and have Gaussian distribution. In this case \(|G(k)|\) has a Rayleigh distribution\(^4\) and phases \( \theta_k \) are randomly distributed.

Another property we need to mention of random fields is the ergodicity. We have only one universe and dealing with ergodic random fields is crucial as ergodicity implies that we can obtain information about a distribution studying a single infinite realisation of the random field. The Ergodic theorem states that the spatial average is equal to the ensemble average. If such a property holds then spatial averages will be equal to expectations over an ensemble of universes to which ours belongs. It can be proved that a Gaussian random density fluctuation field is ergodic if and only if the power spectrum \( P(k) \) is continuous \([43]\).

Gaussian fields have a key feature: they can entirely described by means of the 2-point correlation function. This is a quite important property as this means that the knowledge of the power spectrum is all what we need to study a Gaussian field. In realistic case, such as for the density field in the universe, gravitational instability inevitably leads to non-Gaussianity on small scales. In this case, we would need to measure the 3-point correlation function, also called bispectrum \( B(k) \), or higher moments to observe departures from Gaussianity. As we shall see, the simplest mechanism of cosmic inflation predicts Gaussian primordial fluctuations (see Section 1.4.2); thus, any measurement of non-Gaussianity can be useful to constrain the right model of inflation.

### 1.4.2 Cosmic inflation

The Big Bang model is quite a good theory able to reproduce observations. However, there are some problems with this model that, at the present, can only be solved simultaneously introducing a mechanism in the early universe. This mechanism is called cosmic inflation \([44, 45, 46, 47]\). We should point out that the inflationary hypothesis is not a modification of the Big Bang model rather it is a complementary theory.

\( ^4 \) A random variable \( x \) is distributed according to a Rayleigh distribution if its PDF is \( p(x) = (x/s^2) \exp[-x^2/(2s^2)] \).
Problems in the standard Big Bang model

There are at least three problems recurring in the Big Bang model that we want to discuss here because they can be fixed by means of cosmic inflation: the horizon, the flatness and the origin of primordial fluctuations problems.

The horizon problem concerns the observations of large-scale correlations in the CMB power spectrum. This is a problem because at the time of recombination the particle horizon, eq. (1.35), was much smaller than the scale where signals of correlation are measured. That means distant regions were not in causal contact and thus should not show the same statistical properties. We suspect they have been casually connected in the past and inflation is a mechanism which can provide that.

According to eq. (1.41), the sign of the density parameter $\Omega$ remains always the same throughout the history of the universe. As shown in [20] Section 5.2, the fact that we now observe a flat universe [11] means that the value of $\Omega$ must be initially extremely close to unity leading to a fine-tuning problem. Interestingly, again in [20] it is shown how this flatness problem could be related to the initial condition of the Hubble flow which is directly connected to the density parameter through the Hubble parameter.

Observations of the CMB show that fluctuations must have been at least of order $\delta T/T \sim 10^{-5}$ on some scales. These fluctuations are the seeds for the cosmic structures we observe today. Hence, understanding the origin of these primordial fluctuations is fundamental and the inflation provides a natural way to explain them.

A solution called inflation

Inflation was primarily introduced to solve the flatness and horizon problems. The mechanism for inflation requires an epoch when the universe underwent a phase of accelerated expansion in the general picture of the Big Bang and FLRW model. However, the primordial fluctuation problem requires a more deeper insight into the mechanism of inflation from a quantum mechanical point of view in order to reproduce the CMB temperature fluctuations.

The flatness and horizon problems can be overcome assuming an era of accelerated expansion and here we follow the reasoning discussed in [20], Section 5.1 to show this postulate is able to solve these problems. The homogenous and isotropic patch at the present time $t_0$ has size $\ell_0 \sim ct_0$. At an earlier time $t_i$ this patch was smaller by a factor given by the ratio of the scale factors $a_i/a_0$ that means $\ell_i \sim ct_0 a_i/a_0$ We can compare the size of the homogenous patch of the universe generated at $t = t_i$ with the causal connected region of the universe at that time $\ell_c \sim ct_i$. Assuming the scale factor growing as some power of time, we can make
the approximation $\dot{a} \sim a/t$, and we find:

$$\frac{\ell_i}{\ell_c} \sim \frac{\dot{a}_i}{\dot{a}_0}$$

(1.58)

Similarly, for the density parameter we have from eq. (1.41):

$$\Omega_i - 1 = (\Omega_0 - 1) \left( \frac{a_0 H_0}{a_i H_i} \right)^2 = (\Omega_0 - 1) \frac{\dot{a}_0}{\dot{a}_i}.$$ (1.59)

We can see that in an always decelerating universe $\dot{a}_i/\dot{a}_0 \gg 1$ the homogeneous patch $\ell_i$ is always larger than the causal region $\ell_c$ and for the same reason we have that $\Omega_i - 1 < 10^{-56}$ (see [20]) revealing a fine-tuning problem for the initial conditions. A period of accelerated expansion solves these problems because we can obtain an era where $\dot{a}_i/\dot{a}_0 < 1$. In such a way it is possible to have a homogeneous observable universe from a single connected region and, rewriting eq. (1.59) isolating $\Omega_0$, we obtain the prediction from inflation that $\Omega_0 = 1$ regardless the initial conditions.

**Dynamics of inflation**

In this section we see how it is possible to build a model of inflation able to produce a period of accelerated expansion. As already discussed in Section 1.2.4, in order to obtain an acceleration, we need a component with negative pressure with equation of state parameter $w < -1/3$. The simplest way to achieve this, also corresponding to the easiest inflationary model, is by means of a scalar field $\phi$, usually called inflaton, characterised by the following density and pressure:

$$\rho = \frac{1}{2} \dot{\phi}^2 + V(\phi),$$

$$p = \frac{1}{2} \dot{\phi}^2 - V(\phi).$$ (1.60)

From the equation above we can calculate the equation of motion for the scalar field. We can obtain an equation of state like $p \simeq -\rho$ when assuming that $|\dot{\phi}| \ll V$ and that $|\ddot{\phi}| \ll V'$, where the prime indicates a derivate with respect to $\phi$. These two conditions are called the slow-roll approximation and allow inflaton to drive the accelerated expansion with a scale factor which grows exponentially during the inflationary phase (for more details see again [20]). Slow-roll inflation can be parametrised by means of two parameters linked to derivatives of the inflaton potential (see [48]):

$$\epsilon = \frac{M_P^2}{2} \left( \frac{V'}{V} \right)^2,$$

$$\eta = \frac{M_P^2}{2} \left( \frac{V''}{V} \right).$$ (1.61)
These slow-roll parameters can be related to two important predictions of cosmic inflation about observable quantities. In fact, inflation predicts the power spectrum of primordial (scalar) fluctuations $P_s(k) \propto k^{n_s-1}$ with spectral index given by

$$n_s = 1 - 6\epsilon + 2\eta.$$  \hfill (1.62)

Current constraints on $n_s$ are given by Planck with 68% confidence level: $n_s = 0.968 \pm 0.006$ [11].

Additionally, inflation predicts the existence of primordial gravitational waves (tensor fluctuations) with power spectrum $P_t(k) \propto k^{-2\epsilon}$. Constraints on these gravitational waves are usually given in terms of the tensor to scalar ratio $r = P_t/P_s = -16\epsilon$. A combined analysis of Planck satellite and BICEP data provides an upper limit for this ratio: $r < 0.12$ [49].

A full discussion of how primordial fluctuations from inflation is beyond the scope of this dissertation. However, for the sake of completeness we must at least discuss heuristically how these perturbations formed (we refer the interested reader to [20] for a complete treatment). Primordial fluctuations arise from vacuum state quantum mechanical fluctuations of the inflaton field and are subsequently stretched during the inflation stage and imprinted on super-horizon scales because of the accelerated expansion. When inflation ended, these perturbations can re-enter the horizon when the universe continues in its evolution. Generally speaking, the simplest inflationary model predicts Gaussian primordial fluctuations but there are other theories leaving room for non-Gaussianity. These non-Gaussian features are encoded in the 3-point statistics of the primordial fluctuations such as the primordial bispectrum (e.g. Pettinari’s PhD thesis [50] and reference there in).

Therefore, inflation is a phenomenological theory which let us to avoid recurring a fine-tuning of our cosmological model and set the initial conditions for the formation of the cosmic structures we observe today.

### 1.4.3 Elements of perturbation theory

The tiny primordial perturbations that originated during inflation grew by gravitational instability. The linear theory which describes the growth of these perturbations in an expanding universe must be described by General Relativity, in order to account for relativistic fluid and perturbation modes that have characteristic scales comparable to the size of the horizon. A complete treatment of perturbation theory is beyond the scope of this dissertation but we provide the essential results relevant for us.

In particular, we limit our discussion to the case of a cold dark matter dominated universe when most of the structures are thought to be formed, $z \ll z_{eq}$ and
we restrict our discussion to subhorizon scales. For a \( \Lambda \)CDM model with dark energy equation of state \( w = -1 \), dark energy perturbations are negligible since they are of order of \( (1 + w)\Phi \) (e.g. \[51\]). Therefore we continue to consider cold dark matter perturbations even for \( z \leq 1 \) when the dark energy density starts to become dominant.

In order to continue we need to introduce perturbations in Einstein equations which means perturbing the metric and the energy-density tensor.

### Scalar perturbations and Newtonian gauge

The FLRW metric can be thought as our background metric and small perturbations can be introduced as follows:

\[
\text{ds}^2 = [g^{(0)}_{\mu\nu} + \delta g_{\mu\nu}]dx^\mu dx^\nu. \tag{1.63}
\]

The background metric is simply:

\[
g^{(0)}_{\mu\nu}dx^\mu dx^\nu = a^2(\tau)(d\tau^2 + \gamma_{ij}dx^i dx^j), \tag{1.64}
\]

where \( \gamma_{ij} \) represents the spatial part of the FLRW metric.

The homogeneous and isotropic background allows us to decompose the metric perturbations \( \delta g_{\mu\nu} \) in three kinds: scalar, vector, tensor. This is made possible because of the invariant properties with respect to the group of translations and rotations (see \[52\] and \[20\]).

We mainly discuss here scalar perturbations. This is because vector perturbations decay quickly and tensor perturbations corresponds to gravitational waves which are very important, but not relevant for our purposes. Again, you can find in Bertschinger 2001 \[52\] and Mukhanov’s book \[20\] much more details about the evolutions of these perturbations.

For scalar perturbations, the metric takes the following form:

\[
\text{ds}^2 = a^2(\tau) \left[ \left( 1 + \frac{2\Phi}{c^2} \right) c^2 d\tau^2 + 2B_i dx^i d\tau - \left( 1 - \frac{2\Psi}{c^2} \right) \gamma_{ij} - 2E_{ij} \right] dx^i dx^j, \tag{1.65}
\]

where the comma indicates the derivative with respect to a component of the coordinates, e.g \( B_i = \partial B / \partial x^i \) and \( B \) and \( E \) are some scalar functions.

The freedom to choose an arbitrary reference frame in General Relativity is usually called gauge invariance. We can always reduce the perturbed metric in eq. (1.65) to a new form more suitable for our purposes given this freedom for the choice of the coordinates. By means of an appropriate transformation of coordinates, it is possible to show (see Chapter 7 in \[20\]) that we can write the perturbed metric for
scalar perturbation as follows:

\[ ds^2 = a^2(\tau) \left[ \left( 1 + \frac{2\Phi}{c^2} \right) c^2 d\tau^2 - \left( 1 - \frac{2\Psi}{c^2} \right) \gamma_{ij} dx^i dx^j \right], \tag{1.66} \]

or analogously using the coordinate time \( t \) by means of the transformation \( d\tau = dt/a \).

The metric as in eq. (1.66) is written in the so-called Newtonian gauge. We will use this gauge choice for our purposes in the rest of this dissertation.

**Linear evolution**

We now describe the derivation of the differential equation determining the evolution of a density perturbation \( \delta \) defined as:

\[ \rho(x,t) = \bar{\rho}(t)[1 + \delta(x,t)], \tag{1.67} \]

where \( x \) represent a set of comoving coordinates related to the physical coordinates \( r \) by \( x = a^{-1}r \), \( \rho(x,t) \) is the density field and \( \bar{\rho}(t) \) is the mean density of the universe at a given epoch.

Non-linear structures develop from small initial perturbations through gravitational instability. Basically, matter falls into high-density regions of the universe and the amplitude of the primordial fluctuations is amplified. As already stated, a complete treatment requires a complete relativistic perturbation theory. However we are just interested to describe the evolution of density perturbations of non-relativistic, pressureless (equation of state parameter \( w = 0 \)), collisionless dark matter on scales below the horizon. For these reason we assume a background FLRW universe where inhomogeneities are described by the metric in eq. (1.66) where we also assume General Relativity is the theory of gravity setting \( \Phi = \Psi \). From these assumptions and from the conservation law for the energy-density tensor \( T^\mu_\nu \) and the 00 component of Einstein equations, eq. (1.12), we obtain the following set of equations, expanding all the quantities involved up to the first order:

\[
\begin{align*}
\frac{\partial \delta}{\partial t} + \nabla \cdot \mathbf{v} &= -\nabla \cdot (\delta \mathbf{v}) \\
\frac{\partial \mathbf{v}}{\partial t} + aH \mathbf{v} + \nabla \Phi &= -(\mathbf{v} \cdot \nabla) \mathbf{v} \\
\nabla^2 \Phi &= 4\pi G a \bar{\rho}(t) \delta = \frac{3}{2} H^2(t) a^2(t) \Omega_m(t) \delta,
\end{align*}
\tag{1.68}
\]

where the last equation is called Poisson’s equation and \( \mathbf{v} \) indicates the velocity perturbation to the Hubble flow and it is normally called peculiar velocity.

The system of equations above can be linearised by maintaining only first-order terms and combining those linearised equations, we obtain the following second-
order differential equation for the density perturbations:

\[
\frac{\partial^2 \delta}{\partial t^2} + 2aH \frac{\partial \delta}{\partial t} - \frac{3}{2} H^2 a^2 \Omega_m \delta = 0.
\]

(1.69)

Here we dropped the time dependence in order to keep the notation clear. The coefficients of eq. (1.69) are spatially homogeneous. This allows us to write a solution as:

\[
\delta(x, t) = \delta_+ (x) D_+(t) + \delta_- (x) D_-(t).
\]

(1.70)

The solution \( D_- \) decays with time while \( D_+ \) grows. For a matter dominated universe we have, for instance, \( D_+ \propto a \) and \( D_- \propto a^{-3/2} \).

Neglecting thus \( D_- \) and identifying \( D \equiv D_+ \) we can re-write eq. (1.69) as follows in function of the redshift:

\[
\frac{d^2 D}{dz^2} + 2H(z) \frac{dD}{dz} - \frac{3}{2} \Omega_{m,0} H_0^2 (1 + z)^3 D = 0,
\]

(1.71)

where the function \( D(z) \) is called growth factor and is usually normalised such that it is equal to one at the present epoch.

A general solution of eq. (1.71) has been provided for the first time by Heath (1977) [53]:

\[
D(z) = \frac{H(z)}{H_0} \int_z^{+\infty} \frac{dz'}{H^3(z')} \left[ \int_0^{+\infty} \frac{dz''(1 + z'')}{H^3(z'')} \right]^{-1},
\]

(1.72)

and see also Lahav et al. (1991) [54], Carroll, Press & Turner (1992) [55] and Hamilton (2001) [56] for a similar discussion and for approximated formulation of \( D(z) \).

**Transfer function**

In the previous section we mainly discussed the growth of structures during a matter dominated era. However, before that, we had in the history of the universe an epoch of radiation domination. During this epoch, perturbations inside the horizon could not grow tangibly, as discussed and detailed in Meszaros (1974) [57]. In practice we have that, using the dimensionless definition of the power spectrum, \( \Delta^2(k) \propto \text{constant} \). This basically means \( P(k) \propto k^{-3} \). As the horizon grows and we move to a matter dominated era we can find, imprinted in the peak of the function \( P(k) \), this smooth transition from the primordial shape with \( n = n_s \sim 0.96 \) to \( n = -3 \). All this information is encoded in a function called transfer function \( T(k) \).

In general the transfer function describes the evolution of perturbations as they re-enter the horizon after inflation. The function \( T(k) \) can be very complicated as it
depends on all the possible components found in the universe, such as dark matter, baryons and neutrinos. Several fit functions have been provided in literature for the transfer function (see Bardeen et al. (1986) [58] and Eisenstein & Hu (1998) [59]). By definition, the transfer function is the quantity by which, at a certain \( k \), a linear fluctuation is enhanced or suppressed with respect to a perturbation on very large scale:

\[
T(k) = \frac{\delta(k, z = 0)\delta(0, z = +\infty)}{\delta(k, z = +\infty)\delta(0, z = 0)},
\]

and it is clear that \( T(k) \to 1 \) when \( k \to 0 \).

Given growth factor and transfer function we can now write a complete equation for the power spectrum of density fluctuations accounting for both growth and all the physical phenomena occurring in the history of the universe:

\[
P(k, z) = D^2(z)T^2(k)k^{n_s}.
\]

**Spherical collapse and non-linear evolution**

As the universe evolves, perturbations grow and the evolution becomes non-linear. Small scale evolution can no longer be described by the linearised equation used above.

A first approach we can use in order to try to describe analytically the formation of non-linear structures is considering a spherical overdensity (\( \Omega > 1 \)) in a flat (\( \Omega = 1 \)) FLRW background. In a matter dominated model, we have:

\[
R = \frac{\Omega_{m,0}}{2(\Omega_{m,0} - 1)}(1 - \cos \theta),
\]

\[
t = \frac{\Omega_{m,0}}{2H_0(\Omega_{m,0} - 1)^{3/2}}(\theta - \cos \theta).
\]

for the size of the overdensity and the time. In this simplified model, the spherical perturbation reaches a maximum size at a time \( t_{\text{max}} \) corresponding to \( \theta = \pi \) and eventually collapses at a time \( t = 2t_{\text{max}} \) corresponding to \( \theta = 2\pi \). Assuming the collapsing matter has not any internal pressure, it seems the collapse will end when an infinite density is reached. Although, in reality, the collapse will stop when the virial theorem (see Binney & Tremaine’s Galactic Dynamics [60] ) condition is satisfied:

\[
E_k = -\frac{1}{2}E_p,
\]

where \( E_k \) and \( E_p \) are, respectively, the kinetic energy and the potential energy of the systems. This condition produces the collapse when \( \theta = 3\pi/2 \) corresponding to \( t_{\text{vir}} \approx 1.81t_{\text{max}} \).

For our purposes, in the following we will simply assume \( t = 2t_{\text{max}} \) as the collapse...
time and thus expanding eq. (1.75) up to $O(\theta^5)$, we can find that, at the collapse, the density contrast is $\delta_c \simeq 1.686$. Therefore the linear value of the threshold $\delta_c$ for a perturbation, indicates that it has collapsed. This value is widely used when trying to build an analytical model for describing the evolution of perturbations and the statistics of bound structures as we will see in Section 1.4.4. For completeness we should make clear that the actual value of the threshold is $\delta_c \simeq 178$ for a spherical collapse model. Of course, this model has limitations. For instance, it does not account for anisotropies during the collapse. A common used approximation in this sense is the one introduced by Zel’ dovich [61].

Naturally, the Zel’ dovich approximation also has limits and in order to try to obtain a more complete description of the non-linear evolution, numerical N-body simulations are necessary. The simplest way a numerical simulation can be built is representing the cosmological fluid as made of discrete particles and summing up the gravitational interactions between them using a Newtonian approximation. Once the gravitational interaction has been calculated, positions and velocities of the particles are updated and gravity forces can be calculated again. There are also other more sophisticated techniques involving grids or combining operations on grids and among particles. Gravitational interactions are more relevant when the simulation is only made of dark matter particles. If baryons are included we also need to include in the simulation the hydrodynamical effects important for a fair description of phenomena occurring in galaxies and cluster of galaxies.

We are not going into much detail of numerical simulations here, but we stress that they represent an essential tool in order to study the cosmological model and compare theoretical expectations with observations, in particular on those scales where the linear theory fails or when the complexity of the physical phenomena involved makes a simple analytical approach impractical.

1.4.4 Statistics of dark matter haloes

In $\Lambda$CDM model, where the matter content is dominated by cold dark matter, primordial fluctuations can be important on small scales. As an example, consider an universe with hot dark matter, where hot means relativistic. In this particular case we would have small scales perturbations erased by free-streaming of these hot particles.

In the case of cold dark matter, structures first form on sub-galactic scales and then merge together to give rise to larger objects. We refer in the following to these dark matter structures as haloes. This picture of structure formation is called hierarchical, because the process is bottom-up. During this process of structures formation, it can be possible to provide tools for a statistical description of the
distribution of these structures according to their mass.

The basic idea is that given a probability distribution function for the density perturbations, a dark matter halo forms through gravitational collapse when its density contrast in a certain volume containing a certain mass has exceed some threshold \( \delta_c \) and we have already seen that an extrapolation from linear theory gives a \( \delta_c \) of order of unity. Therefore we can consider that a bound object forms when its density contrast reaches the threshold. The mass of these haloes can be calculated by filtering the initial density field by means of an appropriate window function.

Our aim is estimating the expected number of haloes for a given mass per unit volume. Therefore, hereafter, we consider a filtered density field \( \delta_M \):

\[
\delta_M(x, z) = \int_{0}^{+\infty} d^3 y \, \delta(y, z) W(|x - y|),
\]

(1.77)

where \( W(|x - y|) \) is some window function smoothing the density field on a characteristic scale \( R \) and a mass \( M \propto R^3 \) can be associated to it. Furthermore, we assume a Gaussian distribution of density fluctuations:

\[
p(\delta_M) = \frac{1}{\sqrt{2\pi \sigma_M^2}} \exp \left( -\frac{\delta_M^2}{2\sigma_M^2} \right),
\]

(1.78)

where \( \sigma_M^2 \) represents the mass variance given a window function:

\[
\sigma_M^2(R, z) = \frac{1}{2\pi^2} \int_{0}^{+\infty} dk \, k^2 \, W^2(k, R) P(k, z),
\]

(1.79)

where \( P(k, z) \) is given by eq. (1.74) and \( W(k, R) \) is the window function which we assume throughout this dissertation to be a top-hat filter:

\[
W(k, R) = 3 \frac{\sin(kR) - kR \cos(kR)}{(kR)^3}.
\]

(1.80)

The probability that a point is in a collapsed region at a given epoch is then:

\[
F(\delta_M > \delta_c(z)) = \int_{\delta_c(z)}^{+\infty} p(\delta_M) d\delta_M = \frac{1}{2} - \int_{0}^{\delta_c(z)} p(\delta_M) d\delta_M,
\]

(1.81)

and using eq. (1.78) in eq. (1.81) we eventually can show:

\[
F(M, z) = \frac{1}{2} - \int_{0}^{\delta_c(z)} \frac{1}{\sqrt{2\pi \sigma_M^2}} \exp \left( -\frac{\delta_M^2}{2\sigma_M^2} \right) = \frac{1}{2} \left[ 1 - \text{erf} \left( \frac{\delta_c(z)}{\sqrt{2}\sigma_M} \right) \right]
\]

(1.82)

This simple idea has been described for the first time in Press & Schechter (1974) [62] although the procedure suffers a drawback which can be seen in eq. (1.82). In
fact, $F(0, z)$ should be equal to 1 but, since $\text{erf}(0) = 0$, we have that the result is $1/2$. Press & Schechter acknowledged this problem and temporarily fixed it multiplying the distribution function for a factor two. However, the problem is deeply embedded in the Press & Schechter (PS) approach. Imagine to count the collapsed haloes at a given smoothing scale $R$ and also that the density fluctuation is $\delta(R) < \delta_c$ while on a larger scale $R'$ we have $\delta(R') > \delta_c$. Therefore, the PS approach does not count the point in the chosen region in the function $F(M, z)$ to be part of the halo of mass $M'$ because on the selected smoothing scale $R$ the fluctuation is still below the threshold.

In literature this problem is usually referred as the “cloud-in-cloud” problem and has been tackled soon after the publication of the seminal paper by Press & Schecter, e.g. Bardeen et al. (1986) [58], Peacock & Heavens (1990) [63], Bond et al. (1991) [64]. In these latter works, more theoretical explanations for accounting for the factor 2 in the original PS formula are provided. In particular, Peacock & Heavens (1990) [63] and Bond et al. (1991) [64] proposed an approach called excursion set theory based on smoothing the density field starting from large filtering scale and following the path of $\delta(R)$ calculating the probability of the first time, for a certain scale $R$, it crosses the barrier given by $\delta_c$. This does account for the correct normalisation in $F(M, z)$, although this procedure still encounters some technical issues as described, for instance, in Maggiore & Riotto 2010 [65], related to the characteristic of the barrier and to the fact that the first excursion set theory was based on a sharp k-space filter which makes difficult to assign unambiguously a mass to each smoothed density fluctuation.

Given a distribution $F(M, z)$ we can define the comoving number density of dark matter haloes with mass in the range $[M, M + dM]$ as follows:

$$\left| \frac{\partial F}{\partial M} \right| = \frac{M n(M, z)}{\bar{\rho}(z)} ,$$  \hspace{1cm} (1.83)

where $\bar{\rho}$ is the comoving mean density of the universe defined as:

$$\bar{\rho}(z) = \int_0^{+\infty} dM M n(M, z) ,$$ \hspace{1cm} (1.84)

and the function $n(M, z)$ is called the halo mass function. The mass function can be also written as:

$$\frac{M^2}{\bar{\rho}} n(M) \frac{dM}{M} = \nu f(\nu) \frac{d\nu}{\nu} ;$$ \hspace{1cm} (1.85)

where we dropped the redshift dependence to keep the notation clear and the func-
Figure 1.5 – Peak-background split. Any density fluctuation can be seen as the sum of a large scale fluctuation and a small scale one. Credit: Cosmological Physics, Peacock, 1999 [1].

Assuming spherical collapse, the parameters are $p = 0$, $A(p) = 0.5$, and $q = 1$ and we obtain the mass function found by Press & Schechter (1974) [62]; alternatively, ellipsoidal collapse results in the Sheth & Tormen (1999) mass function where $p \simeq 0.3$, $A(p) \simeq 0.3222$, and $q \simeq 0.75$, which provides better agreement with N-body simulations [66].

**Halo bias**

From spherical collapse theory, we know that the collapse occurs when a mass fluctuation $\delta_M$ reaches a critical value (in this case extrapolated from linear theory) which is the critical density $\delta_c \simeq 1.68$. However, we can expect the number density of objects to be affected by the fact that structures lie in an overdensity rather than in an underdense region. Particularly, given a high-density region, the abundance of haloes in that region will be enhanced with respect to the mean. In order to try to understand this mechanism, we may think the density field as composed of the sum of a background fluctuation on large scale and a peak fluctuation on small scale. This approach is usually called peak-background split in literature (e.g. [1]).

Then, in order to reach the critical density, a small scale fluctuation needs to have $\delta_M = \delta_c - \delta_b$ where $\delta_b \ll 1$ is the value of the background fluctuation at some large scale. In a certain region with a certain $\delta_b$, the probability to have a small
scale fluctuation greater the threshold for the collapse is:

\[ \tilde{F}(\delta_M > \delta_c - \delta_b) = \int_{\delta_c - \delta_b}^{+\infty} p(\delta_M) d\delta_M = \frac{1}{2} - \int_0^{\delta_c - \delta_b} p(\delta_M) d\delta_M = \frac{1}{2} - \int_0^{\delta_c - \delta_b} \frac{1}{\sqrt{2\pi}\sigma_M} \exp\left(-\frac{\delta_M^2}{2\sigma_M^2}\right) d\delta_M = \frac{1}{2} \left[ 1 - \text{erf} \left( \frac{\delta_c - \delta_b}{\sqrt{2}\sigma_M} \right) \right]. \]  

(1.87)

Recalling the derivation of the mass function by means of eq. (1.81) and (1.82) and changing the lower limit in the integral to \( \delta_c - \delta_b \), now we have:

\[ n_L(M) = \frac{\rho_0}{M} \left| \frac{d\tilde{F}}{dM} \right| = \sqrt{\frac{2}{\pi}} \rho_0 \left| \frac{d\sigma_M}{dM} \right| \frac{\delta_c - \delta_b}{\sigma_M^2} \exp\left(-\frac{(\delta_c - \delta_b)^2}{2\sigma_M^2}\right), \]  

(1.88)

where we indicate by \( n_L(M) \) the halo mass function modulated by the local background density. We emphasise that we use the original PS approach only for the sake of simplicity and that we can obtain the same results by means of the excursion set theory (e.g. Mo & White (1996) [67]).

Simply splitting the exponential function in its components we find:

\[ n_L(M) = n(M) \left[ 1 - \frac{\delta_b}{\delta_c} \right] \exp\left(-\frac{\delta_b^2}{2\sigma_M^2}\right) \exp\left(\frac{\delta_c \delta_b}{\sigma_M^2}\right). \]  

(1.89)

where \( n(M) \) is the mass function found in the previous section. Assuming that \( \delta_b \ll 1 \):

\[ n_L(M) \approx n(M) \left[ 1 - \frac{\delta_b}{\delta_c} \right] \left[ 1 - \frac{\delta_b^2}{2\sigma_M^2} \right] \left[ 1 + \frac{\delta_c \delta_b}{\sigma_M^2} \right] \sim f(M) \left[ 1 + \frac{\delta_c \delta_b}{\sigma_M^2} - \frac{\delta_b}{\delta_c} \right] \]  

(1.90)

where in the latter step we have considered only terms to first order in \( \delta_b \).

Then, at first order in \( \delta_b \), the final result we obtain with this approach is that the number density of haloes \( f(M) \) with a given mass \( M \) is locally biased:

\[ n_L(M, x) = n(M) \left[ 1 + b(M) \delta_b(x) \right]. \]  

(1.91)

We define the bias function as:

\[ b_{Lg}(M) = \frac{1}{\delta_c} \left( \frac{\delta_c^2}{\sigma_M^2} - 1 \right). \]  

(1.92)

The subscript \( Lg \) in eq. (1.92) reminds us that the PS formalism is implicitly Lagrangian, that means that clustering is view considering the early times dark matter halo positions where they form. However, for describing spatial clustering properties including the motion induced by gravitational interaction of haloes, it
would be more suitable to adopt a Eulerian point of view where haloes change their positions over time. Obviously, a Lagrangian view is equivalent to the Eulerian one when considering average values based on the mass function which are spatially independent. Nevertheless, clustering statistics as 2-point correlation function can be viewed differently. Essentially, this argument reduces to a change of coordinates from a Lagrangian space $x = q$ to a Eulerian space through $x(q, z) = q + S(q, z)$, where $S(q, z)$ is the Lagrangian displacement field that specifies the motion of the cosmological fluid.

As shown in Mo & White (1996) [67] and Catelan et al. (1998) [68] this change of framework leads to a slight different modification of eq. (1.92):

$$b(M) = 1 + \frac{1}{\delta_c} \left( \frac{\delta^2}{\sigma_M^2} - 1 \right).$$  \hspace{1cm} (1.93)

In the rest of this dissertation we will always refer to the Eulerian bias if not otherwise stated.

The halo bias can be parametrised exactly as the mass function since the form of $b(M)$ depends on the halo mass function used: A common parametrisation of the halo bias (consistent with the mass function) can be taken from [66]:

$$b(M) = 1 + \frac{q\nu - 1}{\delta_c} + \frac{2p}{\delta_c(1 + (q\nu)^p)} ,$$  \hspace{1cm} (1.94)

where $p, q$ and $\nu$ are defined as above for the mass function. The application of these notions to the analysis of clustering statistics will be discussed in Chapter 3 in the context of building a theoretical model for the correlation of intrinsic galaxy sizes and luminosities.
Chapter 2

Weak Gravitational Lensing

Whether you can observe a thing or not depends on the theory which you use. It is the theory which decides what can be observed.

Albert Einstein

The theory of gravitational lensing can be entirely described by General Relativity. Photons travel following geodesics and these geodesics are calculated from the curvature of the space-time due to the content of the universe according to the Einstein equations. This results in background galaxy images that are distorted by the structures that photons pass through during their journey to us. Gravitational lensing distortions can be generally discussed in terms of deviations with respect to a fiducial ray in a light beam. These effects can be described through the introduction of shear, i.e. ellipticity distortions, and magnification, which involves sizes, fluxes and number densities.

Broadly speaking, we have two regimes in gravitational lensing: strong lensing occurs mainly near clusters of galaxies, when the matter content is high and the distortions are quite significant; weak lensing is more of a statistical effect, and can be detected measuring the coherence of galaxy image distortions by means of n-point correlation functions. In the weak lensing regime, typical distortions of galaxy ellipticities and sizes are of order of a few percent. In this Chapter we review the basic properties of weak gravitational lensing.

We start by introducing the geometrical framework connected to General Relativity, assuming photons can be described as light rays. Then we discuss the Jacobi matrix which maps image and source angles, and its application in weak lensing. In the final part of this Chapter we see how weak gravitational lensing can be detected and used to constrain the cosmological model.
We are going to present the main points concerning light distortions in gravitational fields and do not aim to an exhaustive treatment of gravitational lensing here. We refer the reader interested in a deeper study of what is discussed in this Chapter to Schneider, Ehlers & Falco (1992) [69] and the review by Bartelmann and Schneider [3].

2.1 Propagation of light

In most cases of interest in gravitational lensing, photons can be described as plane waves. In this section we introduce the description of photons as light rays; we start with the definition of a light ray, then extend the discussion to a beam and finally we derive the geodesic deviation equation that plays a key role in the gravitational lensing formalism.

2.1.1 Definition of light rays

In general, photons can be represented as plane monochromatic waves:

\[ f = ae^{i\psi} = ae^{i(k_\mu x^\mu + \alpha)}, \]  

(2.1)

where the quantity \( \psi \) is called the eikonal, \( k_\mu \) is called wave-vector, \( x^\mu \) is the four-vector of the coordinates, and \( \alpha \) and \( a \) are constant.

A plane wave described in eq. (2.1) oscillates with angular frequency \( \omega \) given by:

\[ \omega \equiv \left| \frac{d\psi}{ds} \right| = \frac{dx^\mu}{ds} \frac{\partial \psi}{\partial x^\mu} = -g_{\mu\nu} v^\mu k^\nu, \]  

(2.2)

where \( s \) is the proper time, \( v^\mu = \frac{dx^\mu}{ds} \) is the four-velocity and the wave-vector \( k_\mu \) is defined as:

\[ k^\mu = \frac{\partial \psi}{\partial x^\mu}. \]  

(2.3)

Considering \( k^\mu \) as a vector field then the integral curves \( x^\mu(u) \) of such a field, defined as:

\[ \frac{dx^\mu}{du} = k^\mu, \]  

(2.4)

are called light rays.

Taking the derivative of the tangent vector \( k^\mu \) of a light ray covariantly along the ray and remembering that the wave vector is a gradient (for which \( k_{\mu\nu} = k_{\nu\mu} \)) we have from \( k^\mu k_\mu = 0 \) that:

\[ k^\mu k^\nu_{\mu} = k^\mu g^{\gamma\nu} k_{\gamma\mu} = k^\mu g^{\gamma\nu} k_{\mu\gamma} = \frac{1}{2} g^{\gamma\nu} (k^\mu k_\mu)_{\gamma} = 0 \]  

(2.5)
Then \( k_\mu k^\nu = 0 \).

Since the covariant derivative is defined as: \( y^\alpha_{;\beta} = \partial_\beta y^\alpha + \Gamma^\alpha_{\beta\gamma} y^\gamma \) we obtain the same equations for the wave vector:

\[
d\frac{d^2 x^\mu}{du^2} + \Gamma^\mu_{\nu\gamma} \frac{dx^\mu}{du} \frac{dx^\gamma}{du} = 0,
\]

which means that light rays are null geodesics.

### 2.1.2 Geometry of light beams

We now study the system of rays associated with a fixed phase function \( \psi \). Let each ray be parametrised by an affine parameter \( u \) and let the rays be labeled by three parameters \( y^\alpha \) (parameters may be, for example, phase value of the ray, angles distinguishing photons travelling in different directions).

The ray system is given by: \( x^\alpha = f^\alpha(u, y^\alpha) \). Then we obtain:

\[
k^\alpha = \frac{\partial f^\alpha}{\partial u} = -g^{\alpha\beta} \psi_\beta \tag{2.7}
\]

Consider the ray \( \gamma \) given by particular values \( y^\alpha \). The rays infinitesimally near \( \gamma \) are given by \( y^\alpha + \delta y^\alpha \) and the vectors:

\[
\delta x^\alpha = \frac{\partial f^\alpha}{\partial y^\beta} \delta y^\beta \tag{2.8}
\]

connect \( \gamma \) with its neighbouring rays.

Now we have to give the definition of a beam. A beam is a collection of null geodesics which have a vertex point. Most important is the following property. Two rays connected by \( \delta x^\alpha \) belong to the same fixed phase if and only if:

\[
k_\alpha \delta x^\alpha = 0 \tag{2.9}
\]

which is a parametrisation independent property because of the definition of \( \delta x^\alpha \). This property comes from the definition of the wave vector.

By means of these definitions, we can also introduce in these terms the redshift. Recalling Section 1.2.2, the redshift \( z \) is the change of frequency of the photon, \( \omega_S \), emitted by a source and the one \( \omega_O \) actually observed:

\[
1 + z \equiv \frac{\omega_S}{\omega_O} = \frac{(g_{\mu\nu} v^\mu k^\nu)_S}{(g_{\mu\nu} v^\mu k^\nu)_O}, \tag{2.10}
\]

where in the last equality we have used eq. (2.2). From eq. (2.10) we can also notice that there is no fundamental interpretation for the redshift. It can be either
seen as a gravitational dilation or a kind of Doppler effect. In fact, given a certain redshift value, we always have the freedom to choose a reference frame, for example one where both observer and source are at rest (gravitational interpretation) or one where observer and source wave vectors are equal and then a difference in four-velocity (Doppler interpretation).

2.1.3 Fermat’s principle

It would be great to find for light rays something analogous to the least action principle of mechanics. In general, this is not possible for the simple reason that we cannot write a non-vanishing Lagrangian for light rays because photons have zero mass. However, we can write down something similar following to Fermat’s principle.

Although called a principle, it is actually a theorem in the General Relativity context. Let us assume to have an event $S$ called source and a world line $O$ where the observer is located. Fermat’s principle states that a curve $\gamma$ is a light ray (i.e. a null geodesic) if and only if its arrival time $\tau$ is stationary under the first-order variation of $\gamma$ among the set of possible curves from $S$ to $O$.

We can express the stationary condition of the arrival time as $\delta \tau = 0$. Note that we do not talk of the time taken for a light ray to travel a particular path, because it does not have any meaning in General Relativity [69].

A complete proof of Fermat’s principle in a relativistic context is given in Schneider, Ehlers & Falco (1992) [69]. This theorem is fundamentally based on the path integral formalism of quantum electrodynamics introduced by Feynman in 1948 [70]. Basically, the idea is that a photon can take all the possible paths from $S$ to $O$, where each path has a different probability described as a wave with some amplitude and phase, which are to some extent linked to the time taken on each path. Paths with similar travelling times sum their amplitudes up because their phases are almost equal, whereas paths with different travelling time cancel their amplitudes each other. As a result, a photon is more likely to take the least time path (where the definition of time depends on the particular context, as we have seen).

Fermat’s principle can provide an alternative way to describe the propagation of light rays. Here, we have introduced it in order to provide a theoretical link with the more suitable description of photons given by quantum electrodynamics. However, in all cases we consider, the propagation distances of the photons are large with respect to their wavelengths.

For instance, consider a light beam propagating from a source to a detector in your house. The light rays will travel along a straight line, which is, according to a non-relativistic version of Fermat’s principle, the quickest path. When we put a
lens made of some material between source and detector, the geometry of the lens allows for different paths to be taken with the same travelling time. Therefore, the lens simply re-arranges the paths taken by the photons to arrive to the detector.

A gravitational lens works based on the same principle: an inhomogeneity modifying the arrival time of a light beam to an observer which in turn produces image distortions exactly as seeing a candle through a glass of wine.

In practice, Fermat’s principle tells us something about the interpretation of the meaning of a null geodesic, analogously to the interpretation of standard geodesics according to a least action principle. Although Fermat’s principle can give us the equations governing gravitational lensing (e.g. Blandford & Narayan (1986) [71]), in the following we are going to derive the basic lensing equations from the geodesic deviation equation.

2.1.4 The geodesic deviation equation

The propagation of a light beam can be described by the geodesic deviation equation, which we will calculate in this section.

Let us consider a set of light rays starting at an event $S$ and let any smooth, one parametric subfamily of these rays be given by $x^\alpha = f^\alpha(u, y)$ where $y$ labels the rays and $u$ is an affine parameter on each ray exactly as in eq. (2.7) and (2.8):

$$k^\alpha = \frac{\partial f^\alpha}{\partial u}, \quad \delta x^\alpha = \frac{\partial f^\alpha}{\partial y},$$

so that $\delta x^\alpha$ connects nearby rays. As a consequence of eq. (2.11) we have:

$$\frac{D\delta x^\alpha}{du} = \frac{Dk^\alpha}{dy},$$

where the derivative are covariant along the curves with constant values for $y$ and $u$.

The set of light rays we are considering represents a narrow beam. The angular separation at the source $S$ is given by the amount of change of the vector connecting two rays $\delta x^\alpha$ along the path $dL$. We can define $dL$, from eq. (2.2), and we have:

$$dL \equiv v^\alpha dx_\alpha = \omega du,$$

where $v^\alpha$ represents, the observer (or source) four-velocity. The displacement in terms of the affine parameter is therefore given by $du = dL/\omega$. Incidentally, this also provides a physical interpretation for the affine parameter in terms of the displacement of the photon along its path. By means of these definitions, the angular
separation, for example, seen from the observer $\delta \theta_O$, is given by:

$$\left( \frac{D \delta x^\alpha}{d \kappa} \right)_O = \left( \frac{\omega D \delta x^\alpha}{d \mathbf{L}} \right)_O = \omega_O \delta \theta_O^\alpha, \quad (2.14)$$

or, written in a more useful way for our purposes:

$$\delta \theta^\alpha_O = \frac{1}{\omega^O} k^\beta \nabla^\beta \delta x^\alpha, \quad (2.15)$$

where we have just used the fact that $\frac{D}{d \kappa} = k^\beta \nabla^\beta$ from eq. (1.9).

In order to obtain an equation which tells us how a vector $\delta x^\alpha$ is transported along the ray $\gamma$ (which is a null geodesic), we differentiate eq.(2.12) and use the properties of the Riemann tensor:

$$\frac{D^2 k^\alpha}{d x^\mu d x^\nu} - \frac{D^2 k^\alpha}{d x^\nu d x^\mu} = R_{\mu \nu \delta}^\alpha k^\delta, \quad (2.16)$$

we arrive at:

$$\frac{D^2 \delta x^\alpha}{d \kappa d \kappa} = \frac{D^2 k^\alpha}{d \kappa d \kappa} \frac{d x^\mu}{d \kappa} \frac{d x^\nu}{d \kappa} = \frac{D^2 k^\alpha}{d y d u} + R_{\mu \nu \delta}^\alpha \delta x^\mu k^\nu k^\delta. \quad (2.17)$$

Recalling that the wave-vector is parallel transported along a geodesic, i.e. $\frac{D k^\alpha}{d \kappa} = 0$, that means:

$$\delta x^\alpha = R_{\mu \nu \delta}^\alpha k^\nu k^\delta \delta x^\mu. \quad (2.18)$$

We obtained a linear differential equation for the deviation vector $\delta x^\alpha$, called the geodesic deviation equation. This represents a fundamental equation in gravitational lensing, and it is often also called Jacobi equation. From eq. (2.18) we are able to calculate the effects that gravitational lensing produces and to parametrise them appropriately.

### 2.2 The Jacobi matrix

The geodesic deviation equation given in eq. (2.18) is a second-order linear differential equation and, assuming the quantity $R_{\mu \nu \delta}^\alpha k^\nu k^\delta$ to be a smooth function, satisfies continuity conditions. This means that any solution of eq. (2.18) must be related to its initial conditions. Essentially, we have that:

$$(\delta x^\alpha, \delta x^\alpha)(u) = D^\beta_{\alpha}(u)(\delta x^\alpha, \delta x^\alpha)(u_O), \quad (2.19)$$

where on the right-hand side we consider the initial conditions at the observer $u_O = 0$ and $D^\beta_{\alpha}(u)$ is a matrix containing the terms related to the Riemann tensor of eq.
(2.18).

If we assume as initial condition that the observer represents a vertex of the ray bundle, i.e. \( \delta x^a(u_O) = 0 \), we end up with the following relation between the source separation and angular image:

\[
\delta x^a(u_S) = J^a_{\beta}(u_S, u_O) \delta \theta^\beta(u_O), \tag{2.20}
\]

where \( J^a_{\beta} \) is a sub-matrix of \( D^\beta_{\alpha} \) and the factor \( \omega \) (coming from the definition of \( \delta \theta^\beta \)) has been absorbed in its definition. The matrix \( J^a_{\beta} \) is called the Jacobi matrix.

From the geodesic deviation equation in eq. (2.18) and from eq. (2.20) we can see that the Jacobi matrix satisfies the following equation:

\[
\frac{D^2 J(u)}{du^2} = \mathcal{T}(u) J(u), \tag{2.21}
\]

with initial conditions are \( J(0) = 0 \) and \( dJ/du(0) = 1 \) and where \( \mathcal{T} \) is called the optical tidal matrix which describes how the space-time curvature affects the light propagation. The optical tidal matrix is symmetric, i.e., \( \mathcal{T}^T = \mathcal{T} \), and its components depend exclusively on the curvature of the metric (e.g. Seitz et al. (1994) [72], Bartelmann & Schneider (2001) [3]).

Note that this argument of linearity in order to obtain the Jacobi matrix equation has been pointed out initially by Seitz et al. (1994) [72], although more recently some remarks have been made by Reimberg & Abramo (2013) [73] where an explicit construction of the Jacobi matrix is given. Here we need to make some geometrical remarks. First, we define the photon direction as follows: \( n^\alpha = \frac{1}{\omega} k^\alpha - v^\alpha \). We also notice that two rays with the same phase must have \( \delta x^a k^a_S = 0 \).

From these, it can be shown that the vector \( \delta x^a_S \) lies in the galaxy (source) plane and that \( \delta \theta^\alpha_O \) is in the observer plane (see Bonvin (2008) [2]). This is due to the fact that we can always re-parametrise our ray system \( x^\alpha = f^\alpha(u, y^\alpha) \) in terms of different \( u \) and \( y \) and therefore choose a parametrisation such that \( \delta x^a_S v_{Sa} = 0 \).

Furthermore, from \( \delta x^a_S k_{Sa} = 0 \) and by means of the definition of photon direction given above, we can see that \( \delta x^a_S n_{Sa} = 0 \). Therefore, we have that \( \delta x^a_S \) lies in a subspace orthogonal to the source four-velocity and to the photon direction.

We can apply a similar reasoning to the vector \( \delta \theta^\alpha_O \). In fact, from eq. (2.2) and eq. (2.15) it can be seen that \( \delta \theta^\alpha \) is in general orthogonal to \( k^\alpha \) and, consequently, \( \delta \theta^\alpha_O n_{Oa} = 0 \), and also we have that \( \delta \theta^\alpha_O v_{Oa} = 0 \).

In order to obtain an explicit expression for the Jacobi matrix, we need to solve directly eq. (2.18) reducing that second order differential equation to a first order system of equations in the variables \( \delta x^a \) and \( \delta \theta^\alpha \). Following again Bonvin (2008) [2], we build an orthonormal basis at the observer \( u_O \) made by the vec-
tors \((E_1^\alpha, E_2^\alpha, n_0^\alpha, v_0^\alpha)\), where the subspace \((E_1^\alpha(u), E_2^\alpha(u))\) is called screen (given the other two vectors of the basis). It is possible to write the vectors in eq. (2.20) in this basis:

\[
\delta x^\alpha(u) = -\xi_1(u)E_1^\alpha(u) - \xi_2(u)E_2^\alpha - \xi_0[n_0^\alpha(u) + v_0^\alpha(u)]
= -\xi_1(u)E_1^\alpha(u) - \xi_2(u)E_2^\alpha - \xi_0 \frac{u_0(u)}{\omega_0} k^\alpha(u),
\]

where the last term of the right-hand side is derived according to the definition of photon direction given above, and note that the component \(\xi_0\) in eq. (2.22) accounts for a possible deviation also along the direction of propagation. For the vector \(\delta \theta_0^\alpha\) we have:

\[
\delta \theta_0^\alpha = -\theta_1(u_o)E_1^\alpha(u_o) - \theta_2(u_o)E_2^\alpha(u_o).
\]

In this framework, it is straightforward to re-write eq. (2.20), defining \(\hat{\xi} = (\xi_1, \xi_2, \xi_0)\) and \(\hat{\theta} = (\theta_1, \theta_2, 0)\), in the following form:

\[
\hat{\xi}(u_s) = \hat{J}(u_s, u_0) \hat{\theta}(u_0),
\]

where the matrix \(\hat{J}(u_s, u_0)\) is defined as:

\[
\hat{J}(u_s, u_0) = \begin{pmatrix}
E_{1a}(u_s)J_0^\alpha(u_s)E_1^\beta(u_o) & E_{1a}(u_s)J_0^\alpha(u_s)E_2^\beta(u_o) & 0 \\
E_{2a}(u_s)J_0^\alpha(u_s)E_1^\beta(u_o) & E_{2a}(u_s)J_0^\alpha(u_s)E_2^\beta(u_o) & 0 \\
k_0(u_s)J_0^\alpha(u_s)E_1^\beta(u_o) & k_0(u_s)J_0^\alpha(u_s)E_2^\beta(u_o) & 0
\end{pmatrix}.
\]

### 2.2.1 Distance-duality relation

For completeness we are now going to discuss how the Jacobi matrix can be used to describe the cosmological relation between luminosity distance and angular diameter distance. We would like to stress the point that also the reverse situation of eq. (2.20) can be completely described in terms of Jacobi matrix:

\[
\delta x^\alpha(u_o) = J_0^\alpha(u_o, u_s) \delta \theta^\beta(u_s),
\]

Both \(J_0^\alpha(u_s, u_0)\) of eq. (2.20) and \(J_0^\alpha(u_o, u_s)\) of eq. (2.26) are solutions of eq. (2.21). Following Perlick (2004) [74], consider now two solutions of the Jacobi equation, namely \(J_1 \equiv J(u, u_1)\) and \(J_2 \equiv J(u, u_2)\). We can then build the following quantity:

\[
\frac{DJ_1}{du} J_2^T - J_1^T \frac{DJ_2}{du} = \text{constant}.
\]

The fact that the quantity above is constant can be easily shown recalling that the optical tidal matrix \(T\) is symmetric and, therefore, by means of the initial conditions
of eq. (2.21), we have that $J(u_1, u_2) = -J^T(u_2, u_1)$. One consequence of this latter relation is that:

$$\det J(u_1, u_2) = \det J(u_2, u_1), \quad (2.28)$$

Eq. (2.28) is called Etherington’s reciprocity law [75].

As shown in Fig. 2.1, we can see that the two situations described by eq. (2.20) and (2.26) correspond respectively to the case of the angular diameter distance and the luminosity distance already mentioned in Section 1.2.2. In particular, here we can see a more general definition for these two distances in terms of elements of area $dA$ and solid angle $d\Omega$.

The angular diameter distance relates the physical size of the source and angular size seen by the observer:

$$d_A = \sqrt{\frac{dA_S}{d\Omega_O}}. \quad (2.29)$$

The luminosity distance links the luminosity of the source with the flux measured by the observer. Since the number of photons $N$ is conserved during the propagation from the source to the observer, we have:

$$dN \propto \frac{L_S d\tau_S d\Omega_S}{\omega_S} \propto \frac{F_O d\tau_O dA_O}{\omega_O} = \text{constant}, \quad (2.30)$$

where we indicate with $L$ the luminosity and with $F$ the flux. Remembering that $d\tau_O/d\tau_S = \omega_S/\omega_O = 1 + z$, we obtain:

$$d_L = \sqrt{\frac{L_S}{4\pi F_O}} = (1 + z) \sqrt{\frac{dA_O}{d\Omega_S}}. \quad (2.31)$$
Additionally, given eq. (2.15), we arrive at:

\[ \sqrt{\frac{dA_O}{d\Omega_S}} \sqrt{\frac{d\Omega_O}{dA_S}} = \frac{\omega_S}{\omega_O} \sqrt{\det J(u_O, u_S)} = (1 + z), \]  

(2.32)

and this leads to the distance-duality relation between luminosity distance \( d_L(z) \) and angular diameter distance \( d_A(z) \):

\[ d_L(z) = (1 + z)^2 d_A(z). \]  

(2.33)

A remarkable property of eq. (2.33) is that it holds in any spacetime, assuming the conservation of the number of photons. This means that it is valid also in theories of modified gravity.

### 2.3 The magnification matrix

The Jacobi matrix \( J(u_O, u_S) \) is strictly related to the lensing magnification matrix. More specifically, we can understand the weak lensing formalism from Fig.2.2 and by following the notation in Bartelmann & Schneider (2001) [3]:

\[ \beta(\theta) = \theta - \frac{D_{ds}}{D_s} \hat{\alpha}(D_d \theta) \equiv \theta - \alpha(\theta) \]  

(2.34)

where \( \beta \) is the true position of the source, \( \theta \) represents the position where the source is observed, \( \hat{\alpha} \) is the deflection angle; \( \alpha \) is called the scaled deflection angle. \( D_{ds}, \)
Weak Gravitational Lensing

$D_d$ and $D_s$ are all angular-diameter distances. Also, reporting from Fig. 2.2, $\eta$ is the deviation vector at the source and $\xi$ on the lens plane.

The gradient of the lensing map given in eq. (2.34) is called the magnification matrix:

$$A^i_j \equiv \frac{\partial \beta^i}{\partial \theta^j} = \delta^i_j - \frac{\partial \alpha^i}{\partial \theta^j}. \quad (2.35)$$

In weak lensing we have to deal with small angles and we can Taylor expand the lensing map as $\beta \simeq \mathcal{A} \theta$. Therefore we have:

$$\eta = (u_O - u_S) \beta = (u_O - u_S) \mathcal{A} \theta, \quad (2.36)$$

that is

$$J(u_O, u_S) = (u_O - u_S) \mathcal{A}. \quad (2.37)$$

Hence, in order to calculate the magnification matrix we need the Jacobi matrix introduced in Section 2.2. In a totally homogeneous and isotropic universe there would be no detectable gravitational lensing effect. We consider a FLRW metric with scalar perturbations in the Newtonian gauge assuming General Relativity (see Section 1.4.3):

$$ds^2 = a^2(\tau) \left[ \left( 1 + \frac{2\Phi}{c^2} \right) c^2 d\tau^2 - \left( 1 - \frac{2\Phi}{c^2} \right) \gamma_{ij} dx^i dx^j \right]. \quad (2.38)$$

For this case the Jacobi matrix, and consequently the magnification matrix, has been calculated in literature (e.g. Bonvin, Durrer & Gasperini (2006) [76] and Bonvin (2008) [2]) and, in terms of the coordinate $\chi$ of the FLRW metric$^1$, we have:

$$J(\chi_O, \chi_S) = a_S f_K(\chi_O - \chi_S) \begin{pmatrix} 1 - \kappa - \gamma_1 & -\gamma_2 & 0 \\ -\gamma_2 & 1 - \kappa + \gamma_1 & 0 \\ w \cdot E_1 & w \cdot E_2 & 0 \end{pmatrix}, \quad (2.39)$$

where the matrix on the right-hand side is the magnification matrix $\mathcal{A}$, $\gamma_1$ and $\gamma_2$ represent the two component of the shear, $\kappa$ is the convergence, and in the third row of the matrix the vector $\mathbf{w}$ indicates the contribution of the peculiar velocities of the source $\mathbf{v}_S$ and the observer $\mathbf{v}_O$ where in bold we indicate the spatial part of the four-vectors. We can explicitly write the terms in the magnification matrix. As stated in [2], our observable is the redshift $z_S$ of the source, which is also a perturbed quantity, $z_S = \bar{z}_S + \delta z_S$, where at linear order (e.g. [77] and [76] for a derivation):

$$\frac{\delta z_S}{1 + z_S} = (\mathbf{v}_O - \mathbf{v}_S) \cdot \mathbf{n} + (\Phi_O - \Phi_S) - 2 \int_{\chi_O}^{\chi_S} d\chi \Phi', \quad (2.40)$$

$^1$In [2] the magnification matrix is given in function of the conformal time; however remember that we have $cd\tau = -d\chi$ for null geodesics.
where $\partial/\partial \chi \equiv \, '$ and $\mathbf{n}$ is the spatial part of the vector $n^\alpha$ indicating the photon direction.

Accounting for that and defining $\nabla_\perp^2 = \nabla^2 - (\mathbf{n} \cdot \nabla)^2 + 2\chi^{-1} \mathbf{n} \cdot \nabla$, we have that the components of $\mathbf{A}$, at linear order, are (see [2]):

$$
\gamma_1 = \frac{1}{c^2} \int_{\chi_0}^{\chi_S} d\chi \frac{f_K(\chi_S - \chi) f_K(\chi - \chi_0)}{f_K(\chi_S - \chi_0)} \partial_i \partial_j \Phi (E^i_1 E^j_1 - E^i_2 E^j_2),
$$

$$
\gamma_2 = \frac{2}{c^2} \int_{\chi_0}^{\chi_S} d\chi \frac{f_K(\chi_S - \chi) f_K(\chi - \chi_0)}{f_K(\chi_S - \chi_0)} \partial_i \partial_j \Phi E^i_1 E^j_2,
$$

$$
\mathbf{w} = \mathbf{v}_S - \mathbf{v}_O - \frac{1}{c^2} \int_{\chi_0}^{\chi_S} d\chi \nabla \Phi,
$$

$$
\kappa = \kappa_g + \kappa_v + \kappa_{SW} + \kappa_{ISW},
$$

where the pure gravitational lensing term $\kappa_g$ is:

$$
\kappa_g = \frac{1}{c^2} \int_{\chi_0}^{\chi_S} d\chi \frac{f_K(\chi_S - \chi) f_K(\chi - \chi_0)}{f_K(\chi_S - \chi_0)} \nabla_\perp^2 \Phi,
$$

and the Doppler lensing term (see Bacon et al. (2014) [77]) $\kappa_v$ due to the peculiar velocities is:

$$
\kappa_v = \frac{1 + z_S}{H(z_S) f_K(\chi_S - \chi_0)} \mathbf{v}_O \cdot \mathbf{n} + \left( 1 - \frac{1 + z_S}{H(z_S) f_K(\chi_S - \chi_0)} \right) \mathbf{v}_S \cdot \mathbf{n}.
$$

The Sachs-Wolfe $\kappa_{SW}$ and the integrated Sachs-Wolfe $\kappa_{ISW}$ contributions which describe how the propagation of the photons is affected by the gravitational potential and its change with time are:

$$
\kappa_{SW} = \frac{1}{c^2} \left[ 2 \Phi_S - \Phi_O + \frac{1 + z_S}{H(z_S) f_K(\chi_S - \chi_0)} (\Phi_O - \Phi_S) \right]
$$

$$
\kappa_{ISW} = \frac{1}{c^2} \left[ \frac{2}{f_K(\chi_S - \chi_0)} \int_{\chi_0}^{\chi_S} d\chi \Phi + 2 \left( 1 - \frac{1 + z_S}{H(z_S) f_K(\chi_S - \chi_0)} \right) \int_{\chi_0}^{\chi_S} d\chi \Phi' \right].
$$

Note that all the integrations are performed along the radial coordinate rather than over the affine parameter $u$ of the geodesic. This is usually referred as the Born approximation, which basically means that we calculate the lensing signal along the unperturbed geodesic. The importance of this approximation and of the rotational effects due to multiple lens planes (also called lens-lens coupling) has been discussed in Cooray & Hu (2002) [78], where they show that corrections for these effects must be accounted for only for ultra-high precision measurements. In this dissertation we always assume both approximations to be valid.

Bonvin (2008) [2] shows that the only relevant contribution is the one involving
peculiar velocities, especially at low redshift. Bacon et al. 2014 [77] described how this velocity contribution can be exploited to constrain the cosmological model.

In principle, we could also have a rotation term $\phi_r$ in in the off-diagonal terms of eq. (2.39), arising from the fact that generally lensing contributions come from multiple lens planes. For our purposes we can safely neglect this term and we refer to Pen & Mao (2006) [79] for more details about this rotation effect. Neglecting the third row of eq. (2.39) means that when we refer to the magnification matrix we will consider only its 2x2 upper-left sub-matrix:

$$A = \begin{pmatrix} 1 - \kappa - \gamma_1 & -\gamma_2 \\ -\gamma_2 & 1 - \kappa + \gamma_1 \end{pmatrix}. \quad (2.45)$$

### 2.3.1 Effects of lensing on galaxy images

The images of background galaxies are distorted and the information encoded in the magnification matrix describes how this happens. The shear and the convergence produce distortions in the observed ellipticity, size and flux of those galaxies.

Let us assume a Cartesian reference frame $(x, y)$ on the sky. In absence of shear $\gamma_1 = \gamma_2 = 0$, in the magnification matrix of eq. (2.45), we have only the convergence and it is immediately clear that a circular galaxy image with angular size $\theta$ corresponds to a circular source with size $\beta = (1 - \kappa)\theta$.

In case only $\gamma_1$ were present then we would have our source stretched along the x-axis or the y-axis, depending on the orientation of the source and the sign of $\gamma_1$. The term $\gamma_2$ instead only affects the components of the ellipticities along axes at 45 degrees with respect to the x-axis or the y-axis.

The traceless symmetric part of the magnification matrix can be represented as a complex number. That means the shear field can be viewed as:

$$\gamma = \gamma_1 + i\gamma_2 = e^{2i\phi}, \quad (2.46)$$

where $\phi$ is the orientation angle of the image. Note a rotation of the galaxy image of $\phi = \pi$ does not change the shear.

The determinant of this magnification matrix gives the cosmic magnification $\mu$ of a surface area element:

$$\mu = \frac{1}{\det A} = \left[(1 - \kappa)^2 - |\gamma|^2\right]^{-1}, \quad (2.47)$$

and that in the weak lensing regime where $|\kappa|$ and $|\gamma|$ $\ll$ 1, the magnification is approximately $\mu \simeq 1 + 2\kappa$. The magnification affects both number densities and solid angles.

Another important thing to point out from the magnification matrix $A$ of eq.
(2.45) is that shear cannot be directly observed. However, we can write the magnification matrix in terms of the so-called reduced shear $g$:

$$A = (1 - \kappa) \begin{pmatrix} 1 - g_1 & -g_2 \\ -g_2 & 1 - g_1 \end{pmatrix}, \text{ where } g_{1,2} = \frac{\gamma_{1,2}}{1 - \kappa}. \quad (2.48)$$

Of course, in case of weak lensing $|\kappa|$ and $|\gamma| \ll 1$ and therefore $g \simeq \gamma$ to first order. However, corrections for the reduced shear are much more relevant than for Born approximation and lens-lens coupling and since the reduced shear is the observable quantity, statistical lensing predictions must account for that. An accurate analysis of the impact of corrections due to the reduced shear is given in Dodelson, Shapiro & White (2006) [80].

### 2.4 Convergence and shear power spectra

The statistical analysis of weak lensing measurements can be done through the study of the convergence and shear power spectra. We have already found from the derivation of the magnification matrix how convergence and shear can be written in terms of the fluctuation of the potential in the metric, as shown in eq. (2.41).

Additionally, here we are interested in the pure gravitational part of $\kappa$ and therefore we assume the dominant contribution to the convergence coming from the gravitational lensing term, hence $\kappa \simeq \kappa_g$. Recalling that $\rho_{cr}(z) = \rho_{cr,0}E(z)$, it is possible to write Poisson’s equation (see eq. (1.68)) as follows:

$$\nabla^2 \Phi = \frac{3H_0^2}{2a} \Omega_{m,0} \delta. \quad (2.49)$$

This means we can re-write the convergence, eq. (2.42), as follows:

$$\kappa(\theta, \chi) = \frac{3H_0^2 \Omega_{m,0}}{2c^2} \int_0^\chi d\chi' \frac{f_K(\chi - \chi')f_K(\chi')}{f_K(\chi)} \frac{\delta[f_K(\chi')\theta, \chi']}{a(\chi')} \quad (2.50)$$

where we just set $\chi_0 = 0$.

The equation above is valid only for fixed source at the comoving distance $\chi(z)$. When sources are distributed in redshift, eq. (2.50) needs to be averaged over some normalised source distribution $n(\chi)$:

$$\kappa(\theta) = \int_0^{\chi_p} d\chi n(\chi) \kappa(\theta, \chi), \quad (2.51)$$

where $\chi_p$ is the particle horizon given in eq. (1.35). Simply re-arranging the limits
of integration we obtain:
\[ \kappa(\theta) = \frac{3H_0^2\Omega_{m,0}}{2c^2} \int_0^{\chi_p} d\chi g(\chi)\delta[f_K(\chi)\theta, \chi], \tag{2.52} \]
where the lensing efficiency function \( g(\chi) \) is given by:
\[ g(\chi) = \frac{f_K(\chi)}{a(\chi)}\int_\chi^{\chi_p} d\chi' n(\chi') \frac{f_K(\chi'-\chi)}{f_K(\chi')} . \tag{2.53} \]

It is clear from the equations above that the 2-point statistics of the convergence depends on the density power spectrum. On large scales (small \( k \)) the density power spectrum goes as \( P(k) \propto k \), therefore we can safely assume that the power is effectively quite small for scales larger than a certain scale. This allows us to use the Limber approximation \[81\] when calculating the convergence power spectrum. In fact, the Limber approximation shows that when there is no power on scales larger than a particular scale \( L \) the integration needed to obtain the 2-point statistic in Fourier space is remarkably simplified implying only one integration along the line of sight rather than a double one. For our purposes, taking the Fourier transform of eq. (2.52) and calculating the 2-point function, we have:
\[ C_\kappa(\ell) = \frac{9H_0^4\Omega_{m,0}^2}{4c^2} \int_0^{\chi_p} d\chi \frac{g^2(\chi)}{[f_K(\chi)]^2} P \left( k = \frac{\ell}{f_K(\chi)}, \chi \right) , \tag{2.54} \]
where \( P(k, \chi) \) is the density power spectrum and \( \ell \) is the Fourier counterpart of the real space angular variable \( \theta \). However, because of the assumption made in the Limber approximation regime, we expect it to fail on large scales (\( \ell \to 0 \)). This has been discussed in Giannantonio et al. (2012) \[82\] where it is also shown that the Limber approximation works very well from \( \ell \) around a few hundred.

For our purposes the Limber approximation is a fair assumption and the convergence angular power spectrum in eq. (2.54) is all that we need. This is because in the weak lensing regime magnification \( \mu \) and shear \( \gamma \) have the same statistical properties for what concerns the 2-point function (e.g. \[3\]). In Fourier space, a derivative with respect to \( \theta \) is replaced by a multiplication by \( \ell \). As both magnification and shear are related to the second derivatives of the gravitational potential, then their 2-point statistics can only differ for a combination of \( \ell s \) when averages are taken. In particular, as the magnification is related to the convergence, it comes from the Laplacian of the gravitational potential. As shown in Section 3.1.2 of Bartelmann & Schneider (2001) \[3\] the deflection angle \( \alpha \) can be re-written in terms of the projected gravitational potential \( \phi \) as \( \alpha = 2\nabla \phi \), where
\[ \phi(\theta) = \frac{1}{2\pi} \int d^2\theta' \kappa(\theta') \ln |\theta - \theta'| . \tag{2.55} \]
Therefore, from the definition given in eq. (2.41) and (2.42), we have the following:

\[
\gamma_1 = \left( \frac{\partial^2}{\partial \theta_1^2} - \frac{\partial^2}{\partial \theta_2^2} \right) \phi,
\]
\[
\gamma_2 = 2 \frac{\partial^2}{\partial \theta_1 \partial \theta_2} \phi,
\]
\[
\kappa = \left( \frac{\partial^2}{\partial \theta_1^2} + \frac{\partial^2}{\partial \theta_2^2} \right) \phi.
\]

Taking the Fourier transform of the equations above we obtain:

\[
\tilde{\gamma}_1 = (\ell_1^2 - \ell_2^2) \tilde{\phi},
\]
\[
\tilde{\gamma}_2 = 2 \ell_1 \ell_2 \tilde{\phi},
\]
\[
\tilde{\kappa} = (\ell_1^2 + \ell_2^2) \tilde{\phi},
\]

where the tilde indicates the Fourier space components.

From these equations, we can immediately see that \((\ell_1^2 + \ell_2^2)^2 = |\ell|^4\) in the average \(\langle \kappa \rangle\). For the shear we have a different combination of second derivatives and hence we have \((\ell_1^2 - \ell_2^2)^2 + 4\ell_1^2 \ell_2^2 = |\ell|^4\) when calculating \(4\langle \gamma_1^2 + \gamma_2^2 \rangle\). Therefore, apart for a factor 4, the statistical properties are exactly the same, which means \(C_{\gamma}(\ell) = C_\kappa(\ell)\). For this reason we will always refer to the convergence power spectrum even when we discuss cosmic shear.

As can be easily seen from eq. (2.54), the weak lensing signal is sensitive, at the same time, to the geometry of the universe, the growth of structures and the cosmological content of the universe. For this reason weak lensing represents an extremely powerful tool to shed light on how the universe works and what it is made of.

### 2.5 Measuring ellipticities and sizes

Despite its power of study the cosmological model, weak gravitational lensing is often referred as challenging. The most difficult thing is indeed measuring image distortions. Over the years several methods have been proposed to try to extract lensing information from cosmological data.

A simple way to measure ellipticities and sizes of galaxies is by means of the measured surface brightness \(I(\theta)\) and its weighted quadrupole moments \(Q_{ij}\) described in Kaiser, Squires & Broadhurst (KSB) (1995) [83]:

\[
Q_{ij} = \frac{\int d^2 \theta W(\theta) I(\theta) \theta_i \theta_j}{\int d^2 \theta W(\theta) I(\theta)},
\]

(2.58)
assuming to choose a reference frame which has its origin at the centre of the galaxy image.

The surface brightness profile of a galaxy depends on the type of the galaxy itself. For elliptical galaxies, the surface brightness follows a de Vaucouleurs profile:

\[ I(R) = I_e \exp \left( -7.67 \left[ \left( \frac{R}{R_e} \right)^{1/4} - 1 \right] \right), \tag{2.59} \]

where \( R \) is the distance from the center of the image, \( R_e \) is the radius containing half of the luminosity of the galaxy, \( I_e \) is the surface brightness at \( R = R_e \).

Disk galaxies have profile which follows an exponential law:

\[ I(R) = I_0 \exp \left( -\frac{R}{R_h} \right), \tag{2.60} \]

where \( I_0 \) is the central surface brightness and \( R_h \) a scale length of the disk. Of course, the surface brightness profile of disk galaxies with an important central bulge feature can be described by a combination of eq. (2.59) and eq. (2.60).

By means of eq. (2.58), we can define the size \( r \) and the ellipticities \( \epsilon_1, \epsilon_2 \):

\[
\begin{align*}
    r &= (Q_{11} Q_{22} - Q_{12}^2)^{1/2} \\
    \epsilon_1 &= \frac{Q_{11} - Q_{22}}{Q_{11} + Q_{22}} \\
    \epsilon_2 &= \frac{2Q_{12}}{Q_{11} + Q_{22}}.
\end{align*}
\tag{2.61}
\]

The observed size \( r_O \) is related to the intrinsic size \( r_1 \) through the magnification: \( r_O = \mu(\theta) r_1 \). For galaxy ellipticities we have:

\[
\epsilon = \frac{\epsilon_{\text{intr}} + g}{1 + g^* \epsilon_{\text{intr}}},
\tag{2.62}
\]

where the asterisk indicates complex conjugation and quantities are \( \epsilon = \epsilon_1 + i\epsilon_2 \), \( g = g_1 + ig_2 \) and in case of weak lensing, the two relation for sizes and ellipticities reduce to:

\[
\langle r_O \rangle \simeq (1 + \kappa(\theta)) \langle r_1 \rangle \\
\langle \epsilon \rangle \simeq \langle \epsilon_{\text{intr}} \rangle + g(\theta).
\tag{2.63}
\]

So far weak lensing seems straightforward. But, unfortunately this is not the end of the story. In fact, when we try to recover ellipticities and sizes we face several problems. First of all, the estimated values of convergence and reduced shear from a general ellipticity and size measurement can depend on how those size and ellipticities are affected by lensing. This means that in general, the relation given above
Figure 2.3 – The forward process which leads a galaxy image from the sky to the detector. Estimation of shear from acquired images is the inverse problem. (Credit: Handbook for the GREAT08 challenge [84].

needs to be corrected for something which tells us how ellipticities and sizes change with respect to a certain applied convergence or reduced shear. This correction is generally called responsivity $\eta$, $R$ and we have:

$$\langle r_0 \rangle \simeq (1 + \eta \kappa(\theta)) \langle r_I \rangle$$

$$\langle \epsilon \rangle \simeq \langle \epsilon_{\text{intr}} \rangle + Rg(\theta).$$

In particular, given the subject of this thesis, we will discuss extensively the responsivity for sizes $\eta$ in Chapter 4.

For the shear responsivity $R$, generally it can be represented as a 2x2 tensor where the diagonal terms give an indication of how shear estimation in one direction affects ellipticity in the same direction whereas the off-diagonal terms describe how shear estimation in one direction affects ellipticity along the other direction. Generally, the former tend to be similar each other while the latter tend to be small.

Other important challenges in weak lensing arise from technical subtleties such as correction for Point Spread Function (PSF) anisotropies, pixelisation due to the detector resolution, Poisson noise coming from photon counts and noise originated by the fact that galaxies have intrinsic ellipticities.

We are not going to go into much detail here, but a fair representation of the problem is given in Fig. 2.3 where it is shown how the image of a galaxy is affected by cosmic shear and instruments to obtain the final image we use to extract information from.

Other alternatives to the KSB method described above have been proposed. Re-
fregier (2003) discussed the possibility to obtain shear measurements with a method based on the linear decomposition of each galaxy image by means of an orthonormal set of functions formed by Hermite or Laguerre polynomials, which are called shapelets. Hirata & Seljak (2003) proposed a method called Re-Gaussianization [85] based on an elliptical Laguerre expansion.

Another possibility is using a Bayesian shear estimate by means of a model fitting of the galaxy surface brightness profile. As described by Miller et al. (2007) [86], the main advantage of the Bayesian procedure with respect to the KSB approach is that the inclusion of an ellipticity prior should give an unbiased shear measurements. In fact, for each galaxy the Bayesian posterior probability \( p_i(\epsilon | y_i) \) is:

\[
p_i(\epsilon | y_i) = \frac{L(y_i|\epsilon)P(\epsilon)}{\int d\epsilon P(\epsilon)L(y_i|\epsilon)}.
\]

where \( P(\epsilon) \) is the ellipticity prior, \( L(y_i|\epsilon) \) is the likelihood of measuring the i-th value of the data \( y_i \) given an intrinsic ellipticity \( \epsilon \).

Summing over the data will produce:

\[
\left\langle \frac{1}{N}p_i(\epsilon | y_i) \right\rangle = \int dy \frac{L(y_i|\epsilon)P(\epsilon)}{\int d\epsilon P(\epsilon)L(y_i|\epsilon)} \int d\epsilon f(\epsilon)h(y|\epsilon),
\]

where \( f(\epsilon) \) is the probability distribution of intrinsic ellipticities and \( h(y|\epsilon) \) is the probability distribution for \( y \) given \( \epsilon \). Eq. (2.66) will give the true intrinsic ellipticity distribution when assuming that \( h(y|\epsilon) = L(y_i|\epsilon) \) and \( P(\epsilon) = f(\epsilon) \):

\[
\left\langle \frac{1}{N}p_i(\epsilon | y_i) \right\rangle = P(\epsilon) = f(\epsilon).
\]

Therefore, a good representation of the intrinsic distribution of the ellipticities as a prior provides an unbiased estimation of the posterior probability. This method is called lensfit [86] and other Bayesian methods are described in Bernstein et al. (2014) [87] and also reviewed in Bridle et al. (2009) [84].

An other interesting method for measuring shapes and sizes is given in Tewes et al. (2012) [88] and is called MegaLUT. This method corrects the measured galaxy ellipticity for biases through a machine learning code and results can be quite fast.

In conclusion, measuring shapes and sizes is the central problem of lensing measurements. Obtaining unbiased values for the ellipticities and the sizes of the galaxies is of primary importance in order to give the correct cosmological interpretation of the data.


2.6 Weak lensing by clusters

Although in this dissertation we are mainly interested to weak lensing induced by large-structure, we need to discuss briefly about the method to measure shear and convergence from surface density maps of galaxy clusters. Typically, strong lensing effects are associated with the most inner regions of clusters. However, at a certain distance there is a regime where distant background galaxies are weakly distorted as well.

The basic technique used to approach the problem of reconstructing the shear and convergence field from the 2-D density field has been first described by Kaiser & Squires (1993) [89]. The method is based on using eq. (2.57) in order to relate the Fourier transform of shear and convergence. In fact, from eq. (2.57) follows that:

\[ \tilde{\kappa} = \frac{\ell^2}{\ell_1^2 - \ell_2^2} \tilde{\gamma}_1, \]

\[ \tilde{\kappa} = \frac{\ell^2}{2\ell_1\ell_2} \tilde{\gamma}_2. \]  

(2.68)

We can consider a linear combination of these two equivalent definition of \( \tilde{\kappa} \) such as:

\[ \tilde{\kappa} = n \frac{\ell^2}{\ell_1^2 - \ell_2^2} \tilde{\gamma}_1 + m \frac{\ell^2}{2\ell_1\ell_2} \tilde{\gamma}_2, \]  

(2.69)

where \( n + m = 1 \) and then we minimise the variance \( \langle \kappa^2 \rangle \) and find \( n = (\ell_1^2 - \ell_2^2)^2 / \ell^4 \) to obtain:

\[ \tilde{\kappa} = \frac{1}{\ell^2} [((\ell_1^2 - \ell_2^2) \tilde{\gamma}_1 + 2\ell_1\ell_2 \tilde{\gamma}_2]. \]  

(2.70)

The bottom line of this method is that once the shear field is measured and its Fourier transform obtained, we need to solve for \( \tilde{\kappa} \) and then Fourier transform back to real space to have a convergence field map.

An issue related to this technique is that we actually measure the reduced shear \( g = \gamma / (1 - \kappa) \) rather than the shear \( \gamma \). Therefore a transformation of the magnification matrix as \( \mathcal{A} \rightarrow \lambda \mathcal{A} \) would produce the convergence to transform as \( 1 - \kappa' = \lambda(1 - \kappa) \) and the shear as \( \gamma' = \lambda \gamma \) and this means that we can distinguish the effect of such a transformation by measuring the reduced shear.

This issue is commonly called mass-sheet degeneracy because under this transformation the convergence becomes \( \kappa' = \lambda \kappa + (1 - \lambda) \), adding a sheet of surface mass density in the process of reconstruction of the convergence field. From eq. (2.47), under this sort of transformation we have that \( \mu \propto \lambda^2 \), therefore a possible method to remove the mass-sheet degeneracy is measuring the magnification in combination with the shear as recognised by Broadhurst et al (1995) [90]. We will review some magnification measurements in Section 2.8.
Another problem with the Kaiser & Squires method is that the reconstruction should be done ideally on the entire angular sky. However, observations are only made on finite field and this can introduce several systematic effects when obtaining the convergence map through this method. In order to reduce this problem due to boundary effects, Bartelmann (1996) [91] propose to reconstruct the gravitational potential locally through a least-chi squared method. We refer the reader interested in further improvements of this technique based on local reconstruction to Seitz & Schneider (1996) [92], Lombardi & Bertin (1998) [93], Seitz & Schneider (2001) [94].

2.7 Cosmic Shear

Weak lensing measurements on large-scales thus far have focused primarily on the shape distortions, or shear; cosmic shear correlations were first detected in 2000 by several groups, Bacon et al. (2000) [95], Kaiser et al. (2000) [96], Van Waerbeke et al. (2000) [97], Wittman et al. (2000) [98] and have since been significantly improved using surveys such as CFHTLenS (Canada-France-Hawaii Telescope Lensing Survey [99]). Shear measurements of weak lensing are a critical component of future surveys such as Euclid and LSST. Despite the challenging aspects involving cosmic shear, the fact that the typical scatter in ellipticity is of order of $\sigma_\epsilon \sim 0.3$ makes cosmic shear very attractive when the number of galaxies available is quite large.

The angular power spectrum in eq. (2.54) is valid for a 2-dimensional projected analysis of the lensing signal. In order to try to exploit also the redshift information we can perform a tomographic analysis as suggested in Hu (1999) [100]. This method consists in slicing the galaxy survey in redshift bins and then correlating the convergence field in each bin with the others. In this case given for instance two redshift bins labelled as $i$ and $j$, the angular convergence power spectrum will simply be:

$$C_{\kappa ij}(\ell) = \frac{9H_0^4\Omega_m^2}{4\ell^2} \int_0^\chi_p d\chi \frac{g_i^j(\chi)g_j^i(\chi)}{[f_K(\chi)]^2} P\left( k = \frac{\ell}{f_K(\chi)}, \chi \right), \quad (2.71)$$

where the lensing efficiencies $g^i$ are calculated in their respective bins.

Although lensing tomography allows one to extract more cosmological information, in a statistical sense, a full 3D analysis of weak lensing can provide even better results. 3D weak lensing has been described by Heavens (2003) [101] using a method to recover the 3-dimensional shear field through the use of spherical harmonics, Bessel functions and photometric redshifts for the distances of the sources. Measurements of cosmic shear by means of 3D lensing can be found in Kitching et al. (2007) for the COMBO-17 survey [102] and Kitching et al. (2014) for CFHTLenS data [103].
In general, lack of information on the redshift distribution of the sources is always a problem in weak lensing. This is because, as shown in both eq. (2.54) and (2.71) the redshift distribution is a key ingredient of the shear correlation function and errors can propagate to lead to incorrect estimation of the cosmological parameters. Obtaining spectroscopic redshifts for a large survey provides more precision information but is currently not realistically feasible because of the required observation time. Photometric redshifts, instead, provide a quicker alternative even though errors are generally larger. Also, as we will see in Section 2.7.1, photometric redshift errors are crucial when analysing intrinsic correlations of galaxy ellipticities.

In the weak lensing regime, observed ellipticities $\epsilon_{\text{obs}}$ are related to cosmic shear and the intrinsic ellipticity of each galaxy $\epsilon_{\text{intr}}$ as follows:

$$\epsilon_{\text{obs},a} \simeq \epsilon_{\text{intr},a} + \gamma_a,$$  \hspace{1cm} (2.72)

where the label $a = 1, 2$ indicates the vector components of shear and ellipticity.

The 2-point correlation function between observed ellipticities between redshift bins $(i, j)$ is:

$$\langle \epsilon_{\text{obs},a}^i \epsilon_{\text{obs},a}^j \rangle = \langle \epsilon_{\text{intr},a}^i \epsilon_{\text{intr},a}^j \rangle + \langle \epsilon_{\text{intr},a}^i \gamma_a^j \rangle + \langle \gamma_a^i \epsilon_{\text{intr},a}^j \rangle + \langle \gamma_a^i \gamma_a^j \rangle.$$  \hspace{1cm} (2.73)

If galaxies had random intrinsic ellipticities then we could neglect the first three terms of (2.73). In that case any correlation in observed ellipticities arises from weak gravitational lensing and measuring those correlations would give us the cosmic shear correlation function and a direct estimation of the matter power spectrum and eventually of the cosmological parameters. Correlations of intrinsic ellipticities may lead to bias in the estimation of the weak lensing signal from cosmic shear. These intrinsic correlations are commonly referred as Intrinsic Alignments (IAs) and we will discuss their theoretical basis and the observational constraints on them in Section 2.7.1.

Intrinsic alignments and spurious correlations introduced by systematic errors in shear measurements processes can be in principle seen through contamination in B-modes. In fact, analogously to electromagnetism, since the shear field is symmetric, it can be decomposed in its electric, i.e. curl-free, part (which we indicate by E) and its magnetic, i.e. divergence-free, part (which we label by B).

From the magnification matrix we can define the 2x2 trace-free shear tensor field as:

$$\gamma_{ab}(\theta) = \begin{pmatrix} \gamma_1(\theta) & \gamma_2(\theta) \\ \gamma_2(\theta) & -\gamma_1(\theta) \end{pmatrix},$$  \hspace{1cm} (2.74)

where $\theta = (\theta_1, \theta_2)$ and we can build the E and B part in terms of the derivatives of
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\[ \nabla^2 \gamma_E(\theta) = \partial_a \partial_b \gamma_{ab} \]
\[ \nabla^2 \gamma_B(\theta) = \varepsilon^{ad} \partial_b \partial_d \gamma_{ab}, \]

where \( \varepsilon^{ad} \) is the Levi-Civita symbol in two dimensions.

Applying the Fourier transform on the components of the shear tensor, the curl-free and divergence-free part are (e.g. [104], [105]):

\[ \tilde{\gamma}_E(\ell) = \left( \frac{\ell_1^2 - \ell_2^2}{\ell_1^2 + \ell_2^2} \right) \tilde{\gamma}_1(\ell) + 2\ell_1 \ell_2 \tilde{\gamma}_2(\ell) \]
\[ \tilde{\gamma}_B(\ell) = \frac{2\ell_1 \ell_2 \tilde{\gamma}_1(\ell) - (\ell_1^2 - \ell_2^2) \tilde{\gamma}_2(\ell)}{\ell_1^2 + \ell_2^2} . \]

The E-B mode shear power spectrum can be built from the correlation function \( \xi_1(\theta) = \langle \gamma_1(0) \gamma_1(\theta) \rangle \), \( \xi_2(\theta) = \langle \gamma_2(0) \gamma_2(\theta) \rangle \), and \( \xi_\times = \langle \gamma_1(0) \gamma_2(\theta) \rangle \) as follows (e.g. [105]):

\[ C_{EE}(\ell) = \frac{\pi}{2} \int d\theta \theta (\xi_1(\theta) + \xi_2(\theta)) J_0(\ell \theta) + [\xi_1(\theta) - \xi_2(\theta)] J_1(\ell \theta)) \]
\[ C_{BB}(\ell) = \frac{\pi}{2} \int d\theta \theta (\xi_1(\theta) + \xi_2(\theta)) J_0(\ell \theta) - [\xi_1(\theta) - \xi_2(\theta)] J_1(\ell \theta)) \]
\[ C_{EB} = 2\pi \int d\theta \theta \xi_\times(\theta) J_1(\ell \theta) . \]

where \( \theta = \sqrt{\theta_1^2 + \theta_2^2} \). Usually in literature, the correlation functions \( \xi_1 \) and \( \xi_2 \) can be found arranged as \( \xi_+ = \xi_1(\theta) + \xi_2(\theta) \) and \( \xi_- = \xi_1(\theta) - \xi_2(\theta) \). As pointed out in Stebbins (1996) [104] and Kamionkowski et al. (1998) [105], density perturbations produce only scalar perturbations to the metric of the space-time and therefore they are curl-free. Vector and tensor modes are safely negligible; although gravitational waves, i.e. tensor modes, can produce weak lensing signal but, again, this is expected to be very small. We can then say that weak gravitational lensing, in principle, only produce E-modes. However, B-modes can also be found in weak lensing analyses. For instance, correlations of intrinsic galaxy shapes, usually referred as Intrinsic Alignments, can produce B-modes in addition to contamination of the E-modes and biasing of the cosmological parameter estimation (see Section 2.7.1). Other sources of B-modes can be clustering of galaxies [106], correction for Born approximation and coupling of lens systems at different redshifts [78].

Additionally, the presence of B-modes can be seen to indicate the presence of some systematic effects in the data, as described in Section 2.5, and B-modes are often used as a test for such systematics.

Normally, we measure ellipticity with respect to some reference frame on the
sky. We call these ellipticities $\tilde{\epsilon}_1$ and $\tilde{\epsilon}_2$. In order to align the x-axis of this reference frame with the line connecting to galaxies, we must perform a rotation in both points where these galaxies are located, as follows:

$$\epsilon_1 = \tilde{\epsilon}_1 \cos(2\phi) + \tilde{\epsilon}_2 \sin(2\phi)$$
$$\epsilon_2 = -\tilde{\epsilon}_1 \sin(2\phi) + \tilde{\epsilon}_2 \cos(2\phi).$$

(2.78)

It is easy to see that the ellipticity is unchanged under a transformation of 90 degrees. Under a parity transformation, for instance reflecting the line connecting the two galaxies, the component $\epsilon_1$ does not change its sign but the component $\epsilon_2$ does. If we want invariance under such transformations we require that $\xi_x$ and therefore $C_{EB}$ both vanish. However, as highlighted by Schneider et al. (2002) [106], since cosmic shear is measured from a finite set of data, the cross-power term may be not zero due to cosmic variance and, therefore, a measurement of $C_{EB}$ can be useful as lower limit on the error estimate of the other power spectra.

2.7.1 Intrinsic Alignments

Intrinsic ellipticities are not randomly distributed and this can lead to intrinsic correlation of galaxy shapes, also called intrinsic alignments (IAs). Recent review of this topic can be found in Kirk et al. (2015) [107], Krause et al. (2015) [108], Troxel & Ishak (2015) [16]. Historically, soon after the detection of cosmic shear on large scale, several authors pointed out the problem of intrinsic alignments. Brown et al. (2002) [109] measured a non-zero correlation between intrinsic ellipticities. In the same period other authors (e.g. Croft & Metzler (2000) [110], Heavens et al. (2000) [111], Catelan et al. (2001) [112], Crittenden et al. (2001) [113]) have proposed some models for intrinsic alignment of galaxies. Before reviewing these models and also observational results about intrinsic alignments we need to explain the nature of the first three terms (the intrinsic alignment terms) on the right-hand-side in eq.(2.73).

The first term $\langle \epsilon_{i_{\text{intra}}} \epsilon_{j_{\text{intra}}} \rangle$ is called II term (intrinsic-intrinsic correlation); this term represents the correlation between intrinsic ellipticities. Neglecting the II signal leads to an overestimate of the shear correlation function because we would count pairs which are not correlated due to gravitational lensing. By using redshift information one can recognise physically close pairs which are aligned because those likely formed in a similar environment, for instance undergoing the same tidal gravitational field. The second and third terms $\langle \epsilon_{i_{\text{intra}}} \gamma^i \rangle + \langle \gamma^i \epsilon_{j_{\text{intra}}} \rangle$ are called GI terms (density-intrinsic ellipticity correlation); this term is subtler to understand than the previous one and the useful way to gain intuition is by using Fig.2.4.

As can be seen from the figure above, unlike the II correlation, the GI term involves galaxies at different redshifts. The galaxy ellipticities appear negatively
correlated when viewed on the sky. The background galaxy image (red object) is tangentially aligned because of lensing distortion of the light coming from that galaxy; instead the blue object is physically close to the lens (indicated by DM in Fig. 2.4) and is radially aligned because of the mutual gravitational interaction. In this case, by neglecting the GI signal we would underestimate the shear correlation function because we would add negatively correlated pairs up getting a lower shear correlation. Intrinsic alignments may affect seriously cosmological parameters constraints by using weak lensing. For instance Bridle & King (2007) [114] have shown that the dark energy equation of state may be mis-estimated by around 50 per cent for a non-tomographic analysis. This mis-estimate only applies if all cosmological parameters are known except the Dark Energy equation of state. In fact they find that if several cosmological parameters are fitted simultaneously the mis-estimate becomes even worse. A quantification of this bias can be found in Kirk, Bridle & Schneider (2010) [115]. Several authors have faced this subject both on the observational and theoretical side.

Some methods have been proposed in order to mitigate the intrinsic alignment effects. In order to minimise the intrinsic signal, King & Schneider (2002) [116] have proposed to give less weight to a galaxy pairs where the redshift difference is small. This method can be applied only for II correlation (it does not take into account GI correlation which was considered for the first time by Hirata & Seljak (2004) [117]).

In Joachimi & Schneider (2010) [118] the authors demonstrate that by using tomographic cosmic shear data and a particular weighting scheme, it is possible both to null the GI signal and extract it. Since this method makes use of the characteristic dependence of the intrinsic alignment and lensing signal on redshift, high quality redshift measurements are required because the nulling performance strongly depends on the photometric redshift accuracy.

Another method is given in [118], where the authors proposed to use joint cosmic shear and galaxy number density correlations. The main idea of this method is to

\[\text{(Actually, this paper came out in 2004 but there was an error in one of the IA power spectrum equations. Then, the corrected version has been published in 2010. We always refer to the corrected version in this dissertation.)}\]
use both ellipticity and number density statistics in order to get some information about weak gravitational lensing and then constrain cosmological parameters. Indeed, ellipticities depend on gravitational lensing because cosmic shear can modify the shape of the galaxies; also gravitational lensing modifies the flux of the objects and thus reduces or increases number counts of galaxies above a certain limiting magnitude. A large number of nuisance parameters are required for this analysis in order to take into account uncertainties in the knowledge of the bias parameter, intrinsic alignment, photometric redshift. By marginalising over these parameters it is possible to constrain cosmological parameters but since a large number of parameters have to be used, this leads to a loss of constraining power (see also [115]).

Theoretical models of IAs

Regarding the theoretical models, a general state-of-art about modelling intrinsic alignment is given in Catelan et al. (2001) [112] and Hirata & Seljak (2004) [117]. These authors have proposed two different models for the intrinsic alignment depending on the galaxy type.

For early type galaxies (ellipticals) a linear model has been proposed. The attribute linear refers to the fact that in this model the intrinsic ellipticity of a galaxy is assumed to follow the linear relation [117]:

\[
\epsilon_{\text{intr}} = (\epsilon_{\text{intr,+}}, \epsilon_{\text{intr,\times}}) = -\frac{C_1}{4\pi G} (\nabla_x^2 - \nabla_y^2, 2\nabla_x \nabla_y) \Phi_P
\]  

(2.79)

where \(\Phi_P\) is the Newtonian potential at the time of galaxy formation (assumed to be during matter domination) and \(C_1\) is a normalisation constant.

In this model, the observed shape of a galaxy may be determined (at least in part) by the shape of its halo. When a spherical overdensity undergoes gravitational collapse to form a galaxy in a region of constant tidal gravitational field, the difference between the acceleration on the two sides makes the resulting collapse anisotropic (see Fig.2.5).

According to the linear model and by using the Poisson equation \((\Delta \Phi \propto \delta)\) which relates gravitational potential to linear density field, both the II and GI power spectrum are proportional to the linear matter power spectrum [117].

\[
P_{\text{II}}(k) = A^2 \frac{C_1 \bar{\rho}^2 a^4}{D^2} P_\delta^{\text{lin}}(k),
\]  

(2.80)

\[
P_{\text{GI}}(k) = -A \frac{C_1 \bar{\rho} a^2}{D} P_\delta^{\text{lin}}(k).
\]  

(2.81)

For late type galaxies (spirals) a quadratic model has been proposed. In this model galaxy ellipticities are correlated because their angular momentum are cor-
related. Basically, the gravitational tidal field exerts a torque on the protogalaxy generating angular momentum. Close galaxies undergoing the same tidal field may have correlated angular momenta. As a consequence, disks are viewed with similar inclination and ellipticity correlations may arise. The angular momentum that arises from this mechanism is:

\[ L_i \propto \varepsilon_{ijk} I_{jn} D_{kn}, \tag{2.82} \]

where \( \varepsilon_{ijk} \) is the Levi-Civita symbol, \( I \) represents the inertia tensor, \( D_{kn} = \partial_k \partial_n \Psi \) is the tidal gravitational field. According to this model and taking the line of sight to be the \( \hat{x} \)-axis, the observed ellipticity is related to the angular momentum ([112]):

\[ \epsilon_{\text{intr}} \propto (L_y^2 - L_z^2; 2L_y L_z), \tag{2.83} \]

which is a quadratic model in the sense that we have a dependence on the matter power spectrum squared. As pointed out by [117], in the quadratic model there is no GI signal expected at leading order when assuming a Gaussian density field, a linear bias model for galaxies and a linear evolution of the density field.

**Status of observations**

The models described above can reproduce the data only on large scales because on small scales non-linear effects become non-negligible. An example of this is given by Joachimi et al. (2011) [119] where observational data (MegaZ-LRG sample from SSDS) are compared to the linear alignment model described above (see Fig.2.6).

In order to account for this gap on small scales given by non-linear effects, Schneider & Bridle (2010) [120] have built a model by populating a spherical dark matter halo with galaxies, with a number density that follows the density profile of the
Figure 2.6 – Comparison between MegaZ-LRG data and linear alignment model for the projected GI correlation. The dashed line is the model using the linear matter power spectrum; the solid line is the model using non-linear corrections. Credit: Joachimi et al. (2011) [119].

halo. They assume the universe is entirely made up of haloes that contain a single central galaxy with an orientation determined by the curvature of the large scale potential (i.e. the linear alignment model) and satellite galaxies distributed spherically according to the halo dark matter profile that are oriented pointing at the centre of the halo. However their model could be more realistic as they also assume that all haloes are spherical and all satellites point at the halo centre, subject to a random misalignment angle. They also assume that there is only a single population of satellite galaxies and a single population of central galaxies; a more complex approach would include a mix population of spiral and elliptical galaxies with different alignment models for each. A final assumption is that the central galaxies are aligned with their host halo.

More recently, in order to give a better representation of the behaviour of intrinsic alignments on non-linear scales, Blazek et al. (2015) [121] presented a model for tidal alignment of galaxies based on perturbation theory.

As we have seen in Fig.2.6, early type galaxies show some intrinsic alignment signal. Some studies has been carried out for late type galaxies as well (for example see Mandelbaum et al. (2011) [122]); in that case no intrinsic correlation has been observed. That could mean either there is some astrophysical reason for which the correlation vanishes or the signal is too small to be detected. A general summary of all the observation done so far is given in Fig.2.7.

Results for intrinsic alignments have also come from the CFHTLenS data ([123]). These results confirm the importance of including intrinsic alignment in weak lensing
analyses and also the intrinsic alignment correlations for both early type and late type galaxies are consistent with the previous works (see Fig.2.8).

As we have already mentioned, current models cannot reproduce the data behaviour on small scales. This problem is still unsolved and in order to overcome that the halo model approach to intrinsic alignments given in Schneider & Bridle (2010) [120] has been proposed. But even the halo model does not solve the problem effectively.

Hirata et al. (2007) [124] proposed to replace the linear matter power spectrum by the non-linear matter power spectrum in the equations (2.80) and (2.81); this model is called the non-linear alignment model (NLA model) to distinguish it from the linear one. By using this non-linear power spectrum we have the best fit between model and data. We can see that in Fig.2.6 where the solid line is the NLA model. However, as pointed out in [124], "the theoretical justification for believing this new model to trace the matter power spectrum in the non-linear regime is dubious".

Recent observations provided by Singh et al. (2015) [125] show detection of II and GI signal with SDSS BOSS LOWz data, in particular also they find that central galaxy in clusters are aligned with their haloes and satellite galaxies are radially aligned within groups. On the other hand, Sifon et al. (2015) [126] find no detection of radial alignments using galaxy clusters from CFHT data.
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Figure 2.8 – Results from CHFTLenS: (a) contour plots showing the difference between using or not intrinsic alignments in the model; (b) contour plots (the parameter $A$ is the same as given in eq.(2.80) and eq.(2.81)) showing estimation of intrinsic alignment correlations for different samples of galaxies Credit: Heymans et al. (2013) [123].

From the theoretical side, simulations can be useful to constrain the impact of intrinsic alignments, especially hydrodynamical simulations (see for instance [127]). On the observational side, the kind of ellipticity used to measure intrinsic alignments can be extremely important in determining the amplitude of the intrinsic correlations, as shown in Singh & Mandelbaum (2015) [125] again with SDSS BOSS LOWz data. These results show that the study of intrinsic alignments has to be carried out carefully. Although intrinsic alignments are often seen as weak lensing systematics, actually they contain important information about how galaxies formed and evolved and it is worthwhile to study them in any case.

2.8 Cosmic magnification

Along with shape distortions, gravitational lensing also modifies galaxy sizes and fluxes. This is effect is called cosmic magnification and in this Section we are going to briefly review some of these measurements. Note that information on shape and size distortions come from the same surveys, thus it is essential to study how we can exploit the combination of the full signal coming from cosmic shear and cosmic magnification in order to obtain better constrains on the cosmological model.

Gravitational lensing magnification in the weak limit has two effects. First, the flux received from distant sources is increased, resulting in a relatively deeper apparent magnitude limited survey. Secondly, the solid angle is stretched, diluting the surface density of source images on the sky. The net result of these competing effects is an induced cross-correlation between physically separated populations
that depends on how the loss of sources due to dilution is balanced by the gain of sources due to flux magnification. Cosmic magnification, leading to coherent size and brightness distortions, has been also been observed but not to the same extent as shear.

In the weak lensing regime $|\kappa|$ and $|\gamma| \ll 1$, so the magnification is approximately $\mu \simeq 1 + 2\kappa$. Therefore, in weak lensing limit, the observed galaxy sizes and fluxes, $r$ and $F$, are related to their intrinsic values by:

$$r \rightarrow \mu^{1/2}r = (1 + \kappa)r$$
$$F \rightarrow \mu F = (1 + 2\kappa)F.$$  \hfill (2.84)

Magnification has been primarily probed through the cross-correlation between foreground galaxies and background objects selected by their flux, known as flux magnification or magnification bias. This happens because surveys are normally limited in magnitude. Sources beyond the threshold can be magnified and brought into the sample, modifying the magnitude distribution. Generally, this can arise also when there is a particular size cut in the survey. The implications of these survey cuts on the number counts of sources are discussed in Schmidt et al. (2009a) [128] and the implication for cosmic shear can be found in Schmidt et al. (2009b) [129]. In Chapter 4 we will discuss how the magnification bias affects the intrinsic size-magnitude distribution of a sample of galaxies in the context of modelling the intrinsic correlations of galaxy sizes and magnitudes.

### 2.8.1 Number counts

We are now going to briefly describe how this cross-correlation can be used to detect magnification. Following the notation in Scranton et al. (2005) [15], we call $N_0(F)dF$ the number of sources with a flux in the range $[F, F + dF]$ and $n(F)dF$ the corresponding number of lensed sources after the magnification $\mu$. We assume that the number counts follow a power law such as $n_0(F) \propto F^{-s(F)}$. Therefore, the effects of dilution of the surface density and stretching of the solid angle can be described as follows:

$$N(F)dF = \frac{1}{\mu} \left( \frac{F}{\mu} \right) \frac{dF}{\mu} = A_0\mu^{-2} \left( \frac{F}{\mu} \right)^{-s(F/\mu)} dF,$$ \hfill (2.85)

where $A_0$ is a normalisation constant.

We assume in the weak lensing regime, i.e. $\mu \sim 1$, that the index $s$ does not change in the interval $[F, F\mu]$. We can then write the number counts as:

$$N(F)dF = \mu^{s-2}N_0(F)dF.$$ \hfill (2.86)
All of this can be reformulated in terms of magnitudes:

\[ N(m)dm = \mu^{2.55-1}N_0(m)dm = \mu^{\alpha-1}N_0(m)dm. \]  

(2.87)

This can be exploited when cross-correlating background sources and foreground lenses. Let us call the source number density \( N_S(\theta) \). Then we can write:

\[ \frac{N_S(\theta) - \langle N_S \rangle}{\langle N_S \rangle} \simeq 2(\alpha - 1)\kappa(\theta), \]  

(2.88)

where the factor 2 comes from the deviation from unity of magnification, \( \mu = 1 + 2\kappa \), as shown below eq. (2.47). For the foreground galaxies \( N_G(\theta) \), we can link their number counts fluctuations to the underlying density field \( \delta \) assuming that galaxies are tracers of the matter distribution through a parameter of bias \( b \), which we assume here constant for simplicity:

\[ \frac{N_G(\theta) - \langle N_G \rangle}{\langle N_G \rangle} \simeq b\delta(\theta). \]  

(2.89)

The quasar-galaxy cross-correlation hence becomes:

\[ w_{GS}(\theta, m) = 2(\alpha(m) - 1)\kappa(\theta)\delta(0). \]  

(2.90)

In practice, quasars are considered over a large range of magnitudes. Therefore the slope of the power law \( \alpha \) could change as it is function of \( m \). For this reason people use a mean value for the slope, calculated as follows:

\[ \langle \alpha - 1 \rangle = \frac{\int dm N(m)[\alpha(m) - 1]}{\int dm N(m)}. \]  

(2.91)

We can also write the cross-correlation in terms of power spectra by means of the Limber approximation as we have done for the convergence power spectrum in eq. (2.54). Given a redshift distribution for the sources \( n_S(\chi) \) and for the foreground galaxies \( n_G(\chi) \):

\[ C_{GS}(\ell) = \frac{3H_0^2\Omega_m\sigma_b}{c^2} b \int_0^{\chi_p} d\chi \frac{g_S(\chi)n_G(\chi)}{f_K(\chi)} P\left( \frac{\ell}{f_K(\chi)}, \chi \right), \]  

(2.92)

where we have:

\[ g_S(\chi) = \frac{f_K(\chi)}{a(\chi)} \int_\chi^{\chi_p} d\chi' n_S(\chi') \frac{f_K(\chi' - \chi)}{f_K(\chi')}. \]  

(2.93)

This was first detected using background quasars by Scranton et al. (2005) [15] with Sloan Sky Digital Survey (SDSS) data (see Fig. 2.9) and other background sources, such as Lyman-break galaxies, have since been used to study the mass profiles of
Figure 2.9 – Detection of cosmic magnification with SDSS data from Scranton et al. (2005) [15]. Measurements represents the quasar-galaxy cross-correlation as a function of quasar g-band magnitude given in eq. (2.90) where the label Q indicates that the sources are quasars. Errorbars represent 1σ errors.

dark matter haloes (e.g. Hildebrandt et al. (2009) [130], Van Waerbeke et al. (2010) [131], Hildebrandt et al. (2011) [132], Ford et al. (2012) [133], Hildebrandt et al. (2013) [134], Bauer et al. (2014) [135]). Such galaxy-galaxy lensing can be combined with shear measurements as a complementary weak lensing probe which allows us to control systematics and cosmological parameters constraints [136, 137].

2.8.2 Moments of the magnitude distribution

Another method to measure magnification instead of using number counts is looking at the shift of the moments of the magnitude distribution. For instance, one can measure the shift in the mean magnitude:

$$\langle m \rangle = \frac{I_1}{I_0} = \frac{\int_{m_a}^{m_b} dm N(m) m}{\int_{m_a}^{m_b} dm N(m)}$$

where the limits of integration simply account for the fact that in practice one measure the number counts only in a finite magnitude bin \([m_a, m_b]\). It is easy to see that a shift can be only seen if the slope \(\alpha(m)\) is not constant in the considered magnitude interval.

A galaxy can act as a lens, magnifying the flux \(F_{\text{obs}} = \mu F\), but it is also respon-
sible for a potential reddening of the background source due to the presence of dust. This latter effect can be included in the flux magnification as follows:

\[ F_{\text{obs}} = \mu e^{-\tau_F} F, \]  

(2.95)

where \( \tau_F \) is a function called optical depth which gives an indication for dust extinction at a given wavelength \( \lambda \). The magnitude shift at that wavelength then will be:

\[ \delta m_{\lambda} = -2.5 \log_{10} \mu + \frac{2.5}{\ln 10} \tau_F, \]  

(2.96)

recalling that \( m \propto -2.5 \log_{10} F \).

This basically means that, generally, magnification and extinction are in competition, with the main difference that magnification is fully achromatic while dust extinction depends on the wavelength. Multiband measurements can therefore help to separate these two effects. The magnitude shift in eq. (2.96) can be written in terms of the magnitude distribution as shown in Menard et al. (2010) [138]:

\[ N(m) \propto N_0(m - \delta m_{\text{ind}}), \]  

(2.97)

where the induced magnitude shift (when dust effects are neglected) is \( \delta m_{\text{ind}} = -2.5 \log_{10} \mu \). This means that there is a shift in the mean magnitude:

\[ \delta m_{\text{obs}} = \langle m \rangle - \langle m_0 \rangle, \]  

(2.98)

which is the difference between the lensed and the unlensed mean magnitude. Generally, for measurements done in a magnitude bin \([m_a, m_b]\), the observed shift \( \delta m_{\text{obs}} \) of the mean magnitude of the sources is different from the induced shift \( \delta m_{\text{ind}} \) by the foreground population of galaxies. However, when the shift is quite small compared to the limits magnitudes of the bin, we can use the following approximation given by Menard et al. (2010) [138] and Hildebrandt (2015) [139]:

\[ \delta m_{\text{obs}} \approx C_S \delta m_{\text{ind}}, \]  

(2.99)

where:

\[ C_S = 1 - \frac{1}{I_0} \{ [N(m_b)m_b - N(m_a)m_a] - \langle m_0 \rangle [N(m_b) - N(m_a)] \}. \]  

(2.100)

Hildebrandt (2015) [139] has shown that potentially these cross-correlations are affected not only by dust extinction but also by several other problems such as photometric noise and colour selection. This means that measuring only the slope of the source number counts is not sufficient. The main reason for this is that it
Figure 2.10 – Projected surface mass density measured around groups using the estimator by Schimdt et al. (2012) [140]. The solid and dotted lines represent respectively the best-fitting model for the mass distribution of the clusters using magnification and shear.

requires a perfect knowledge of the unlensed number counts whereas actually number counts are always affected by noise. Also, it is commonly assumed that properties such as the colour distribution characterising the the source sample are not changing under the effect of cosmic magnification. However, Hildebrandt (2015) [139] shows that the main contamination comes from dust extinction, which has then to be modelled in the simple theory of the magnitude shift reviewed above.

2.8.3 Cosmic magnification with sizes and magnitudes

Cosmic magnification can also be detected directly using galaxy sizes and magnitudes along with galaxy shape in order to reconstruct the mass distribution of galaxy clusters (Bartelmann et al. (1996) [91]) because size information is already available from lensing surveys.

The first measurement of weak lensing magnification using galaxy sizes and fluxes has been presented by Schmidt et al. (2012) [140] with COSMOS Hubble Space Telescope galaxies where they measured the projected surface density using an estimator which combines sizes and magnitudes. They find a signal-to-noise only a factor of around 2 less than shear; this can be seen in Fig. 2.10 where measurements of shear and magnification for the same sample of galaxies are shown, when adopting the best-fitting model for the mass distribution of the clusters using a Navarro-Frenk-White (NFW) profile (Navarro, Frenk & White (1996) [141], see also Section 3.2.3).
Huff & Graves (2014) [142] also detected magnification using sizes measured by means of the Fundamental Plane relation for early-type galaxies in SDSS data and, more recently, Duncan et al. (2016) [143] presented the first measurement of individual cluster mass estimates using weak lensing size and flux magnification from the HST STAGES survey data which seems to be promising in terms of signal-to-noise for future exploiting of size and flux magnification in cosmology.

Casaponsa et al. (2013) [144] studied the extent that size magnification can be used as a complement to cosmic shear, investigating how observational limitations can affect this kind of measurement. By means of simulated images of galaxies and using the \textit{lensfit} measurement method, they find that unbiased estimation of the convergence is possible using galaxies with angular sizes larger than the point spread function by a factor at least 1.5 and a signal-to-noise ratio greater than 10.

There are several good reasons for using size and magnitude information along with cosmic shear. The first reason is that we have size and magnitude information already available from a lensing survey, and, ideally, one should exploit all of the statistical power to constrain the cosmological model. Furthermore, using different weak lensing probes can be important to mitigate the impact of shape distortion systematics. Heavens, Alsing & Jaffe (2013) [17] showed that combining size and shape information from weak lensing measurements could, in principle, improve our current constraints on cosmological parameters compared to using only cosmic shear. Additionally, they show that size measurements can be largely uncorrelated with shape measurements if the square root of the area of the galaxy image is used as size estimator; in this case, as shown in Fig. 2.11, the Figure of Merit for the dark energy equation of state parameter can be greatly improved, although the analysis in [17] does not account for intrinsic correlation of galaxy sizes.

### 2.8.4 Intrinsic size and magnitude correlations

Recently, Alsing et al. (2015) [18] extended the analysis of Heavens, Alsing & Jaffe (2013) [17] in order to quantify the convergence dispersion expected from size measurements and the possible impact of intrinsic correlation of galaxy sizes and fluxes. They find that the gain in the constraining power depends on the amplitude of these intrinsic correlations. By simply assuming that these intrinsic size and flux correlations are proportional to the matter power spectrum, they show that combining size and shape information can improve our constraining on cosmological parameters. More in detail, in Alsing et al. (2015) [18] they assume, for the intrinsic size and magnitude spatial correlations, a size-magnitude bias of $\beta = 0.05$, such that
Figure 2.11 – Relative improvement in marginal errors of dark energy equation of state parameters: shape alone is shown in red (outer), and with size added in blue (inner). The Figure of Merit is increased by over a factor of 4. Credit: Heavens, Alsing & Jaffe (2013) [17].

their model for these intrinsic correlations is:

$$P_{\kappa \kappa}(k, z) = \beta^2 P_{\delta \delta}^{\text{lin}}(k, z),$$

$$P_{\delta \kappa}(k, z) = \beta \sqrt{P_{\delta \delta}(k, z) P_{\delta \delta}^{\text{lin}}(k, z)},$$

(2.101)

where $P_{\delta \delta}^{\text{lin}}$ is the linear matter power spectrum and $P_{\delta \delta}$ is the matter power spectrum including non-linear corrections.

Particularly, including intrinsic alignments and intrinsic size and flux correlations, and using a scheme of nuisance parameters for the marginalisation, the Figure of Merit for the dark energy equation of state parameters, even though being worse than the ideal case with no intrinsic correlations, are still better than the shear plus intrinsic alignments case, with an improvement between 25 and 65 per cent, as shown in Fig. 2.12.

From the observational point of view, it is important to study the amplitude of intrinsic size correlations or at least place upper bounds on them. In the literature there are several studies that try to study whether the size of a galaxy is function of the environment. Although there are no 2-point correlation function measurements yet of size correlations, there are several attempts to understand whether there exist differences in size between cluster and field galaxies. These attempts have been made especially from the point of view of studying galaxy formation and evolution and mostly regarded the study of the size-stellar mass relation in different environments.

There are some recent claims of dependence of galaxy size on the environment
which also depend on the galaxy type. For early-type galaxies, some authors [145, 146] find no evidence for a dependence on the environment of the galaxy size-mass relation at low redshifts, whereas in [147] the authors find that early-type galaxies have smaller sizes in clusters than in the field. At higher redshift some studies [148, 149, 150, 151] find for early-type galaxies are larger in high density environments such as clusters and groups than in the field; however other authors, e.g. [146] do not find such a dependence. Additionally, for early-type galaxies there are also studies which seem to indicate that those galaxies are brighter in denser environments [152, 153, 154].

For late-type galaxies, some authors claim that there is a dependence of the mean size on the environment at low redshift [145, 155, 156] even though this dependence does not seem to appear at higher redshift, e.g. [150]. Regarding luminosities, in [152, 153, 154] the authors claim that late-type galaxies are brighter in denser environments but also other studies, e.g. [157], do not find such a dependence.

Additionally, recently in [158], the authors find no significant difference in the size distribution of cluster and field galaxies of a given morphology.

The general observational picture regarding size correlations is therefore quite unclear and it is essential to have a model with a physical motivation in order to deal with the intrinsic size correlations. The main topic of this dissertation is to develop such a physical model in order to account for the intrinsic correlations of galaxy sizes and fluxes and we will discuss this in Chapter 3 and 4.
Chapter 3

A first step: halo model for intrinsic size correlations

The mathematics is not there till we put it there.

Arthur Eddington

In this Chapter we investigate the degree to which intrinsic size correlations arise in a simple halo model, assuming the observed galaxy sizes correlate closely with the mass of the haloes and sub-haloes [159]. Larger and more massive galaxies live in more massive haloes, and even if the sub-halo population is largely independent of the halo mass, the sizes of the largest sub-haloes will still be limited by the total halo mass. We use this simple model to predict what would be observed for a magnification estimator based solely on the galaxy sizes, and how the intrinsic signal correlates with the true lensing convergence.

This simple model for how galaxy sizes may be intrinsically correlated is based on the halo model formalism [160, 161, 78, 162]. We first describe our implementation of the halo model itself, and will discuss its implications for galaxy sizes. The purpose of this Chapter is to develop a model for the intrinsic contribution to lensing magnification based on the halo model. For simplicity we start only showing the formalism for the size contribution for a two dimensional lensing analysis. In Chapter 4 we will see how the content of this Chapter can be easily extended to galaxy intrinsic luminosity correlations and lensing tomography.

The content of this Chapter is based on the work presented in Ciarlariello et al. 2015 [163].

A complete specification of the halo model requires knowing the halo mass function and the distribution of sub-halo masses within a halo; it also requires knowing the probability density profile of how sub-haloes are distributed in a halo and understanding the statistics of how haloes are distributed on large scales, usually
parameterised by the mass dependent bias function.

### 3.1 Magnification and intrinsic size correlations

We have seen in Section 2.3 that the determinant of the magnification matrix gives the cosmic magnification $\mu$ of a surface area element:

$$\mu = \frac{1}{\det A} = [(1 - \kappa)^2 - |\gamma|^2]^{-1}, \quad (3.1)$$

and that in the weak lensing regime $|\kappa|$ and $|\gamma| \ll 1$, so the magnification is approximately $\mu \simeq 1 + 2\kappa$.

Given eq. (3.1) for the relation between magnified and intrinsic surface area element, we can derive the relation between magnified and intrinsic angular sizes. The angular size, $\lambda$, of an object becomes

$$\lambda_O = (1 + \kappa)\lambda_I, \quad (3.2)$$

where the subscripts stand for the observed ($O$) and intrinsic ($I$) angular size of the galaxy; we define the intrinsic angular size to be the square root of the solid angle of the galaxy image. As pointed out by [17], this definition for the galaxy size is only expected to be uncorrelated with shape for galaxies with exponential profiles. In the weak lensing limit, this can be written

$$\ln \frac{\lambda_O}{\bar{\lambda}} \simeq \kappa + \ln \frac{\lambda_I}{\bar{\lambda}}, \quad (3.3)$$

where $\bar{\lambda}$ is the mean angular size at a given redshift. Then we use as our estimator the following one [140, 17, 77, 18]:

$$\hat{\kappa} = \ln \frac{\lambda_O}{\bar{\lambda}} - \left\langle \ln \frac{\lambda_O}{\bar{\lambda}} \right\rangle, \quad (3.4)$$

which has zero mean. Note that, relative to their average at a given redshift, the physical size of a galaxy $r$ is essentially a proxy for its observed angular size $\lambda$ because the angular diameter distance $D_A(z)$ is the same for the average:

$$\frac{r(z)}{\bar{r}(z)} = \frac{\lambda D_A(z)}{\bar{\lambda} D_A(z)} = \frac{\bar{\lambda}}{\lambda}. \quad (3.5)$$

For any given galaxy, its observed size will be determined more by its intrinsic size than by its magnification, so any individual measurement will be dominated by this intrinsic size dispersion. But by averaging many such measurements over a patch where the magnification is coherent, one can reach a regime where the magnification
dominates. However, this assumes that the average intrinsic sizes are uncorrelated; if there are intrinsic correlations in sizes, so that \( \langle r \rangle_{\text{patch}} \neq \bar{r} \) then this could be wrongly interpreted as magnification. The magnification estimator will effectively have two contributions, the true convergence and the intrinsic contribution:

\[
\hat{\kappa} = \kappa + \kappa_1.
\]

Here, \( \kappa_1 \) is the contribution to the size magnification estimator arising from the intrinsic sizes; in particular,

\[
\kappa_1 \equiv \ln \frac{\lambda_1}{\bar{\lambda}} - \left\langle \ln \frac{\lambda_i}{\bar{\lambda}} \right\rangle.
\]

The primary observables are the two point moments of the estimator, which has three contributions; in Fourier space, these are written as

\[
C_{\hat{\kappa}}(\ell) = C_{\kappa}(\ell) + 2C_{\kappa\kappa_1}(\ell) + C_{\kappa_1}(\ell).
\]

The lensing auto-correlation is well understood, and here we investigate the other terms in a simple halo model.

### 3.2 Elements of the halo model

The halo model assumes the mass in the Universe is distributed into distinct haloes, whose large scale distribution is described by mass dependent two-point (and potentially higher order) correlations. A central galaxy is associated with the halo centre, and satellite galaxies are distributed around it with some profile probability density. The satellites are associated with sub-haloes, which have a distribution of mass which in principle depends on the mass of the halo in which they sit. In the following we discuss the basic elements of the halo model. Throughout we will indicate halo masses and sizes with \( M, R \) and sub-halo (or satellite) masses and sizes with \( m_{\text{sh}}, r \) and the mass associated with the central galaxy is given by \( M_c = M - \sum_i m_{\text{sh},i} \). Of course, the latter is just a simple assumption: the mass associated with the central galaxy could be different from that. If we assume the gas fraction to be constant across each halo, we would have just a constant shift in the \( M_c \) and the central-central correlations are may be unchanged. However, the central-satellite terms may not. Additionally, we need to consider the fact that the gas fraction can be halo mass dependent and potentially be important.
3.2.1 Halo mass function

We quickly recall some of the relations found in Section 1.4.4. The comoving number density of collapsed haloes with mass between $M$ and $M + dM$ is described by the halo mass function $n(M, z)$

$$\frac{M^2}{\bar{\rho}} n(M) \frac{dM}{M} = \nu f(\nu) \frac{d\nu}{\nu}; \quad (3.9)$$

The function $f(\nu)$ can be written as \[66\]

$$\nu f(\nu) = A_M [1 + (q\nu)^{-p}] \left( \frac{q\nu}{2\pi} \right)^{1/2} \exp \left( -\frac{q\nu}{2} \right). \quad (3.10)$$

It has been found in \[164\] that, for $0.1 \leq \Omega_m \leq 1$ and $-1 \leq w \leq -0.3$, the critical density at a given redshift is accurately given by the following fitting function:

$$\delta_c(z) = \frac{3}{20} (12\pi)^{2/3} (1 + \alpha \log_{10} \Omega_m(z)), \quad (3.11)$$

where $\alpha(w)$ is a function of the dark energy equation of state parameter $w$:

$$\alpha(w) = 0.353 w^4 + 1.044 w^3 + 1.128 w^2 + 0.555 w + 0.131. \quad (3.12)$$

The results showed below in this Chapter are calculated assuming a cosmological constant ($w = -1$) for which $\alpha = 0.013$. In the work done in this thesis, we use the Sheth & Tormen (1999) \[66\] formulation of the mass function.

3.2.2 Sub-halo mass function

For clustering statistics, it is sufficient to simply know how many galaxies are populating a halo of a given mass, known as the Halo Occupation Distribution (HOD) \[165\].

However, for our purposes we also need to quantify the physical properties of satellite galaxies, so we require a sub-halo mass function. We use the parameterisation introduced by \[166, 167\]:

$$\frac{dN(m_{sh}|M, z)}{dm_{sh}} = (1 + z)^{1/2} A_M M m_{sh}^{\alpha} \exp \left[ -\beta \left( \frac{m_{sh}}{M} \right)^3 \right], \quad (3.13)$$

with the parameters $A_M = 9.33 \times 10^{-4}$, $\alpha = -1.9$ and $\beta = 12.2715$. Recently, \[168\] have shown that this sub-halo mass function does not strongly depend on the choice of the cosmological parameters. $N(M, z)$, or the HOD, is simply the integral of the sub-halo mass function of those galaxies above an observable mass or luminosity threshold.
A first step: halo model for intrinsic size correlations

The assumption of this sub-halo mass function is that the number of substructures per host halo mass is universal; more massive haloes host proportionately more satellite galaxies. However, there still remains mass dependence in the exponential cut-off; more massive sub-haloes only exist in more massive haloes. This latter fact implies a weak size correlation between satellites and their central galaxy hosts, which strengthens if the less massive satellites are not observed.

3.2.3 Radial profile

In addition to knowing how many sub-haloes there are, we need to know how they are distributed around the centre of the halo. We assume a Navarro-Frenk-White (NFW) profile [141] both for the distribution of mass in the halo and for the probability of finding any given sub-halo at a particular distance from the centre of the halo. In principle, the sub-halo probability distribution may depend on the sub-halo mass, $m$, and be significantly different from the halo mass distribution.

The NFW profile is given by [141]

$$\rho_{\text{NFW}}(x|M) = \frac{\rho_s}{x/r_s(1 + x/r_s)^2},$$

(3.14)

where $x$ is the distance from the centre of the halo and $r_s$ is the scale radius of the halo. Its concentration is defined as $c = R_{200}/r_s$ where $R_{200}$ is the virial radius of the halo. The virial radius is defined as the radius enclosing an overdensity equal to $200 \rho_{\text{cr}}$. In particular, including the redshift dependence of the critical density, we have:

$$R_{200}(z) = \left( \frac{3 M}{4 \pi \rho_{\text{cr}}(z) \Delta} \right)^{1/3},$$

(3.15)

where $\Delta = 200$ is the redshift independent overdensity parameter and $\rho_{\text{cr}}(z) = \rho_{\text{cr},0} E(z)^2$, where $\rho_{\text{cr}}(z)$ and $E(z)$ are defined as in Section 1.2.3.

The normalisation $\rho_s$ is given by:

$$\rho_s = \frac{M}{4 \pi r_s^3} \left[ \ln(1 + c) - \frac{c}{1 + c} \right],$$

(3.16)

and we use the following model for the concentration from [169]:

$$c(z) = c_0 (1 + z)^{-0.71} \left( \frac{M}{M_{c_0}} \right)^{-0.086},$$

(3.17)

where $c_0 = 7.26$ and $M_{c_0} = 10^{12} M_\odot h^{-1}$. This implicitly assumes that the concentration is a deterministic function of the halo mass, with no scatter.

We can convert from a matter distribution to a sub-halo probability distribution by simply dividing by the total halo mass, $u(x|M) = \rho_{\text{NFW}}(x|M)/M$. Below we
work in Fourier space for calculating power spectra, where it is useful to have the Fourier transform of the normalised density profile given in eq. (3.14):

\[ u(k|M) = \int_0^R dx \frac{4\pi x^2 \sin(kx)}{kx M} \rho_{\text{NFW}}(x|M) \, , \]  

(3.18)

In principle we should also specify the radial profiles and mass-concentration relations for the sub-haloes, as in [166, 167]; however, below we assume a simple relation of the satellite radii to the sub-halo mass, so the sub-halo profiles are not required.

### 3.2.4 Large scale halo distribution

The final element in the halo model description is to specify the large scale distribution of haloes; this is usually done through specifying two-point (and higher) moments to match the expected linear or weakly non-linear behaviour. Here we focus on matching the two-point moments by assuming a simple deterministic bias that is mass dependent.

In the halo model, the two-point correlation function can be written

\[ \xi(x) = \xi_{1h}(x) + \xi_{2h}(x) \, , \]  

(3.19)

where the first term describes the contribution from each halo whereas the second term gives the contribution on large scales from halo correlations. The mass function and probability density profiles are needed to evaluate both terms, but the two-halo term also requires the halo correlation function \( \xi_{1h}(x|M_1, M_2) = b(M_1)b(M_2)\xi_{\text{lin}}(x) \) where \( \xi_{\text{lin}}(x) \) is the linear mass correlation function and \( b(M, z) \) is the bias parameter.

This approximation is justified because on large scales the density correlation function has to follow the linear correlation function. There is an explicit constraint on \( b(M) \), as pointed out by [161], because on large scales the amplitude of the two-halo term of the mass-weighted density power spectrum has to match the amplitude of the linear power spectrum. This gives a constraint for the halo model bias:

\[ \int_0^\infty dM n(M) b(M) \frac{M}{\bar{\rho}} = 1 \, , \]  

(3.20)

so that, on the very largest scales where the mass profile of the haloes is unimportant, the mass distribution matches linear theory.
3.2.5 Size-mass relation

As we are interested in the sizes of galaxies and how they are correlated, we must have a process for relating the observed size of a galaxy to the halo model. For this, we use the size-virial radius relation found by Kravtsov (2013) [159] where abundance matching was used to relate simulated halo masses to the properties of observed galaxies; by this means he found a linear relation between the virial radius $R_{200}$ of the haloes and the radius enclosing half of the galaxy mass $r_{1/2}$:

$$r_{1/2} = 0.015 R_{200}.$$  \quad (3.21)

[159] finds that this relation holds over eight orders of magnitude in stellar mass and for all morphological types. The virial radius $R_{200}$ is the theoretical quantity, while the half-mass radius $r_{1/2}$ is a quantity can be measured. In its work, Kravtsov (2013) [159] obtained $r_{1/2}$ for late-type galaxies from the de-projected stellar surface density profiles from several other studies; then the half-mass radius was determined using the cumulative mass profile of each disk. On the other hand, for early-type galaxies he used a relation between half-mass radius and stellar mass originally derived in [170].

The relation in eq. (4.21) is consistent with the model developed by [171] in which galaxy disc sizes are determined by the angular momentum they acquire during the collapse. As also stated in [159], it is remarkable that the relation given in eq. (4.21) seems to be valid even for early-type galaxies, showing that angular momentum is extremely important in the process of galaxy formation. Additionally, $r_{1/2}$ can be related to the effective radius of a galaxy $R_e$, which is the radius enclosing half of the light of the galaxy, through $r_{1/2} = 1.34 R_e$ [159]. In the following we identify $r_{1/2}$ with $r(m)$ in order to keep the notation concise.

3.2.6 Mass threshold

In order to translate the halo model into observable quantities, we need to model the galaxy selection effects. For simplicity, we will assume that we have a survey complete to some intrinsic luminosity threshold. Assuming the luminosity directly relates to stellar mass, we require a relationship between halo mass and galactic stellar mass for selecting a minimum halo mass for our calculations. We use the relation given by [172]:

$$\frac{M_*}{M} = C \times \left[ \left( \frac{M}{M_0} \right)^{-a} + \left( \frac{M}{M_0} \right)^{b} \right]^d.$$  \quad (3.22)
where $C = 0.129$, $M_0 = 10^{11.4} M_{\odot}$, $a = 0.926$, $b = 0.261$ and $d = 2.440$, $M$ is the mass of the host halo and $M_*$ is the mass of the galaxy embedded in the halo. The halo masses $M$ have been calculated using numerical simulations and the galaxy stellar mass $M_*$ by means of Sloan Sky Digital Survey (SDSS) data. The relation in eq. (3.22) is obtained assuming a one-to-one correspondence between sub-haloes and galaxies by using the abundance matching technique, that is the hypothesis that the cumulative halo mass function is equal to the cumulative galaxy mass function.

By means of eq. (3.22) we choose minimum masses for both sub-haloes and haloes equal to $m_{\text{sh, min}} = M_{\text{min}} = 10^{11} M_{\odot} h^{-1}$ that corresponds to a minimum galaxy mass equal to $2 \times 10^9 M_{\odot} h^{-1}$. Setting this limit for the minimum halo mass is also in agreement with the Halo Occupation Distribution (HOD) model analysed in [165].

### 3.3 Translating to observations

Given the halo model assumptions, we can work out its implications for observables. We first look at background quantities before moving on to the two-point quantities of primary interest. Here, following the approach given in [173], we build up from the simplest halo model quantities to the size-weighted galaxy distribution and how it impacts the magnification estimator defined above.

#### 3.3.1 Halo density

We begin with the discrete distribution of the haloes, which is described by

$$n_h(x) = \int_0^\infty dM \sum_i \delta_D(M - M_i) \delta_D^{(3)}(x - x_i)$$

(3.23)

where we have integrated over the possible halo masses. The sum within the integral has expectation given by the mass function defined above,

$$\left\langle \sum_i \delta_D(M - M_i) \delta_D^{(3)}(x - x_i) \right\rangle = n(M) = \frac{dN_h}{dM dV}.$$  

(3.24)

The total halo density is given by the integral, $\bar{n}_h = \int_0^\infty dM n(M)$. 

3.3.2 Halo matter density

If we assume that the mass distribution is dominated by that mass associated with the haloes (ignoring that in sub-haloes), the dark matter density field is given by:

$$\rho(\mathbf{x}) = \sum_i \rho_{\text{NFW}}(\mathbf{x} - \mathbf{x}_i, M_i) = \sum_i M_i u(\mathbf{x} - \mathbf{x}_i, M_i) ,$$  \hspace{1cm} (3.25)

where the sum is over the haloes and $u(\mathbf{x}, M)$ is the density profile normalised to the halo mass. We can obtain a continuous density field from the discrete one given in eq. (3.25) by introducing Dirac delta functions:

$$\rho(\mathbf{x}) = \int_0^\infty dM n(M) M \sum_i \delta_D(M - M_i) \delta^{(3)}(\mathbf{x}' - \mathbf{x}_i) \times u(|\mathbf{x} - \mathbf{x}'|, M) .$$ \hspace{1cm} (3.26)

Taking the ensemble average we obtain the mean matter density:

$$\bar{\rho} \equiv \langle \rho(\mathbf{x}) \rangle = \int_0^\infty dM n(M) M ,$$ \hspace{1cm} (3.27)

where we used the fact that $\int d^3x u(|\mathbf{x} - \mathbf{x}_i|, M) = 1$ (since the function $u$ is normalised for each halo).

3.3.3 Galaxy density

In the halo model, it is assumed that the galaxy density is composed of two terms, the central galaxies positioned at the halo centre and satellite galaxies distributed around the halo centre. Analogously to the halo density defined above, we can write the galaxy density as

$$n_g(\mathbf{x}) = \int_0^\infty dM \sum_i \delta_D(M - M_i) \sum_j \delta^{(3)}(\mathbf{x} - \mathbf{x}_i - \mathbf{x}_j),$$ \hspace{1cm} (3.28)

where the $\sum_j$ is over the central and possible satellite galaxies and $\mathbf{x}_j$ represents their position relative to the halo centre; $\mathbf{x}_j = 0$ for the central galaxy, while for the satellite galaxies, these positions are described by the satellite probability profile.

The average number of satellites for a halo of a given mass is $\langle N_{\text{sat}}|M \rangle$ which is related to the halo occupation distribution; it is an integral of the sub-halo mass function defined above:

$$\langle N_{\text{sat}}|M \rangle = \int_{m_{\text{sh,min}}}^M dm_{\text{sh}} \frac{dN(m_{\text{sh}}|M)}{dm_{\text{sh}}} ,$$ \hspace{1cm} (3.29)
and the HOD has one more than this to account for the central galaxy. Again, we assume that substructures inside a halo follow a spatial distribution $u_d(|x - x_c|, M)$ (which we assume to be of the form given in Eq. 3.14) depending on the halo mass and where $x_c$ are the coordinates of the centre of the halo. After averaging over the sub-halo ensembles, the galaxy density can be written as

$$n_g(x) = \sum_i \delta_D^{(3)}(x - x_i) + \langle N_{\text{sat}}|M_i \rangle u_d(x - x_i|M_i) ,$$

which can again be written as

$$n_g(x) = \int dM \int d^3x' \sum_i \delta_D(M - M_i) \delta_D^{(3)}(x' - x_i) \times \left[ \delta_D^{(3)}(x - x') + \langle N_{\text{sat}}|M \rangle u_d(|x - x'|, M) \right].$$

After averaging over the positions of the haloes, we find

$$\bar{n}_g = \int_0^\infty dM n(M) (1 + \langle N_{\text{sat}}|M \rangle).$$

This could alternatively be written as

$$\bar{n}_g = \int_{M_{\text{min}}}^{\infty} dM n(M) \int_{m_{\text{sh, min}}}^{M} dm_{\text{sh}} \times \left( \delta_D(m_{\text{sh}} - M_c) + \frac{dN(m_{\text{sh}}|M)}{dm_{\text{sh}}} \right),$$

where we have introduced minimum halo and galaxy masses which will arise in realistic observations and assumed the central galaxy mass is comparable to that of the halo itself. Though a fraction of the total halo mass will reside in the sub-haloes, we do not expect this to greatly impact the central galaxy size.

### 3.3.4 The galaxy size field

Our assumption is that the observed half-radius is related to the sub-halo mass, as described above. We weight the galaxy density defined in Sec. 3.3.3 by the radius, and normalise by the total galaxy density to define a galaxy size field as

$$r(x) = \bar{n}_g^{-1} \int_0^{\infty} dM \sum_i \delta_D(M - M_i) \sum_j \delta_D^{(3)}(x - x_i - x_j) r(m_{j, \text{sh}})$$

where $r(m_{j, \text{sh}})$ is the radius associated with the mass of the central galaxy or satellite galaxies. For the central galaxy, we should formally base its radius on the residual mass, that is, subtracting the integrated mass in sub-haloes from the total halo mass.
However, even for the smallest haloes in our model, the total mass in the sub-haloes only accounts for around 10% of the halo mass, so this correction makes a small change in the inferred central radius. We checked that our correlation results are not affected by making this correction and for simplicity we adopt the radius based on the total mass.

The galaxy size field given by eq.(3.34) can be averaged over the sub-halo ensembles to find

\[
\bar{r}(x) = \bar{n}_g^{-1} \sum_i \left[ r(M_i) \delta_D^{(3)}(x - x_i) + \bar{r}_{\text{sat}}(M_i) \langle N_{\text{sat}} | M_i \rangle u_d(x - x_i) \right] 
\]

(3.35)

where \( \bar{r}_{\text{sat}}(M) \equiv \int_{M_{\text{min}}}^{M} \frac{dN(m_{\text{sh}} | M)}{dm_{\text{sh}}} r(m_{\text{sh}}) dm_{\text{sh}} / \langle N_{\text{sat}} | M \rangle \) is the average satellite radius for satellites in a halo of mass \( M \).

With this, it is straightforward to derive the distribution of radii and derive the average galaxy size:

\[
\bar{r} = \bar{n}_g^{-1} \int_{M_{\text{min}}}^{\infty} dM n(M) \int_{m_{\text{min}}}^{M} \frac{dN(m_{\text{sh}} | M)}{dm_{\text{sh}}} \left( \delta_D(m_{\text{sh}} - M) + \frac{dN(m_{\text{sh}} | M)}{dm_{\text{sh}}} \right) r(m_{\text{sh}}). 
\]

(3.36)

In Fig.3.1 radii distributions, calculated by means of the size-mass relation given by eq. (4.21) found by [159] combined with the halo mass function, for centrals, satellites and total galaxy population are shown and the mean values for half-mass radius for each type of structures are indicated at redshift \( z = 0 \).

### 3.3.5 The local estimator field

The magnification estimator, defined as

\[
\hat{\kappa} = \ln \frac{\lambda_0}{\lambda} - \left\langle \ln \frac{\lambda_0}{\lambda} \right\rangle, 
\]

(3.37)

acts on the observed angular sizes of galaxies, potentially combining galaxies over a range of redshifts. It is possible however to consider a local definition of the estimator field that when summed over redshift becomes the two-dimensional projected estimator.

The intrinsic contribution to the magnification estimator arises because the ob-
Figure 3.1 – Distribution of object for given half-mass radius for both haloes and subhaloes as well as for the total population at redshift $z = 0$ and for minimum halo mass $M_{\text{min}} = 10^{11} M_\odot h^{-1}$. Symbols indicate the mean half-mass radii for each population (square for satellites, circle for centrals, triangle for the total population). In the main plot the number densities are shown for the entire range of half-mass radii; for a better view of the mean values the inset plot represents the number densities in the range between $r_{1/2} = 1.7 \text{kpc} h^{-1}$ and $r_{1/2} = 1.8 \text{kpc} h^{-1}$. 
served size depends on the true galaxy size. These are related through the angular
diameter distance, and for objects at a given redshift \( \lambda = r(z)/D_A(z) \), so that,
\[
\ln \frac{\lambda}{\bar{\lambda}} = \ln \frac{r(z)}{\bar{r}} + \ln \frac{\bar{r}}{D_A(z)\bar{\lambda}}.
\]
(3.38)

For objects at a given redshift, their observed size field and true size field are related
by a constant term, which cancels when considering the fluctuation field. Their
fluctuations are identical,
\[
\ln \frac{\lambda}{\bar{\lambda}} - \langle \ln \frac{\lambda}{\bar{\lambda}} \rangle_z = \ln \frac{r(z)}{\bar{r}} - \langle \ln \frac{\bar{r}}{\bar{r}} \rangle_z.
\]
(3.39)

Thus, the intrinsic contribution is effectively
\[
\kappa_I(z) = \ln \frac{r(z)}{\bar{r}} - \langle \ln \frac{\bar{r}}{\bar{r}} \rangle_z.
\]
(3.40)

Note that in both cases, dividing by the mean radius (or angular size) makes the
argument of the logarithm dimensionless, but any scale would be equivalent, as the
divisors cancel when subtracting the field average. It is the clustering of relative
sizes which contributes to the magnification estimator.

This Chapter is primarily concerned with statistics of the angular sizes of galax-
ies, projected over a broad redshift distribution. Statistics related to the true phys-
ical sizes of galaxies potentially would be biased by individual photometric redshift
errors, and so it would be essential to treat these carefully in any 3-D or tomographic
analysis of the physical size correlations.

A given realisation of halo and sub-halo positions results in a estimator-weighted
density field as
\[
\kappa_I(x) = n_{\text{sh}}^{-1} \int_0^\infty dM \sum_i \delta_D(M - M_i) \times \sum_j \delta_D^{(3)}(x - x_i - x_j) \kappa_I(m_{j,\text{sh}}).
\]
(3.41)

where:
\[
\kappa_I(m_{\text{sh}}, z) = \ln \left( \frac{r(m_{\text{sh}})}{\bar{r}} \right) - \langle \ln \frac{\bar{r}}{\bar{r}} \rangle_z.
\]
(3.42)

By definition, the expectation of this estimator is zero, \( \langle \kappa_I \rangle = 0 \); the expectation
value of the log-size field at a given redshift is
\[
\langle \ln \left( \frac{r}{\bar{r}} \right) \rangle \equiv \bar{n}_g^{-1} \int_{M_{\text{min}}}^{\infty} dM \, n(M) \int_{m_{\text{sh},\text{min}}}^{M} dm_{\text{sh}} \times \left( \delta_D(m_{\text{sh}} - M) + \frac{dN(m_{\text{sh}}|M)}{dm_{\text{sh}}} \right) \ln \left( \frac{r(m_{\text{sh}})}{\bar{r}} \right).
\] (3.43)

### 3.4 Two-point statistics

Our focus here is to understand the implications of size correlations on two-point statistics, and in particular in comparing how the power spectrum of the magnification estimator relates to that of the true magnification once size correlations are included. Thus, we must calculate the power spectrum of the intrinsic size correlations and their cross correlation with the true magnification.

As discussed above, in the halo model two-point correlations receive contributions from pairs of galaxies inhabiting the same halo and from where they inhabit two different haloes. The same holds for the power spectrum:

\[
P(k) = P_{1h}(k) + P_{2h}(k).
\] (3.44)

It is straightforward to calculate the power spectrum of the matter density fluctuation $\delta \rho/\bar{\rho}$ using the halo model formalism developed above [160]:

\[
P_{1h}(k) = \int_0^{\infty} dM n(M) \left( \frac{M}{\bar{\rho}} \right)^2 u^2(k, M),
\] (3.45)

\[
P_{2h}(k) = \bar{b}_p^2 P_{\text{lin}}(k),
\]

where

\[
\bar{b}_p = \int_0^{\infty} dM n(M) b(M) \frac{M}{\bar{\rho}} u(k, M).
\] (3.46)

The number density fluctuation is similar, but accounts for the central and satellite galaxy contributions separately. The one-halo term includes terms from the central-satellite and the satellite-satellite pairs within the same halo:

\[
P_{1h}(k) = \bar{n}_g^{-2} \int_0^{\infty} dM n(M) (\langle N_{\text{sat}}|M \rangle \bar{\rho}) u(k, M)
\]

\[
+ \langle N_{\text{sat}} (N_{\text{sat}} - 1)|M \rangle u^2(k, M)
\].

The two-halo term has three contributions, including central-central, central-satellite and satellite-satellite terms:

\[
P_{2h}(k) = \bar{b}_g^2 P_{\text{lin}}(k),
\] (3.48)
where

\[
\bar{b}_n = \bar{n}_g^{-1} \int_0^\infty dM n(M) b(M) (1 + \langle N_{\text{sat}} | M \rangle u(k, M)) .
\] (3.49)

### 3.4.1 Magnification estimator power spectrum

In this subsection we present our model for the correlation between log-size of galaxies. In the one-halo terms, we only include the cross-correlations between different galaxies, so there is no central-central contribution.

#### One-halo terms

Applying the halo model formalism, we obtain the following power spectra for the auto-correlation:

\[
P_{\kappa_1-\text{sat}}(k) = \bar{n}_g^{-2} \int_{M_{\text{min}}}^\infty dM n(M) \\
\times \left[ \int_{m_{\text{sh, min}}}^M dm_{\text{sh}} \frac{dN(m_{\text{sh}} | M)}{dm_{\text{sh}}} \kappa_1(m_{\text{sh}}) u_4(k, M) \right]^2
\] (3.50)

We also have contribution from central-satellite correlation terms:

\[
P_{\kappa_1-\text{cs}}(k) = \frac{2}{\bar{n}_g^2} \int_{M_{\text{min}}}^\infty dM n(M) \kappa_1(M) \\
\times \int_{m_{\text{sh, min}}}^M dm_{\text{sh}} \frac{dN(m_{\text{sh}} | M)}{dm_{\text{sh}}} \kappa_1(m_{\text{sh}}) u_4(k, M)
\] (3.51)

#### Two-halo terms

Applying the halo model formalism, we obtain the following power spectra for the auto-correlation:

\[
P_{\kappa_1}^{2h}(k) = (\bar{b}_{\kappa_1,c} + \bar{b}_{\kappa_1,s})^2 P_{\text{lin}}(k),
\] (3.52)

where:

\[
\bar{b}_{\kappa_1,c} = \bar{n}_g^{-1} \int_{M_{\text{min}}}^\infty dM n(M) b(M) \kappa_1(M)
\] (3.53)

and

\[
\bar{b}_{\kappa_1,s} = \bar{n}_g^{-1} \int_{M_{\text{min}}}^\infty dM n(M) b(M) \int_{m_{\text{sh, min}}}^M dm_{\text{sh}} \frac{dN(m_{\text{sh}} | M)}{dm_{\text{sh}}} \\
\times \kappa_1(m_{\text{sh}}) u_4(k, M).
\] (3.54)

These biases are perhaps the most important result of our model, as the two-halo terms dominate on the scales where lensing is most easily interpreted. In Fig. 3.2 we
show how the central and satellite biases evolve as a function of redshift. The central bias ranges from 0.1 at high redshifts, down to a few times $10^{-2}$ at low redshifts, while the satellite bias is considerably smaller ($\sim 10^{-3}$), becoming negative at low redshifts.

As the sample will be dominated by central galaxies at this mass threshold, it is worth trying to understand its amplitude better in the limit where there are only central galaxies. Recall the definition of the intrinsic convergence field is the log of the radius minus its average (see eq. (3.40)). Examining the expression for the central bias, we see that it is effectively a weighted average of $\kappa_I$, where the number density weight is modified by a bias function, $b(M)$. Were it not for this bias factor, this integral is the usual density averaging, meaning that the two terms in $\kappa_I$ would exactly cancel by definition.

If $b(M)$ were constant, independent of the mass, the central bias would also be zero. The central bias thus depends on how $b(M)$ changes as a function of mass. In particular, since the bias increases for larger mass haloes, where the radii are larger than average, this implies $\bar{b}_{\kappa_I,c}$ is positive. Its magnitude depends on how fast $b(M)$ increases over the mass range that dominates the estimator, $M_{\min} < M < 10^{14} M_\odot h^{-1}$.

The picture is somewhat more complex when the satellite population becomes
more important. The satellite distribution is weighted somewhat to lower mass
galaxies (Fig. 1), so the mean of the log radius becomes smaller. This fact tends
to increase the central bias. Meanwhile, the weighting towards lower mass tends to
cancel the increase in \( b(M) \), reducing the amplitude of \( \bar{b}_{\text{cts}} \). For the lowest redshifts,
the up-weighting of the low masses is enough to make \( \bar{b}_{\text{nl}} \) negative.

### 3.4.2 Density-size cross power spectra

For the cross-correlation density-size we obtain for both central and satellites:

\[
P_{\rho_{1h}}^{1h\,-\text{-sat}}(k) = \bar{\rho}^{-1} \bar{n}_{g}^{-1} \int_{0}^{\infty} \text{d}Mn(M)M
\times \int_{m_{\text{sh, min}}}^{M} \text{d}m_{\text{sh}} \frac{\text{d}N(m_{\text{sh}} | M)}{\text{d}m_{\text{sh}}} \kappa_{1}(m_{\text{sh}}) u(k, M) u_{d}(k, M)
\]

where \( \bar{b}_{\rho} \) is given in eq. (3.46) (using the constraints given in eq. (3.27) and eq.
(3.20)) and the other bias factors are given above.

In this work, we are assuming all of the lensing mass is associated with the haloes,
and ignore mass associated with sub-clumps. On large scales, this should be a
good approximation, but potentially it fails to take into account further correlations
between size and density on scales within haloes. It would be straight forward to
extend this work to include this effect in the halo model.

### 3.5 Discussion

#### 3.5.1 Model assumptions

We evaluate our results in the context of a flat \( \Lambda \)CDM cosmology with parameters
consistent with best-fit Planck data [11]; in particular, we assume a total matter
density \( \Omega_{m,0} = 0.32 \), cosmological constant density \( \Omega_{\Lambda,0} = 0.68 \), baryon density
\( \Omega_{b,0} = 0.049 \) and Hubble constant \( H_{0} = 100 \text{h km s}^{-1} \text{Mpc}^{-1} \), where \( h = 0.67 \). In
addition, we assume the spectral index of the matter power spectrum is \( n_{s} = 0.96 \)
and it is normalised such that \( \sigma_{8} = 0.83 \).

We adopt the transfer function given in [59] and non-linear evolution of the
matter power spectrum (for estimating lensing convergence power spectrum) is cal-
culated with HALOFIT from [174] recently revised by [175].

For the redshift distribution of lensed sources, we adopt the commonly used
Figure 3.3 – Power spectra for two different redshift distributions, CFHTLenS-like with mean redshift $\langle z \rangle \simeq 0.8$ and Euclid-like with $\langle z \rangle \simeq 0.96$. 
A first step: halo model for intrinsic size correlations

We consider two different sets of parameters for this redshift distribution form; following [120], to simulate a Euclid-like survey we assume $a = 2$, $b = 1.5$, $z_0 = 0.64$ which gives a mean redshift around 0.96. For a CFHTLenS-like survey, we use parameters from [176]: $a = 0.836$, $b = 3.425$, $z_0 = 1.171$ which give a mean redshift approximately $z \simeq 0.8$. For the shallow survey we used $a = 0.6$, $b = 1.5$, $z_0 = 0.55$ in order to obtain a mean redshift around $z \simeq 0.5$.

### 3.5.2 Comparison of power spectra

In Fig. 3.3 we show the contributions to the power spectrum of $\hat{\kappa}$ for the CFHTLenS and Euclid-like surveys. As can be seen, intrinsic size correlations are relevant even for a very deep survey such as Euclid, where their contamination increases from 10% on the largest scales to being comparable to the convergence on $\ell \sim 100$. For the CFHTLenS-like survey, with $\langle z \rangle \simeq 0.8$, the contamination is even larger, beginning at 25% of the convergence signal on large scales.

For these surveys, the largest intrinsic contribution comes from the cross correlation between the intrinsic sizes and the convergence, while the intrinsic auto-correlation is sub-dominant except at the smallest scales. On the largest scales, $\kappa$ and $\kappa_1$ are strongly correlated as the ratio $\langle \kappa \kappa_1 \rangle / \sqrt{\langle \kappa \rangle \langle \kappa_1 \rangle \langle \kappa_1 \rangle}$ is of order 80%.

For a shallower redshift distribution, the intrinsic contamination can dominate the signal. To demonstrate this, in Fig. 3.4 we show the contributions for a survey with $\langle z \rangle \simeq 0.5$; there we see the intrinsic and convergence spectra are comparable, and significantly correlated. With multiple bins, the convergence dominates in high redshift bins, but remains correlated with the intrinsic sizes in lower redshift bins; unlike the convergence, the intrinsic sizes will be relatively uncorrelated between bins.

In Fig. 3.5 we show the contribution to the size-size power spectrum and size-convergence power spectrum arising from centrals and satellites, and also how the spectra arise from the one-halo and two-halo terms. The spectra are dominated by the two-halo contributions on the scales of interest, and on these scales the central galaxy contribution is most significant; this follows from what was seen previously for the central and satellite biases. On smaller scales, the one-halo term and the contribution from satellites both become more important.

Recall that at low redshift, the satellite bias becomes negative, because satellite galaxies have sizes generally smaller than the total mean value. In the size-size correlation, this leads to the satellites being negatively correlated with the central
A first step: halo model for intrinsic size correlations

Figure 3.4 – The power spectra for a more shallow survey, with mean redshift around $z \sim 0.5$. Here the intrinsic effects are more significant than for surveys centred at higher redshifts.

galaxy population. They also contribute negatively to the size-convergence spectrum, though with an amplitude much smaller than the positive amplitude arising from the central sizes.

Formally the size-size power spectrum should be positive definite; however here we have omitted correlations of galaxies with themselves. As a result, on small scales the negative cross-correlation between central and satellite galaxies can actually dominate. On such scales, probing the typical galaxy sizes, our model is not expected to be physical; on these scales, galaxies will begin to overlap and they would not be observed as distinct.

We have presented a simple model for calculating intrinsic correlations for galaxy sizes using halo model formalism. This is a first calculation and necessarily neglects some effects which could be very relevant. One important issue that should be factored in is scatter in the mass-radius relation; this could considerably weaken the correlations we see in the sizes. Galaxy sizes may also be environmental dependent and affected by baryonic physics in ways that are hard to fold into the simple halo model.

We also have restricted our analysis to size contribution to magnification and to a simple mass threshold in the selection of galaxies. We have done this in order to present the necessary formalism we need to use to extend this analysis to a model for more realistic surveys, one might consider how these effects would impact a flux limited sample, or one with a cut-off in the observed angular size of galaxies.
Figure 3.5 – Contributions to the intrinsic size (upper plots) and intrinsic size-convergence (lower plots) power spectra for CFHTLenS-like survey ($\langle z \rangle \approx 0.8$). We plot the absolute values; the central-satellite contribution for $C_{\kappa_i}$ and the satellite contribution for $C_{\kappa\kappa_i}$ as well as the one-halo term are negative. On the scales of interest, the correlations are dominated by the two-halo contributions for the central galaxies.
This preliminary study indicates that, as for measurements of galaxy shapes, it may not be possible to ignore intrinsic correlations when interpreting measurements of galaxy sizes and magnitudes. These effects, and particularly correlations between convergence and intrinsic properties, are potentially an important systematic for magnification measurements and could significantly bias the resulting cosmological constraints if they are not accounted for. On the other hand, they represent a new observable that could potentially tell us more about how galaxies form.

Correlations of galaxy magnitudes are also used to detect magnification and these are similarly expected to be correlated with halo masses; it is worth investigating how magnitudes are correlated with both convergence and galaxy sizes and this is a straightforward extension of the halo model we have developed in this Chapter.

In the Chapter 4 we extend the work presented so far to more realistic situations including survey limits and the luminosity intrinsic contribution to magnification.
Chapter 4

Modelling size and luminosity correlations on realistic surveys

All science is full of statements where you put the best face on your ignorance, where you say: true enough, we know awfully little about this, but more or less irrespective of the stuff we don’t know about, we can make certain useful deductions.

Hermann Bondi

The simple model developed for intrinsic size correlations in Chapter 3 can be easily extended to intrinsic magnitude correlations.

In this Chapter, we investigate a theoretical model which allow us to combine both intrinsic size and magnitude information and we study the impact of both effects on the estimated lensing convergence when size and magnitudes are used. The halo model developed for intrinsic size correlations is applied to magnitudes and galaxy luminosities are correlated with the mass of the haloes and subhaloes following the relation given in Vale & Ostriker (2008) [177]. By means of this relation and the relation found by Kravtsov (2013) [159] we introduce a model for the intrinsic size-magnitude distribution based on the halo mass function.

In order to try to model more realistic surveys, we also include in our model size and magnitude thresholds. The effect of this threshold is twofold. On one side the intrinsic size-magnitude distribution is modified and this results in a change in the mean size and magnitude. On the other hand, cuts in the survey change the lensing responsivity and this also depends on whether one uses sizes or magnitudes and whether one is performing lensing tomography or not.
The work presented in this Chapter is based on Ciarlariello & Crittenden (2016) (to be submitted soon).

### 4.1 Magnification estimators

In the weak lensing limit, the observed galaxy sizes and fluxes, \( r \) and \( F \), are related to their intrinsic values through the lensing convergence \( \kappa \) by:

\[
\begin{align*}
    r_O &= (1 + \kappa) r_I \\
    F_O &= (1 + 2\kappa) F_I
\end{align*}
\]  

(4.1)

where the subscripts stand for the observed (O) and intrinsic (I) quantities. As pointed out by Heavens et al. (2013) [17], a galaxy size defined as the square root of the galaxy image is only expected to be uncorrelated with shape for galaxies with exponential profiles. In order to get an estimator for the lensing convergence, we define the logarithm of the angular galaxy size and use the definition of apparent magnitude for galaxy fluxes, following [140], as follows:

\[
\begin{align*}
    \lambda &= \ln \frac{r}{\text{[arcsec]}} \\
    m &= m_{\text{ref}} - 2.5 \log_{10} \frac{F}{F_{\text{ref}}}
\end{align*}
\]  

(4.2)

where \( \lambda \) is logarithm galaxy size, \( m \) is the galaxy magnitude and \( m_{\text{ref}} \) is the magnitude for a reference flux. Then we use as our point estimators the following (e.g. Schmidt et al. (2012) [140], Heavens et al. (2013) [17], Bacon et al. (2014) [77]):

\[
\begin{align*}
    \hat{\kappa}_{\text{size}} &= \lambda_O - \langle \lambda_O \rangle \\
    \hat{\kappa}_{\text{mag}} &= m_O - \langle m_O \rangle
\end{align*}
\]  

(4.3)

For any given galaxy, its observed size and magnitude will be determined more by its intrinsic values than by its magnification, so any individual measurement will be dominated by this intrinsic dispersion. But by averaging many such measurements over a patch where the magnification is coherent, one can reach a regime where the magnification dominates.

However, this assumes that the average intrinsic sizes and magnitudes are uncorrelated; if there are intrinsic correlations, so that \( \langle r \rangle_{\text{patch}} \neq \bar{r} \) and \( \langle m \rangle_{\text{patch}} \neq \bar{m} \) then this could be wrongly interpreted as magnification. The magnification estimator for either sizes or magnitudes will effectively have two contributions, the true lensing...
convergence and the intrinsic contribution:

\[
\hat{\kappa}_{\text{size}} = \eta_\lambda \kappa + \kappa_1^\lambda, \\
\hat{\kappa}_{\text{mag}} = \eta_m \kappa + \kappa_1^m.
\] (4.4)

Here, \(\kappa_1\) is the intrinsic contribution to the magnification estimator arising from the intrinsic galaxy sizes and magnitudes and the quantities \(\eta_\lambda\) and \(\eta_m\) are called lensing responsivities \([18]\); in an ideal case (a survey with no cuts) we would have \(\eta_\lambda = 1\) and \(\eta_m \simeq -2.17\). In general, these values will depend on both the survey population and observational thresholds and can be redshift dependent, e.g. Alsing et al. (2015) \([18]\) and Schmidt et al. (2012) \([140]\).

4.1.1 Incorporating selection effects

So far we have focused on estimating lensing from an ideal survey, implicitly assuming that the population of objects is not affected by lensing. In realistic surveys, the galaxies are only included if they exceed some thresholds for detection, either in magnitude, size or both. In such a case, magnification can bring new objects into the survey, affecting the number density of objects and their average sizes and magnitudes. This effect is generally called magnification bias \([128]\).

The average properties of the galaxies that enter into a sample are assumed to depend on the convergence as

\[
\langle \lambda_0 \rangle = \langle \lambda_1 \rangle + \eta_\lambda \kappa \\
\langle m_0 \rangle = \langle m_1 \rangle + \eta_m \kappa.
\] (4.5)

Below we show how the mean values and, consequently, the responsivities \(\eta_\lambda\) and \(\eta_m\) are changed when dealing with the realistic case of a survey with size and magnitude limits.

Magnification bias: galaxy number density

Here we briefly discuss the effect of magnification bias on the galaxy number density, following the treatment of Schmidt et al. (2009) \([128]\); we then extend this to the properties relevant for convergence estimation using sizes and magnitudes. When there is either a magnitude or size threshold, magnification increases the number density as objects are brought in the sample; at the same time, the solid angle is stretched leading to a dilution of the number density. The dilution effect changes the number density, and the result of these two competing effects could be that cross-correlations can be induced between galaxy populations that are physically separated.
We denote the observed and intrinsic distributions of galaxies as functions of log-sizes, magnitudes and positions by \( \Phi_O(\lambda_O, m_O, \theta_O) \) and \( \Phi_I(\lambda_I, m_I, \theta_I) \) respectively. Conservation of the total number of galaxies implies:

\[
d^2\theta_I d\lambda_I dm_I \Phi_I(\lambda_I, m_I, \theta_I) = d^2\theta_O d\lambda_O dm_O \Phi_O(\lambda_O, m_O, \theta_O),
\]

where the relations below describe the change of size, magnitudes and area after a lensing transformation in an ideal case (below \( q \simeq -2.17 \) is the ideal magnitude responsivity):

\[
\begin{align*}
\lambda_O &= \lambda_I + \kappa \\
m_O &= m_I + q\kappa \\
d^2\theta_O &= (1 + 2\kappa)d^2\theta_I.
\end{align*}
\]

The number density of objects in a survey is given by,

\[
n_O(\theta_O) = \int d\lambda_O dm_O \Phi_O(\lambda_O, m_O, \theta_O) S(\lambda_O, m_O),
\]

where \( S(\lambda_O, m_O) \) denotes the selection function of the survey. For simplicity we assume the selection function to be spatially constant and a step function describing magnitude and size limits \( (m_{\lim} \text{ and } \lambda_{\lim}) \):

\[
S(\lambda_I, m_I) = \Theta(\lambda_I - \lambda_{\lim})[1 - \Theta(m_I - m_{\lim})]
\]

where the function \( \Theta(x) \) is the Heaviside function. \( S(\lambda_I, m_I) \) is essentially the selection function \( S(\lambda_O, m_O) \) evaluated at \( \kappa = 0 \).

Provided these relations, we can Taylor-expand the observed selection function with respect the convergence to find:

\[
n_O(\theta_O) = (1 - 2\kappa) \int d\lambda_I dm_I \Phi_I(\lambda_I, m_I, \theta_I) \times \\
\times \left[ S(\lambda_I, m_I) + \frac{\partial S}{\partial \lambda_I} \kappa + q \frac{\partial S}{\partial m_I} \kappa \right].
\]

If the function \( S(\lambda_I, m_I) \) is taken to be a step function, its derivatives are Dirac delta functions of either size or magnitude. Finally, we obtain:

\[
n_O(\theta_O) = n_I(\theta_I)[1 + (\zeta_1 + \zeta_2 - 2)\kappa(\theta_I)],
\]
where \( n_I(\theta_i) \) indicates the intrinsic number density of sources and:

\[
\zeta_1 = \int_{-\infty}^{m_{\text{lim}}} dm_I f_1(\lambda_{\text{lim}}, m_I)
\]

\[
\zeta_2 = -q \int_{\lambda_{\text{lim}}}^{+\infty} d\lambda_I f_1(\lambda_I, m_{\text{lim}}).
\]

(4.12)

where we have defined:

\[
f_1(\lambda_I, m_I) \equiv \frac{\Phi_1(\lambda_I, m_I)}{n_I}.
\]

(4.13)

These are defined in terms of the intrinsic galaxy properties, which may be known if one has a model of these; however, in practice, it could be quite difficult to evaluate them because estimating the intrinsic properties from data may introduce further systematics.

### Magnification bias: mean size and magnitude

By means of the galaxy number density results from the previous section, we can calculate how mean values for sizes and magnitudes are affected by magnification bias. Eventually, we will obtain the equation for the responsivities when selection cuts are used in a region. In the following we begin with the calculation for the mean size. Analogous results hold also for magnitudes and we provide equation at the end of this section. The observed mean log-size is a region of the sky is given by:

\[
\langle \lambda_O \rangle(\theta_O) = \frac{1}{n_O} \int d\lambda_O d\lambda_O \Phi(\lambda_O, m_O, \theta_O) \times \lambda_O S(\lambda_O, m_O).
\]

(4.14)

Translating everything in intrinsic quantities and accounting for the magnification bias effect in the galaxy number density, we obtain:

\[
\langle \lambda_O \rangle(\theta_O) = \frac{1 - 2\kappa}{n_I[1 + (\gamma_1 + \gamma_2 - 2)\kappa]} \int d\lambda_I d\lambda_I \Phi_1(\lambda_I, m_I, \theta_I) \times \nonumber \\
\times (\lambda_I + \kappa) \left[ S(\lambda_I, m_I) + \frac{\partial S}{\partial \lambda_I} \kappa + q \frac{\partial S}{\partial m_I} \kappa \right].
\]

(4.15)

Carrying on the calculations, neglecting second order terms:

\[
\langle \lambda_O \rangle(\theta_O) = \langle \lambda_I \rangle(\theta_I) + \kappa(\theta_I) + (\alpha_1 + \alpha_2)\kappa(\theta_I) + \\
- \langle \lambda_I \rangle(\zeta_1 + \zeta_2)\kappa(\theta_I),
\]

(4.16)
where:

\[
\alpha_1 = \int_{-\infty}^{m_{\text{lim}}} dm_1 f_1(\lambda_{\text{lim}}, m_1) \lambda_{\text{lim}} \\
\alpha_2 = -q \int_{\lambda_{\text{lim}}}^{+\infty} d\lambda_1 f_1(\lambda_1, m_{\text{lim}}) \lambda_1
\]

(4.17)

Finally, recalling eq. (4.62), we obtain the equation for the size responsivity:

\[
\eta_\lambda = 1 + (\alpha_1 + \alpha_2) - \langle \lambda_1 \rangle (\zeta_1 + \zeta_2),
\]

(4.18)

and, analogously, the equation for the magnitude responsivity:

\[
\eta_m = q + (\beta_1 + \beta_2) - \langle m_1 \rangle (\zeta_1 + \zeta_2),
\]

(4.19)

where we have:

\[
\beta_1 = \int_{-\infty}^{m_{\text{lim}}} dm_1 f_1(\lambda_{\text{lim}}, m_1) m_1 \\
\beta_2 = -q \int_{\lambda_{\text{lim}}}^{+\infty} d\lambda_1 f_1(\lambda_1, m_{\text{lim}}) m_{\text{lim}}.
\]

(4.20)

### 4.2 Modelling intrinsic correlations

In this Chapter we follow the prescription adopted in Chapter 3 where the basic elements of the halo model for intrinsic correlations have been presented. We refer to Chapter 3 for all the details.

Recall that we use the halo mass function \(n_{\text{com}}(M, z)\) given by [66], describing the comoving number density of collapsed haloes; the distribution of satellite galaxies is described by the sub-halo mass function \(dN(m_{\text{sh}}, M, z)/dm_{\text{sh}}\) from [166], which depends on the host halo mass and accounts for the fact that more massive sub-haloes only exist in more massive haloes. We also assume that sub-haloes are distributed around the centre of the halo according to a Navarro-Frenk-White (NFW) profile [141]; in particular, since we work with power spectra, we use the Fourier transform of the NFW profile normalized to the halo mass indicated with \(u(k|M)\). In principle, we should also specify a density profile for sub-haloes, as in [166]; however we assume simple relations of satellite radii and luminosities to the sub-halo mass, so the sub-halo profiles are not required.

For the correlation on large scales we use the bias model (consistent with the mass function) from [66].
4.2.1 From haloes to galaxies

We need a method for relating the observed size and magnitude of galaxies to the halo and sub-halo masses in the halo model. For this, we use the relations found by [159] (as we did in Chapter 3) and by [177] to relate halo masses to galaxy sizes and luminosities respectively. Both of these relations were found by means of abundance matching, which relates simulated halo masses to the properties of observed galaxies.

By this means [159] found a linear relation between the virial radius $R_{200}$ of the haloes and radius enclosing half of the galaxy mass $r_{1/2}$ holding over eight orders of magnitude in stellar mass and for all morphological types:

$$r_{1/2} = 0.015R_{200}.$$  \hspace{1cm} (4.21)

In this Chapter we use the effective radius $R_e$ which is defined as the radius in which half of the light of the galaxy image is contained and is simply related to $r_{1/2}$ through the relation given in [159]:

$$R_e = \frac{r_{1/2}}{1.34}.$$  \hspace{1cm} (4.22)

In the following we identify $R_e$ with $r(m_{sh})$ in order to keep the notation concise. Regarding luminosities, [177] used abundance matching to fit the mass-luminosity relation for individual galaxies, using the following double power-law:

$$L = L_0 \left(\frac{M}{M_0}\right)^a \left[1 + \left(\frac{M}{M_0}\right)^b\right]^{1/k}.$$  \hspace{1cm} (4.23)

In principle abundance matching can be used for any choice of waveband, provided the luminosity function is well-constrained. [177] found for the $K$ waveband, $L_0 = 1.37 \times 10^{10} L_\odot h^{-2}$, $M_0 = 6.14 \times 10^9 M_\odot h^{-1}$, $a = 21.03$, $b = 20.74$ and $k = 0.0363$, while for the $B_j$ waveband the parameters are: $L_0 = 4.12 \times 10^9 L_\odot h^{-2}$, $M_0 = 1.66 \times 10^{10} M_\odot h^{-1}$, $a = 6.653$, $b = 6.373$ and $k = 0.111$ [177]. In the following, we primarily assume our galaxy luminosities are provided in the $B_j$ waveband, as these provide the more conservative results, but we also provide the size-magnitude probability distribution in the $K$ waveband for comparison. Although we are aware that weak lensing data are usually given in the $r$-band or the $i$-band, here we provide results for our model in the $B_j$ (and occasionally $K$) band only because we can fully exploit the relation in eq. (4.23) given in [177], which are indeed given in these two wavebands. By means of a well-calibrated eq. (4.23) for the $r$-band and the $i$-band, the methods and the results provided in this work can be easily extended to more realistic cases.

We should also account for the fact that the observed luminosity of a galaxy in a
given waveband has to be corrected to be converted in the rest-frame luminosity of that galaxy. In order to calculate this correction, called \textit{k-correction}, we would need to have the parameters for the relation given in eq. (4.23) in several wavebands as well. However, in this work we simply use eq. (4.23) for a given band, even though we remark that this issue has to be addressed in future works in order to obtain a better prediction for the intrinsic correlations of galaxy properties.

Apparent and absolute magnitudes ($M_{\text{abs}}$) are related by:

$$m - M_{\text{abs}} = 25 + 5 \log_{10} \frac{d_L}{\text{Mpc}},$$  \hspace{1cm} (4.24)

and the absolute magnitudes are defined as,

$$M_{\text{abs}} - M_{\text{abs},\odot} = -2.5 \log_{10} \frac{L}{L_{\odot}},$$  \hspace{1cm} (4.25)

where we need to remember that each quantity is defined in a certain waveband. In particular, $M_{\text{abs},\odot}$ is the solar absolute magnitude in a well defined waveband. Finally we obtain the equation which can link apparent magnitudes with luminosities and, eventually, halo masses:

$$m = 25 + 5 \log_{10} \frac{d_L(z)}{\text{Mpc}} - 2.5 \log_{10} \frac{L}{L_{\odot}} + M_{\text{abs},\odot},$$  \hspace{1cm} (4.26)

where $M_{\text{abs},\odot}$ is the solar absolute magnitude in the chosen waveband.

The intrinsic estimators given in eq. (4.62) are defined for observed angular sizes and magnitudes. However, it is possible to link the estimators directly to the physical sizes and luminosities of the galaxy when a local definition in redshift of the estimator is considered. For the intrinsic size estimator, the angular size and the true size are related through the angular diameter distance $\lambda_I = r(z)/D_A(z)$, so that:

$$\lambda_I = \ln \frac{r(z)}{\text{Mpc}} - \ln \frac{D_A(z)}{\text{Mpc}}.$$  \hspace{1cm} (4.27)

For objects at a given redshift their observed size field and true size field are related by a constant term which cancels when considering fluctuations:

$$\delta^I_I(m_{\text{sh}}, z) \equiv \lambda_I - \langle \lambda_I \rangle_z = \ln \frac{r(z)}{\text{Mpc}} - \left< \ln \frac{r}{\text{Mpc}} \right>_z.$$  \hspace{1cm} (4.28)

and the same for magnitudes:

$$\delta^m_I(m_{\text{sh}}, z) \equiv m_I - \langle m_I \rangle_z = -2.5 \log_{10} \frac{L(z)}{L_{\odot}} + 2.5 \left< \log_{10} \frac{L}{L_{\odot}} \right>_z.$$  \hspace{1cm} (4.29)
4.2.2 Modelling the size-magnitude distribution

In our model, the intrinsic size and magnitude distribution is essentially given by the integral of the halo and sub-halo mass functions; here we use the physical halo mass function corrected for the co-moving volume:

\[ n(M, z) = \frac{dN}{dMd\Omega} = \frac{dN}{dM dV_{\text{com}}} \frac{dV_{\text{com}}}{d\Omega} = n_{\text{com}}(M, z) \frac{c}{H_0} \left(1 + z\right)^2 D_A^2(z) \frac{E(z)}{E(z)} \]

(4.30)

Here, \( D_A(z) \) is the angular diameter distance and \( E(z) \) describes the evolution of the Hubble parameter.

We also model the additional scatter in the galaxy size-halo mass and luminosity-halo mass relations (in terms of magnitudes) using Gaussian distributions. For the scatter, we use the values found by [159] for the size-virial radius relation of galaxies of 0.2 dex, corresponding to an intrinsic scatter of \( \sigma_\lambda \approx 0.46 \). For the luminosity-halo mass relation we use the value found by [178], that is \( \sigma_{\log_{10} L} \sim 0.2 \); translating into magnitudes, we find an intrinsic scatter of \( \sigma_m = 0.5 \). Notice that we simply assume the intrinsic scatters independent on the halo mass. For the present, we will assume that these scatters are independent.

The resulting size-magnitude distribution is given by:

\[
\Phi_1(\lambda_1, m_1, z) = \int_{+\infty}^{M_\ast} dM n(M, z) \times \\
\times \int_{M_\ast}^{M} dm_{\text{sh}} \left( \delta_D(m_{\text{sh}} - M_\ast) + \frac{dN(m_{\text{sh}}|M_\ast, z)}{dm_{\text{sh}}} \right) \times \\
\times N(\lambda_1|\lambda(m_{\text{sh}}, z), \sigma_\lambda) N(m_1|m(m_{\text{sh}}, z), \sigma_m)
\]

(4.31)

where \( M_\ast = 10^{10} M_\odot h^{-1} \) indicates the minimum mass for haloes hosting a galaxy (see [172]) and \( M_\ast \) accounts for the fact that the central galaxy has somewhat less mass than the full halo and:

\[
N(\lambda_1|\lambda(m_{\text{sh}}, z), \sigma_\lambda) = \frac{1}{\sqrt{2\pi \sigma_\lambda}} \exp \left(-\frac{(\lambda_1 - \lambda(m_{\text{sh}}, z))^2}{2\sigma_\lambda^2}\right)
\]

\[
N(m_1|m(m_{\text{sh}}, z), \sigma_m) = \frac{1}{\sqrt{2\pi \sigma_m}} \exp \left(-\frac{(m_1 - m(m_{\text{sh}}, z))^2}{2\sigma_m^2}\right)
\]

(4.32)

are the Gaussian distributions which account for the intrinsic scatter in the size and luminosity-halo mass relations.

In Fig. 4.1 size-magnitude distributions in the B_1 waveband are shown for different redshifts. As expected, the observed sizes and fluxes are larger at lower redshifts. Also plotted is the size-magnitude distribution integrated over redshift,
Figure 4.1 – Size-Magnitude distribution in B_j-band for different redshifts. Cuts are \( m_{\text{lim}} = 24 \) and \( \lambda_{\text{lim}} = -2.5 \).
which in shape resembles that of the mean redshift. In Fig. 4.2, we also plot the integrated size-magnitude distribution in the $K$ waveband, derived from the $K$-band luminosity-mass function given above. Results in those figure are shown for magnitudes between $24 < m < 20$ and sizes between $−2.5 < \lambda < −0.5$. We assume the same cuts throughout this work if not otherwise stated.

4.2.3 Survey thresholds and responsivities

To obtain more realistic results, we need to model the galaxy selection effects. In practise, it can be necessary to perform cuts in surface brightness of galaxies, as this quantity is conserved by weak gravitational lensing processes. We assume a magnitude-limited survey with $m_{\text{lim}}$. The limiting magnitude value refers to a given waveband and here we assume values for either for $B_j$ or $K$ waveband corresponding to the luminosity-halo mass relations given by [177]. It is also necessary to model the selection effect for galaxy sizes, and assume a limiting size of $r_{\text{min}}$ in arc seconds.

Given a model for the size-magnitude distribution and assuming a selection function, it is also possible to calculate the redshift distribution of the survey:

$$p(z) = \frac{\int_{\text{cuts}} d\lambda dm \Phi(\lambda, m, z)}{\int dz \int_{\text{cuts}} d\lambda dm \Phi(\lambda, m, z)},$$

(4.33)

Assuming, for the $B_j$-band, sharp cuts at $\lambda_{\text{lim}} = −2.5$ and $m_{\text{lim}} = 24$ our model in eq. (4.33) can roughly reproduce the expected redshift distribution with mean redshift $\langle z \rangle \sim 0.96$ parametrised by:

$$p(z) \propto z^a \exp \left[ − \left( \frac{z}{z_0} \right)^b \right],$$

(4.34)
which is normalized to unity in order to have the number density distribution given by: \( n(z) = \bar{n} p(z) \). We consider, following [120], a set of parameters to simulate a Euclid-like survey spanning 15000 square degrees with an average galaxy number density per steradian \( \bar{n} = 30 \text{ arcmin}^{-2} \) and we assume the parameters to be \( a = 2 \), \( b = 1.5 \), \( z_0 = 0.64 \) which gives a mean redshift around 0.96.

Given these survey thresholds, we can derive the size and magnitude responsivities of the model as a function of redshift as described above. These are shown in Fig. 4.3. We can see that the size responsivity approaches the ideal \( (\eta_\lambda = 1) \) at low redshifts; this reflects the fact that the magnitude cut, rather than the size cut, is of primary importance at these redshifts, as can be seen in Fig. 4.1. The magnitude responsivity is significantly different from the ideal \( (\eta_m = -2.17) \), reflecting the fact that many galaxies can be pulled into the sample by magnification.

To calculate the responsivity for a survey distributed over a redshift range, we must first integrate the size-magnitude distribution function over redshift. The responsivity can be derived from:

\[
     f_1(\lambda_1, m_1) = \frac{\int dz \, \Phi(\lambda, m, z)}{\int dz \int_{\text{cuts}} d\lambda dm \, \Phi(\lambda, m, z)}.
\]  

(4.35)

Recall however, this is a mean responsivity and different redshifts respond differently; the redshift dependence re-weights the effective convergence as described below.
4.2.4 Conditional averages

As the statistical properties of galaxies in haloes depend on their mass, it is useful to understand how the average properties of galaxies that are selected depend on their mass. In particular, we are interested in the probability that a galaxy of a given mass and redshift enters into the sample, and how the mean sizes and magnitudes are affected. For selection thresholds \( \lambda_{\text{lim}} \) and \( m_{\text{lim}} \), a galaxy with sub-halo mass \( m_{\text{sh}} \) and at redshift \( z \), the probability of observing the galaxy is

\[
P_{\text{obs}}(m_{\text{sh}}, z) = \int_{\lambda_{\text{lim}}}^{+\infty} d\lambda N(\lambda | \lambda(m_{\text{sh}}, z), \sigma_\lambda) \int_{-\infty}^{m_{\text{lim}}} dm N(m | m(m_{\text{sh}}, z), \sigma_m) \tag{4.36}
\]

Similarly, the weighted averages, \( \tilde{\lambda}(m_{\text{sh}}, z) \) and \( \tilde{m}(m_{\text{sh}}, z) \), can be defined as follows:

\[
\tilde{\lambda}(m_{\text{sh}}, z) = \int_{\lambda_{\text{lim}}}^{+\infty} d\lambda \lambda N(\lambda | \lambda(m_{\text{sh}}, z), \sigma_\lambda) \int_{-\infty}^{m_{\text{lim}}} dm N(m | m(m_{\text{sh}}, z), \sigma_m),
\]
\[
\tilde{m}(m_{\text{sh}}, z) = \int_{-\infty}^{m_{\text{lim}}} dm m N(m | m(m_{\text{sh}}, z), \sigma_m) \int_{\lambda_{\text{lim}}}^{+\infty} d\lambda N(\lambda | \lambda(m_{\text{sh}}, z), \sigma_\lambda) \tag{4.37}
\]

4.3 Correlation statistics

4.3.1 Galaxy size and luminosity fields

Following [163] we can define the galaxy density field given a model for the joint size-magnitude distribution:

\[
n_{g}(x) = \int_{\lambda_{\text{lim}}}^{+\infty} d\lambda \int_{-\infty}^{m_{\text{lim}}} dm \times \left[ \sum_{i} \delta_D(\lambda - \lambda_i) \delta_D(m - m_i) \sum_{j} \delta_{D}^{(3)}(x - x_i - x_j) \right] \tag{4.38}
\]

where the \( \sum_{j} \) is over the central and possible satellite galaxies and \( x_j \) represents their position relative to the halo centre; \( x_j = 0 \) for the central galaxy, while for the satellite galaxies, these positions are described by the satellite probability profile. By means of the relations described in section (4.2.1) we weight the galaxy density field by size and magnitude to define, respectively, the galaxy size and luminosity.
fields as:

$$\lambda(x) = \bar{n}_g^{-1} \int_{\lambda_{\text{lim}}}^{+\infty} d\lambda \int_{m_{\text{lim}}}^{m_{\text{lim}}} dm \times$$

$$\times \sum_i \delta_D(\lambda - \lambda_i) \delta_D(m - m_i) \sum_j \delta_D^3(x - x_i - x_j) \lambda(m_{\text{sh},j})$$

$$m(x) = \bar{n}_g^{-1} \int_{\lambda_{\text{lim}}}^{+\infty} d\lambda \int_{-\infty}^{m_{\text{lim}}} dm \times$$

$$\times \sum_i \delta_D(\lambda - \lambda_i) \delta_D(m - m_i) \sum_j \delta_D^3(x - x_i - x_j) m(m_{\text{sh},j})$$

(4.39)

where $\lambda(m_{\text{sh},j})$ and $m(m_{\text{sh},j})$ are, respectively, the angular size and the magnitude associated with the mass of the central galaxy or satellite galaxies and:

$$\bar{n}_g(z) = \int_{M_{\star}}^{+\infty} dM n(M, z) \times$$

$$\times \int_{M_{\star}}^{M} dm_{\text{sh}} \left( \delta_D(m_{\text{sh}} - M_{\star}) + \frac{dN(m_{\text{sh}} | M, z)}{dm_{\text{sh}}} \right) P_{\text{obs}}(m_{\text{sh}}, z)$$

(4.40)

and for the mean size and magnitude we have:

$$\langle \lambda \rangle(z) = \frac{1}{\bar{n}_g} \int_{M_{\star}}^{+\infty} dM n(M, z) \int_{M_{\star}}^{M} dm_{\text{sh}} \times$$

$$\times \left( \delta_D(m_{\text{sh}} - M_{\star}) + \frac{dN(m_{\text{sh}} | M, z)}{dm_{\text{sh}}} \right) \bar{\lambda}(m_{\text{sh}}, z)$$

$$\langle m \rangle(z) = \frac{1}{\bar{n}_g} \int_{M_{\star}}^{+\infty} dM n(M, z) \int_{M_{\star}}^{M} dm_{\text{sh}} \times$$

$$\times \left( \delta_D(m_{\text{sh}} - M_{\star}) + \frac{dN(m_{\text{sh}} | M, z)}{dm_{\text{sh}}} \right) \bar{m}(m_{\text{sh}}, z).$$

(4.41)

Generally, regardless we deal with size or magnitude fluctuations as defined in eq. (4.28) and (4.29), a given realization of halo and sub-halo positions results in an estimator-weighted density field as:

$$\delta_1(x) = \bar{n}_g^{-1} \int_{\lambda_{\text{lim}}}^{+\infty} d\lambda \int_{-\infty}^{-m_{\text{lim}}} dm \times$$

$$\times \sum_i \delta_D(\lambda - \lambda_i) \delta_D(m - m_i) \sum_j \delta_D^3(x - x_i - x_j) \delta_1(m_{\text{sh}})$$

(4.42)

The main difference with Chapter 3 is that a fixed halo mass threshold has been used whereas here we try to model the fields including the cuts in the survey. These cuts affect not only the lensing responsivity but also the statistics of intrinsic sizes and magnitudes.

However, given the one-point statistics for the size and the magnitude field, we
can formally use the same equation for the halo model of intrinsic size correlations given in [163] simply replacing $\lambda(m_{sh}) \rightarrow \tilde{\lambda}(m_{sh})$ and $m(m_{sh}) \rightarrow \tilde{m}(m_{sh})$ in the equations provided there when applied to angular sizes and magnitudes.

In order to keep the notation short, we only use $\delta_I$ to indicate the intrinsic contribution either from size or magnitude without including responsivities. Just replace it with $\delta^\lambda_I$ or $\delta^m_I$ to obtain the respective equations for size and magnitude power spectra. Additionally, in order to keep in mind the effects of the cuts in the size-magnitude distribution, we indicate the generic intrinsic field fluctuation as $\tilde{\delta}_I$ to indicate we are including the cuts in the survey.

### 4.4 Two-point statistics

In this section we present our model for the correlation between intrinsic correlation of galaxy sizes and magnitudes. In the one-halo terms, we only include the cross-correlations between different galaxies, so there is no central-central contribution.

#### 4.4.1 Auto-correlation terms

As in Chapter 3, applying the halo model formalism power spectra we can calculate the one-halo and two-halo terms for the auto-correlation of intrinsic properties. The satellite-satellite contribution is:

$$
P_1^{\text{sat}}(k) = \tilde{n}_{g}^{-2} \int_{M_*}^{\infty} dM \frac{dN}{dM} \left( \int_{M_*}^{M} dm_{sh} \frac{dN(m_{sh}|M)}{dm_{sh}} \tilde{\delta}_I(m_{sh}) u_d(k, M) \right)^2 \tag{4.43}
$$

We also have contribution from central-satellite correlation terms:

$$
P_1^{\text{cen-sat}}(k) = \frac{2}{\tilde{n}_{g}^2} \int_{M_*}^{\infty} dM \frac{dN}{dM} \tilde{\delta}_I(M_c) \left( \int_{M_*}^{M} dm_{sh} \frac{dN(m_{sh}|M)}{dm_{sh}} \tilde{\delta}_I(m_{sh}) u_d(k, M) \right) \tag{4.44}
$$

For the two-halo term we have:

$$
P_2^2(k) = (\tilde{b}_{\delta_I,\text{cen}} + \tilde{b}_{\delta_I,\text{sat}})^2 P_{\text{lin}}^\text{in}(k) \tag{4.45}
$$

where:

$$
\tilde{b}_{\delta_I,\text{cen}} = \tilde{n}_{g}^{-1} \int_{M_*}^{\infty} dM n(M) b(M) \tilde{\delta}_I(M_c) \tag{4.46}
$$
and
\[
\bar{b}_{\delta_{I,sat}} = \bar{n}_g \int_{M_*}^{\infty} dM n(M) b(M) \int_{M_*}^{M} dm_{sh} \frac{dN(m_{sh}|M)}{dm_{sh}} \times \delta_I(m_{sh}) u_d(k, M).
\]

### 4.4.2 Density-size cross power spectra

For the cross-correlation density-size we obtain for both central and satellites:
\[
P_{1h-sat}^{\rho \delta_I}(k) = \bar{n}_g^{-1} \int_{0}^{\infty} dM n(M) M \times \int_{M_*}^{M} dm_{sh} \frac{dN(m_{sh}|M)}{dm_{sh}} \delta_I(m_{sh}) u_d(k, M)
\]
\[
P_{2h}^{\rho \delta_I}(k) = \bar{b}_\rho (\bar{b}_{\delta_{I,cen}} + \bar{b}_{\delta_{I,sat}}) P^{lin}(k)
\]

where \(\bar{b}_\rho\) is the same term defined in eq. (3.46) in Chapter 3 and the other bias factors are given above.

As in [163], we are assuming all of the lensing mass is associated with the haloes, and ignore mass associated with sub-clumps. Potentially this approximation fails to take into account further correlations between size and density on scales within haloes but it should be good to understand the large-scale behaviour.

### 4.4.3 Size-Magnitude cross-power spectra

In order to combine size and magnitude correlation can be difficult as we need to consider a redshift interval over which the responsivity \(\eta_\lambda\) and \(\eta_m\) are constant. Although this does not seem to be the case over a large redshift range, when measuring correlations in narrower redshift bins we combine that information. In that case we also have the cross-correlation of sizes and magnitudes.

The one-halo terms for the size-magnitude cross-power spectrum are:
\[
P_{1h-sat}^{\delta_{I} \delta_{m}}(k) = \bar{n}_g^{-2} \int_{M_*}^{\infty} dM n(M) \times \left[ \int_{M_*}^{M} dm \frac{dN(m_{sh}|M)}{dm_{sh}} \delta_I(m_{sh}) u_d(k, M) \right] \times \left[ \int_{M_*}^{M} dm \frac{dN(m_{sh}|M)}{dm_{sh}} \delta_m(m_{sh}) u_d(k, M) \right]
\]

The contribution from central-satellite correlation terms is made of two parts,
depending on whether we consider the size or the magnitude of the central galaxy:

\[
\tilde{P}_{\delta_1^c \delta_1^m}^{2h-cen\rightarrow sat}(k) = \frac{1}{\bar{n}_g^2} \int_{M_*}^{\infty} dM n(M) \times \\
\times \left[ \tilde{\delta}_1^\lambda (M_c) \int_{M_*}^{M} dm_{sh} \frac{dN(m_{sh}|M)}{dm} \tilde{\delta}_1^m (m_{sh}) u_{d}(k, M) + \right. \\
\left. + \tilde{\delta}_1^m (M_c) \int_{M_*}^{M} dm_{sh} \frac{dN(m_{sh}|M)}{dm} \tilde{\delta}_1^\lambda (m_{sh}) u_{d}(k, M) \right] \\
\] (4.50)

For the two-halo terms we have three contributions: from centrals, from satellites and from the central-satellite term.

\[
P^{2h-cen\rightarrow cen\rightarrow sat}(k) = \tilde{b}_{\delta_1^c \delta_1^c} \tilde{b}_{\delta_1^m \delta_1^m} P^{lin}(k) \\
P^{2h-sat\rightarrow sat}(k) = \tilde{b}_{\delta_1^c \delta_1^c} \tilde{b}_{\delta_1^m \delta_1^m} P^{lin}(k) \\
P^{2h-cen\rightarrow sat\rightarrow cen}(k) = (\tilde{b}_{\delta_1^c \delta_1^c} \tilde{b}_{\delta_1^m \delta_1^m} + \tilde{b}_{\delta_1^c \delta_1^c} \tilde{b}_{\delta_1^m \delta_1^m}) P^{lin}(k) \\
\] (4.51)

4.4.4 2D lensing

In order to compare intrinsic size and magnitude correlations with weak lensing convergence power spectra we have to integrate the projected correlations over the
Figure 4.5 – Convergence power spectra for Euclid-like survey when redshift dependence of the responsivity values is included.

Figure 4.6 – 2D-only size power spectra in $B_j$-band for a Euclid-like survey. Shot-noise is shown separately (dashed line).
redshift distribution:

\[ \kappa_1(\theta) = \int d\chi p(\chi) \delta_1(\chi, \theta, \chi), \]  

(4.52)

where \( p(\chi) = n(\chi)/\bar{n} \) and we dropped the notation for size and magnitude as the projection works in the same fashion for both and \( n(\chi) \) is the redshift distribution described in section 4.1 and \( \eta \) indicates either the size or magnitude responsivity. Our primary observable will be the two-point moment of the estimator, which has three contributions; in Fourier space, these can be written as

\[ C_\kappa(\ell) = C_{\eta\kappa}(\ell) + C_{11}(\ell) + C_{G1}(\ell) + \sigma_{\text{shot}}^2, \]  

(4.53)

where \( \sigma_{\text{shot}} \) is the shot-noise term and \( \bar{n} \) is the total number of galaxies per steradian. The first term on the right-hand side is the responsivity-weighted lensing convergence power spectrum \( C_{\eta\kappa} \), which is well understood and can be calculated by means of the Limber approximation [81]:

\[ C_{\eta\kappa}(\ell) = \left( \frac{3H_0^2\Omega_m}{2c^2} \right)^2 \int_0^{\chi_{\text{hor}}} d\chi \frac{g^2(\chi)}{[f_K(\chi)]^2} P_\delta \left( \frac{\ell}{f_K(\chi)}, \chi \right), \]  

(4.54)

where \( P_\delta \) is the matter power spectrum, \( \chi \) is the comoving distance along the line of sight, \( \chi_{\text{hor}} \) is the comoving horizon distance and \( f_K(\chi) \) is the comoving angular
diameter distance. The weighting function

\[
g(\chi) = \frac{f_K(\chi)}{a(\chi)\bar{n}} \int_{\chi}^{\chi_{\text{hor}}} d\chi' \eta_{\epsilon}(\chi') n(\chi') \frac{f_K(\chi' - \chi)}{f_K(\chi')},
\]

where \(a\) is the dimensionless scale factor, \(c\) is the speed of light, \(H_0\) is the Hubble constant, \(\Omega_{m,0}\) is the present matter density parameter and \(n(\chi)d\chi\) is the effective number density of galaxies per steradian in \(d\chi\), normalized so that \(\int n(\chi)d\chi = \bar{n}\), where \(\bar{n}\) is the total number density of galaxies per steradian. The radial function \(f_K(\chi)\) depends on \(K\), the inverse square of curvature radius in units of \(H_0/c\), as follows:

\[
f_K(\chi) = \begin{cases} 
\sqrt{K} \sin(\sqrt{K} \chi) & K > 0 \\
\chi & K = 0 \\
\sqrt{-K} \sinh(\sqrt{-K} \chi) & K < 0.
\end{cases}
\]

For simplicity, below we will assume \(K = 0\).

Notice that we also included the responsivity function \(\eta_{\epsilon}(\chi')\) in eq. (4.55). This is because of the redshift dependence of the responsivity due to the cuts in the survey. The effect is a different weighting of the convergence power spectrum depending whether we are measuring size or magnitude correlations. This difference is shown in Fig. (4.5).

The other terms in eq. (4.53) are the intrinsic contributions. For both sizes and magnitudes separately we have that the II and the GI terms are:

\[
C_{\Pi}(\ell) = C_{\kappa\lambda}(\ell) \\
C_{\text{GI}}(\ell) = 2C_{\kappa\lambda}(\ell)
\]

Again we assume Limber’s approximation the intrinsic terms (for both size and magnitude) in the II and GI terms are calculated as follows:

\[
C_{\kappa\lambda}(\ell) = \int_{\chi_{\text{hor}}}^{\infty} d\chi \frac{p^2(\chi)}{\chi^2} P_{\delta\lambda}(\ell, \chi) \\
C_{\kappa\lambda}(\ell) = \int_{0}^{\chi_{\text{hor}}} d\chi \frac{p^2(\chi)}{\chi^2} P_{\delta\lambda}(\ell, \chi) \\
C_{\kappa\lambda}(\ell) = \frac{3H_0^2\Omega_{m,0}}{2c^2} \int_{0}^{\chi_{\text{hor}}} d\chi \frac{g(\chi)p(\chi)}{\chi^2} P_{\kappa\lambda}(\ell, \chi) \\
C_{\kappa\lambda}(\ell) = \frac{3H_0^2\Omega_{m,0}}{2c^2} \int_{0}^{\chi_{\text{hor}}} d\chi \frac{g(\chi)p(\chi)}{\chi^2} P_{\kappa\lambda}(\ell, \chi)
\]

### 4.4.5 Lensing tomography

Weak lensing tomography (e.g [100]), is a technique that aims to exploit the redshift information present in weak lensing data. We cut the distribution given in eq. (4.34)
at \( z_{\text{max}} = 3 \), normalize it to unity and we divide \( p(z) \) in \( N_{\text{bin}} = 10 \) tomographic bins of width \( \Delta z_{\text{bin}} = 0.3 \) such that:

\[
p^i(z) = \begin{cases} 
p(z) & z_i < z \leq z_{i+1} \\
0 & \text{otherwise}
\end{cases}
\]

for every \( i = 1, \ldots, N_{\text{bin}} \) and \( z_i \) and \( z_{i+1} \) indicate the bin boundaries and:

\[
\int_0^{z_{\text{max}}} dz p^i(z) = 1,
\]

in order to have \( n^i(z) = \bar{n}^i p^i(z) \) where \( \bar{n}^i \) is the number density of galaxies per steradian in the \( i \)th redshift bin which is given by:

\[
\int_{z_i}^{z_{i+1}} dz n(z) = \bar{n} \int_{z_i}^{z_{i+1}} dz p(z) = \bar{n}^i,
\]

Also, for simplicity we do not include any photometric redshift error; this is an important issue when considering intrinsic correlations in weak lensing; the width of the redshift bins we assume in our calculations are larger than the expected photometric redshift errors and therefore the effects due to this issue are expected to be small for our purposes. However, we remind to future works the study of the implication of photo-z errors for intrinsic size and magnitude correlations.

For narrower bins of a tomographic analysis, we can take the responsivities as constant, allowing for a more direct combination of size and magnitude information. In each bin, our estimator then will be:

\[
\hat{\kappa}_I^\lambda \equiv \frac{1}{\eta_\lambda} (\lambda_I - \langle \lambda_I \rangle) , \\
\hat{\kappa}_I^m \equiv \frac{1}{\eta_m} (m_I - \langle m_I \rangle) .
\]

Indicating with \( i \) and \( j \) two different redshift bins, equations from previous sections
are then slightly modified in this case:

\[
C^{ij}_\kappa (\ell) = \left( \frac{3H_0^2 \Omega_{m,0}}{2c^2} \right)^2 \int_0^{X_{\text{hor}}} d\chi \frac{g^i(\chi)g^j(\chi)}{|f_K(\chi)|^2} \frac{\delta_\ell^j}{\delta_\ell^i} \left( \frac{\ell}{\chi}, \chi \right)
\]

\[
C^{ij}_\kappa \lambda (\ell) = \frac{1}{\eta_\kappa \eta_\lambda} \int_0^{X_{\text{hor}}} d\chi \frac{p^i(\chi)p^j(\chi)}{\chi^2} \frac{\delta_\ell^j}{\delta_\ell^i} \left( \frac{\ell}{\chi}, \chi \right)
\]

\[
C^{ij}_\kappa m (\ell) = \frac{1}{\eta_\kappa \eta_m} \int_0^{X_{\text{hor}}} d\chi \frac{p^i(\chi)p^j(\chi)}{\chi^2} \frac{\delta_\ell^j}{\delta_\ell^i} \left( \frac{\ell}{\chi}, \chi \right)
\]

\[
C^{ij}_\kappa \lambda m (\ell) = \frac{3H_0^2 \Omega_{m,0}}{2c^2} \frac{1}{\eta_\lambda} \int_0^{X_{\text{hor}}} d\chi \frac{p^i(\chi)g^j(\chi) + g^i(\chi)p^j(\chi)}{\chi^2} \times \frac{\delta_\ell^j}{\delta_\ell^i} \left( \frac{\ell}{\chi}, \chi \right)
\]

\[
C^{ij}_\kappa \lambda \lambda m (\ell) = \frac{3H_0^2 \Omega_{m,0}}{2c^2} \frac{1}{\eta_m} \int_0^{X_{\text{hor}}} d\chi \frac{p^i(\chi)g^j(\chi) + g^i(\chi)p^j(\chi)}{\chi^2} \times \frac{\delta_\ell^j}{\delta_\ell^i} \left( \frac{\ell}{\chi}, \chi \right)
\]

The tomographic lensing kernel is defined as follows:

\[
g^i(\chi) = \frac{f_K(\chi)}{a(\chi)} \int_\chi^{X_{\text{hor}}} d\chi' p^i(\chi') \frac{f_K(\chi' - \chi)}{f_K(\chi')}
\]

A simple noise-weighted estimator that combines the two estimators for galaxy sizes and magnitudes has already been presented in [140] for the surface density and [18] for the convergence. Following their notation, we can write the total intrinsic contribution to the convergence as linear combination of two terms:

\[
\hat{\kappa}_1 = \alpha_\lambda \hat{\kappa}_1^\lambda + \alpha_m \hat{\kappa}_1^m,
\]

where the two coefficients are constrained so that \( \alpha_\lambda + \alpha_m = 1 \) in order to have an estimator unbiased for the convergence. Then we obtain:

\[
\hat{\kappa} = \kappa + \kappa_1.
\]

Minimising the variance of the full estimator given above, we obtain explicit forms for \( \alpha_\lambda \) and \( \alpha_m \):

\[
\alpha_\lambda = \frac{\eta_\lambda^2 \sigma_m^2 - \eta_\lambda \eta_m \sigma_{\lambda m}^2}{\eta_\lambda^2 \sigma_\lambda^2 + \eta_m^2 \sigma_m^2 - 2 \eta_\lambda \eta_m \sigma_{\lambda m}^2}
\]

\[
\alpha_m = \frac{\eta_m^2 \sigma_\lambda^2 - \eta_\lambda \eta_m \sigma_{\lambda m}^2}{\eta_\lambda^2 \sigma_\lambda^2 + \eta_m^2 \sigma_m^2 - 2 \eta_\lambda \eta_m \sigma_{\lambda m}^2}.
\]
\[ \Delta \sigma^2 = \sigma^2_{\lambda} + \sigma^2_{\lambda m} + \alpha^2_{\lambda m} \]

where the variances are calculated given a model for the size-magnitude distribution:

\[ \sigma^2_{\lambda} = \int_{\lambda_{\text{lim}}}^{+\infty} \int_{m_{\text{lim}}}^{-\infty} d\lambda dm f(\lambda, m)(\lambda - \langle \lambda \rangle)^2 \]

\[ \sigma^2_{\lambda m} = \int_{\lambda_{\text{lim}}}^{+\infty} \int_{m_{\text{lim}}}^{-\infty} d\lambda dm f(\lambda, m)(m - \langle m \rangle)^2 \]

\[ \sigma^2_{\lambda m} = \int_{\lambda_{\text{lim}}}^{+\infty} \int_{m_{\text{lim}}}^{-\infty} d\lambda dm f(\lambda, m)(\lambda - \langle \lambda \rangle)(m - \langle m \rangle). \]

For the model we propose for the joint size-magnitude distribution, we obtain for the variances and for the coefficients the values shown in Table 4.1. The values in Table 4.1 are comparable with the observations made in Alsing et al. (2015) [18]. However, we notice that the values in [18] are measured from CFHTLenS data and also that the size-magnitude distribution calculated in our model is narrower than the one observed in [18]. In Table 4.1 we follow the notation used in [18]; the sign of \( \sigma^2_{\lambda m} \) is negative because reflects the fact that more massive haloes corresponds to larger galaxy sizes and lower galaxy magnitudes.

Therefore we have that the II term is:

\[ C_{\text{II}}^{ij}(\ell) = \alpha^i_{\lambda} \alpha^j_{\lambda} \left( C^{ij}_{\kappa \kappa \lambda}(\ell) + \delta^{ij} \frac{\sigma^2_{\lambda}}{\bar{n}^i(\eta^i_{\lambda})^2} \right) + \]

\[ + \alpha^i_{m} \alpha^j_{m} \left( C^{ij}_{\kappa \kappa \lambda}(\ell) + \delta^{ij} \frac{\sigma^2_{m}}{\bar{n}^i(\eta^i_{m})^2} \right) + \alpha^i_{\lambda} \alpha^j_{\lambda} C^{ij}_{\kappa \kappa \lambda}(\ell) + \alpha^i_{m} \alpha^j_{m} C^{ij}_{\kappa \kappa \lambda}(\ell) + \alpha^j_{\lambda} \alpha^i_{\lambda} C^{ij}_{\kappa \kappa \lambda}(\ell), \]

and for the GI term:

\[ C_{\text{GI}}^{ij}(\ell) = \alpha^i_{\lambda} C^{ij}_{\kappa \kappa \lambda} + \alpha^i_{\lambda} C^{ij}_{\kappa \kappa m}(\ell) + \alpha^i_{m} C^{ij}_{\kappa \kappa \lambda}(\ell) + \alpha^i_{m} C^{ij}_{\kappa \kappa m}(\ell) \]

Since the redshift bins are not overlapping and no photometric redshift errors are included we have that the II term vanishes in bin cross-correlation and contributes only when \( i = j \) and for the GI contributions the term \( g^i(\chi)p^j(\chi) \) is zero when considering \( i < j \).
Figure 4.8 – Optimal noise-weighting combination tomographic power spectra in B_j-band for a Euclid-like survey by using 5 redshift bins. Each redshift bin has $\Delta z = 0.3$ (Bin 1 ranges from $z = 0$ to $z = 0.3$ and so on). Responsivity values are different in each bin. Lensing convergence is indicated with GG and the II term includes the size-magnitude cross-correlation (shot-noise is shown separately).
Figure 4.9 – Tomographic power spectra in B_j-band for a Euclid-like survey by using 5 redshift bins for sizes alone. Lensing convergence is indicated with GG, size-convergence correlation with $\lambda G$, and size-size correlation with $\lambda\lambda$ (shot-noise is shown separately).
Figure 4.10 – Tomographic power spectra in $3_{\lambda}$-band for a Euclid-like survey by using 5 redshift bins for magnitude alone. Lensing convergence is indicated with GG, magnitude-convergence correlation with $mG$, and magnitude-magnitude correlation with $mm$ (shot-noise is shown separately).
4.5 Discussion

4.5.1 Model assumptions

As in Chapter 3, we evaluate our results in the context of a flat $\Lambda$CDM cosmology with parameters consistent with best-fit Planck data [179]; in particular, we assume a total matter density $\Omega_{m,0} = 0.32$, cosmological constant density $\Omega_{\Lambda,0} = 0.68$, baryon density $\Omega_{b,0} = 0.049$ and Hubble constant $H_0 = 100 \, h \, \text{km s}^{-1} \, \text{Mpc}^{-1}$, where $h = 0.67$. In addition, we assume the spectral index of the matter power spectrum is $n_s = 0.96$ and it is normalised such that $\sigma_8 = 0.83$.

We adopt the transfer function given in [59] and non-linear evolution of the matter power spectrum (for estimating lensing convergence power spectrum) is calculated with HALOFIT from [174] recently revised by [175].

We calculated the lensing tomography when combining size and magnitude (Fig. (4.8)), for size-only (Fig. (4.9)), and for magnitude-only (Fig. (4.10)).

4.5.2 Results

In Chapter 3 we examined the issue of intrinsic size correlations in a halo model, where the sizes of galaxies were assumed to be a simple function of the sub-halo mass. In this Chapter, we have extended this analysis by examining the correlations in galaxy brightness, and by introducing intrinsic scatter in the mass-size and mass brightness relations. We have also put realistic selection effects into our predictions to account for the reduced responsivity of the mean properties of galaxies to convergence.

Overall, we find these improvements in the modelling have not affected the main conclusion of [163] discussed in Chapter 3, that intrinsic correlations in the galaxy properties used to trace magnification are an important systematic to measurements of the convergence power spectrum; if ignored, they can significantly bias the cosmological interpretation of the convergence measurements.

The principle determining factor of the importance of the intrinsic correlations is their estimator weighted bias. Because of the steeper relationship between the sub-halo mass and the luminosity reflected in the Vale and Ostriker (2008) relation, we expect a higher bias for the magnitude correlations compared to that expected for sizes, as can be seen in Figure 4.4. The intrinsic contamination to magnitude correlations can actually be comparable to the convergence signal itself, as shown in Figure 4.7, and dominant in narrow tomographic bins. However, as discussed in this Chapter, in order to improve our model, an important point for future work will be to introduce in the model the effect of the $k$-correction accounting for the relation between the observed and rest-frame luminosity of the galaxies.
The addition of scatter in the mass-size and mass-luminosity relations does not directly affect expectations of the two-point correlations. However, it does impact the probability that a galaxy of a given mass will be selected, and therefore the bias weighting of the sample. Our modelling of the distribution indicates a significant sensitivity to the size cuts, consistent with observations and indicating a responsibility of the means to magnification significantly suppressed compared to the ideal values.

In the absence of intrinsic systematics, it can be beneficial to combine sizes and magnitudes together into a single noise-weighted estimator [18]. However, given the difference in the expected intrinsic correlations, combining size and magnitudes may make the systematic contamination worse than for sizes alone. But this may be mitigated depending on how the intrinsic correlations are marginalised over.

The tomographic analysis in Fig. (4.8), (4.9) and (4.10) shows that, like intrinsic shape correlations, intrinsic size and brightness correlations are a serious problem within narrow bins, and ameliorating them requires exploiting cross-correlations between bins where the II contributions are negligible. However, at low redshifts, and in neighbouring bins, the GI terms can also be a comparable systematic.

Regarding the cosmological parameter inference, we do not perform this kind of analysis in this work. However, it would be possible to give some comment and compare our findings with the results obtained by [18]. The results shown in Fig. 2.12 in Section 2.8 are obtained assuming a bias equal to $\beta = 0.05$. As shown in Fig. 4.4 in this Chapter, the bias factors for the intrinsic size and magnitude contributions that we calculate by means of our theoretical model are not too far from the value of the bias assumed in [18]. However, it seems that our calculations predict a slightly higher bias (see for example the magnitude bias in Fig. 4.4) and hence an impact on estimating the dark energy equation of state parameter potentially higher than expected in [18], suggesting a more pessimistic scenario for dark energy constraints. It is therefore fundamental to measure large scale correlations of intrinsic galaxy sizes and magnitudes in order to try to understand what the values of this bias factors are and whether they can be compared to our theoretical predictions.

Our theoretical results emphasise the need to better quantify these intrinsic correlations, particularly on small scales where the halo model is approximate and potentially is missing important physics. Hydrodynamic simulations have more realistic small scale physics, but may not have the full dynamic range essential for weak lensing analyses. Semi-analytic models, based on simulated merging trees and constrained to match related galaxy observables, may improve the situation.

Equally essential is to focus on measuring these effects in large scale surveys, focusing on low redshifts and large scales where the intrinsic signal is expected to dominate over shot noise and the convergence signal. We are presently investi-
gating whether these correlations can be observed in the SDSS. Measurements of such correlations are observationally challenging; they are subject to many of the same systematics as shape measurements. However, the magnification estimators have the additional complication of needing to accurately assess the mean sizes and magnitudes and their responsivities to lensing under the selection function.
Chapter 5

Conclusions

Weak gravitational lensing occurs when light from distant background galaxies is distorted by the large scale structures between them and the observer. Galaxy images are sheared and magnified and their observed shapes, sizes and fluxes are changed. By measuring correlations of those effects it is possible to better understand our universe in terms of cosmological models and constraints on cosmological parameters.

From the observational point of view, our ability to study cosmology from weak lensing relies on how well we can measure galaxy image distortions with current instruments. From the theoretical side, another important issue is the impact of correlations of intrinsic galaxy properties. Intrinsic correlations can arise when intrinsic properties of galaxies are linked to the underlying density field where galaxies evolve.

Intrinsic correlations of galaxy ellipticities, also called Intrinsic Alignments (IAs), have extensively been studied in literature both theoretically, developing models as well as by means of N-body simulations, and observationally through direct measurements. As a result of those studies it has been found that constraints on cosmological parameters can be strongly biased if IAs are neglected; it is then necessary to account for intrinsic correlations of galaxy properties in order to obtain unbiased constrained on cosmological parameters from weak lensing.

Size and magnitude information comes for free from a weak lensing survey though

\[\text{Steven Weinberg}\]
their interpretation can be challenging. In order to use that information to probe the cosmological model then we absolutely need to understand the potential impact of analogous correlations of sizes and magnitudes. Recently, some authors (Heavens et al. 2013 [17], Alsing et al. 2015 [18]) pointed out the importance of adding size information to ellipticities measurements in order to help to obtain better constraints on cosmological parameters. Intrinsic correlations are important to study in order to understand to what extent they can impact weak lensing measurements and because they could provide insights on our general understanding of how galaxies formed.

The main goal of this thesis is studying and developing a model for intrinsic size and magnitude correlations. The motivation for this kind work are in the research for potential systematics in weak lensing analyses that involve the use of cosmic magnification in combination with cosmic shear.

5.1 Simple halo model for intrinsic size correlations

In Chapter 3 we have presented a simple model for calculating intrinsic correlations for galaxy sizes using a simple model based on the halo model formalism. We also modelled the intrinsic size correlations including small scales correlations given by sub-haloes hosted in each halo. The sub-halo mass function we used to account for sub-halo size distribution in haloes is the one given by Giocoli et al. 2010 [166] which they found analysing cosmological N-body simulations. In order to link halo to galaxy sizes we used the relation found by Kravtsov (2013) [159].

We have restricted our analysis to size contribution to magnification and we have neglected some effects as the scatter in the mass-radius relation and the fact that galaxy sizes may also depends on the environment and affected by baryonic physics. We also assumed a simple mass threshold in the selection of galaxies.

This preliminary model indicates that, as for cosmic shear, the intrinsic correlations of galaxy sizes are potentially an important systematic for cosmic magnification, in particular correlations between convergence and intrinsic size.

We studied the contributions to the power spectrum of for the CFHTLenS and Euclid-like surveys. We found that intrinsic size correlations are relevant even for a very deep survey such as Euclid, where their contamination increases from 10% on the largest scales to being comparable to the convergence on $\ell \sim 100$. For the CTHHTLenS-like survey, with $\langle z \rangle \simeq 0.8$, the contamination is even larger, beginning at 25% of the convergence signal on large scales.
5.2 Including cuts in the surveys: effects on intrinsic correlations

In Chapter 4 we extended our model including intrinsic magnitude correlations. We used the same formalism developed in Chapter 3 and in order to link halo to galaxy luminosities we followed Vale & Ostriker (2008) [177] that allow us to connect galaxy luminosities to halo masses by using the abundance matching technique, similarly as done by Kravtsov (2013) [159] for galaxy sizes. We also built a model for the size-magnitude probability distribution and we calculated how lensing magnification changes the observed mean size and magnitude accounting for selection effects of a general survey.

We studied these models for a Euclid-like survey. Overall, we find these improvements in the modelling have not affected the main conclusion of [163] discussed in Chapter 3, that intrinsic correlations in the galaxy properties used to trace magnification are an important foreground to measurements of the convergence power spectrum; if ignored, they can significantly bias the cosmological interpretation of the convergence measurements.

We found that the intrinsic contamination of magnitude correlations can be more relevant than the contamination by size correlations. The main reason for this is the steeper relationship between halo masses and galaxy luminosity as given by Vale & Ostriker (2008) [177]. However, in our model, we do not include for the effect of the \textit{k-corrections}, that is the relation between the observed and rest-frame luminosity of the galaxies. In order to do this we would need relation between halo mass and luminosity for several wavebands. We leave the investigation of this important issue to future work.

We also include scatter in the halo mass-galaxy size and halo mass-galaxy luminosity relations. While this scatter does not affect the conclusion regarding the results for the two-point correlations, it does have an impact in terms of selection effects of the sample. Given the model for the size-magnitude distribution, we obtain values for responsivities and variances that are consistent with observations.

We used a noise-weighted estimator in order to combine size and magnitude information following [18]. However, given the difference in the expected intrinsic correlations, combining size and magnitudes may make the systematic contamination worse than for sizes alone.

The tomographic analysis shows that intrinsic size and brightness correlations can represent an important contamination within narrow bins. Cross-correlations between bins can provide better results, even though, at low redshifts and in neighbouring bins, the GI term can also be a comparable systematic. Also, our results suggest that the shot-noise contribution can be too high to allow a detection of in-
trinsic correlations at low redshift where the intrinsic contributions are expected to be dominant. However, this could depend on the characteristic of the survey, on the model for the size-magnitude distribution and on the fact that we are using sharp cuts in the sample.

5.3 Future Research

It is essential to focus on measuring these effects in large scale surveys, focusing on low redshifts and large scales where the intrinsic signal is expected to dominate over shot noise and the convergence signal. We are presently investigating whether these correlations can be observed in the SDSS. Although the work done so far in this direction is preliminary, we could already see that a crucial point in measuring size correlations is the chosen definition of size. Galaxy sizes can be measured by fitting a surface brightness profile and extracting a scale length that can be used as size measurement. The problem is that different profiles can give different values for the galaxy sizes. An other important is represented by potential cross-correlation of galaxy size and PSF and as pointed out by [144]; a solution could be, in this case, considering only galaxies with sizes larger than the PSF. It is therefore important to carefully study how to extract the right size information out of the data in order to understand how to interpret the measurement of potential intrinsic size correlations. Furthermore, the magnification estimators have the additional complication of needing to accurately assess the mean sizes and magnitudes.

In order to compare observations and theory, the analysis of cosmological N-body simulations can be helpful. First of all, we can use N-body simulations to study non-linear scales in order to go beyond the simple halo model that we assume in our theory. Secondly, we could improve our understanding of galaxy formation and evolution processes making a comparison of possible measured intrinsic correlations of galaxy properties with results from simulations including baryonic matter.

Our theoretical results indicate that we need to continue to study these intrinsic correlations, in particular on small scales where the halo model gives approximate predictions. In order to do that hydrodynamic simulations or, alternatively, semi-analytic models, based on simulated merging trees and constrained to match related galaxy observables, can be used to have more realistic small scale physics included in the prediction for intrinsic correlations.

In conclusion, the study of these intrinsic correlations of galaxy properties not only would help cosmologists to obtain unbiased constraints on cosmological parameters, but it is also interesting on its own, because their detection could provide a wealth of information for understanding how galaxies form and evolve.
References


References


References


[151] Delaye, L. *et al.* Larger sizes of massive quiescent early-type galaxies in clusters than in the field at 0.8 < z < 1.5. MNRAS **441**, 203–223 (2014). URL http://adsabs.harvard.edu/abs/2014MNRAS.441..203D.


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<td>Department: ICG</td>
<td>First Supervisor: PROF. ROBERT CRITTENDEN</td>
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<tr>
<td>a) Have all of your research and findings been reported accurately, honestly and within a reasonable time frame?</td>
<td>YES</td>
</tr>
<tr>
<td>b) Have all contributions to knowledge been acknowledged?</td>
<td>YES</td>
</tr>
<tr>
<td>c) Have you complied with all agreements relating to intellectual property, publication and authorship?</td>
<td>YES</td>
</tr>
<tr>
<td>d) Has your research data been retained in a secure and accessible form and will it remain so for the required duration?</td>
<td>YES</td>
</tr>
<tr>
<td>Question</td>
<td>Answer</td>
</tr>
<tr>
<td>----------</td>
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</tr>
<tr>
<td>e) Does your research comply with all legal, ethical, and contractual requirements?</td>
<td>YES</td>
</tr>
</tbody>
</table>

**Candidate Statement:**
I have considered the ethical dimensions of the above named research project, and have successfully obtained the necessary ethical approval(s)

**Ethical review number(s) from Faculty Ethics Committee (or from NRES/SCREC):**
- B7-A1E8-09D7-9A00
- B7-A1E8-09D7-9A00

If you have not submitted your work for ethical review, and/or you have answered ‘No’ to one or more of questions a) to e), please explain below why this is so:

**Signed (PGRS):** SANDRO CIARLARIELLO  
**Date:** 31 MARCH 2016