Primordial non-Gaussianity in the large-scale structure of the Universe

by

Matteo Tellarini

This thesis is submitted in partial fulfilment of the requirements for the award of the degree of Doctor of Philosophy of the University of Portsmouth.

March, 2016
Copyright

© Copyright 2016 by Matteo Tellarini. All rights reserved.

The copyright of this thesis rests with the Author. Copies (by any means) either in full, or of extracts, may not be made without the prior written consent from the Author.
Abstract

Primordial fluctuations are expected to be produced in the very early Universe, sourcing the anisotropies in the cosmic microwave background and seeding the formation of structures. In this thesis we study the effect of density perturbations produced during inflation on the large-scale galaxy bispectrum.

We start by reviewing the basic concepts of modern cosmology and introducing the tools used in this research: Newtonian perturbation theory, statistics of random fields, the mass function of collapsed halos and the halo bias model. We then briefly describe how models of inflation source local-type non-Gaussian distributed primordial density perturbations.

We apply these tools to justify the bivariate model for the halo density in the presence of primordial non-Gaussianity and derive some known results, like the scale-dependent halo bias. The aim is to show that the statistics of large-scale structure can be used to probe local-type non-Gaussianity of the primordial density field, complementary to existing constraints from the cosmic microwave background.

Parametrising the amount of primordial non-Gaussianity with the leading-order non-linear parameter $f_{NL}$ and the next-order one, $g_{NL}$, we will investigate how galaxy and matter bispectra can distinguish between them, despite their effects being nearly degenerate in the power spectra. We determine a connection between the sign of the halo bispectrum on large scales and the parameter $g_{NL}$ and construct a combination of halo and matter bispectra that is sensitive to $f_{NL}$.

After that, we will focus on local-type non-Gaussianity with $f_{NL}$ only. It is known that the non-linear evolution of the matter density introduces a non-local tidal term in the halo bias model. Furthermore, we will show that the bivariate model in the Lagrangian frame leads to a novel non-local convective term in the Eulerian frame which can lead to non-negligible corrections in the halo bispectra, in particular on large scales or at high redshift.

Finally, we address the problem of modelling redshift space distortions in the

ii
galaxy bispectrum, finding novel contributions with the characteristic large scale amplification induced by local-type non-Gaussianity. Therefore, redshift space distortions can potentially lead to a biased measurement of $f_{NL}$, if not properly accounted for. Moreover, we propose an analytic template for the monopole which can be used to fit against data on large scales, extending models used in recent measurements.

We conclude the thesis with some discussion of future developments. Observational constraints will also be discussed, based on idealised forecasts on $\sigma_{f_{NL}}$ – the accuracy of the determination of $f_{NL}$. Our findings suggest that the constraining power of the galaxy bispectrum in current surveys would provide $f_{NL}$ measurements competitive with constraints from the cosmic microwave background and future surveys could improve this further.
Declaration

Whilst registered as a candidate for the above degree, I have not been registered for any other research award. The results and conclusions embodied in this thesis are the work of the named candidate and have not been submitted for any other academic award.

Word count

Words in text: 32939
Words in headers: 351
Words outside text (captions, etc.): 2713
Number of headers: 91
Number of floats/tables/figures: 40
Number of math inlines: 1754
Number of math displayed: 355
Acknowledgements

First of all I am grateful to my supervisory team, David Wands, Ashley J. Ross and Gianmassimo Tasinato, for getting me started to research in Cosmology and guiding me through these years. I have really appreciated their always opened doors to my questions, no matter how naive and silly they were, for their patience with me and my English and never putting me under pressure.

Moreover, I wish to special thank Gianmassimo Tasinato for hosting me in Swansea. It has been a fantastic experience to work and discuss with him.

Also, I acknowledge a number of people whose comments and advice helped me a lot to develop my knowledge in the subject and improve the quality of our publications: Kazuya Koyama, Héctor Gil-Marín, Yuting Wang, Davide Bianchi, Benedict Kalus, Gong-Bo Zhao, Francesco Pace, Cornelius Rampf, Tommaso Giannantonio and Nina Roth.

I wish to thank a bunch of ICG people and visitors for their kindness towards me and the time spent together – from a simple chat in the coffee room to hanging out sometimes: David Bacon, Davide Bianchi, Humberto Borges, Ben Bose, Julie Bower, Marco Bruni, Angela Burden, Dario Cannone, Diego Capozzi, Thiago Caranes, Javier Chagoya, Sandro Ciarlariello, Marco Crisostomi, James Etherington, Christian Fidler, Héctor Gil-Marín, Holly Green, Daniel Goddard, Matthew D. Hull, Benedict Kalus, Kazuya Koyama, Claudia Maraston, Felipe A. Oliveira, Francesco Pace, Guido W. Pettinari, Jennifer E. Pollack, Alkistis Pourtsidou, Cornelius Rampf, Rossana Ruggeri, Najla Said, Valentina Salvatelli, Fulvio Sbisa, Dario Scovacricchi, Marco Surace, Jimmy Tarr, Thomas Tram, Vincent Vennin, Eleonora Villa, Matthew Withers, Alex Zucca.

A special thank goes to Emilio Franceschini for making my first year in Portsmouth the most enjoyable I could ever hope and for being a good friend, no matter how far away we live.

I wish to show my gratitude to Laura Minghetti and Alberto Bertini for hosting me many times and to my grandparents – Paolina Manzini, Luciano Ronchi, Marisa Neretti, Natale Tellarini – and my sister, Chiara Tellarini. They
all were a constant support from home.

I am deeply thankful to my mother, Monica Ronchi, and my father, Paolo Tellarini, for all the efforts they made to help me all over these years, which I will never forget. To a great extent this thesis is merit of the care they put in growing me and my education.

Finally, my gratitude goes to Elisa Bertini for discovering together amazing places across UK and loving me all over these years, despite all the difficulties. Facing them together made us stronger. My future is with her.
Preamble

The work presented in this thesis was entirely carried out at the Institute of Cosmology and Gravitation, University of Portsmouth, United Kingdom, over the period October 2012 - March 2016.

The following chapters are based on published works:


Quant à moi, mes bras sont rompus
Pour avoir étreint des nuées.

C. BAUDELAIRE
# Table of Contents

**Abstract**  

**Declaration**  

**Acknowledgements**  

**Dissemination**  

1 **Introduction**  

1.1 Units and abbreviations  

2 **The homogeneous and isotropic Universe**  

2.1 Friedmann-Robertson-Walker metric  

2.2 Dynamics of the Universe  

2.3 Hot Big Bang cosmology  

2.4 Beyond the Hot Big Bang  

2.5 Inflation  

2.6 Summary  

3 **From perturbations to structure formation**  

3.1 Perturbation theory  

3.1.1 Linear solutions  

3.1.2 Non-linear solutions  

3.1.3 Lagrangian viewpoint  

3.2 The statistics of random fields  

3.2.1 Applications to the matter density field  

3.3 Large-scale structure  

3.3.1 Mass function  

3.3.2 Local Lagrangian biasing model  

3.3.3 Lagrangian-to-Eulerian transformation  

---  

ix
3.3.4 Local Eulerian bias model ........................................ 47
3.3.5 Non-local Eulerian bias model ..................................... 48
3.3.6 Halo power spectrum and bispectrum ............................ 49
3.4 Summary .................................................................... 50

4 Primordial non-Gaussianity ................................................. 52
4.1 Inflation and cosmological perturbations .......................... 52
4.1.1 CMB constraints ....................................................... 56
4.2 Primordial non-Gaussianity in LSS ................................. 57
4.2.1 Perturbation theory revisited ....................................... 58
4.2.2 Non-Gaussian mass function ...................................... 59
4.2.3 Peak-background split with PNG .................................. 61
4.2.4 Bivariate model ....................................................... 63
4.2.5 Scale-dependent bias in the halo power spectrum .......... 64
4.3 Summary .................................................................... 67

5 Primordial non-Gaussianity in the bispectra of LSS .......... 68
5.1 The power spectrum and the bispectrum: the single source case .................................................. 70
5.1.1 The two point function of halo and matter densities ........ 73
5.1.2 The three point functions of halo and matter densities .... 74
5.1.3 Methods to disentangle \( g_{NL} \) from \( f_{NL} \) ...................... 77
5.2 The power spectrum and the bispectrum: the multiple source case .................................................. 78
5.2.1 The two point functions of halo and matter densities .... 81
5.2.2 The three point functions of halo and matter densities .... 81
5.3 Stochastic halo bias, and combinations of power spectra and bispectra ............................................ 83
5.3.1 Stochastic halo bias and power spectra ........................ 83
5.3.2 Stochastic halo bias and the bispectra ......................... 85
5.4 Distinguishing \( g_{NL} \) with large-scale structure .................. 86
5.5 Summary .................................................................... 98

6 Non-local bias in the halo bispectrum with primordial non-Gaussianity .................................................. 100
6.1 Non-local Eulerian bias .................................................. 101
6.2 Three-point functions of halo and matter overdensities .... 105
6.2.1 Matter bispectrum ..................................................... 105
6.2.2 Halo bispectrum ....................................................... 106
6.3 Analytic estimates ......................................................... 112
6.4 Summary .................................................................... 115
<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>Galaxy bispectrum, primordial non-Gaussianity and redshift space distortions</td>
<td>118</td>
</tr>
<tr>
<td></td>
<td>7.1 Galaxy overdensity in the Eulerian frame</td>
<td>119</td>
</tr>
<tr>
<td></td>
<td>7.2 Redshift space distortions</td>
<td>121</td>
</tr>
<tr>
<td></td>
<td>7.2.1 Galaxy overdensity in redshift space</td>
<td>122</td>
</tr>
<tr>
<td></td>
<td>7.2.2 Galaxy bispectrum monopole</td>
<td>125</td>
</tr>
<tr>
<td></td>
<td>7.3 Conclusions</td>
<td>128</td>
</tr>
<tr>
<td>8</td>
<td>Conclusions</td>
<td>130</td>
</tr>
<tr>
<td></td>
<td>8.1 Breaking the degeneracy between $g_{NL}$ and $f_{NL}$</td>
<td>131</td>
</tr>
<tr>
<td></td>
<td>8.2 Non-local bias in the halo bispectrum</td>
<td>132</td>
</tr>
<tr>
<td></td>
<td>8.3 Galaxy bispectrum in redshift space</td>
<td>133</td>
</tr>
<tr>
<td></td>
<td>8.4 Measuring $f_{NL}$ with redshift surveys: forecasts</td>
<td>134</td>
</tr>
<tr>
<td></td>
<td>8.4.1 Methodology</td>
<td>135</td>
</tr>
<tr>
<td></td>
<td>8.4.2 Results</td>
<td>138</td>
</tr>
<tr>
<td></td>
<td>8.4.3 Conclusions</td>
<td>139</td>
</tr>
<tr>
<td>A</td>
<td>Inequalities among non-Gaussian parameters</td>
<td>141</td>
</tr>
<tr>
<td>B</td>
<td>Displaced Gaussian random fields</td>
<td>143</td>
</tr>
<tr>
<td>C</td>
<td>BSS halo bispectrum</td>
<td>145</td>
</tr>
<tr>
<td>D</td>
<td>Halo-matter bispectra</td>
<td>147</td>
</tr>
<tr>
<td>E</td>
<td>Position-dependent power spectrum</td>
<td>150</td>
</tr>
<tr>
<td>F</td>
<td>$D$ factors</td>
<td>154</td>
</tr>
<tr>
<td>G</td>
<td>Basic numbers for BOSS, eBOSS, DESI, Euclid</td>
<td>157</td>
</tr>
<tr>
<td></td>
<td>Bibliography</td>
<td>161</td>
</tr>
</tbody>
</table>
# List of Tables

1.1 List of acronyms .................................................. 4

2.1 Behaviour of a universe dominated by only one form of energy . 11

2.2 Cosmological density parameters .......................... 13

2.3 Extrapolation of the density parameter at different times .... 16

3.1 Schematic sources of the terms contributing to the halo bispectrum with Gaussian initial conditions .................. 49

6.1 Schematic sources of the terms contributing to the halo bispectrum with non-Gaussian initial conditions .................. 108

8.1 Forecasts for $\sigma_{f_{\delta}}$ from the bispectrum of BOSS, eBOSS, DESI and Euclid .................. 139

G.1 Basic numbers for BOSS LRGs .......................... 157

G.2 Basic numbers for eBOSS LRGs .......................... 158

G.3 Basic numbers for eBOSS QSOs .......................... 158

G.4 Basic numbers for eBOSS ELGs .......................... 159

G.5 Basic numbers for DESI .......................... 159

G.6 Basic numbers for Euclid .......................... 160
# List of Figures

2.1 A possible potential for the slow-roll inflaton ............... 20

3.1 Linear matter power spectrum .......................... 38

3.2 Visual representation for the bispectrum .................... 40

3.3 Press-Schechter and Sheth-Tormen mass functions .......... 43

3.4 Peak-Background Split .................................. 45

3.5 Shape dependence of the terms contributing to the halo bispectrum with Gaussian initial conditions .................. 50

4.1 \( \alpha(k, z) \) function ........................................ 58

4.2 Halo-matter power spectrum at redshift \( z = 1 \) and various \( f_{NL} \) ........ 66

5.1 Halo bispectrum in the squeezed configuration, for halo mass \( M = 10^{13} M_\odot, \) redshift \( z = 1, f_{NL} = -1 \) ............... 90

5.2 Halo bispectrum in the squeezed configuration, for halo mass \( M = 10^{13} M_\odot, \) redshift \( z = 1, f_{NL} = 0 \) ............... 90

5.3 Halo bispectrum in the squeezed configuration, for halo mass \( M = 10^{13} M_\odot, \) redshift \( z = 1, f_{NL} = +1 \) ............... 91

5.4 Halo bispectrum in the squeezed configuration, for halo mass \( M = 10^{14} M_\odot, \) redshift \( z = 1, f_{NL} = -1 \) ............... 92

5.5 Halo bispectrum in the squeezed configuration, for halo mass \( M = 10^{14} M_\odot, \) redshift \( z = 1, f_{NL} = 0 \) ............... 92

5.6 Halo bispectrum in the squeezed configuration, for halo mass \( M = 10^{14} M_\odot, \) redshift \( z = 1, f_{NL} = +1 \) ............... 93

5.7 Halo bispectrum in the squeezed configuration, for halo mass \( M = 10^{14} M_\odot, \) redshift \( z = 2, f_{NL} = -1 \) ............... 93

5.8 Halo bispectrum in the squeezed configuration, for halo mass \( M = 10^{14} M_\odot, \) redshift \( z = 2, f_{NL} = 0 \) ............... 94

5.9 Halo bispectrum in the squeezed configuration, for halo mass \( M = 10^{13} M_\odot, \) redshift \( z = 2, f_{NL} = +1 \) ............... 94

xiii
<table>
<thead>
<tr>
<th>Section No.</th>
<th>Section Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.10</td>
<td>Halo bispectrum $B_{hhk}$ in the squeezed configuration for $f_{NL} = -1$</td>
<td>95</td>
</tr>
<tr>
<td>5.11</td>
<td>$C_{fnl}$ in the squeezed configuration for different halo masses and redshifts</td>
<td>96</td>
</tr>
<tr>
<td>5.12</td>
<td>Amplitude of halo bispectra as a function of scale</td>
<td>97</td>
</tr>
<tr>
<td>6.1</td>
<td>The Eulerian bias coefficients as a function of mass and redshift, assuming $f_{NL} = 1$</td>
<td>104</td>
</tr>
<tr>
<td>6.2</td>
<td>Shape dependence of term $A = A_1 + A_2$ contributing to the halo bispectrum</td>
<td>107</td>
</tr>
<tr>
<td>6.3</td>
<td>Shape dependence of terms from B to F contributing to the halo bispectrum</td>
<td>109</td>
</tr>
<tr>
<td>6.4</td>
<td>Shape dependence of terms from G to K contributing to the halo bispectrum</td>
<td>110</td>
</tr>
<tr>
<td>6.5</td>
<td>Shape dependence of terms from N to R contributing to the halo bispectrum</td>
<td>111</td>
</tr>
<tr>
<td>6.6</td>
<td>Relative difference between the BSS model for the halo bispectrum and the one derived without invoking the spherical collapse approximation</td>
<td>112</td>
</tr>
<tr>
<td>7.1</td>
<td>Relative difference between the galaxy bispectrum monopole with PNG and the one without</td>
<td>128</td>
</tr>
</tbody>
</table>
Chapter 1

Introduction

With the development of modern cosmology on the one hand and the limits of the energies testable in ground-based particle accelerators on the other, a fascinating idea has emerged: the Universe could be used as a cosmological collider.

Fundamental interactions are studied by colliding relativistic particles and tracking the outcomes with surrounding detectors. If we replace the high-energy interactions in our laboratory with those which occurred in the very early Universe and the produced particles with galaxies, the analogy is evident but with the subtle difference that our detectors are now located about 13.7 billion years after the interactions took place. Of course, this way of testing energies beyond the limit currently reachable on Earth relies on interpreting the imprints left in late-time observables. This thesis fits within that picture.

Inflation is a possible description of the very early Universe, providing a viable mechanism to source density perturbations. Growing with time, they give rise to the Cosmos as we observe it today. Different realisations of this mechanism reveal a rich phenomenology that can be parametrised in convenient ways. Throughout the thesis, we will assume a specific parametrisation that accounts for a class of models known as local-type models. As we will show, these leave an imprint in the galaxies, potentially manifest in their large-scale clustering.

The large amount of data that will be available with future galaxy surveys is capable of constraining to high-accuracy the parameters of local-type models, possibly discriminating among different scenarios. Thus a deep understanding of the physics that initial density perturbations underwent to become the clustered structures that we observe today is of crucial importance to predict a template that may be fitted to data. Failing to do this properly would inevitably bias
our measurements.

The present thesis tries to address some of these problems, first showing a possible way to break the degeneracy between two non-Gaussian parameters, then improving the bias model by recognising the presence of novel terms and, finally, modelling the distortions that peculiar velocities introduce into redshift survey data.

Accurate theoretical models, together with our ability to handle large data and identify any potential source of observational systematics is fundamental to measure local-type models parameters and improve considerably our understanding of the physics of inflation.

The outline of the thesis is as follows:

- Chapter 2 contains a brief introduction to modern cosmology. Starting from the Friedmann-Robertson-Walker metric, the Hot Big Bang cosmology will be reviewed, followed by inflation – a possible solution to the problem of initial conditions for the Big Bang model;

- Chapter 3 reviews the structure formation process within Newtonian perturbation theory. The results for the density and velocity fields will be presented, together with their statistical properties. The mass function within the Press-Schechter approach will then be introduced: it naturally fits the idea of structures biased with respect to the underlying matter distribution and allows us to estimate their statistical properties;

- Chapter 4 briefly describes how inflation generates the primordial fluctuations and why their statistics is non-Gaussian. The observational search for departures from Gaussianity is then discussed, together with the imprint that primordial non-Gaussianity leaves in the formation of objects and their clustering;

- Chapter 5 studies how the halo bispectra can distinguish between different primordial non-Gaussian parameters in local-type models;

- Chapter 6 investigates the non-local bias features in the halo bispectrum. Some of these are specifically introduced by primordial non-Gaussianity;

- Chapter 7 addresses the problem of modelling redshift space distortions in the galaxy bispectrum when local-type non-Gaussianity is considered;

- Chapter 8 presents the conclusions of the research done and the outline of future work. The prospects for observational constraints on local-type
non-Gaussianity will also be discussed, based on idealised forecasts for the
galaxy bispectrum.

At the end of the thesis, appendices $\text{A}$ and $\text{B}$ contain ancillary materials
to clarify some of the results of chapters 5 and 6 respectively. Long equations
are quoted in appendices $\text{C}$, $\text{D}$ and $\text{F}$ to make the main text less cumbersome. Appendix $\text{E}$ contains some preliminary results on a new observable known as
position-dependent power spectrum. In appendix $\text{G}$ the basic numbers for a
selection of current and future galaxy surveys are listed.

1.1 Units and abbreviations

The speed of light is set to unity, $c = 1$, although sometimes it will be explicitly
written to avoid any confusion. The energies are measured in eV. The Boltz-
mann constant is assumed to be $k_B = 1$, so that in chapter 2 the temperature
will be often considered as an alternative measure of energy.

Vectors will always be indicated in bold face, $\mathbf{x}$ for example, while $x$ refers
to its modulus. Therefore, integration is understood in the following way:

$$\int d\mathbf{x} \equiv \int d^3x .$$

A list of often used acronyms is presented in table 1.1.
<table>
<thead>
<tr>
<th>Acronyms</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>FRW</td>
<td>Friedmann-Robertson-Walker</td>
</tr>
<tr>
<td>GR</td>
<td>General Relativity</td>
</tr>
<tr>
<td>CMB</td>
<td>Cosmic Microwave Background</td>
</tr>
<tr>
<td>LSS</td>
<td>Large-Scale Structure</td>
</tr>
<tr>
<td>BBN</td>
<td>Big Bang Nucleosynthesis</td>
</tr>
<tr>
<td>HBB</td>
<td>Hot Big Bang</td>
</tr>
<tr>
<td>GUT</td>
<td>Grand Unified Theories</td>
</tr>
<tr>
<td>LPT</td>
<td>Lagrangian Perturbation Theory</td>
</tr>
<tr>
<td>EPT</td>
<td>Eulerian Perturbation Theory</td>
</tr>
<tr>
<td>PDF</td>
<td>Probability Distribution Function</td>
</tr>
<tr>
<td>PS</td>
<td>Press-Schechter (mass function)</td>
</tr>
<tr>
<td>ST</td>
<td>Sheth-Tormen (mass function)</td>
</tr>
<tr>
<td>LV</td>
<td>Lo Verde et al (mass function)</td>
</tr>
<tr>
<td>ES</td>
<td>Excursion Set formalism</td>
</tr>
<tr>
<td>PBS</td>
<td>Peak-Background Split</td>
</tr>
<tr>
<td>PNG</td>
<td>Primordial non-Gaussianity</td>
</tr>
<tr>
<td>nG</td>
<td>non-Gaussian</td>
</tr>
<tr>
<td>BSS</td>
<td>Baldauf-Senatore-Seljak (bispectrum model)</td>
</tr>
<tr>
<td>RSD</td>
<td>Reshift Space Distortions</td>
</tr>
<tr>
<td>FoG</td>
<td>Fingers of God</td>
</tr>
<tr>
<td>LRG</td>
<td>Luminous Red Galaxy</td>
</tr>
<tr>
<td>ELG</td>
<td>Emission Line Galaxy</td>
</tr>
<tr>
<td>QSO</td>
<td>Quasar</td>
</tr>
</tbody>
</table>
Chapter 2

The homogeneous and isotropic Universe

Modern cosmology started with the Einstein equations of General Relativity (GR) \[1\]. They were solved for the case of a homogeneous and isotropic Universe filled with a perfect fluid in 1922 by A. Friedmann \[2\].

Remarkably, observations are now able to confirm that this solution is more than an academic exercise. Although we are familiar with preferred directions and inhomogeneity in the solar system and the local Universe, at sufficiently large scales the Cosmos appears to be the same wherever we look, with matter homogeneously distributed all around. Statistical homogeneity and isotropy, implying that no preferred locations exist, are the fundamental principles of modern cosmology.

In this chapter, the cosmological model arising from these principles within the general theory of relativity is introduced. First, the metric and measures of distances in an evolving space-time will be reviewed, while the kinematics of particles follows from geodesic motion. The equations governing the dynamics of the Universe will then be derived.

As a logical consequence of this formulation, the Universe appears to have originated from an initial hot and highly dense state. Based upon this picture, it has undergone three major stages, which will be described later. The agreement of a different range of predictions with observations supports this model as a cornerstone of modern cosmology. However, it faces an initial conditions problem: why is the Universe homogeneous and isotropic? A possible solution from the physics of the very early Universe has been proposed. Its simplest realization will be introduced in the final part of the chapter.
2.1 Friedmann-Robertson-Walker metric

From the theory of maximally symmetric spaces, homogeneity and isotropy allows us to fix the line element in the following form \[3\]

\[
ds^2 = -dt^2 + a^2(t) \left[ dr^2 + S_K(r)^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right],
\]

(2.1)

where the spatial coordinates \((r, \theta, \phi)\) are the comoving coordinates of a point and the variable \(t\) is the time measured by an observer at rest in the comoving frame \([r, \theta, \phi] = \text{const.}\), i.e the proper time. The dimensionless scale factor \(a(t)\) describes how distances expand or contract with time, while

\[
S_K(r) = \begin{cases} 
R_0 \sin \left( \frac{r}{R_0} \right) , & K = +1 \\
 r , & K = 0 \\
R_0 \sinh \left( \frac{r}{R_0} \right) , & K = -1 
\end{cases}
\]

(2.2)

where \(K\) is the curvature constant, taking values \(K = -1, 0, +1\) for a spherical, flat and hyperbolic space respectively, and \(R_0\) is the radius of curvature of the Universe at the present time, which has dimension of length. The line element of eq. (2.1) is known as the Friedmann-Robertson-Walker (FRW) metric.

A coordinate system alternative to \((r, \theta, \phi)\) can be chosen by replacing the radial coordinate \(r\) with \(x \equiv S_K(r)\). The FRW metric with coordinates \((x, \theta, \phi)\) reads

\[
ds^2 = -dt^2 + a^2(t) \left[ dx^2 + \frac{dx^2}{1 - \frac{K}{R_0} x^2} + x^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right].
\]

(2.3)

Also, by introducing the conformal time \(d\tau = dt/a(t)\), the FRW metric of eq. (2.3) can be re-written as

\[
ds^2 = a^2(\tau) \left[ -d\tau^2 + \frac{dx^2}{1 - \frac{K}{R_0} x^2} + x^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right].
\]

(2.4)

Suppose that an observer is at some point with comoving coordinates \((r_0, \theta_0, \phi_0) = (0, 0, 0)\) and a light source at \((r_E, \theta_E, \phi_E)\). The homogeneity and isotropy of space leave complete freedom on the choice of the position \(r\) and direction \((\theta, \phi)\), without any loss of generality. If the photon emitted by the source at time \(t_E\) reaches the observer at some later time \(t_\infty\), the comoving distance between the emitter and the observer is defined as

\[
f(r_E) \equiv \int_{t_\infty}^{t_E} \frac{dt}{a(t)} = \int_0^{r_E} dr = r_E
\]

(2.5)

with the physical distance at time \(t\) being

\[
d_p(t) = a(t)f(r_E) = a(t) r_E.
\]

(2.6)
Working within the alternative coordinate system \((x, \theta, \phi)\), the comoving distance is

\[
f(x_E) \equiv \int_{t_E}^{t_0} \frac{dt}{a(t)} = \int_0^{x_E} \frac{dx}{\sqrt{1 - K x^2}} = \begin{cases} R_0 \arcsin \left( \frac{x_E}{R_0} \right), & K = +1 \\ x_E, & K = 0 \\ R_0 \arcsinh \left( \frac{x_E}{R_0} \right), & K = -1 \end{cases} \quad (2.7)
\]

and, thus, the physical distance reads

\[d_p(t) = a(t)f(x_E). \quad (2.8)\]

The physical distance that light could have travelled since the beginning of the Universe at \(t = 0\), i.e the particle horizon, is

\[d_{hor}(t) = a(t) \int_0^t \frac{dt'}{a(t')} = \tau(t) - \tau(0). \quad (2.9)\]

A finite value for \(d_{hor}\) implies a past light cone bounded by a horizon. Therefore two events at time \(t\) cannot have a common cause if the physical distance between them is more than twice \(d_{hor}(t)\) and, in that case, they are said to be out of causal contact.

In GR, the equation of geodesic motion is \[d^2 x^\mu \big/ d\eta^2 + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\eta} \frac{dx^\beta}{d\eta} = 0, \quad (2.10)\]

where \(\eta\) is the affine parameter. By computing the Christoffel symbols \(\Gamma^\mu_{\alpha\beta}\) for a FRW metric and labelling with \(p\) the modulus of the momentum \(p^i = dx^i/d\eta\), the geodesic equation becomes

\[\dot{p} + \frac{\ddot{a}}{a} p = 0, \quad (2.11)\]

which leads to the conclusion

\[p \propto \frac{1}{a(t)}. \quad (2.12)\]

In an expanding universe, eq. (2.12) states that freely moving massive particles will come to rest with the comoving frame, while the wavelength of massless particles increases with time. The wavelength shift \(z\) can be quantified as

\[z \equiv \frac{\lambda_O - \lambda_E}{\lambda_E}, \quad (2.13)\]

where \(\lambda_E\) is the wavelength of the photon at the emission, \(\lambda_O\) at observation. If \(z < 0\) it is called blueshift, when \(z > 0\) redshift.

A relation between \(z\) and the scale factor is obtained by exploiting the emission and observation of two consecutive wave crests; for a source and an observer
fixed at the same comoving coordinates previously considered, eq. (2.5) guarantees
\[
\int_{t_{OE}}^{t_O} \frac{dt}{a(t)} = \int_{t_{OE} + \delta t_{OE}}^{t_O + \delta t_O} \frac{dt}{a(t)} \quad \Rightarrow \quad \int_{t_{OE}}^{t_{OE} + \delta t_{OE}} \frac{dt}{a(t)} = \int_{t_{OE}}^{t_{OE} + \delta t_{OE}} \frac{dt}{a(t)} .
\] (2.14)

If the time intervals \(\delta t_O\) and \(\delta t_E\) are smaller than the time required to the scale factor to vary appreciably, \(a(t)\) can be taken as a constant and then
\[
1 + z = \frac{\lambda_O}{\lambda_E} = \frac{a(t_O)}{a(t_E)} .
\] (2.15)

Equation (2.15) rephrases the result of eq. (2.12) for massless particles, by quantifying the wavelength shift. In an expanding Universe \(a(t_O) > a(t_E)\) and thus photons are redshifted along their path to the observer.

In general, measuring distances in an evolving Universe is a tricky matter. Two possible ways can be introduced: the angular diameter distance and the luminosity distance.

Starting with the former, consider an object of physical diameter \(D\), aligned for simplicity along the \(\theta\) coordinate such that its edges are located at comoving coordinates \((r_E, \theta_E, \phi_E)\) and \((r_E, \theta_E + \Delta \theta_E, \phi_E)\). By setting a constant time \(t\) in the FRW metric, the proper length is \(\int ds = D = a(t_E)S_K(r_E)\Delta \theta_E\). The angle subtended by the object at the location of the observer \((r = 0)\) is then
\[
\Delta \theta_E = \frac{D}{a(t_E)S_K(r_E)} ,
\] (2.16)

and the angular diameter distance \(d_A\) is defined as
\[
d_A \equiv \frac{D}{\Delta \theta_E} = a(t_E)S_K(r_E) .
\] (2.17)

For the luminosity distance, it is known that the distance of an object with absolute luminosity \(L\) can be computed in an Euclidean static Universe, once the flux \(F\) is measured by an observer at some time \(t_O\), thanks to the inverse square law,
\[
d_L^2 = \frac{L}{4\pi F} .
\] (2.18)

However in an expanding Universe the detected flux will depend not only on how the photons are spread over the surface of the sphere with area \(4\pi a(t_O)^2S_K^2(r_E)\) at the observation time \(t_O\) (as usual \(r_E\) is the comoving radial coordinate of the source). Indeed, the expansion introduces two additional effects: first of all, the energy of the photons is decreased by the redshift factor \((1 + z)\) [see eq. (2.13)]. Second, two photons emitted in the same direction and separated by the time interval \(\Delta t_E\) at the source, and thus at physical distance \(c\Delta t_E\), will be detected by the observer separated at a stretched distance \(c\Delta t_E(1 + z)\). Hence, the time
interval at the arrival will be $\Delta t(1 + z)$. By taking into account all these effects, the measured flux is thus

$$F = \frac{L}{4\pi a^2(t_\Omega) S_K^2 (r_E)^2 (1 + z)^2}, \quad (2.19)$$

and, in analogy with the Euclidean static case, the luminosity distance is defined as

$$d_L \equiv a(t_\Omega) S_K^2 (r_E) (1 + z). \quad (2.20)$$

By comparing eqs. (2.17) and (2.20), the relation $d_A = d_L (1 + z)^{-2}$ is easily found.

With the quantities defined so far and after some manipulations, it is possible to expand the luminosity distance $d_L$ in terms of the redshift $z$ \[4\]. Assuming the observer to be at present time $t_0$, the relation is

$$H_0 d_L = z + \frac{1}{2} (1 - q_0) z^2 + \ldots \, , \quad (2.21)$$

where $H_0$ and $q_0$ are the today values of the Hubble and deceleration parameters, respectively. The former is defined as $H \equiv \dot{a}(t)/a(t)$, the latter as $q \equiv -\ddot{a}(t)/(a(t)H^2)$. For small $z$, eq. (2.21) reduces to the famous Hubble’s law

$$z = H_0 d_L ; \quad (2.22)$$

the Hubble diagram of standard candles is one of the most direct probes that the Universe is expanding [5]. Indeed, since the present day value of the Hubble parameter is \[6\]

$$H_0 = 100h \frac{\text{km}}{\text{s Mpc}} \quad \text{where} \quad h = 0.6727 \pm 0.0066 \, , \quad (2.23)$$

eq. (2.21) implies that distant light sources are receding from us more quickly than the near ones, with the deceleration parameter $q_0$ accounting for deviations from the linear relation. The FRW metric accommodates this evidence through the scale factor $a(t)$; it is common to assume its present value $a(t_0) = 1$, while at some initial time it was $a(t = 0) = 0$.

Two scales are associated with the value $H_0$:

$$H_0^{-1} = 9.78 h^{-1} \text{Gyr} \, , \quad (2.24)$$

$$c H_0^{-1} = 3.00 h^{-1} \text{Gpc} \, . \quad (2.25)$$

The Hubble time $H_0^{-1}$ indicates the time scale for the Universe to vary appreciably, while the Hubble length $c H_0^{-1}$ is the distance that light can travel during a Hubble time. Roughly speaking, eqs. (2.24) and (2.25) give us an estimate of the age and size of the present Universe, respectively.

\footnote{Here and after the subscript 0 will always be used to indicate present day values.}
2.2 Dynamics of the Universe

\textit{Einstein equations} specify the relation between the curvature and the energy in the Universe, through a set of ten coupled, non-linear, partial differential equations \([1]\):

\[
\mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (\mathcal{R} + 2\Lambda) = 8\pi G T_{\mu\nu},
\]

where \(R_{\mu\nu}\) is the Ricci tensor, \(g_{\mu\nu}\) the metric, \(R\) the Ricci scalar, \(\Lambda\) a constant, \(G\) the Newton constant and \(T_{\mu\nu}\) the energy-momentum tensor. The choice of the FRW metric fixes all quantities on the left-hand side, while assuming a perfect fluid the energy-momentum tensor is

\[
T_{\mu\nu} = (\rho + P) u^\mu u^\nu + P g_{\mu\nu},
\]

where \(\rho\) is the energy density in the rest frame of the fluid, \(P\) is the isotropic pressure and \(u^\mu\) the 4-velocity in the comoving frame. Moreover, the energy-momentum tensor is constrained by the \textit{continuity equation}:

\[
\nabla_\nu T^{\mu\nu} = 0.
\]

A perfect fluid in a FRW metric is a good description of the large-scale, homogeneous and isotropic Universe.

The 0 − 0 component of the Einstein equations is known as the \textit{Friedmann equation} \([4]\):

\[
H^2 = \frac{8\pi G}{3} \sum_i \rho_i - \frac{K}{R_0^2 a^2},
\]

where the index \(i\) runs over all the possible types of energy, including the cosmological constant energy density term \(\rho_\Lambda = \Lambda / 8\pi G\); eq. (2.28) relates the expansion rate \(H\) of the Universe to its energy content and curvature. The \(i\) − \(i\) component of Einstein equations is called \textit{evolution equation}

\[
2\frac{a}{a} + \frac{a^2}{a^2} + \frac{K}{R_0^2 a^2} = -8\pi G \sum_i P_i,
\]

while the combination of eqs. (2.28) and (2.29) gives the \textit{acceleration equation}

\[
\ddot{a} = \frac{4\pi G}{3} \sum_i (\rho_i + 3P_i).
\]

The continuity equation \(\nabla_\nu T^{\mu\nu} = 0\) for a perfect fluid reads

\[
\dot{\rho} + 3H(\rho + P) = 0,
\]

which implies that the expansion of the universe can lead to local changes in the energy density. Note that eqs. (2.28), (2.29) and (2.31) are not independent.

There is supporting evidence that the Universe is made up of three different components of energy: pressure-less matter (most of which is at present in the
form of a cold, non-visible component, i.e. cold dark matter [7], radiation [8] and a cosmological constant-like term [9, 10]. By considering an equation of state, relating the pressure $P$ to the energy density $\rho$,

$$P = \omega \rho ,$$

(2.32)

through the constant parameter $\omega$, the pressure-less matter and radiation components are described by $\omega = 0$ and $\omega = 1/3$ respectively, while for the cosmological constant $\omega = -1$, as required by eq. (2.31) to get $\dot{\rho} = 0$. Solving eq. (2.31) immediately yields the evolution of the density to be

$$\rho(a) \propto a^{-3(1+\omega)} ,$$

(2.33)

which can be used to integrate the Friedmann eq. (2.28). For a universe dominated by a single energy component with constant $\omega$ then

$$a(t) \propto t^{\frac{2}{3(1+\omega)}} .$$

(2.34)

Interestingly, the cosmological constant ($\omega = -1$) leads to an exponential solution, while in other cases ($\omega \neq 1$) the expansion rate is

$$H(t) = \frac{2}{3(1+\omega)} t .$$

(2.35)

A summary of these results is presented in table 2.1.

<table>
<thead>
<tr>
<th>$\rho_i$</th>
<th>$\omega$</th>
<th>$\rho(a)$</th>
<th>$a(t)$</th>
<th>$H(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Matter</td>
<td>0</td>
<td>$a^{-3}$</td>
<td>$\frac{2}{3}$</td>
<td></td>
</tr>
<tr>
<td>Radiation</td>
<td>$\frac{1}{3}$</td>
<td>$a^{-4}$</td>
<td>$\frac{1}{2}$</td>
<td>$\sqrt{\frac{2}{3}}$</td>
</tr>
<tr>
<td>Cosmological constant</td>
<td>$-1$</td>
<td>const.</td>
<td>$e^{Ht}$</td>
<td></td>
</tr>
</tbody>
</table>

Using eq. (2.33), the Friedmann equation can be rephrased as

$$H^2 = \frac{8\pi G}{3} \sum_i \rho_i \left( \frac{a}{a_0} \right)^{-3(1+\omega_i)} - \frac{K}{R_0^2 a^2} ,$$

(2.36)

where $a_0 = a(t_0) = 1$ refers to the present day value of the scale factor. It is now convenient to introduce the critical energy density, i.e. the energy content for a flat Universe ($K = 0$),

$$\rho_c \equiv \frac{3H^2}{8\pi G} \quad \text{whose value today is} \quad \rho_{c0} = 1.88 h^2 \times 10^{-26} \text{kg m}^{-3} ,$$

(2.37)
and define the density parameter for the $i$-th energy component respect to $\rho_c$, \[ \Omega_i = \frac{\rho_i}{\rho_c} \quad (2.38) \]

Another way of writing the Friedmann eq. (2.28) then follows
\[ 1 - \Omega(t) = -\frac{K}{R_0^2 a(t)^2 H(t)^2}, \quad (2.39) \]

which makes manifest the relation between the curvature $K$ and the total energy density $\Omega = \sum_i \Omega_i$. Moreover, eq. (2.39) can be used to cancel the $K/R_0^2$ term in eq. (2.36) so that for a Universe made of matter, radiation and $\Lambda$ the Friedmann equation reads
\[ \frac{H^2}{H_0^2} = \frac{\Omega_{r0}}{(a/a_0)^4} + \frac{\Omega_{m0}}{(a/a_0)^3} + \Omega_{\Lambda0} - 1 - \Omega_0 \quad (2.40) \]

where $\Omega_{i0}$ is the present day value of the $i$-th density parameter. The last result can be used to write the age of the Universe as the solution of the integral
\[ t = \frac{1}{H_0} \int_0^a \frac{da}{\sqrt{\Omega_{r0} \left( \frac{a}{a_0} \right)^4 + \Omega_{m0} \left( \frac{a}{a_0} \right)^3 + \Omega_{\Lambda0} \left( \frac{a}{a_0} \right)^2 + (1 - \Omega_0)}} \quad (2.41) \]

and the particle horizon [eq. (2.9)] as
\[ d_{\text{hor}}(t) = \frac{a(t)}{H_0} \int_0^{a(t)} \frac{da}{\sqrt{\Omega_{r0} + \Omega_{m0} \left( \frac{a}{a_0} \right) - (1 - \Omega_0) \left( \frac{a}{a_0} \right)^2 + \Omega_{r0} \left( \frac{a}{a_0} \right)^4}}. \quad (2.42) \]

The present values of the density parameters are reported in table 2.2: they are consistent with a flat ($K = 0$), accelerated expanding Universe,
\[ \Omega_0 = \Omega_{r0} + \Omega_{m0} + \Omega_{\Lambda0} \simeq 1, \quad (2.43) \]

solving numerically the integrals of eqs. (2.41) and (2.42) with these values returns the actual age $t_0 \simeq 13.7 \text{ Gyr}$ and horizon $d_{\text{hor}}(t_0) \simeq 14 \text{ Gpc}$ of the Cosmos.

### 2.3 Hot Big Bang cosmology

A couple of considerations follow from the Hubble’s law [eq. (2.22)] and the Friedmann eq. (2.40). First of all, if light sources (like galaxies) are currently...
Table 2.2: Cosmological density parameters [11].

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total matter density</td>
<td>$\Omega_m h^2$</td>
<td>0.1277$^{+0.0080}_{-0.0079}$</td>
</tr>
<tr>
<td>Radiation density</td>
<td>$\Omega_\gamma h^2$</td>
<td>2.47 $\times 10^{-5}$</td>
</tr>
<tr>
<td>Cosmological constant</td>
<td>$\Omega_\Lambda$</td>
<td>0.716 ± 0.055</td>
</tr>
</tbody>
</table>

receding from each other, they must have been closer together in the past. Interpreting the redshift as a Doppler’s shift, the Hubble’s law can be thought as a relation between the velocity $v$ and the distance $d$ of a source: $v = H_0 d$. Assuming that no forces acted to accelerate or decelerate their motion, a rough approximation of the time that has elapsed since they were in contact is given by the present Hubble time. Thus, the observed galaxy redshifts naturally suggest that the expanding FRW Universe originated from a high-density, high-temperature initial state: this picture is called *Hot Big Bang* (HBB) model.

On top of that, the scaling relations of eq. (2.33) indicate that the expansion rate $H$, as described in eq. (2.40), has been driven by a radiation dominated phase at early times, a matter dominated stage at intermediate times and, finally, by a $\Lambda$-like domination. These three different epochs are sketched below.

Radiation era

At early times the Universe was dominated by radiation, with energy content described by a state parameter $\omega = 1/3$ and density scaling as $a^{-4}$.

A fluid of photons, neutrinos, antineutrinos, protons, neutrons, electrons and positrons in thermal equilibrium via weak interactions existed at about $t \lesssim 1$ s (or temperature $T \gtrsim 1$ MeV). Equilibrium is maintained while the rate of interactions per unit time $\Gamma$ is $\Gamma > H$. However, as the Universe expands and its temperature drops to $T \simeq 1$ MeV, the interaction rate of neutrinos falls below $H$ and they decouple. Then, the proton to neutron ratio practically freezes out and, at $T \simeq 0.1$ MeV, the free neutrons are all in bound states: mainly $^4\text{He}$ nuclei and smaller fractions in deuterium, $^3\text{He}$, $^7\text{Li}$. The formation of light elements during the radiation era is called *Big Bang Nucleosynthesis* (BBN): since the predicted abundances are in agreement with observations, BBN is a corner-stone of the HBB model [4][12].
Matter era

After the matter-radiation equality at \( z_m \approx 3300 \), the Universe enters into a matter dominated stage: the energy density scales as \( a^{-3} \) and is described by a state parameter \( \omega = 0 \).

At the beginning of the matter dominated era, the photon-baryon plasma is in thermal equilibrium through the Thompson scattering of photons with electrons. As the Universe expands and the mean energy density of photons drops below the ionization energy of hydrogen (\( Q = 13.6 \) eV), neutral atoms start to form, a phase called recombination. To effectively take place, the recombination process strictly depends on the baryon-to-photon ratio \( \eta \), as high-energy photons in the tail of the blackbody spectrum may well prevent the formation of neutral atoms for sufficiently low values of \( \eta \). By defining the recombination as the time at which half of the baryonic content of the Universe is in a neutral state, it can be shown that recombination happens at about \( z_{\text{rec}} \approx 1370 \) (or \( T \approx 0.12 \) eV), under the simplifying assumption that all the baryons are in form of ionized/neutral hydrogen\(^2\) \[13\].

Because of the loss of free electrons, the photon-electron scattering rate falls soon below the Hubble rate \( H \) and photons decouple. The last scattering epoch (ls) is the surface at which photons last interacted, which happens at \( z_{ls} \approx 1100 \). After that, they free-stream and Universe becomes transparent to radiation: these relic photons form the Cosmic Microwave Background (CMB). Its detection is another confirmation of the HBB model. Measurements of the CMB show fluctuations in the temperature \( \delta T/T < 10^{-4} \) \[14\], indicating that even at early stages the Universe was highly homogeneous and isotropic.

After that, clouds of neutral hydrogen and helium together with cold dark matter evolve under gravitational collapse. Proto-halos first (\( z > 10 \)) and protogalaxies (\( z \lesssim 10 \)) then begin to form and evolve through subsequent merging processes, according to the hierarchical clustering picture \[15\]. As stars and galaxies are born, the Universe starts to be re-ionized, an effect that can be measured through the CMB polarisation.

\( \Lambda \)-like era

The observed magnitude-redshift relation of Type Ia supernovae \[9,10\] were one of the first clear indications of the accelerated expansion of the Universe. This happens at redshift \( z \approx 0.66 \), before the matter-\( \Lambda \) equality which occurs at \( z = (\Omega_{\Lambda 0}/\Omega_{m 0})^{1/3} - 1 \approx 0.3 \).

\(^2\)In reality, hydrogen accounts for about \( \approx 74\% \) of the baryonic component.
This stage seems to be well described by a fluid with constant density and state parameter $\omega = -1$. However, caution must be made by interpreting this exactly as a cosmological constant. Indeed, it could well be a dynamical form of dark energy [16] or a large-scale modification of the theory of gravity [17]. Future experiments will hopefully unveil the nature of this $\Lambda$-like expansion.

2.4 Beyond the Hot Big Bang

The HBB cosmology successfully fits a range of different observations: the primordial abundance of light elements (BBN), the relic background radiation (CMB) and the expansion of the Universe.

However, it also poses four problems, which are briefly formulated below. Since these cannot be explained within this model, solutions must go beyond the HBB cosmology.

Flatness problem

All current observations are compatible with a flat Universe, consistently measuring [6,18,20]

$$|1 - \Omega_0| < 0.02.$$  \hspace{1cm} (2.45)

This bound together with eq. (2.39) allows us to extrapolate back in time the value of the total density parameter,

$$1 - \Omega(t) = \frac{1 - \Omega_0}{\left(\frac{a(t)}{a(t_0)}\right)^2}.$$  \hspace{1cm} (2.46)

the results are shown in table 2.3 for a selection of different stages. The extrapolation shows dramatic consequences: in order to explain the measured value of eq. (2.45), $\Omega(t)$ must have been fine-tuned to one part in $10^{61}$ at the Planck time! The argument can be inverted by saying that a tiny difference at that stage would have led to a completely different Universe, with $K \neq 0$.

The idea that the cosmos depends upon an extremely fine-tuned initial condition is unsatisfactory and should instead be replaced with a physical mechanism capable of flattening the early Universe.

Horizon problem

On the one hand, the HBB model predicted the existence of the CMB and its detection confirmed the assumption of homogeneity and isotropy of the Universe even at early times. On the other hand, the fact that points in all directions
Table 2.3: Extrapolation of the density parameter at different times

| Epoch                        | $T$     | $a(t)/a(t_0)$ | $|1 - \Omega(t)|$ |
|-----------------------------|---------|---------------|-----------------|
| Matter-radiation equality   | 0.5 eV  | $\sim 3 \times 10^{-4}$ | $\lesssim 10^{-5}$ |
| Big Bang Nucleosynthesis    | 1 MeV   | $\sim 2 \times 10^{-10}$ | $\lesssim 10^{-17}$ |
| GUT time                    | $10^{12}$ TeV | $\sim 2 \times 10^{-28}$ | $\lesssim 10^{-53}$ |
| Planck time                 | $10^{16}$ TeV | $\sim 2 \times 10^{-32}$ | $\lesssim 10^{-61}$ |

share the same temperature to one part in $10^4$ seems to be at odds with the finite horizon distance of the FRW model.

Since at the time of last scattering the horizon was $d_{\text{hor}}(t_{ls}) \approx 0.4$ Mpc and the angular-diameter distance to that surface is $d_A \approx 13$ Mpc, two points on the CMB map subtended by an angle bigger than

$$\theta_{\text{hor,ls}} = \frac{d_{\text{hor}}(t_{ls})}{d_A} \approx 2^\circ$$

were out of causal contact at last scattering. Therefore, the uniformity of the CMB temperature cannot be explained within the HBB model.

**Monopole problem**

In the HBB picture, the universe started from an initial hot and highly dense state, with the expansion progressively lowering both the temperature and density. In analogy with the electro-weak phase transition tested in colliders, particle physicists have made the hypothesis that the cosmos underwent through several phase transitions: starting from a symmetric state, as the universe cooled, spontaneous symmetries breaking determined the actual broken symmetric state at low energies [4].

In this context, Grand Unified Theories predict that the GUT phase transition creates point-like topological defects which act as magnetic monopoles, with rest mass $m_M \sim 10^{12}$ TeV [4]. The predicted number density at the time of production is about $n_M \sim 10^{82}$ m$^{-3}$ [13]; clearly this prediction appears to be at odds with the lack of detection of any topological defects.

A physical mechanism to dilute the monopole abundance is thus needed, if we believe in unification of forces.

**Origin of perturbations problem**

Finally, during matter domination, inhomogeneity and anisotropies are evident in form of fluctuations in the CMB temperature $\delta T / T \sim 10^{-5}$ and a rich zool-
ogy of structures, from stars up to clusters of galaxies. Of course, a structure formation paradigm based on gravitational collapse can be implemented in the FRW model, potentially explaining all the objects that are observed in the Universe. However, a mechanism seeding the origin of matter and temperature perturbations is missing.

2.5 **Inflation**

In this section, it will be shown how a simple assumption can solve the flatness, horizon and monopole problems at once, while the solution for the origin of perturbations will be presented in chapter 4.

*Inflation* is an elegant way to get rid of all the HBB model issues previously described and is defined as a stage of accelerated expansion of the scale factor $a(t)$, which took place before the standard radiation dominated epoch. Looking back at eq. (2.30), the condition $\ddot{a} > 0$ is satisfied for any $\omega < -\frac{1}{3}$ and a cosmological constant-like term (with $\omega \approx -1$) is assumed to drive inflation. Remember that the properties of a $\Lambda$-dominated universe are

$$
\rho_\Lambda = \frac{\Lambda}{8\pi G}, \quad H = \sqrt{\frac{\Lambda}{3}} = \text{const.}, \quad a(t) = a_i e^{H(t-t_i)},
$$

where $t_i$ and $a_i$ are respectively the time and scale factor at the beginning of inflation. For simplicity and concreteness, a pre-inflationary stage will be assumed to be like a standard radiation-dominated epoch and $t_i = t_{\text{GUT}} \sim 10^{-36}$ s. A toy model for the evolution of the scale factor follows

$$
a(t) = \begin{cases} 
a_i \left(\frac{t}{t_i}\right)^{\frac{1}{2}}, & t \leq t_i \\
a_i e^{H(t-t_i)}, & t_i < t \leq t_f \\
a_i e^{H(t_f-t_i)} \left(\frac{t}{t_f}\right)^{\frac{1}{2}}, & t > t_f, \end{cases}
$$

(2.47)

with $t_f$ and $a_f$ being respectively the time and scale factor at the end of inflation. Clearly, in the time interval $t_f - t_i$, the scale factor increases by a factor

$$
\frac{a(t_f)}{a(t_i)} = e^N,
$$

where $N$ is the number of $e$-foldings, defined as

$$
N \equiv H(t_f - t_i).
$$

As will be shown below, this simple inflation model is already able to solve the flatness, horizon and monopole problems.
From eq. (2.39), the total density parameter at the beginning and end of inflation are related to $N$ through

$$\frac{|1 - \Omega(t_f)|}{|1 - \Omega(t_i)|} = e^{2N}.$$  

Assuming $|1 - \Omega(t_i)| \sim 1$, it follows from table 2.3 that at GUT time $e^{-2N} \lesssim 10^{-53}$, which yields immediately to $N \gtrsim 60$. Therefore, inflation is an efficient mechanism to flatten the universe if it produces at least about 60 $e$-foldings.

The solution of the horizon problem is found by considering the evolution of the horizon distance. Since the horizon is defined as

$$d_{\text{hor}}(t) = a(t) \int_0^t \frac{dt'}{a(t')} ,$$

before inflation started ($t_i \sim t_{\text{GUT}}$) the horizon size is

$$d_{\text{hor}}(t_i) = a_i \int_0^{t_i} \frac{dt}{a_i(t/t_i)^{1/2}} = 2t_i \approx 6 \cdot 10^{-28} \text{ m},$$

while at the end of inflation ($N \approx 60$) the horizon is

$$d_{\text{hor}}(t_f) = a(t_i) \left( \int_0^{t_i} \frac{dt}{a(t_i)(t/t_i)^{1/2}} + \int_{t_i}^{t_f} \frac{dt}{a(t_i)e^{H(t-t_i)}} \right)$$

$$\approx e^N (2t_i + H^{-1})$$

$$\approx e^N 3t_i \sim 1 \text{ pc} .$$

In practice, since the horizon increases by a factor $e^N$, inflation acts by taking submicroscopic scales to about a parsec. Indeed, this gives a solution to the horizon problem: if the horizon distance at the last scattering surface without inflation is about $0.4 \text{ Mpc}$, inflation makes it $d_{\text{hor}}(t_{ls}) \approx e^N 0.4 \text{ Mpc} \sim 10^{26} \text{ Mpc}$. This value guarantees that all the CMB photons were in causal contact before last scattering, giving a causal mechanism for the uniformity of its temperature. Finally, inflation efficiently dilutes the magnetic monopoles, if they are produced before or during this stage. The expected number density of monopoles at the GUT time is $n_M(t_{\text{GUT}}) \approx 10^{83} \text{ m}^{-3}$, assuming $t_i \simeq t_{\text{GUT}}$ and considering the scaling $n_M \propto a^{-3}$, the present number density is

$$n_M(t_0) = \left( \frac{a_i}{a_0} \right)^3 n_M(t_i) = e^{-3N} \left( \frac{a_f}{a_0} \right)^3 n_M(t_i) \sim 10^{-61} \text{ Mpc}^{-3} .$$

Such a low value explains why magnetic monopoles have not been detected.

**Standard inflation**

Building upon an early accelerated expansion of $a(t)$ to solve the HBB model issues, most inflationary models are based on a scalar field $\phi(t, \mathbf{x})$, called the
*inflaton*. Below, it will be briefly shown how this field can give rise to a $\Lambda$-like term and how the Universe consistently enters into the radiation dominated epoch, at the end of the accelerated phase.

We assume the inflaton $\phi(t, x)$ is a real scalar field, moving along a potential $V(\phi)$. Its Lagrangian is

$$
\mathcal{L} = \frac{1}{2} \dot{\phi} \partial_\mu \phi^{\mu} + V(\phi),
$$

with energy-momentum tensor

$$
T_{\mu\nu} = \phi_{,\mu} \phi_{,\nu} - g_{\mu\nu} \left( \frac{1}{2} \dot{\phi} \dot{\phi} + V(\phi) \right).
$$

In the case of a homogeneous field $\phi(t)$, the inflaton behaves as a perfect fluid and the previous equation yields

$$
T_{00} = \rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi),
$$
$$
T_{ii} = P_\phi = \frac{\dot{\phi}^2}{2} - V(\phi).
$$

The Friedmann equation for homogeneous $\phi(t)$ reads

$$
H^2 = \frac{8\pi G}{3} \left( \frac{1}{2} \dot{\phi}^2 + V(\phi) \right),
$$

while the equation of motion is

$$
\ddot{\phi} + 3H \dot{\phi} + V_{,\phi} = 0,
$$

where $V_{,\phi} = dV/d\phi$. Assuming

$$
\dot{\phi}^2 \ll V(\phi),
$$

it follows from eqs. (2.50) and (2.51) that $\omega = p_\phi/\rho_\phi \approx -1$, and the Friedmann equation reads

$$
H^2 \simeq \frac{8\pi G}{3} V(\phi).
$$

Thus the almost exponential expansion of the scale factor,

$$
- \frac{\dot{H}}{H^2} \ll 1,
$$

is now realised. With the additional assumption

$$
\ddot{\phi} \ll 3H \dot{\phi},
$$

the equation of motion reduces to

$$
3H \dot{\phi} \simeq -V_{,\phi}(\phi),
$$

19
at the expense of throwing away a solution mode, which is a decaying mode in an overdamped system.

For eqs. (2.54) and (2.57) to hold requires that $\epsilon, |\eta| \ll 1$, where we define the slow-roll parameters as \cite{21}

$$
\epsilon \equiv -\frac{\ddot{H}}{H^2} \simeq \frac{1}{16\pi G} \left( \frac{V}{V'} \right)^2,
$$

\hspace{1cm}

$$
\eta \equiv \frac{1}{8\pi G} \left( \frac{V_{,\phi \phi}}{V} \right).
$$

Slow-roll inflation takes place when $\epsilon \ll 1$, while $|\eta| \ll 1$ ensures that it lasts for a sufficient period of time, with the number of $e$-foldings given by

$$
N = \ln \left( \frac{a(t_f)}{a(t_i)} \right) = \int_{t_i}^{t_f} H \, dt \simeq -8\pi G \int_{\phi_i}^{\phi_f} \frac{V}{V'} \, d\phi.
$$

This is the simplest realization of the early, almost-exponential acceleration of the scale factor and is known as single-field, slow-roll inflation (or slow-roll inflaton).

A concrete example of a potential $V(\phi)$ is shown in fig. 2.1. The inflaton starts in a false vacuum state at $\phi = 0$, where $V(0) = V_0$, then it slowly rolls toward the true vacuum at $\phi = \phi_0$. Here it oscillates around the minimum $V = 0$, with the Hubble friction term in eq. (2.53) damping the oscillations. The energy density of the inflaton is converted into relativistic particles, when the inflaton particles decays into radiation, which reheats the universe. The reheating stage avoids the dilution of particles, in contrast to what happens to magnetic monopoles. Then, the Universe enters into the radiation-dominated epoch and the standard HBB model applies.

![Figure 2.1: A possible potential for the slow-roll inflaton. Image from 13](image.png)
2.6 Summary

In this chapter, the metric and the basic equations describing the dynamics of the homogeneous and isotropic Universe have been introduced. The FRW cosmology is a good description of the Cosmos from the time of BBN up to today, with the CMB directly confirming the homogeneity and isotropy at early times, $z_{ls} \simeq 1100$. Measurements of the density parameters indicate that the present Universe is filled with matter (mostly in the form of a cold, invisible component), an unknown form of dark energy effectively acting as a cosmological constant $\Lambda$, and a much smaller amount of radiation.

In the HBB picture, the Universe originated from a hot, highly-dense initial state, about $13.7 \text{ Gyr}$ ago. As it expanded and temperature and density dropped, it went through three stages of radiation, matter and $\Lambda$-like domination respectively, with expansion rate and density evolution described by the Friedmann, the acceleration and the continuity equations.

The HBB model successfully accounts for the abundance of light elements, the presence of the CMB and the expansion of the Cosmos but fails to explain the flatness, the uniformity of the CMB temperature and the absence of monopole relics from a GUT phase transition. The hypothesis of an almost exponential expansion of the scale factor taking place before the radiation era solves many of the problems of initial conditions for the HBB model. We have reviewed the simplest implementation of this idea with a slowly rolling scalar field.

The origin of CMB fluctuations and structures has been intentionally left as an open problem. In chapter 4 it will be shown how inflation can explain these as well.
Chapter 3

From perturbations to structure formation

Going beyond the FRW cosmology, it is possible to imagine that, given a sufficiently large initial fluctuation in the matter density field, it will grow with time under gravitational collapse and eventually form virialized objects, potentially arranged in bound systems with other objects. This mechanism is a simple though fair representation of the structure formation process as it is understood today in modern cosmology to explain the rich zoology of objects that are observed in the Universe.

The reason why initial fluctuations arise will be addressed in the next chapter, while the first part of this one is dedicated to how the matter density evolves when perturbations remain small enough. A criteria for whether the gravitational collapse occurs follows within this perturbative approach.

After that, it will be explained why, in cosmology, fields like the matter density must be described in a statistical sense and, then, the relevant statistics for the purpose of this thesis will be introduced. These are key quantities in order to infer the properties of random fields, that can actually be compared with observations.

Resuming the discussion on structure formation, an approach to deal with non-linearities beyond the perturbative regime will be presented in the second part of the chapter. It allows us to predict the number density of halos made of dark matter, in agreement with N-body simulations. Halos are then populated by galaxies.

A possible way to relate the halo density to the underlying matter distribution will be discussed at the end of the chapter and will be used to predict the
3.1 Perturbation theory

With the recent detection of gravitational waves \cite{22}, the missing piece of an already well-established theory of gravity has been found. General Relativity has passed all the solar system tests and accurately predicted the observed signal from the merging of two compact objects \cite{23}. Moreover, a perturbed FRW metric allows us to describe the photon-baryon fluid through the Boltzmann equation and the corrections to the photons’ path from the last scattering surface \cite{21}, which fits the CMB data well with only a small number of free parameters \cite{6,19}. So far, no deviations from GR have been measured even on large scale but more stringent tests will be possible with future observations. For all these reasons GR is up to now a successful theory of gravity.

GR recovers the Newtonian theory under the weak field approximation and slow motion \((v \ll c)\) \cite{1} and, in general, it is expected to reproduce all the Newtonian results plus additional terms \cite{24}. As will be shown below, there are however regimes in which Newtonian gravity is still applicable and believed to be a good approximation \cite{25}.

Matter evolution in a GR framework is nowadays an active field of research (see for instance \cite{26}) but the inclusion of these effects is beyond the scope of this thesis. However, all the results here presented can be recovered in a proper GR treatment, plus corrections arising where the Newtonian approximation is poor \cite{24,26}.

Dark matter and baryons after decoupling are well described by pressureless dust particles. Assuming the existence of perturbations at some initial time, the density fluctuations at the comoving position \(x\) and conformal time \(\tau\) can be defined as

\[
\delta(x, \tau) \equiv \frac{\rho(x, \tau) - \bar{\rho}(\tau)}{\bar{\rho}(\tau)},
\]

(3.1)

where \(\bar{\rho}\) is the average matter density. If the fluctuations to be modelled are well inside the horizon and the peculiar velocities are much smaller than the speed of light, then we expect the Newtonian fluid equations in an expanding Universe to hold \cite{27}:

\[
\dot{\delta} + \nabla \cdot [(1 + \delta) \mathbf{v}] = 0,
\]

(3.2)

\[
\dot{\mathbf{v}} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\mathcal{H}\mathbf{v} - c_s^2 \nabla \delta - \nabla \phi,
\]

(3.3)

\[
\nabla^2 \phi = 4\pi G\bar{\rho} \delta,
\]

(3.4)

\footnote{Entropy perturbations do not occur in a single-component fluid}
where \( x \) is now the comoving Eulerian position, the dots stand for \( \partial/\partial \tau \), \( v = dx/d\tau \) is the peculiar velocity field, \( \phi(x) \) the Newtonian gravitational potential, \( c_s = \sqrt{dP/d\rho} \) the sound speed and \( H(\tau) = d\ln a/d\tau = aH \) is the conformal Hubble parameter. Equations (3.2) to (3.4) are the continuity, Euler and Poisson equations respectively. This system can be solved in terms of \( \delta \) and the divergence of the velocity field \( \theta(x, \tau) \equiv \nabla \cdot v \), in a regime where the vorticity \( \omega(x, \tau) = \nabla \times v \) is vanishing \[28\]. Throughout this thesis, it will be assumed that \( \delta \) and \( \theta \) can be split into a linear and non-linear part

\[
\delta(x, \tau) = \delta_{\text{lin}} + \delta_{\text{nonlin}} , \tag{3.5}
\]

\[
\theta(x, \tau) = \theta_{\text{lin}} + \theta_{\text{nonlin}} . \tag{3.6}
\]

At linear level the system of eqs. (3.2) to (3.4) reads

\[
\dot{\delta}_{\text{lin}} + \nabla \cdot v_{\text{lin}} = 0 , \tag{3.7}
\]

\[
\dot{v}_{\text{lin}} = -H v_{\text{lin}} - c_s^2 \nabla \delta_{\text{lin}} - \nabla \phi_{\text{lin}} , \tag{3.8}
\]

\[
\nabla^2 \phi_{\text{lin}} = 4\pi G \bar{\rho} a^2 \delta_{\text{lin}} , \tag{3.9}
\]

where \( v = v_{\text{lin}} + v_{\text{nonlin}} \) and \( \phi = \phi_{\text{lin}} + \phi_{\text{nonlin}} \) have been decomposed analogously to eqs. (3.5) and (3.6). By taking the divergence of the Euler equation,

\[
\nabla \cdot \dot{v}_{\text{lin}} = -H \nabla \cdot v_{\text{lin}} - c_s^2 \nabla^2 \delta_{\text{lin}} - \nabla^2 \phi_{\text{lin}} , \tag{3.10}
\]

and using the Poisson equation \( \nabla^2 \phi_{\text{lin}} = 4\pi G \bar{\rho} a^2 \delta_{\text{lin}} \) and continuity equation \( \nabla \cdot v_{\text{lin}} = \theta_{\text{lin}} = -\dot{\delta}_{\text{lin}} \), a closed form for \( \delta_{\text{lin}} \) can be derived:

\[
\ddot{\delta}_{\text{lin}} + H \dot{\delta}_{\text{lin}} - \left( c_s^2 \nabla^2 + 4\pi G \bar{\rho} a^2 \right) \delta_{\text{lin}} = 0 . \tag{3.11}
\]

Using the following convention for the Fourier transform,

\[
\delta(x, \tau) = \int_{-\infty}^{+\infty} \frac{dk}{(2\pi)^3} \delta(k, \tau) e^{ik \cdot x} , \tag{3.12}
\]

\[
\delta(k, \tau) = \int_{-\infty}^{+\infty} dx \, \delta(x, \tau) e^{-ik \cdot x} , \tag{3.13}
\]

eq (3.11) can be rewritten in Fourier space as

\[
\ddot{\delta}_{\text{lin}}(k) + H \dot{\delta}_{\text{lin}}(k) + \left( c_s^2 k^2 - 4\pi G \bar{\rho} a^2 \right) \delta_{\text{lin}}(k) = 0 . \tag{3.14}
\]

The evolution of density fluctuations crucially depends on their size. Indeed, the terms inside the parenthesis introduce a physical scale,

\[
\lambda_J = \frac{2\pi a}{k_J} = c_s \sqrt{\frac{\pi}{G \bar{\rho}}} , \tag{3.15}
\]
known as the Jeans length. It defines a criteria to understand whether or not a density fluctuation of size $\lambda$ is free to grow. If $\lambda < \lambda_J$, then eq. (3.14) is solved by a pressure-supported sound wave, while for $\lambda > \lambda_J$ perturbations evolve under gravitational collapse.

Important considerations follow by writing the Jeans length as

$$\lambda_J = \frac{2\sqrt{2}}{3} \pi \omega \frac{1}{H},$$

where the Friedmann eq. (2.36) for a flat Universe and the equation of state $P = \omega \rho$ [see eq. (2.32)] with $\epsilon_\delta = \omega = \text{constant}$ are assumed. During the radiation dominated era ($\omega = 1/3$) density perturbations do not grow, as the Jeans length is of order the Hubble size,

$$\lambda_{Jr} = \frac{2}{3} \sqrt{2} \frac{\pi}{3} \frac{H}{r},$$

(3.17)

During the matter era ($\omega = 0$), fluctuations in the dark matter and baryons follow different paths. Dark matter candidates can be in form of thermal (like WIMPs) or non thermal (like axions) relics. The former decouples well before BBN, as they are expected to have mass $m_{dm} \gg 1\text{MeV}$, while the latter are already decoupled at production [4]. This means that in the matter era the Jeans length for dark matter is $\lambda_{Jdm} \ll H^{-1}$ [13].

Instead, baryons are tightly coupled to photons till recombination, a stage at which the density of radiation respect to baryons is still comparable $\rho_r/\rho_b \sim 1$. Therefore, before decoupling, density perturbations in ordinary matter do not grow since

$$\lambda_{J} \ll \lambda_{Jb} \sim \lambda_{Jr},$$

(3.18)

where $\lambda_{Jb}$ is the Jeans length for baryons. Then, once photons decouple, it suddenly drops to $\lambda_{Jb} \simeq \lambda_{Jdm} \ll H^{-1}$. Without pressure support, baryons start to fall into the pre-existing dark matter potential well and the structure formation process begins.

Focusing only on scales larger than the Jeans length, the pressure term $c_s^2 \nabla \delta$ can be dropped from the Euler equation; the system of eqs. (3.2) to (3.4) now describes only growing (and decaying) density fluctuations. After some manipulations they can be re-written as a system of two coupled integro-differential equations in Fourier space

$$\dot{\delta}_k + \theta_k = -\int \frac{d\mathbf{k}_1}{(2\pi)^3}\int \frac{d\mathbf{k}_2}{(2\pi)^3} \delta^D(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) \frac{(\mathbf{k}_1 + \mathbf{k}_2) \cdot \mathbf{k}_1}{k_1^4} \delta(\mathbf{k}_2) \theta(\mathbf{k}_1),$$

(3.19)

$$\dot{\theta}_k + \mathcal{H}\theta_k + 4\pi G a^2 \rho_0 \delta_k = \int \frac{d\mathbf{k}_1}{(2\pi)^3}\int \frac{d\mathbf{k}_2}{(2\pi)^3} \delta^D(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) \times$$
where $\delta_k = \delta(k, \tau)$, $\theta_k = \theta(k, \tau)$ and $\delta^D$ is the Dirac delta function. Equation (3.19) comes from the continuity eq. (3.2), while eq. (3.20) from the divergence of the Euler eq. (3.3). The Poisson eq. (3.4) has been used to substitute $\nabla^2 \phi$.

In the next two subsections the linear and non linear solutions will be presented.

3.1.1 Linear solutions

The linearised system of equations eqs. (3.7) to (3.9) in Fourier space with $c_s^2 = 0$ yields

\begin{equation}
\dot{\delta}_{\text{lin}} = -\theta_{\text{lin}},
\end{equation}

\begin{equation}
\ddot{\delta}_{\text{lin}} + \mathcal{H} \dot{\delta}_{\text{lin}} - \frac{3}{2} \Omega_m H^2 \delta_{\text{lin}} = 0,
\end{equation}

where the subscript $k$ has been dropped to simplify the notation. Looking for solutions in the form $\delta_{\text{lin}}(k, \tau) = D(\tau)\delta_{\text{lin}}(k, 0)$, where $D(\tau)$ is the linear growth factor, gives the second-order differential equation

\begin{equation}
\ddot{D} + \mathcal{H} \dot{D} - \frac{3}{2} \Omega_m H^2 D = 0,
\end{equation}

whose solutions are a growing mode and a decaying mode,

\begin{equation}
\delta_{\text{lin}}(k, \tau) = D_+(\tau)A(k) + D_-(\tau)B(k).
\end{equation}

Expression for $D_\pm$ are easily found when the Universe is dominated by a single component. The results as a function of time $t$ and comoving position $x$ are

\begin{equation}
\delta_{\text{lin}}(x, t) = A_1(x) + B_1(x) \ln t,
\end{equation}

\begin{equation}
\delta_{\text{lin}}(x, t) = A_2(x)t^{2/3} + B_2(x)t^{-1},
\end{equation}

\begin{equation}
\delta_{\text{lin}}(x, t) = A_3(x) + B_3(x)e^{-2H\Lambda t},
\end{equation}

respectively for the radiation [eq. (3.25)], matter [eq. (3.26)] and $\Lambda$ [eq. (3.27)] epoch. In the following, only the growing mode $D_+$ will be considered.

A case of interest is a Universe filled only with matter $\Omega_m$ and cosmological constant $\Omega_\Lambda$: as measurements confirm, this is a good description of the Cosmos at late times, after the matter-radiation equality. In a $\Lambda$CDM cosmology, the linear growth factor has the integral representation

\begin{equation}
D_+(a(\tau)) = \frac{5}{2} \Omega_m H(a) \int_0^a \frac{da'}{a'^3H(a')} ,
\end{equation}

26
where the Friedmann equation now reads

\[
\frac{H(a)}{H_0} = \sqrt{\frac{\Omega_{m0}}{a^3} + \frac{1 - \Omega_{m0} - \Omega_{\Lambda0}}{a^2} + \Omega_{\Lambda0}}.
\]  

(3.29)

A common choice for the normalization of \(D_+\) is

\[
D(a) = \frac{D_+(a)}{D_+(a_0 = 1)},
\]  

(3.30)

so that in the following \(D\) will always stand for the linear growth factor, normalized to one at present.

Finally, the divergence of the velocity field easily follows from the continuity eq. (3.21):

\[
\theta_{\text{lin}}(k, \tau) = -\frac{d\delta_{\text{lin}}}{d\tau} = -\frac{\delta_{\text{lin}} dD(a)}{D} = -fH\delta_{\text{lin}},
\]  

(3.31)

where \(f\) is the logarithmic derivative

\[
f = \frac{d\ln D}{d\ln a},
\]

which can be approximated as

\[
f(\Omega_m, \Omega_\Lambda) \approx \Omega_m^{\frac{2}{7}} + \Omega_\Lambda^{\frac{1}{7}} \left(1 + \Omega_m^2\right).
\]  

(3.32)

### 3.1.2 Non-linear solutions

The non-linear solutions to the system of eqs. (3.19) and (3.20) can be expanded perturbatively using the \(n\)-th power of the linear density contrast \(\delta_{\text{lin}}(k, \tau)\) as a basis [28,29,31]

\[
\delta_{\text{nonlin}}(k, \tau) = \sum_{n=1}^{\infty} \int \frac{dk_1}{(2\pi)^3} \ldots \int \frac{dk_{n-1}}{(2\pi)^3} \int \frac{dk_n}{(2\pi)^3} \delta^D(k - k_1 - \ldots - k_n) \times
\]

\[
\times F_n(k_1, \ldots, k_n, \tau)\delta_{\text{lin}}(k_1, \tau) \ldots \delta_{\text{lin}}(k_n, \tau),
\]  

(3.33)

\[
\theta_{\text{nonlin}}(k, \tau) = -fH\sum_{n=1}^{\infty} \int \frac{dk_1}{(2\pi)^3} \ldots \int \frac{dk_{n-1}}{(2\pi)^3} \int \frac{dk_n}{(2\pi)^3} \delta^D(k - k_1 - \ldots - k_n) \times
\]

\[
\times G_n(k_1, \ldots, k_n, \tau)\delta_{\text{lin}}(k_1, \tau) \ldots \delta_{\text{lin}}(k_n, \tau).
\]  

(3.34)

In general, the kernels \(F_n\) and \(G_n\) are time dependent. However, since they are weakly sensitive to the underlying cosmology, they are usually computed for the case of a flat, matter-only Universe\(^2\) where they are constant in time [28]. The kernels can be derived to be [31,32]

\[
F_n(p_1, \ldots, p_n) = \sum_{m=1}^{n-1} \frac{\mathcal{G}_m(p_1, \ldots, p_m)}{(2n + 3)(n - 1)} \left[ (2n + 1) \frac{k \cdot k_1}{k_1^2} F_{n-m}(p_{m+1}, \ldots, p_n) + \frac{k^2 (k_1 \cdot k_2)}{k_1^2 k_2^2} G_{n-m}(p_{m+1}, \ldots, p_n) \right],
\]  

(3.35)

\(^2\)The flat (\(\Omega_m = 1, \Omega_\Lambda = 0\)) case is known as Einstein-De Sitter universe, where \(\mathcal{H} = 2/\tau\), \(D = a\) and \(f = 1\).
\[ G_n(p_1, \ldots, p_n) = \sum_{m=1}^{n-1} \frac{G_m(p_1, \ldots, p_m)}{(2n-3)(n-1)} \left[ \frac{3 \cdot k_1 \cdot k_2}{k_1^2} F_{n-m}(p_{m+1}, \ldots, p_n) + \right. \\
\left. + n \frac{k^2(k_1 \cdot k_2)}{k_1^2 k_2^2} G_{n-m}(p_{m+1}, \ldots, p_n) \right], \] (3.36)

where \( k_1 \equiv p_1 + \ldots + p_m, \ k_2 \equiv p_{m+1} + \ldots + p_n \) and \( k \equiv k_1 + k_2 \). Since \( F_1 = G_1 = 1 \), the first-order results of section 3.1.1 can be included in the expansion and the full perturbative results are

\[ \delta(k, \tau) = \sum_{n=1}^{\infty} \delta^{(n)}(k, \tau), \] (3.37)

\[ \theta(k, \tau) = \sum_{n=1}^{\infty} \theta^{(n)}(k, \tau), \] (3.38)

with \( \delta_{\text{lin}} = \delta^{(1)} \) and \( \theta_{\text{lin}} = \theta^{(1)} \). Throughout the thesis, the non-linear evolution will be considered only at second order, so that \( \delta_{\text{nonlin}} \simeq \delta^{(2)} \) and \( \theta_{\text{nonlin}} \simeq \theta^{(2)} \), where

\[ \delta^{(2)}(k, \tau) = \int \frac{dk_1}{(2\pi)^3} \int \frac{dk_2}{(2\pi)^3} \delta^D(k - k_1 - k_2) F_2(k_1, k_2) \delta_{\text{lin}}(k_1, \tau) \delta_{\text{lin}}(k_2, \tau), \] (3.39)

\[ \theta^{(2)}(k, \tau) = -fH \int \frac{dk_1}{(2\pi)^3} \int \frac{dk_2}{(2\pi)^3} \delta^D(k - k_1 - k_2) \times \] \[ \times G_2(k_1, k_2) \delta_{\text{lin}}(k_1, \tau) \delta_{\text{lin}}(k_2, \tau), \] (3.40)

and the corresponding kernels for \( \Omega_m = 1 \) are

\[ F_2(k_1, k_2) = \frac{5}{7} + \frac{1}{2} \frac{k_1 \cdot k_2}{k_1 k_2} \left( \frac{k_1}{k_1} + \frac{k_2}{k_2} \right) + \frac{2}{7} \frac{(k_1 \cdot k_2)^2}{k_1^2 k_2^2}, \] (3.41)

\[ G_2(k_1, k_2) = \frac{3}{7} + \frac{1}{2} \frac{k_1 \cdot k_2}{k_1 k_2} \left( \frac{k_1}{k_2} + \frac{k_2}{k_1} \right) + \frac{4}{7} \frac{(k_1 \cdot k_2)^2}{k_1^2 k_2^2}. \] (3.42)

### 3.1.3 Lagrangian viewpoint

\( \delta(k, \tau) \) and \( \theta(k, \tau) \) are the Fourier transform of the corresponding quantities in the comoving Eulerian frame, with \( \mathbf{x} \) being the position vector of a particle. Opposed to that, in the Lagrangian reference frame the position of a fluid element (or particle) does not change with time. The initial spatial coordinate \( q \) in the Lagrangian picture is related to the evolved Eulerian coordinate through the formula

\[ x(q, \tau) = q + \Psi(q, \tau), \] (3.43)

with \( \Psi \) being the displacement field, i.e. the mapping between the two coordinate systems that contains all the information about densities and velocities.
Since the matter density is a 3-scalar, one has that the mass enclosed in an infinitesimal volume element is

\[ M = \rho(x, z)a(z)^3 \sqrt{|g_x|} d^3x = \rho(q, z)a(z)^3 \sqrt{|g_q|} d^3q, \] (3.44)

where in Newtonian gravity the determinant of the Eulerian metric \( g_x \) is simply \( |g_x| = 1 \), but the Lagrangian space has a non-trivial metric \( g_q \) even in Newtonian theory. In eq. (3.44) the conformal time has been replaced with redshift \( z \) as an equivalent measure of time; hereafter we will use \( \tau \) and \( z \) interchangeably. By defining the coordinate Jacobian

\[ J(q, z) \equiv \begin{vmatrix} \frac{dx}{dq} \end{vmatrix} = \sqrt{|g_q|}, \] (3.45)

and using eq. (3.43), we obtain

\[ J(q, z) = \det (\delta^K_{ij} + \Psi_{i,j}(q, z)) \] (3.46)

where \( \Psi_{i,j} = \partial \Psi_i / \partial q_j \) and \( \delta^K_{ij} \) is the Kronecker delta function. At the initial redshift \( z_{in} \), the Eulerian and Lagrangian frame are equivalent, i.e. \( \Psi = 0 \), and hence \( J = 1 \). If in addition one assumes that the mass per volume element is conserved and the initial density was uniform, \( \rho(q, z_{in}) = \bar{\rho}(z_{in}) \) in the limit \( z_{in} \to \infty \), then

\[ M = a^3(z)\rho(x, z)dx = a^3(z_{in})\bar{\rho}(z_{in})dq \] (3.47)

which implies

\[ a^3(z)\rho(x(q, z), z)J(q, z)dq = a^3(z_{in})\bar{\rho}(z_{in})dq \] (3.48)

and hence

\[ J = \left| \frac{dx}{dq} \right| = \frac{a^3(z_{in})\bar{\rho}(z_{in})}{a^3(z)\rho(q, z)} = \frac{\bar{\rho}(z)}{\rho(q, z)} = [1 + \delta^L(q, z)]^{-1}, \] (3.49)

where \( \delta^L(q, z) \) is the Lagrangian fully non-linear density contrast. Hereafter the superscript \( L \) is used to distinguish the Lagrangian quantities from the Eulerian ones, which will now appear with the superscript \( E \).

Working in analogy with the Eulerian perturbation theory (EPT), the displacement and the Lagrangian density field can be expanded in series:

\[ \Psi(q, \tau) = \sum_{n=1}^{\infty} \Psi^{(n)}(q, \tau) \] (3.50)

\[ \delta^L(q, \tau) = \sum_{n=1}^{\infty} \delta^{L,(n)}(q, \tau) \] (3.51)

29
and the Jacobian can now be rewritten as
\[
\mathcal{J} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \sum_{n=1}^{\infty} \begin{pmatrix} \Psi_{1,1}^{(n)} & \Psi_{1,2}^{(n)} & \Psi_{1,3}^{(n)} \\ \Psi_{2,1}^{(n)} & \Psi_{2,2}^{(n)} & \Psi_{2,3}^{(n)} \\ \Psi_{3,1}^{(n)} & \Psi_{3,2}^{(n)} & \Psi_{3,3}^{(n)} \end{pmatrix} = \sum_{n=0}^{\infty} \mathcal{J}^{(n)}. \tag{3.52}
\]

Up to second order, one finds
\[
\mathcal{J}^{(0)} = 1, \tag{3.53}
\]
\[
\mathcal{J}^{(1)} = \nabla_{\mathbf{q}} \cdot \Psi^{(1)}, \tag{3.54}
\]
\[
\mathcal{J}^{(2)} = \nabla_{\mathbf{q}} \cdot \Psi^{(2)} + \frac{1}{2} \left[ \left( \nabla_{\mathbf{q}} \cdot \Psi^{(1)} \right)^2 - \sum_{i,j} \Psi_{i,j}^{(1)} \Psi_{j,i}^{(1)} \right], \tag{3.55}
\]

where we use \(\nabla_{\mathbf{q}}\) to indicate the gradient operator in Lagrangian coordinates and distinguish it from the one in the Eulerian frame, \(\nabla\).

Equation \((3.49)\) can be rewritten perturbatively as follows
\[
1 + \sum_{n=1}^{\infty} \delta \mathcal{L}^{(n)} = \frac{1}{\sum_{n=0}^{\infty} \mathcal{J}^{(n)}}, \tag{3.56}
\]

and, assuming all the \(\mathcal{J}^{(n)}\) with \(n > 0\) to be small perturbations around \(\mathcal{J}^{(0)} = 1\), up to second order one has
\[
1 + \delta \mathcal{L}^{(1)} + \delta \mathcal{L}^{(2)} + \cdots = 1 - \mathcal{J}^{(1)} + \left( \mathcal{J}^{(1)} \mathcal{J}^{(2)} \right) + \cdots. \tag{3.57}
\]

Using eqs. \((3.54)\) and \((3.55)\), the Lagrangian density is related to the displacement field through
\[
\delta \mathcal{L}^{(1)} = -\mathcal{J}^{(1)} = -\nabla_{\mathbf{q}} \cdot \Psi^{(1)}, \tag{3.58}
\]
\[
\delta \mathcal{L}^{(2)} = \mathcal{J}^{(1)} \mathcal{J}^{(2)} = \nabla_{\mathbf{q}} \cdot \Psi^{(2)} + \frac{1}{2} \left[ \left( \nabla_{\mathbf{q}} \cdot \Psi^{(1)} \right)^2 + \sum_{i,j} \Psi_{i,j}^{(1)} \Psi_{j,i}^{(1)} \right]; \tag{3.59}
\]

the previous equations clearly indicate that \(\Psi\) is the dynamical quantity in the Lagrangian picture.

Before showing how \(\Psi\) is solved in Lagrangian perturbation theory (LPT), we can already establish the transformation law between the Lagrangian and Eulerian density field by noting that the physical density must be equal in either frames:
\[
\delta \mathcal{E}(\mathbf{x}, z) = \delta \mathcal{L}(\mathbf{q}, z) = \delta \mathcal{L}(\mathbf{x} - \Psi, z) \tag{3.60}
\]

\[
= \delta L(\mathbf{x} - \Psi, z) \tag{3.61}
\]

30
\[ \delta^E(x, z) = \delta^L(x, z) - \Psi(x, z) \cdot \nabla \delta^L(x, z) + \ldots, \] (3.62)

where in the last line the displacement \( \Psi \) has been assumed to be small, i.e. \( x \simeq q \). Thus, up to second order, it follows that

\[ \delta^E(1) = \delta^L(1), \] (3.63)
\[ \delta^E(2) = \delta^L(2) - \Psi^{(1)} \cdot \nabla \delta^L(1). \] (3.64)

From the Lagrangian-to-Eulerian transformation of the density field in eqs. (3.63) and (3.64), we learn that the first-order density perturbation has the same form in either frames, while at second order the density differs because of the convective term proportional to the displacement \( \Psi \) \(^3\). This is a consequence of the fact that the Lagrangian and Eulerian coordinates agree at leading order [see eq. (3.43)].

As stated above, the quantity of interest in the Lagrangian scheme is the displacement field, i.e. particle trajectories are studied rather than the density and velocity fields (cf. sections 3.1.1 and 3.1.2). The master equation of LPT is obtained by writing the equation of motion for particles trajectories \( x(\tau) \) in an expanding Universe \(^2\),

\[ \frac{d^2x}{d\tau^2} + \mathcal{H} \frac{dx}{d\tau} = -\nabla \phi, \] (3.65)

as a function of \( \Psi \) in the \((q, \tau)\) frame. By taking the divergence of eq. (3.65), using eq. (3.43) and the Poisson eq. (3.4), one obtains

\[ \nabla \cdot \left[ \frac{d^2 \Psi}{d\tau^2} + \mathcal{H} \frac{d\Psi}{d\tau} \right] = -\frac{3}{2} \mathcal{H}^2 \Omega_m \delta^E. \] (3.67)

From eq. (3.47), we have that

\[ a^3(z) \rho(x, z) dx = a^3(z_{in}) \bar{\rho}(z_{in}) \mathcal{J}(x, z)^{-1} dx \] (3.68)

which implies

\[ \mathcal{J}^{-1}(x, z) = \left| \frac{dq}{dx} \right| = [1 + \delta^E(x, z)], \] (3.69)

where \( \delta^E \) is the Eulerian fully non-linear density contrast; thus,

\[ \delta^E(x(q, z), z) = \frac{1 - \mathcal{J}(x(q, z), z)}{\mathcal{J}(x(q, z), z)}. \] (3.70)

\(^3\)See also \(^{34}\) where this term is referred to as a shift term.
On top of that, the $\nabla$ operator in Eulerian coordinates reads in the Lagrangian frame
\[
\frac{\partial}{\partial x_i} = \left[ \frac{dq}{dx} \right]_{ij} \frac{\partial}{\partial q_j} = [\delta^K_{ij} + \Psi_{ij}]^{-1} \frac{\partial}{\partial q_j} .
\] (3.71)

Substituting eqs. (3.70) and (3.71) into eq. (3.67), the master equation of LPT is finally found [28]:
\[
\mathcal{J}(q, \tau) \left[ \delta^K_{ij} + \Psi_{ij}(q, \tau) \right]^{-1} \left[ \frac{d^2 \Psi_{i,j}(q, \tau)}{d\tau^2} + \mathcal{H} \frac{d\Psi_{i,j}(q, \tau)}{d\tau} \right] = \\
= \frac{3}{2} \mathcal{H}^2(\tau)\Omega_m(\tau) \mathcal{J}(q, \tau) - 1 .
\] (3.72)

After some manipulations, eq. (3.72) can be written in the linearised form [29]
\[
\frac{d^2 \Psi^{(1)}_{i,j}(q, \tau)}{d\tau^2} + \mathcal{H} \frac{d\Psi^{(1)}_{i,j}(q, \tau)}{d\tau} = \frac{3}{2} \mathcal{H}^2(\tau)\Omega_m(\tau)\Psi^{(1)}_{i,j} ,
\] (3.73)
based on the perturbative expansion of eq. (3.51). Given that the dynamical variable of eq. (3.73) is only time, we can look for a solution in the separable form $\Psi^{(1)}(q, \tau) = D_1(\tau)\tilde{\Psi}^{(1)}(q)$. Thus, eq. (3.73) now reads
\[
\ddot{D}_1 + \mathcal{H} \dot{D}_1 - \frac{3}{2} \Omega_m\mathcal{H}^2 D_1 = 0 ,
\] (3.74)
in complete analogy with eq. (3.23). Focussing only on the growing mode, $D_1$ is the same as the linear growth function of Eulerian perturbation theory [see eq. (3.28)]. Using eqs. (3.58) and (3.63), we find
\[
\nabla_q \cdot \Psi^{(1)}(q, \tau) = D_1(\tau)\nabla_q \cdot \tilde{\Psi}^{(1)}(q) = -\delta^{(1)}(q, \tau) = -\delta^{E^{(1)}}(x, \tau) .
\] (3.75)

Also, assuming the linear part of the displacement field to be irrotational ($\nabla_q \times \Psi^{(1)} = 0$), a scalar field $\tilde{\phi}^{(1)}$ satisfying $\Psi^{(1)}(q, \tau) = -\nabla_q \tilde{\phi}^{(1)}$ must exist. Since
\[
\nabla_q \cdot \Psi^{(1)}(q, \tau) = -\nabla^2_q \tilde{\phi}^{(1)}(q, \tau) = -\delta^{E^{(1)}}(x, \tau) ,
\] (3.76)
we can identify $\tilde{\phi}^{(1)}$ from the Poisson eq. (3.4):
\[
\tilde{\phi}^{(1)} = \frac{\phi^{(1)}}{4\pi Ga^2\bar{\rho}} ,
\] (3.77)
where $\phi^{(1)}$ is the linear part of the Newtonian gravitational potential, $\phi = \sum_{n=1}^{\infty} \phi^{(n)}$ (see also eqs. (3.8) and (3.9)).

At second order, the master equation of LPT reads [29]
\[
\left( \frac{d^2 \Psi^{(2)}_{i,j}(q, \tau)}{d\tau^2} + \mathcal{H} \frac{d\Psi^{(2)}_{i,j}(q, \tau)}{d\tau} - \frac{3}{2} \mathcal{H}^2(\tau)\Omega_m(\tau)\Psi^{(2)}_{i,j} \right) =
\]
\[ = -\frac{3}{2} H^2(\tau) \Omega_m(\tau) \left[ \frac{1}{2} \left( \Psi_{k,k}^{(1)} \right)^2 - \frac{1}{2} \Psi_{i,j}^{(1)} \Psi_{j,i}^{(1)} \right] ; \tag{3.78} \]

again, the solution is in the separable form \( \Psi^{(2)}(q, \tau) = D_2(\tau) \tilde{\Psi}^{(2)}(q) \), where the time evolution of the second-order growth factor \( D_2 \) is described by

\[ \ddot{D}_2 + \mathcal{H} \dot{D}_2 - \frac{3}{2} \Omega_m \mathcal{H}^2 D_2 = -\frac{3}{2} \Omega_m \mathcal{H}^2 D_1^2 , \tag{3.79} \]

while the spatial evolution of \( \Psi^{(2)}(q, \tau) \) is encoded in

\[ \tilde{\Psi}_{k,k}^{(2)}(q) = \frac{1}{2} \sum_{i \neq j} \left[ \tilde{\Psi}_{i,i}^{(1)}(q) \tilde{\Psi}_{j,j}^{(1)}(q) - \tilde{\Psi}_{i,j}^{(1)}(q) \tilde{\Psi}_{j,i}^{(1)}(q) \right] . \tag{3.80} \]

In the flat ΛCDM Universe, \( D_2 \) is approximated by \[ D_2(\tau) \simeq -\frac{3}{7} D_1^2(\tau) \Omega_m^{1/4} . \tag{3.81} \]

to better than 0.6%. However, since the dependence on \( \Omega_m \) is very weak, we will simply assume \( D_2 \simeq -\frac{3}{7} D_1^2 \), and thus

\[ \Psi_{k,k}^{(2)}(q, \tau) \simeq -\frac{3}{14} \left( \nabla q \cdot \Psi^{(1)}(q, \tau) \right)^2 - \sum_{i,j} \Psi_{i,j}^{(1)}(q, \tau) \Psi_{j,i}^{(1)}(q, \tau) \right] . \tag{3.82} \]

Hence, from eqs. \( (3.59) \) and \( (3.82) \), we have that the second-order Lagrangian density reads

\[ \delta L^{(2)} = -\nabla q \cdot \Psi^{(2)} + \frac{1}{2} \left[ \left( \nabla q \cdot \Psi^{(1)} \right)^2 + \sum_{i,j} \Psi_{i,j}^{(1)} \Psi_{j,i}^{(1)} \right] \tag{3.83} \]
\[ = \left( \nabla q \cdot \Psi^{(1)} \right)^2 - \frac{2}{7} \left[ \left( \nabla q \cdot \Psi^{(1)} \right)^2 - \sum_{i,j} \Psi_{i,j}^{(1)} \Psi_{j,i}^{(1)} \right] . \tag{3.84} \]

This result can be re-written in an alternative way by invoking the first-order result for an irrotational displacement \( -\nabla q \cdot \Psi^{(1)} = \nabla^2 \tilde{\phi}^{(1)} = \delta^{L(1)} \):

\[ \delta^{L(2)} = \left( \delta^{L(1)} \right)^2 - \frac{2}{7} \left[ \left( \delta^{L(1)} \right)^2 - \sum_{i,j} \left( \nabla^{-2} \delta^{L(1)} \right)_{ij} \left( \nabla^{-2} \delta^{L(1)} \right)_{ij} \right] . \tag{3.85} \]

Then, introducing the trace-free tidal tensor

\[ s_{ij} \equiv \left( \nabla_i \nabla_j - \frac{1}{3} \delta^K_{ij} \nabla^2 \right) \nabla^{-2} \delta , \tag{3.86} \]

the second term in the square bracket of eq. \( (3.85) \) can be replaced with the tidal term \( s^2 = s_{ij} s^{ij} \), so that

\[ \delta^{L(2)} = \frac{17}{21} \left( \delta^{L(1)} \right)^2 + \frac{2}{7} s^2 . \tag{3.87} \]
To summarize the results of this section, we found that the linearly growing mode of the density field has a simple separable form in Lagrangian space

$$\delta^{(1)}(q, z) = C(q)D(z);$$  (3.88)

it drives the subsequent non-linear evolution of the density field, so that the non-linearly evolved field in Lagrangian coordinates may be written as

$$\delta^L(q, z) = \delta_{\text{lin}}(q, z) + \delta_{\text{nonlin}}^L(q, z),$$  (3.89)

where we dropped the superscript $^L$ from the linear density to enforce that the first-order density perturbation $\delta^{(1)}$ has the same form in both the Lagrangian and Eulerian frame [see eq. (3.63)]. Also, as we will work only up to second-order in perturbation theory, we will approximate

$$\delta_{\text{nonlin}}^L(q, z) \simeq \delta^{(2)}(q, z) = \frac{17}{21} (\delta_{\text{lin}}(q, z))^2 + \frac{2}{7} s^2(q, z).$$  (3.90)

Finally, the non-linearly evolved density field of eq. (3.89) in Eulerian coordinates is

$$\delta^E(x, z) = \delta_{\text{lin}}(x, z) + \delta_{\text{nonlin}}^E(x, z),$$  (3.91)

where at second order, using eqs. (3.64) and (3.87), one gets [36]:

$$\delta_{\text{nonlin}}^E(x, z) \simeq \delta^{E(2)}(x, z) = \frac{17}{21} (\delta_{\text{lin}}(x, z))^2 + \frac{2}{7} s^2(x, z) - \Psi(x, z) \cdot \nabla \delta(x, z).$$  (3.92)

Equation (3.92) is a useful alternative way to write the second-order solution for $\delta^{E(2)}$, which will be used in sections 3.3.5 and 6.1. Note indeed that by Fourier transforming eq. (3.92), one obtains again eqs. (3.39) and (3.41).

### 3.2 The statistics of random fields

Before continuing the discussion about matter evolution and formation of structures, it is worth thinking about the meaning of the fields $\delta(x, \tau)$ and $\theta(x, \tau)$ just introduced. The time dependence is described by the laws of physics but there is no expectation for the models to tell us at which position $x$ over the sky a certain overdensity is placed. The reason is that the initial conditions of the Universe and, therefore, $\delta(x, \tau)$ and $\theta(x, \tau)$ are random realizations drawn from some suitable probability distribution functions.

In principle, to gain some information about the properties of these distributions one has to average over different statistical realizations and we face the
problem that we only observe one of them. However it can be shown that the
ensemble average is equivalent to a spatial average for modes with size well be-
low the horizon (or the survey) size. Thus, the predicted statistical properties
of the fields $\delta(x, \tau)$ and $\theta(x, \tau)$ (or others) can actually be compared to data.
For modes of size about the horizon scale (or the survey size) however, this is
no longer true and there is a limit in the precision with which we can measure
the statistics on these scales. This limit is known as cosmic variance. In this
section, the standard tools to deal with the statistics of random fields will be
introduced.

A random field $\varphi(x)$ is associated with a probability distribution function
(PDF), assumed to be statistically homogeneous and isotropic$^4$. At each posi-
tion $x$, a value for $\varphi$ comes with probability given by the PDF. The probability
of finding two fluctuations at some distance $r$ is given by the joint ensemble
average
\[
\xi(r) \equiv \langle \varphi(\mathbf{x}) \varphi(\mathbf{x}+\mathbf{r}) \rangle,
\]
which is known as the 2-point correlation function. As a consequence of the
statistical homogeneity and isotropy, $\xi$ depends only on the modulus of $r$. In
Fourier space the correlator reads
\[
\langle \varphi(k_1) \varphi(k_2) \rangle = \int d\mathbf{x} \int d\mathbf{r} \langle \varphi(\mathbf{x}+\mathbf{r})\varphi(\mathbf{x}) \rangle e^{-i\mathbf{k}_1 \cdot (\mathbf{x}+\mathbf{r})} e^{-i\mathbf{k}_2 \cdot \mathbf{x}}
= (2\pi)^3 \delta^D(\mathbf{k}_1 + \mathbf{k}_2) \int d\mathbf{r} \xi(\mathbf{r}) e^{-i\mathbf{k}_1 \cdot \mathbf{r}}
= (2\pi)^3 \delta^D(\mathbf{k}_1 + \mathbf{k}_2) P_{\varphi\varphi}(k_1),
\]
where $P_{\varphi\varphi}(k)$ is the power spectrum. Extending to higher orders, one gets
\[
\langle \varphi(k_1) \varphi(k_2) \varphi(k_3) \varphi(k_4) \rangle = (2\pi)^3 \delta^D(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) B_{\varphi\varphi\varphi\varphi}(k_1, k_2, k_3, k_4),
\]
\[
\langle \varphi(k_1) \varphi(k_2) \varphi(k_3) \varphi(k_4) \rangle = (2\pi)^3 \delta^D(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) T_{\varphi\varphi\varphi\varphi}(k_1, k_2, k_3, k_4),
\]
and so on, where $B_{\varphi\varphi\varphi\varphi}$ is the bispectrum and $T_{\varphi\varphi\varphi\varphi}$ the trispectrum. These
spectra are the Fourier transform of the corresponding $n$-point correlation func-
tions.

The real-space correlators evaluated at the same coordinate define the mo-

$^4$An example where statistical isotropy is broken will be given in chapter 7.
ments of the distribution. The second-order moment is known as the variance

\[ \sigma^2_\varphi \equiv \langle \varphi(x)^2 \rangle = \int \frac{dk_1}{(2\pi)^3} \int \frac{dk_2}{(2\pi)^3} \langle \varphi(k_1)\varphi(k_2) \rangle e^{ik_1 \cdot x} \]

\[ = \int \frac{dk_1}{(2\pi)^3} \int \frac{dk_2}{(2\pi)^3} \delta^D(k_1 + k_2)P_{\varphi\varphi}(k_1)e^{i(k_1 + k_2) \cdot x} \]

\[ = \int \frac{dk_1}{(2\pi)^3} \delta_D(k_1) P_{\varphi\varphi}(k_1) e^{i(k_1 + k_2) \cdot x} \]

(3.97)

where \( P_{\varphi\varphi}(k_1) = P_{\varphi\varphi}(k_1)k_1^3/(2\pi^2) \) is the dimensionless power spectrum; it is straightforward to extend this to higher orders:

\[ \langle \varphi(x)^n \rangle = \int \frac{dk_1}{(2\pi)^3} \ldots \int \frac{dk_n}{(2\pi)^3} \langle \varphi(k_1) \ldots \varphi(k_n) \rangle e^{i(k_1 + \ldots + k_n) \cdot x} . \]

(3.98)

The cumulants of the PDF are defined as the moments normalized by an appropriate power of the variance,

\[ S_n \equiv \frac{\langle \varphi^n(x) \rangle}{\langle \varphi^2(x) \rangle^n} ; \]

(3.99)

in particular, \( S_3 \) and \( S_4 \) are the skewness and kurtosis of the distribution respectively. Later, the reduced cumulant \( \kappa_n \), which differs from \( S_n \) by factors of \( \sigma_\varphi \), will be used as well,

\[ \kappa_n \equiv \frac{\langle \varphi^n(x) \rangle}{\sigma^2_\varphi} . \]

(3.100)

A special case is a random field drawn from a Gaussian distribution, i.e. a Gaussian random field, here indicated with \( \varphi_G \). Indeed, all the odd \( n \)-point correlators vanish\(^5\), while for \( n \) even and larger than two they are built as powers of the power spectrum \((n = 2)\). More formally, one gets

\[ \langle \varphi_G(k_1) \ldots \varphi_G(k_{2p+1}) \rangle = 0 \]

(3.101)

\[ \langle \varphi_G(k_1) \ldots \varphi_G(k_{2p}) \rangle = \sum_{\text{all pair associations}} \prod_{p \text{ pairs } (i,j)} \langle \varphi_G(k_i)\varphi_G(k_j) \rangle \]n (3.102)

which is the Wick’s theorem. Looking at the definition of the 2-point correlator in Fourier space [eq. (3.94)], it is clear that eq. (3.102) is non vanishing only for disconnected pairs.

3.2.1 Applications to the matter density field

The tools introduced in the previous section can now be used to investigate the statistical properties of the matter density field \( \delta(k, \tau) \). But, first, smoothing of the field is needed. The reason can be understood from two different view

\(^5\)Strictly speaking one could have \( \langle \varphi_G \rangle \neq 0 \) but it is common practice in cosmology to subtract the mean of the field.
points: observationally, surveys have a limited resolution over the finite volume they can access and, theoretically, one wants to model cosmological matter fluctuations down to a certain scale. Being rather extreme, \( \delta \) is not meant to resolve the structure of dark matter particles! In both cases, the matter density field must be smoothed on a certain mass scale \( M \) (or equivalently on a length scale \( R_M = R(M) = (3M/4\pi \rho_m)^{1/3} \)),

\[
\delta_M(k, z) = W_M(k) \delta(k, z).
\] (3.103)

A common choice for the window function \( W_M(k) \) is the Fourier transform of top-hat filter in real space \(^{[37]}\),

\[
W_M(k) = 3 \frac{\sin(kR_M) - kR_M \cos(kR_M)}{(kR_M)^3},
\] (3.104)

which will be assumed throughout the thesis.

Let’s now make the assumption that the linear density field is a Gaussian field\(^6\), \( \delta_{\text{lin}} = \delta^{(1)} = \delta_G \). At the \( n \)-th perturbative order, \( \delta^{(n)}(k, z) \) is the convolution of \( n \) functions \( \delta^{(1)} \) with an appropriate kernel [see eq. (3.33)]. This suggests that for \( n \geq 2 \) the density field is no longer statistically Gaussian, although the presence of \( \delta^{(1)} \) still allows us to use the Wick’s theorem to compute the \( n \)-point correlators.

Thus, the matter power spectrum is given by

\[
\langle \delta_M(k_1) \delta_M(k_2) \rangle = \langle \delta^{(1)}_M(k_1) \delta^{(1)}_M(k_2) \rangle + \left[ \langle \delta^{(1)}_M(k_1) \delta^{(2)}_M(k_2) \rangle + \text{cyc.} \right] + \\
+ \left[ \langle \delta^{(2)}_M(k_1) \delta^{(3)}_M(k_2) \rangle + \text{cyc.} \right] + \\
+ \left[ \langle \delta^{(3)}_M(k_1) \delta^{(2)}_M(k_2) \rangle + \text{cyc.} \right] + \ldots
\]

\[
= (2\pi)^3 \delta^D(k_1 + k_2) W_M(k_1)^2 \left[ P(k_1) + P^{\text{one loop}}(k_1) + \ldots \right],
\] (3.105)

where \( P(k_1) \) is the linear matter power spectrum (see fig. [3.1]), \( W_M(k_1)^2 P(k_1) \) schematically comes from the term \( \langle \delta^{(1)}_M \delta^{(2)}_M \rangle \), while \( \langle \delta^{(1)}_M \delta^{(3)}_M \rangle \) gives a vanishing contribution because of the Wick’s theorem. The terms \( \langle \delta^{(1)}_M \delta^{(3)}_M \rangle \) and \( \langle \delta^{(2)}_M \delta^{(2)}_M \rangle \) define the so-called one-loop power spectrum \( P^{\text{one loop}}(k_1) \) filtered on mass scale \( M \) by \( W_M(k_1)^2 \), which improves the validity of the perturbation theory up to higher values of \( k_1 \) (see for instance \[^{[38]}\]). In the thesis, however, only the leading order (or tree-level) will be considered, so that

\[
\langle \delta_M(k_1) \delta_M(k_2) \rangle \simeq \langle \delta^{(1)}_M(k_1) \delta^{(1)}_M(k_2) \rangle = (2\pi)^3 \delta^D(k_1 + k_2) P(k_1)
\] (3.106)

\(^6\)This assumption will be fully justified in section 4.1.
and the variance of the smoothed, linear density fluctuations is
\[
\sigma^2(M, z) = \langle \delta_M^2 \rangle = \int \frac{dk}{k} \frac{k^3}{2\pi^2} W_M^2(k, R) P(k) .
\] (3.107)

It is now interesting to consider the next-order case, i.e. the matter bispectrum:
\[
\langle \delta_M(k_1) \delta_M(k_2) \delta_M(k_3) \rangle = \langle \delta_M^{(1)}(k_1) \delta_M^{(1)}(k_2) \delta_M^{(1)}(k_3) \rangle + \left[ \langle \delta_M^{(1)}(k_1) \delta_M^{(1)}(k_2) \delta_M^{(2)}(k_3) \rangle + 2 \text{cyc.} \right] + \ldots
\] (3.108)

Since the first term is zero because of the Wick’s theorem, the leading order is schematically given by \( \langle \delta_M^{(1)} \delta_M^{(2)} \rangle \). Using the result of eq. (3.33), the matter bispectrum reads
\[
B(k_1, k_2, k_3, M) = 2W_M(k_1)W_M(k_2)W_M(k_3) \times
\] 
\[
\times \left[ P(k_1)P(k_2)F_2(k_1, k_2) + 2 \text{cyc.} \right] .
\] (3.109)

Therefore, the non-linear evolution of \( \delta(k, z) \) sources a non-vanishing bispectrum even if the linear density field is Gaussian. Related to that, the third-order...
moment is
\[
\langle \delta_M^3 \rangle = \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p'}{(2\pi)^3} \int \frac{d^3p''}{(2\pi)^3} W_M(p)W_M(p')W_M(p'') \langle \delta_M(p)\delta_M(p')\delta_M(p'') \rangle
\]
(3.110)
which can be used to compute the skewness \[27,41\]
\[
S_3 = \frac{34}{7} + \frac{d \ln \sigma^2(R(M),z)}{d \ln R}.
\]
(3.111)

\(S_3\) measures the asymmetry between underdense \((\delta < 0)\) and overdense \((\delta > 0)\) regions because of gravitational evolution. The next order cumulant, the kurtosis, is related to the trispectrum and leads to a more complicated result that can be found in \[28\].

In the following, we will often make use of the graphical representation introduced by Jeong and Komatsu \[42\] to show the shape dependence of the bispectrum. The amplitude of \(B\), or parts of it, are plotted as a function of \(k_2/k_1\) and \(k_3/k_1\) in a colour map, under the condition \(k_3 \leq k_2 \leq k_1\). This requirement avoids multiple visualizations of the same triangle. From all the possible choices of \(k_2/k_1\) and \(k_3/k_1\), some specific configurations can be identified, which are shown in fig. 3.2: equilateral \((k_1 = k_2 = k_3)\), isosceles \((k_1 > k_2 = k_3\) or \(k_1 = k_2 > k_3)\), folded \((k_1 = 2k_2 = 2k_3)\), squeezed \((k_1 \approx k_2 \gg k_3)\) and elongated \((k_1 = k_2 + k_3)\).

### 3.3 Large-scale structure

Wherever we look, the Universe appears to be filled with structures of various shapes and sizes. Starting from the Solar System, planets with radius ranging from a few thousand up to tens of thousand of kilometres orbit around a star, the Sun, of diameter about \(14 \times 10^5\) km. The distance between the Earth and the Sun is measured in a more convenient unit, i.e. the Astronomical Unit \(1\text{AU} \approx 1.5 \times 10^8\) km. The Solar System has a size of order tens of AU and is located in a peripheral region of a spiral galaxy, know as the Milky Way, made of billions of stars. Distances outside the Solar system are usually measured in parsec (pc), i.e. the distance at which 1 AU subtends an angle of 1 arc second: \(1\text{pc} \approx 2 \times 10^5\text{AU}\). Indeed, the Sun is located approximately 8 kpc away from the centre of the galaxy \[43\]. Zooming out, also the Milky Way happens to be in a bound system of about 30 galaxies, all lying within a sphere of radius 1Mpc. This is called the Local Group, while larger groups form clusters. Catalogues indicate that galaxies are spread all around, organized in filaments and walls passing through nearly empty regions called voids, with diameter sizes ranging from 2 Mpc/h up 60 Mpc/h \[44\].
Figure 3.2: Explanation of the visual representation for the bispectrum introduced in [42]. The triangular-shaped region that hosts the colour map is due to the condition \( k_3 \leq k_2 \leq k_1 \). This requirement avoids double visualizations of the same triangular configuration. For the allowed values of \( k_2/k_1 \) and \( k_3/k_1 \) we recognise same specific configurations: point (a) is for the squeezed limit \( (k_1 \simeq k_2 \gg k_3) \), (b) for the equilateral configuration \( (k_1 = k_2 = k_3) \) and (c) for the folded one \( (k_1 = 2k_2 = 2k_3) \). The elongated triangles \( (k_1 = k_2 + k_3) \) resides on the left edge, while the upper and right edges correspond to isosceles triangles \( (k_1 > k_2 = k_3 \text{ or } k_1 = k_2 > k_3) \). General configurations are in the inner region.
All these objects are made of baryonic matter which constitutes a small fraction ($\Omega_b \approx 5\%$) of the total energy content of the Universe. Many pieces of evidence indicate that a much larger fraction ($\Omega_{dm} \approx 26\%$) is made of an unknown kind of matter, invisible to radiation but, potentially, weakly interacting. As briefly sketched in section 3.1, perturbations in the dark matter efficiently grow during the matter era, even when baryons are still coupled to radiation. Dark matter structures evolve non-linearly via merging of small initial seeds, until virialized objects form, called dark matter halos, of size approximately $\lesssim 10 \text{ Mpc}/h$. They are then populated by galaxies. The web of galaxies together with halos is known as the Large-Scale Structure (LSS) of the Universe [15].

In the second part of the chapter, a possible way to describe halo formation out of the density field $\delta$ will be reviewed. A common assumption to express the functional relating the halo density fluctuation $\delta_h$ to the matter density, i.e. $\delta_h = F(\delta)$, will also be presented and used to predict the halo power spectrum and bispectrum.

### 3.3.1 Mass function

In the commonly accepted picture of structure formation, a dominant non-visible component of matter is assumed to collapse into dark matter clumps, the virialized parts of which are called dark matter halos.

Identifying the regions where dark matter halos form involves the definition of a new concept: the mass function, i.e. the number of objects with mass within the range $M$ to $M + dM$ at redshift $z$. As a full non-linear analysis of all the collapsing regions is not feasible, Press and Schechter [45] found a good prescription in qualitative agreement with $N$-body simulations.

The Press-Schechter (PS) prescription simply states that virialized objects of mass $M$ form in regions in which the smoothed density field exceeds a suitable threshold. All the information needed to find the mass spectrum is encoded in the initial PDF of fluctuations, linear theory to describe the evolution of matter perturbations and a suitable threshold $\delta_c$ for collapse. It is important to underline that the PS prescription identifies halos in the Lagrangian frame, as the gravitational clustering is not included at all. Clearly, this does not affect the mass spectrum but plays a role in the discussion of section 3.3.2.

By defining the quantity $\nu = \delta_c / \sigma$, the mass function can be written in a general form as [45]

$$n_h(M,z) = f(\nu) \frac{\bar{\rho}}{M^2} \left| \frac{d \ln \sigma^{-1}}{d \ln M} \right|. \quad (3.112)$$
The threshold $\delta_c$ is usually assumed to be the linearly growing density amplitude for a spherically collapsed object. For example, in a flat $\Lambda$CDM universe the threshold for a spherical collapsed halo is \[\delta_c(z) = \frac{3(2\pi)^{2/3}}{20} \left[ 1 + 0.0123 \log \Omega_m(z) \right], \] (3.113) reducing to $\delta_c \approx 1.686$ when $\Omega_m = 1$. Hence $\delta_c$ is weakly dependent on the value of $\Omega_m$ and $\Omega_\Lambda$ at the time of collapse \[47\].

When a Gaussian PDF of fluctuations is assumed, then \[f_{\text{PS}}(\nu) = \sqrt{\frac{2}{\pi \nu}} e^{-\frac{\nu^2}{2}}, \] (3.114) and by substituting eq. (3.114) into eq. (3.112) the PS mass function is obtained. In the following, the term "mass function" will be loosely used to indicate also $f(\nu)$. However, the last equation comes with two issues. First of all, a factor 2 (the fudge factor) has been added by hand; it accounts for the contribution to collapse from underdense regions, but comes with no dynamical motivation \[45\]. Second, regions below the threshold on some scale can exceed it at larger scales: this miscounting of low-mass clumps is known as the cloud-in-cloud problem \[48\].

The formalism developed in \[48\], called Excursion Set (ES) theory, accounts for both issues. By choosing a fixed point $q$ and progressively reducing the filter radius from $R \to \infty$, it can be shown that the problem of exceeding the threshold is equivalent to a barrier crossing problem for a stochastic process, whose solution is the PS mass function.

A comparison with $N$-body simulations (see fig. 3.3) shows that the PS mass function predicts too many low-mass halos and too few high-mass halos. Fitting the functional form \[f_{\text{ST}}(\nu) = A(p) \sqrt{\frac{2\gamma}{\pi}} \left[ 1 + \left( \frac{\gamma \nu^2}{1} \right)^{p} \right] \nu e^{-\gamma \frac{\nu^2}{2}}, \] (3.115) against simulations provides a good agreement when $\gamma = 0.707$ and $p = 0.3$, while $A(p) = 0.322$ is found under the requirement that all the mass is collapsed into halos. Equation (3.115) is known as the Sheth-Tormen (ST) mass function \[51\] and the ES formalism shows that $f_{\text{ST}}(\nu)$ is the solution of a barrier crossing problem for a stochastic process with moving barrier \[52\]. It generalizes the PS results by accounting for ellipsoidal collapse. Interestingly, both the PS and ST functions depend only on the variable $\nu$, and for this reason they are known as universal mass functions.

\[\text{\textasteriskcentered} A \text{ better agreement is found by replacing } \delta_c = 1.686 \to 1.42, \] \[49,50\]
Figure 3.3: The plot shows the mass function $f(\nu)$. The red points represent numerical data from $N$-body simulations, the solid line the PS mass function, while the dashed and dotted lines the improved fits of the ST mass function [51] and [53] respectively. Figure taken from [54].

A further extension of ES theory generalizes the barrier crossing problem to a non-Markovian process, resulting in the Maggiore and Riotto mass function [55–57]. Adding memory can well describe for instance the case in which mode correlations exist in the initial conditions (see section 4.1). Beyond that, non-universal mass functions calibrated with high-precision against simulations exist as well (see for instance [58]).

### 3.3.2 Local Lagrangian biasing model

The concept of bias between the matter density field and its tracers was introduced in [59], to explain the excess of spatial correlations for the Abell clusters. As we observe galaxies only above a certain brightness threshold, i.e. the detection limit of our telescopes, we typically observe only the most luminous galaxies. Assuming a relation between luminosity and halo mass, this implies we observe a biased galaxy field which can be estimated analytically [37, 60, 61].

The Peak-Background Split (PBS) formalism gives a simple physical interpretation [60], consisting of splitting the density fluctuations into long and short wavelengths. As dark halos are objects of radius typically < 10 Mpc/$h$, the modes with wavelength $\leq 10$ Mpc/$h$ can be considered the cause of collapsed
halos locally,
\[ \delta_{\text{lin},s}(\mathbf{q}, M) = \int_{k \geq l^{-1}} d\mathbf{k} \delta_{\text{lin},M}(\mathbf{k}) e^{-i\mathbf{k} \cdot \mathbf{q}}, \]
while larger wavelengths,
\[ \delta_{\text{lin},l}(\mathbf{q}, M) = \int_{k \leq l^{-1}} d\mathbf{k} \delta_{\text{lin},M}(\mathbf{k}) e^{-i\mathbf{k} \cdot \mathbf{q}}, \]
have the net effect of perturbing the local threshold, as shown in fig. 3.4 (the linear density field \( \delta_{\text{lin}} \) is again assumed to be a Gaussian field as in section 3.2.1).

Indeed, the collapse is now achieved when the small-scale component reaches \( \delta_c - \delta_{\text{lin},l}(\mathbf{q}) \) and the variable \( \nu \) has now a local value
\[ \nu = \frac{\delta_c}{\sigma} \rightarrow \nu(\mathbf{q}) = \frac{\delta_c - \delta_{\text{lin},l}(\mathbf{q})}{\sigma}. \]

This modulation in the collapsing properties induces a bias, as halos are more likely to form in overdense regions. Indeed, by defining the halo overdensity as
\[ \delta_{\text{h}}(\mathbf{q}, z, M) = \frac{n_h(\mathbf{q}, z, M) - \bar{n}_h}{\bar{n}_h}, \]
and treating \( \delta_{\text{lin},l} \) as a perturbation, \( \delta_{\text{h}} \) can be Taylor expanded \[51, 61\]
\[ \delta_{\text{h}}(\mathbf{q}, z, M) = \sum_{j=1}^\infty \frac{b_{\text{L}}^j(z)}{j!} (\delta_{\text{lin},l}(\mathbf{q}, z))^j, \]
where \( b_{\text{L}}^j(z) \) are the Lagrangian bias coefficient,
\[ b_{\text{L}}^j = \left[ \frac{1}{n_h} \left( \frac{\partial n_h}{\partial \delta_{\text{lin},l}} \right)^j \right]_{\delta_{\text{lin},l}=0}. \]

This choice for the functional \( \delta_{\text{h}} = \mathcal{F}(\delta) \) is called local Lagrangian biasing model\[8\].

In this picture the formation sites of halos are identified from the initial density field and the model is capable of describing the statistics of the objects, while their dynamics is then captured in the transformation to the Eulerian space, by applying eq. (3.43). This local Lagrangian biasing scheme comes with a prescription to calculate the bias, based upon the knowledge of the halo mass function. Assuming the ST mass function, the linear and non-linear Lagrangian bias coefficients read respectively \[64\]
\[ b_{\text{L}}^1 = \frac{\gamma \nu^2 - 1}{\delta_c} + \frac{2p}{1 + (\gamma \nu^2)^p} \frac{1}{\delta_c}, \]
\[ b_{\text{L}}^2 = \frac{\gamma \nu^2 - 3}{2\delta_c^2} + \frac{p}{1 + (\gamma \nu^2)^p} \frac{2\gamma \nu^2 + 2p - 1}{\delta_c^2}, \]
\[ b_{\text{L}}^3 = \frac{\gamma \nu^2 - 5}{3\delta_c^3} + \frac{3p}{1 + (\gamma \nu^2)^p} \frac{3\gamma \nu^2 + 3p - 1}{3\delta_c^3}. \]

\[8\]Generalizations of this picture exists as well, see for instance \[62, 63\].
which reduce to the PS predictions when $\gamma = 1$ and $p = 0$:

\[ b_1^L = \frac{\nu^2 - 1}{\delta_c}, \]  
\[ b_2^L = \frac{\nu^2 (\frac{\nu^2 - 3}{2\delta_c^2})}{\nu^2 - 1}. \]  

How the halo overdensity and the bias coefficients are transformed into the Eulerian frame will be explained in section \[3.3.3\].

Figure 3.4: The plot taken from [15] shows the decomposition of the overdensity field into long and short wavelengths according to the PBS approach. The former changes the background, so that the collapse threshold is perturbed and the clustering of halos is modulated.

A somewhat different approach is to assume a local relation between the final number density of objects with mass $M$ at redshift $z$ and the evolved density field, known as local Eulerian biasing\[9\] [65],

\[ \delta_{\text{E}}(x, z, M) = \sum_{j=1}^{\infty} b_j^E(z) (\delta_M(x, z))^j, \]  

where $\delta_M$ is the fully non-linear density contrast in Eulerian space, as defined in eq. (3.37), smoothed on the mass scale $M$. In this picture, the halos are drawn on top of the density field without any memory of their past history, the opposite of the local Lagrangian bias model which assumes that the information about halos is already encoded in the initial density field. However, many recent results show that the local Eulerian biasing model is not sufficiently accurate when compared to simulations \[36, 66, 69\]. In particular, other physically motivated

\[9\] In principle, a term with $j = 0$ yielding just a constant $b_0$ should be included in the expansion. Its value is fixed by the condition $\langle \delta_{\text{E}}^0 \rangle = 0$, but it is usually dropped as it does not contribute at all when connected moments of the density contrast are considered. Also, in Fourier space, the $b_0$ term vanishes for modes $k \neq 0$.  

45
contributions consistent with the symmetries of the dark matter equations of motion [70], like a tidal term, should be included. These new terms break the assumption of locality implicit in eq. (3.126). In section 3.3.3 it will be shown how the tidal term arises naturally in the local Lagrangian bias model when transformed into the Eulerian frame. In any case, either a local or non-local Eulerian model offer only a parametrization of the bias, and in practice the bias coefficients are usually fitted against the data or N-body simulations.

### 3.3.3 Lagrangian-to-Eulerian transformation

Galaxy surveys map the distribution of galaxies, which are located in collapsed halos, displaced with respect to their Lagrangian positions according to eq. (3.43). Equation (3.120) describes the excess of halos in Lagrangian space but this needs to be transformed to the Eulerian frame to account for their dynamics.

In section 3.1.3, we have shown that under the conservation of mass in an infinitesimal volume element and for a uniform initial density [see eq. (3.47)]:

\[
M = a^3(z)\rho(x, z)dx = a^3(z_{in})\bar{\rho}(z_{in})dq .
\]  

(3.127)

By introducing the Jacobian of the transformation \( J \) [eq. (3.45)], it then follows

\[
a^3(z)\rho(x, z)dx = a^3(z_{in})\bar{\rho}(z_{in})J(x, z)^{-1}dx ,
\]  

(3.128)

which implies

\[
J^{-1} = \left| \frac{dq}{dx} \right| = \frac{a^3(z)\rho(x, z)}{a^3(z_{in})\bar{\rho}(z_{in})} = \frac{\rho(x, z)}{\bar{\rho}(z)} = [1 + \delta^E(x, z)] ,
\]  

(3.129)

where \( \delta^E(x, z) \) is the Eulerian fully non-linear density contrast.

However, unlike the matter density, the halo density contrast is not a 3-scalar since it is conventionally defined as a coordinate density [33]. Hence the number of halos in a given volume element is given by

\[
N_h = n_h^L(q, z)dq = n_h^F(x, z)dx .
\]  

(3.130)

Thus one finds

\[
1 + \delta^E_h(x, z) = [1 + \delta^L_h(q, z)] \left| \frac{dq}{dx} \right|
\]  

(3.131)

and using the coordinate Jacobian eq. (3.129), the transformation rule is

\[
1 + \delta^E_h(x, z) = [1 + \delta(x, z)] [1 + \delta^L_h(q, z)] .
\]  

(3.132)

It is important to emphasize that eq. (3.132) is obtained under the condition that halos are neither created nor destroyed. This is a standard assumption...
when transforming the halo density to the Eulerian frame and the predicted statistics of the halo density field (which clearly relies on it) well reproduce the simulations in all the references considered, within the error bars\textsuperscript{10}.

### 3.3.4 Local Eulerian bias model

To solve the transformation eq. (3.132) for $\delta^E_h(x, z)$, the Eulerian matter density $\delta(x, z)$ is known in the regime of validity of the Newtonian perturbation theory (see section 3.1) while the Lagrangian space halo density contrast, $\delta^L_h(q, z)$, is given by eq. (3.120) in terms of the linearly growing density contrast $\delta_{\text{lin}}(q, z)$ in Lagrangian coordinates.

The spherical collapse approximation has been used in many works to expand $\delta_{\text{lin}}(q, z)$ in terms of the the non-linear matter density $\delta(x, z)$ (see for instance \cite{61, 72, 73}). This is a special case in which $\Psi = 0$ and the velocity field vanishes at the centre of the symmetrical collapse so that the Eulerian and Lagrangian coordinates coincide at all times, $x \equiv q$. Following \cite{61} (see also \cite{63}), the linearly growing density at the centre of the spherical collapse can be related to the non-linear density field, $\delta$, through

$$
\delta_{\text{lin}}^L(q) = \sum_{i=0}^{\infty} a_i \delta(x)^i = a_1 \delta + a_2 \delta^2 + \ldots ,
$$

(3.133)

where the coefficients are

$$
a_1 = 1, \quad a_2 = -\frac{17}{21}, \quad \ldots .
$$

(3.134)

Equation (3.133) is local in either Eulerian or Lagrangian coordinates since they coincide at the centre of the symmetrical collapse and leads to the Eulerian halo density

$$
\delta^E_h(x) = b^E_1 \delta + b^E_2 \delta^2 + \ldots ,
$$

(3.135)

where the Eulerian bias coefficients are

$$
b^E_1 = 1 + b^L_1,
$$

$$
b^E_2 = \frac{4}{21} b^L_1 + \frac{b^L_2}{2} .
$$

(3.136)

Clearly eq. (3.135) is compatible with the local Eulerian bias model of eq. (3.126), although there are many supporting pieces of evidence that this local expansion is not accurate enough (see the discussion at the end of section 3.3.2).

\textsuperscript{10}The author is not aware of studies investigating any departure from this assumption and the subsequent effects on the theoretical predictions.
3.3.5 Non-local Eulerian bias model

When the spherical collapse approximation is dropped, the local Eulerian bias model of eq. (3.126) is no longer compatible with the transformation of $\delta_L^h(\mathbf{q}, z)$ into the Eulerian frame.

Indeed the linearly growing density contrast $\delta_{\text{lin}}(\mathbf{q}, z)$ in Lagrangian coordinates that appear in $\delta_L^h(\mathbf{q}, z)$ [see eq. (3.120)] can be expanded up to second-order in terms of the non-linear matter density, $\delta$, and the tidal tensor, $s^2$, thanks to eqs. (3.89) and (3.90):

$$\delta_{\text{lin}}(\mathbf{q}, z) \simeq \delta(\mathbf{x}, z) - \frac{17}{21} (\delta(\mathbf{x}, z))^2 - \frac{2}{7} s^2. \quad (3.137)$$

Thus, the expression for the Eulerian halo overdensity up to second order in terms of the density contrast is obtained [36,74]:

$$\delta^E_h(\mathbf{x}) = b^E_1 \delta + b^E_2 \delta^2 - \frac{2}{7} b^L_1 s^2, \quad (3.138)$$

where the Eulerian bias coefficients are defined as in eq. (3.136). The last term of eq. (3.138) is a non-local, non-linear term which ameliorates the comparison with simulations [36] and generalises the result obtained under the spherical collapse approximation [see eq. (3.135)]. Hence, a local Lagrangian bias model naturally brings to a non-local Eulerian expansion for the halo density [74].

Finally, by performing a Fourier transform of the full (non-spherical collapse) result of eq. (3.138) with respect to Eulerian coordinates, one gets

$$\delta^E_h(\mathbf{k}) = b^E_1 \delta + b^E_2 \delta \ast \delta - \frac{2}{7} b^L_1 s^2, \quad (3.139)$$

where $\ast$ stands for a convolution,

$$\delta \ast \delta = \int \frac{d\mathbf{k}_1}{(2\pi)^3} \delta(\mathbf{k}_1) \delta(\mathbf{k} - \mathbf{k}_1), \quad (3.140)$$

and

$$\delta(\mathbf{k}) = \delta_G(\mathbf{k}) + \int \frac{d\mathbf{q}}{(2\pi)^3} F_2(\mathbf{q}, \mathbf{k} - \mathbf{q}) \delta_G(\mathbf{q}) \delta_G(\mathbf{k} - \mathbf{q}), \quad (3.141)$$

$$s^2(\mathbf{k}) = \int \frac{d\mathbf{q}}{(2\pi)^3} S_2(\mathbf{q}, \mathbf{k} - \mathbf{q}) \delta_G(\mathbf{q}) \delta_G(\mathbf{k} - \mathbf{q}), \quad (3.142)$$

with the new kernel defined as

$$S_2(\mathbf{k}_1, \mathbf{k}_2) = \frac{(\mathbf{k}_1 \cdot \mathbf{k}_2)^2}{k_1^2 k_2^2} - \frac{1}{3}. \quad (3.143)$$

Remember that the standard second-order Newtonian kernel $F_2$ is generated by the non-linear gravitational evolution, while $S_2$ by the tidal term.
3.3.6 Halo power spectrum and bispectrum

The halo overdensity of eq. (3.139) can now be used to compute the tree-level power spectrum and bispectrum as explained in section 3.2.1.

Using leading order term in eq. (3.139), the halo power spectrum is simply the linear matter power spectrum of eq. (3.106) (see also fig. 3.1) rescaled by a constant factor

\[ P_{hh}(k, M, z) = b_1^2 W_M(k)^2 P(k, z), \quad (3.144) \]

while the halo bispectrum reads

\[
B_{hhh}(k_1, k_2, k_3, M, z) = W_M(k_1)W_M(k_2)W_M(k_3) \times \\
\times \left[ b_1^2 \left( 2P(k_1, z)P(k_2, z)F_2(k_1, k_2) + 2 \text{ cyc.} \right)_L + b_1^2 b_2 \left( 2P(k_1, z)P(k_2, z) + 2 \text{ cyc.} \right)_L \\
- \frac{2}{7} b_1^2 b_2^2 \left( 2P(k_1, z)P(k_2, z)S_2(k_1, k_2) + 2 \text{ cyc.} \right)_M \right];
\]

the E superscript in the Eulerian bias factors of eq. (3.136) has been suppressed to simplify the notation. Hence, in the absence of primordial non-Gaussianity the bispectrum is only generated by the non-linear gravitational evolution, \( F_2 \) (term A), the non-linear bias, \( b_2 \) (term L), and, in particular, the last term (M), \( S_2 \), is due to the non-local tidal term, \( s^2 \), in the expression for the halo overdensity [see eq. (3.139)]. In table 3.1 we indicate the schematic source of each term A, L, M.

Adopting the graphical representation introduced in section 3.2.1, fig. 3.5 shows the shape dependence of each of the terms A, L, M in eq. (3.145) for different choices of \( k_1 \), normalizing each of them to the maximum value it takes in the \((k_2/k_1, k_3/k_1)\)-space. This normalisation eliminates the redshift-dependence and, as a result, one should not make a comparison of the amplitude between plots. Note that the absolute value is plotted for M, as it can be positive or

Table 3.1: Schematic sources of the terms A, L, M contributing to the halo bispectrum with Gaussian initial conditions [cf. eq. (3.145)]. Permutations of the components are subentend in the right column.

<table>
<thead>
<tr>
<th>Term</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>( \langle \delta^{(2)}\delta^{(1)}\delta^{(1)} \rangle )</td>
</tr>
<tr>
<td>L</td>
<td>( \langle \delta^{(1)} \ast \delta^{(1)} \delta^{(1)} \delta^{(1)} \rangle )</td>
</tr>
<tr>
<td>M</td>
<td>( \langle s^2 \delta^{(1)} \delta^{(1)} \rangle )</td>
</tr>
</tbody>
</table>
Figure 3.5: Shape dependence of the terms $A$, $L$, $M$ contributing to the halo bispectrum of eq. (3.145), for $k_1 = 0.01, 0.05, 0.1 \, h\text{Mpc}^{-1}$; the sources of $A$, $L$, $M$ are schematically indicated in table 3.1. Each term is normalized to the maximum value it can take in the $k_2/k_1, k_3/k_1$-space. This normalisation eliminates the redshift-dependence and as a result one should not make a comparison of the amplitude between plots because of the different scaling. Note that the last row shows the absolute value of term $M$, since it can take negative values. The violet strip indicates where it is changing sign.

negative. Indeed, the violet region that cuts the plots into two parts is where $M$ changes sign.

In the following chapters we will drop the window functions from the expressions for the halo power spectrum and bispectrum: this is a safe procedure as far as the radius $R_M$ of the filter [see eq. (3.103)] is much smaller than the clustering scale we are interested in.

### 3.4 Summary

In this chapter, the perturbative solutions to the Newtonian fluid equations in an expanding Universe have been reviewed. The time evolution of fields like $\delta(x, \tau)$ and $\theta(x, \tau)$ is understood but our models are not expected to predict their values at some specific position over the sky. The reason is that the initial conditions are stochastic as will be shown in the next chapter. Therefore the fields have to be described in a statistical sense, as realizations drawn from some probability distribution functions. Predicted statistics like the power spectrum and bispectrum are what we can really compare with observations in order to
unveil some properties of the distribution.

Whatever the statistics, overdense regions collapse into structures. Galaxies are arranged in a cosmic web, called the Large-Scale structure, of which most of the mass is in form of dark matter halos. The Press-Schechter approach allows us to compute the number of halos that form out of the density field, describing the highly non-linear evolution that leads to the formation of a virialized object through a set of simple prescriptions. Remarkably, the predictions are in good agreement with simulations.

However, halos trace the underlying matter distribution in a biased way as can be understood via the peak-background split argument. The short density modes drive the local collapse of matter into halos while the long modes perturb the local background, enhancing the formation of objects in overdense regions. Collapsed halos are thus biased tracers of the dark matter density. Assuming a local relation between the halo overdensity and density field in the Lagrangian frame, a prediction for the halo power spectrum and bispectrum has been presented for Gaussian initial conditions, after having appropriately transformed the Lagrangian quantities into the Eulerian frame.

In the next chapter, the origin of perturbations in the CMB temperature and, in particular, in the matter density will be explained within the inflationary paradigm. Interestingly, non-Gaussian fluctuations sourced by inflation potentially leave an imprint in the clustering of halos, which is the topic of this thesis.
Chapter 4

Primordial non-Gaussianity

While in chapter 2 the properties of a homogeneous and isotropic (background) cosmology have been described, in chapter 3 it has been shown how initial matter density perturbations can lead to the formation of structures that are observed in the Universe. However, the origin of initial perturbations has been left as an open problem, which will be addressed in the first part of this chapter.

The intrinsic fluctuations on quantum scales together with the accelerated expansion that may have taken place in the very early Universe suggest the possibility of stretching tiny perturbations up to macroscopic scales, thus producing primordial fluctuations that can have measurable effects, for instance in the power spectrum of CMB temperature fluctuations or matter density.

Whether or not the primordial spectrum of perturbations is fully described by the 2-point correlation function is crucial to understand the properties of the mechanism that generated them. Any higher order connected correlators are classified as primordial non-Gaussianity (PNG), with different inflationary set-ups predicting different templates. This thesis describes the consequences that a specific kind of PNG has on LSS.

Indeed, primordial perturbations not only seed the formation of structures but also leave an imprint in their clustering which can be used to measure PNG, as will be explained in the second part of the chapter.

4.1 Inflation and cosmological perturbations

As shown in section 2.5, inflation is a powerful mechanism to flatten the universe by blowing submicroscopic scales up to macroscopic sizes. However, on quantum scales, the space-time is intrinsically inhomogeneous (for example because of the
creation and annihilation of virtual particles) and, therefore, quantum fluctuations in the inflaton field are also expanded to macroscopic scales. Once their wavelengths exceed the horizon, they remain frozen in and, as the Universe expands, will eventually re-enter the horizon, sourcing the CMB anisotropies and seeding fluctuations in the matter density that lead to structure formation, as seen in chapter 3.

By including quantum fluctuations about a homogeneous field, any light, weakly-coupled scalar field during inflation can be written as

$$\phi(t, x) = \phi_0(t) + \delta\phi(t, x) ,$$

(4.1)

where $\delta\phi$ is taken to be Gaussian distributed, an assumption that is well justified by observations (see section 4.1.1). It can be shown that the power spectrum of $\delta\phi$ on super-horizon scales ($k_1 < aH$) is

$$\langle \delta\phi(k_1)\delta\phi(k_2) \rangle \simeq (2\pi)^3 \delta_D(k_1 + k_2) P_{\delta\phi}(k_1) ,$$

(4.2)

where

$$P_{\delta\phi}(k_1) \approx \frac{H_*^2}{2k_1^3}$$

(4.3)

and $H_*$ is the Hubble scale at Hubble-exit, $k = a_*H_*$. Thus the dimensionless power spectrum is

$$P_{\delta\phi}(k) = \frac{4\pi k^3}{(2\pi)^3} P_{\delta\phi}(k) \simeq \left( \frac{H_*}{2\pi} \right)^2 .$$

(4.4)

The field fluctuation $\delta\phi$ is related to the curvature perturbation $\zeta$, whose statistics can be compared with observations. $\zeta$ is defined as the difference between uniform-expansion hypersurfaces and uniform-matter hypersurfaces, and measures the inhomogeneity. A convenient way for calculating the curvature perturbation is given by the $\delta N$ formalism, in which $\zeta$ is expressed as the difference between the local integrated expansion and the global integrated expansion [75]:

$$\zeta(t, x) = \delta N = N(t, x) - N(t) ,$$

(4.5)

where the integrated expansion $N$ was introduced in eq. (2.61). Thus, the curvature perturbation at some time $t_f$ after inflation ended is given by the integrated expansion from an initially flat hypersurface $\delta N(t_f, x) = 0$ and measures the effects of the inhomogeneous expansion due to field fluctuations $\delta\phi$ during inflation on a uniform-density hypersurface, $\rho(t_f, x) = \rho(t_f)$, after inflation. For the purpose of this simple discussion, the quantity $N$ hides all the dynamics of the field $\phi$ with the potential $V$. 

53
Since in single-field inflation $N$ depends only on $\phi$, the curvature perturbation can be written as a local expansion \cite{76,77}

$$
\zeta = N'\delta\phi + \frac{1}{2}N''\delta\phi^2 + \frac{1}{6}N'''\delta\phi^3 + \ldots ,
$$

(4.6)

with $N' \equiv dN/d\phi$. Hence the power spectrum, bispectrum and trispectrum read respectively

$$
P_\zeta(k_1) = (N')^2P_\delta(k_1) ,
$$

(4.7)

$$
B_\zeta(k_1,k_2,k_3) = \frac{6}{5}f_{\text{NL}}[P_\zeta(k_1)P_\zeta(k_2) + 2 \text{ cyc.}] ,
$$

(4.8)

$$
T_\zeta(k_1,k_2,k_3,k_4) = \frac{54}{25}g_{\text{NL}}[P_\zeta(k_2)P_\zeta(k_3)P_\zeta(k_4) + 3 \text{ cyc.}] +
\tau_{\text{NL}}[P_\zeta(k_3)P_\zeta(k_4)P_\zeta(k_{13}) + 11 \text{ cyc.}] ,
$$

(4.9)

where $k_{13} = |k_1 + k_3|$ and, at leading order, the non-linear parameters are given by

$$
f_{\text{NL}} = \frac{5}{6}\left(\frac{N''}{N'}\right)^2 , \quad \tau_{\text{NL}} = \frac{36}{25}f_{\text{NL}}^2 , \quad g_{\text{NL}} = \frac{25}{54}\left(\frac{N''}{N'}\right)^3 .
$$

(4.10)

On top of that, a possible scale dependence of $P_\zeta$ is usually parametrised by introducing the scalar index $n_s$:

$$
n_s - 1 = \frac{d\ln P_\zeta}{d\ln k} .
$$

(4.11)

The primordial non-Gaussian parameters of eq. (4.10) are clearly sourced by quadratic and higher-order terms that appear in the local expansion of eq. (4.6)\footnote{Another source would be the non-Gaussianity of the field perturbation $\delta\phi$ itself \cite{78} which will not be considered here.}

Since $N$ is model dependent, the results for selected inflationary set-ups will be quoted below.

In single-field slow-roll inflation, it can be shown that \cite{76,77}

$$
f_{\text{NL}} = \frac{5}{6}(3\epsilon - \eta) \ll 1
$$

(4.12)

in the squeezed limit of the bispectrum ($k_1 \simeq k_2 \gg k_3$). Since this is not true in other configurations (although the amplitude is still of order the slow-roll parameter), it is worth to introduce a more general definition for the non-linear parameter

$$
f_{\text{NL}}^{\text{effective}}(k_1,k_2,k_3) \equiv \frac{B_\zeta(k_1,k_2,k_3)}{2[P_\zeta(k_1)P_\zeta(k_2) + P_\zeta(k_2)P_\zeta(k_3)]} ,
$$

(4.13)

which clearly exhibits its shape-dependence in general. For local type non-Gaussianity of eq. (4.6) this reduces to the $k$-independent parameter of eq. (4.10).
The curvaton model is an alternative scenario that can potentially source a large non-Gaussianity in the squeezed limit. In this realisation the inflaton is supposed to produce negligible perturbations, while another light scalar field, the curvaton, is now responsible for producing the primordial density perturbations. During the inflating phase, the curvaton remains a spectator field but it acquires a spectrum of fluctuations on super-horizon scales [eq. (4.4)]. Once inflation ends and the Hubble rate drops below the curvaton mass scale, the curvaton starts to oscillate around its minimum and generates the perturbations. When the curvaton decays into radiation, the fluctuations in the curvaton energy density are transferred into the radiation density, with an efficiency parameter \( r \leq 1 \). The resulting PNG can then be written as eqs. (4.8) and (4.9), where
\[
f_{\text{NL}} = \frac{5}{4r} \left( 1 - \frac{4r}{3} - \frac{2r^3}{3} \right), \tag{4.14}
g_{\text{NL}} = -\frac{25}{6r} \left( 1 - \frac{r}{18} - \frac{10r^2}{9} - \frac{r^3}{3} \right). \tag{4.15}
\]
Schematically, the predictions are \(|f_{\text{NL}}| \sim |g_{\text{NL}}| \sim 1\) or \(|f_{\text{NL}}| \sim |g_{\text{NL}}| \gg 1\).

Modulated reheating is an alternative example for generating primordial perturbations at the end of inflation, by modulating the decay of the inflaton into radiation with a local function of another field and thus producing fluctuations in different regions of the Universe. In a simple realization \( f_{\text{NL}} \simeq 5 \) and \( g_{\text{NL}} \simeq 100/3 \) (or \( f_{\text{NL}} \simeq 5/2 \) and \( g_{\text{NL}} \simeq 25/3 \)) are predicted, with local-type PNG [77].

Motivated by the local expansion of eq. (4.6), the definitions of the non-Gaussian parameters [eq. (4.10)] and the previous results, one can write
\[
\zeta = \zeta_G + \frac{3}{5} f_{\text{NL}} (\zeta_G^2 - \langle \zeta_G^2 \rangle) + \frac{27}{75} g_{\text{NL}} \zeta_G^3, \tag{4.16}
\]
where \( \zeta_G \equiv N' \delta \phi \) is the Gaussian curvature perturbation, \( f_{\text{NL}} \) and \( g_{\text{NL}} \) are constant parameters and the average \( \langle \zeta_G^2 \rangle \) has been subtracted to guarantee \( \langle \zeta \rangle = 0 \). In this special case, the bispectrum peaks in the squeezed limit.

Inflationary models, like the curvaton and modulated reheating, that are well described by the previous expansion are called local-type models. Equation (4.16) is a good description of single-field, slow-roll inflation only in the squeezed limit, as in general \( f_{\text{NL}}(k_1, k_2, k_3) \) and \( g_{\text{NL}}(k_1, k_2, k_3) \) are shape-dependent, although with very small amplitudes. However, a hierarchy emerges between the different predictions for \( f_{\text{NL}} \) and \( g_{\text{NL}} \): an almost negligible amount of non-Gaussianity is produced in the single-field slow-roll inflaton, while the local-type models considered may predict non-Gaussian parameters of order one or larger.

55
All the inflationary set-ups described above are known as single-source models, since the primordial perturbations are seeded by fluctuations in a single field, and interestingly predict the quartic, non-Gaussian parameter to be determined by the second-order one: \( \tau_{NL} = \left( \frac{6}{5} f_{NL} \right)^2 \) [see eq. (4.10)], known as the Suyama-Yamaguchi equality \[79\]. This not true for multi-source models \( \tau_{NL} \geq \left( \frac{6}{5} f_{NL} \right)^2 \) and a simple example will be considered in chapter 5, where both the inflaton and the curvaton are assumed to contribute to density perturbations \[80\].

Apart from this exception, for the rest of this thesis only local-type non-Gaussianity of eq. (4.16) will be considered.

### 4.1.1 CMB constraints

Currently the most accurate constraints on inflation come from the CMB, i.e. from a 2D map at redshift \( z \approx 1100 \) whose temperature fluctuations have been measured with high accuracy.

The dimensionless power spectrum of \( \zeta \) is found to be well described by two parameters,

\[
P_\zeta(k) = \frac{4\pi k^3}{(2\pi)^3} P_\zeta(k) = A_s \left( \frac{k}{k_{\text{pivot}}} \right)^{n_s-1},
\]

where \( A_s \) is the amplitude of scalar perturbations at the pivot scale (an arbitrary scale) and \( n_s \) accounts for scale dependence [see eq. (4.11)]. The Planck data set gives \[6\]

\[
\ln(10^{10} A_s) = 3.064 \pm 0.023,
\]

\[
n_s = 0.9667 \pm 0.0040,
\]

which rules out a scale independent \( (n_s = 1) \) spectrum at more than 5\( \sigma \). The parametrization of eq. (4.17) is an ansatz but the fact that millions of CMB pixels are well described by a couple of parameters is remarkable and consistent with the inflationary paradigm of Gaussian distributed quantum fluctuations that lead to nearly Gaussian, nearly scale invariant perturbations in \( \zeta \).

So far, no detection of PNG from the bispectrum of the CMB map has been reported. WMAP9 set the constraint on local \( f_{NL} \) to be \(-3 < f_{NL} < 77\) at 95\% CL \[19\], while the most recent data from the Planck satellite experiment requires \(-9.2 < f_{NL} < 10.8\) \[81\]. This is compatible with the predictions of simple slowly-rolling single-field models which predict \( |f_{NL}| < 1 \).

The error bars could be further decreased by including the higher-resolution polarization data \[82\], but the constraining power of the CMB is close to having been fully exploited, being limited on small scales by Silk damping and on large
scales by cosmic variance. The question is then which other observables may be suitable to distinguish $f_{NL} \sim 0$ from $f_{NL} \sim 1$, i.e. able to achieve $\sigma_{f_{NL}} \lesssim 1$ – the accuracy of the determination of local $f_{NL}$. This is what theoretical predictions require in order to distinguish between the curvaton or modulated reheating scenarios and slow-roll inflaton, for instance.

Constraints on other primordial non-Gaussian parameters are in general weaker: Planck found $\tau_{NL} \leq 2800$ and $g_{NL} = (-9.0 \pm 7.7) \times 10^4$ [81]. It may be unlikely that we live in a universe with a large hierarchy between $f_{NL}$ and $g_{NL}$ [83], but this possibility cannot be excluded a priori and there are theoretical models able to predict this pattern (see for example [84] or the discussion in [85]). Since $f_{NL}$ and $g_{NL}$ control distinct features of the PDF (skewness and kurtosis) it is very important to have the best observational constraints on each of them. Complementary observables have to be considered to set more stringent bounds on these higher-order non-Gaussian parameters.

By investigating the imprint of PNG on the clustering of objects, many works have shown that LSS is certainly a good candidate to shrink the $f_{NL}$ bounds even further and, potentially, improve the constraints on other non-Gaussian parameters.

The rest of the chapter will show how to link the initial condition of eq. (4.16) to the matter density field and how the halo power spectrum is affected. The effect of PNG on the bispectrum will be the central matter of chapters 5 to 7 and the main body of the thesis.

### 4.2 Primordial non-Gaussianity in LSS

The density contrast at the start of the matter-dominated era is determined by the primordial metric perturbation $\zeta$ of eq. (4.16). In Fourier space one has [21]

$$\delta_{\text{in}}(k, z) = \frac{2k^2 c^2 T(k) D(z)}{5 \Omega_m H_0^2} \frac{g(z = 0)}{g(z = \infty)} \zeta(k),$$

(4.20)

where the transfer function $T(k)$ accounts for the damping of sub-Hubble-scale modes in the radiation era and $T(k) \to 1$ as $k \to 0$. The term $g(z = 0)/g(z = \infty)$ is the growth suppression factor between the present day ($z = 0$ or $a = 1$) and some initial time ($z = \infty$ or $a = 0$), where $g(z) = D(z)/\alpha(z)$ and $g(z = 0)/g(z = \infty) \approx 0.75$ for the currently favoured $\Lambda$CDM model [73,86]. Conventionally, eq. (4.20) is written in terms of a primordial Newtonian potential

$$\delta_{\text{in}}(k, z) = \alpha(k, z) \Phi_{\text{in}}(k),$$

(4.21)
where, from eqs. (4.20) and (4.21), one identifies

$$\Phi_{\text{in}} = \frac{3}{5} \zeta_{\text{inf}}, \quad \alpha(k, z) \equiv \frac{2k^2c^2T(k)D(z)}{3\Omega_mH_0^2} \frac{g(z = 0)}{g(z_\infty)}$$

(4.22)

and therefore the Newtonian potential with local-type non-Gaussianity is

$$\Phi_{\text{in}}(q) = \varphi_G(q) + f_{\text{NL}} \left( \varphi_G^2(q) - \langle \varphi_G^2 \rangle \right) + g_{\text{NL}} \varphi_G^3(q).$$

(4.23)

$\varphi_G(q)$ is a Gaussian field, seeded by free field fluctuations during inflation, and $f_{\text{NL}}$ and $g_{\text{NL}}$ are constant dimensionless parameters quantifying the magnitude of non-Gaussian corrections due to non-linear evolution of the primordial metric perturbation [77]. Note that the non-Gaussian correction is a local function of the Gaussian field at a given initial (Lagrangian) position, $q$. The function $\alpha(k, z)$ is plotted in fig. 4.1.

### 4.2.1 Perturbation theory revisited

Following the discussion of section 3.1.2, the first- and second-order solutions of $\delta$ and $\theta$ in the presence of local-type PNG will be now presented.

The linearly growing mode of eq. (4.21) contains first-, second- and third-order terms with respect to $\varphi_G(q)$:

$$\delta_{\text{lin}}(k, z) = \delta_G(k, z) + f_{\text{NL}} \alpha(k, z)(\varphi_G \ast \varphi_G)_k + g_{\text{NL}} \alpha(k, z)(\varphi_G \ast \varphi_G \ast \varphi_G)_k,$$

(4.24)
where the convolution $\ast$ is defined by

$$\langle \varphi_G \ast \varphi_G \rangle_k = \int \frac{dk_1}{(2\pi)^3} \varphi_G(k_1) \varphi_G(k - k_1), \quad (4.25)$$

and for convenience the first-order (Gaussian) density contrast $\delta_G(k, z)$ has been introduced. It follows from eqs. (4.21) and (4.23) that

$$\delta_G(k, z) = \alpha(k, z) \varphi_G(k). \quad (4.27)$$

Since the perturbative solutions of eqs. (3.33) and (3.34) are built using $\delta_{\text{lin}}$ as a basis, at second-order one finds [73,87]

$$\delta^{(2)}(k, z) = \int \frac{dk_1}{(2\pi)^3} \int \frac{dk_2}{(2\pi)^3} \delta^D(k - k_1 - k_2) \times$$

$$\left[ F_2(k_1, k_2) + f_{NL} \frac{\alpha(k)}{\alpha(k_1)\alpha(k_2)} \right] \delta_G(k_1, z) \delta_G(k_2, z), \quad (4.29)$$

$$\theta^{(2)}(k, z) = -f_H \int \frac{dk_1}{(2\pi)^3} \int \frac{dk_2}{(2\pi)^3} \delta^D(k - k_1 - k_2) \times$$

$$\left[ G_2(k_1, k_2) + f_{NL} \frac{\alpha(k)}{\alpha(k_1)\alpha(k_2)} \right] \delta_G(k_1, z) \delta_G(k_2, z). \quad (4.31)$$

The 3rd order kernels $F_3$ and $G_3$ also include corrections of order $g_{NL}$ and $f_{NL} F_2$.

However the cubic results are not quoted as they will not be used in this thesis.

4.2.2 Non-Gaussian mass function

When the PDF of primordial fluctuations is non-Gaussian, the Sheth-Tormen mass function [see eq. (3.115)] must be replaced with a more accurate mass function. Several mass functions that account for non-Gaussian initial conditions have been proposed in the literature (see for a concise list [72]). Throughout the thesis the Lo Verde et al (LV) mass function [88] will be considered; it is suited for a potential of the form

$$\Phi_{\text{in}} = \varphi_G(q) + f_{NL} \left( \varphi_G^2(q) - \langle \varphi_G^2 \rangle \right) + g_{NL} \left( \varphi_G^3(q) - 3\varphi_G(q)\langle \varphi_G^2 \rangle \right), \quad (4.32)$$

where the additional term $-3\varphi_G^2\langle \varphi_G^2 \rangle$ compared to eq. (4.23) has been introduced in the definition of $g_{NL}$ so that the power spectrum of $\Phi_{\text{in}}$ remains unchanged to first order in $g_{NL}$. Without this convention the amplitude of scalar fluctuations $A_s$ used to set up $\varphi_G$ would be different from the observed value of $A_s$ for $g_{NL} \neq 0$.

The LV mass function can be derived in the barrier crossing model by building up the initial PDF as a series of higher-order reduced cumulants (defined
in eq. (3.100) as \( \kappa_n = \langle \delta_{M}^{2} \rangle_{c}/\sigma^{n} \) times the Gaussian distribution: this is known as the Edgeworth expansion. In order to have a positive definite distribution, it is important to specify the order of the cumulant at which the series is truncated \[89\]. However, for the weakly non-Gaussian regime, the Edgeworth expansion converges rapidly, providing a useful approximation as long as \( 1 \gg \kappa_{3} \gg \cdots \gg \kappa_{n} \). Within this regime, the LV mass function is simply the Press-Schechter mass function plus first-order corrections in \( f_{NL} \) and \( g_{NL} \):

\[
n(M) = \frac{2\rho}{M} \frac{d\ln \sigma^{-1}}{dM} e^{-\nu^{2}/2} \left[ \nu + f_{NL} \left( \kappa_{3}^{(1)}(M) \frac{\nu H_{3}(\nu)}{6} - \frac{d\kappa_{3}^{(1)}}{dM}(M) \frac{H_{2}(\nu)}{6} \right) + g_{NL} \left( \kappa_{4}^{(1)}(M) \frac{\nu H_{4}(\nu)}{24} - \frac{d\kappa_{4}^{(1)}}{dM}(M) \frac{H_{3}(\nu)}{24} \right) \right],
\]

where \( H_{n} \) is \( n \)-th Hermite polynomial and the reduced cumulants for the smoothed density field are

\[
\kappa_{3}(M) = f_{NL} \kappa_{3}^{(1)}(M) \approx f_{NL} \left( 6.6 \cdot 10^{-4} \right) \left[ 1 - 0.016 \ln \left( \frac{M}{h^{-1}M_{\odot}} \right) \right],
\]

\[
\kappa_{4}(M) = g_{NL} \kappa_{4}^{(1)}(M) \approx g_{NL} \left( 1.6 \cdot 10^{-7} \right) \left[ 1 - 0.021 \ln \left( \frac{M}{h^{-1}M_{\odot}} \right) \right].
\]

The approximated results above are taken from \[88\]. It is important to underline that the LV mass function is not in a universal form, as an explicit dependence on the halo mass \( M \) appears. However, the cumulants \( \kappa_{3} \) and \( \kappa_{4} \) are only weakly dependent on the halo mass (see eqs. (4.34), (4.35)), so that it is reasonable to drop \( M \) and take them as constants.

Moreover, for primordial non-Gaussianity of the form of eq. (4.32), the variance needs to be replaced with

\[
\sigma^{2} = \langle \delta_{\text{lin}}^{2} \rangle = \int \frac{d^{3}p}{(2\pi)^{3}} \int \frac{d^{3}p'}{(2\pi)^{3}} W_{M}(p) \alpha(p, z) W_{M}(p') \alpha(p', z) \langle \Phi_{\text{lin}}(p) \Phi_{\text{lin}}(p') \rangle
\approx \sigma_{G}^{2} \left( 1 + \kappa_{2}(M) \right),
\]

In \[88\] a fitting function for \( \kappa_{2} \), i.e. the non-Gaussian correction to \( \kappa_{2} \), is given as well; although \( \kappa_{2} \propto f_{NL}^{2} \), it gives a negligible correction to the variance of Gaussian fluctuations,

\[
\sigma_{G}^{2} = \langle \delta_{G}^{2} \rangle = \int \frac{d^{3}p}{(2\pi)^{3}} \int \frac{d^{3}p'}{(2\pi)^{3}} W_{M}(p) \alpha(p, z) W_{M}(p') \alpha(p', z) \langle \varphi_{G}(p) \varphi_{G}(p') \rangle,
\]

for any realistic value of \( f_{NL} \) and it will be neglected\[8\].

\footnote{In general \( \kappa_{2} \propto f_{NL}^{2}/f_{NL}^{2}. \) The model of eq. (4.32) satisfies the Suyama-Yamaguchi equality: \( \tau_{NL} = (6/5)^{2} f_{NL}^{2} [79]. \)}
Comparing the result of eq. (4.33) with the universal form of the mass function [eq. (3.112)] and the PS mass function [eq. (3.114)], it follows that

\[
\begin{align*}
\mathcal{f}_{LV}(\nu) &= \mathcal{f}_{PS}(\nu) \left[ 1 + \frac{1}{6} \left( \kappa_3(M) H_3(\nu) - \frac{d\kappa_3(M)/dM}{d\ln \sigma^{-1}/dM} H_2(\nu) \right) - \frac{1}{24} \left( \kappa_4(M) H_4(\nu) - \frac{d\kappa_4/dM}{d\ln \sigma^{-1}/dM} H_3(\nu) \right) \right].
\end{align*}
\]

This mass function will be used in chapter 5. Like the PS mass function, it has been derived under the assumption of spherical collapse, but replacing \( \delta_c = 1.686 \to 1.42 \) improves the agreement with simulations.

In chapters 6 and 7, where the cubic parameter will be set to zero \( (g_{NL} = 0) \) and the non-spherical collapse effects will be considered in more details, the mass function will be taken to be

\[
\begin{align*}
\mathcal{f}(\nu) &= \mathcal{f}_{ST} \mathcal{f}_{LV} \mathcal{f}_{PS} = \mathcal{f}_{ST}(\nu) \left[ 1 + \frac{1}{6} \left( \kappa_3(M) H_3(\nu) - \frac{d\kappa_3(M)/dM}{d\ln \sigma^{-1}/dM} H_2(\nu) \right) - \frac{1}{24} \left( \kappa_4(M) H_4(\nu) - \frac{d\kappa_4/dM}{d\ln \sigma^{-1}/dM} H_3(\nu) \right) \right].
\end{align*}
\]

Equation (4.39) will be used to give an improved description of objects formed from ellipsoidal collapse with non-Gaussian initial conditions.

### 4.2.3 Peak-background split with PNG

The PBS argument of section 3.3.2 can be now applied to the Newtonian potential \( \Phi_{in} \) of eq. (4.40). Here the \( f_{NL} \)-only case will be considered,

\[
\Phi_{in} = \varphi_G(q) + f_{NL} \left( \varphi_G^2(q) - \langle \varphi_G^2 \rangle \right),
\]

and the following discussion applies to chapters 6 and 7. The non-trivial generalisation to the case when both \( f_{NL} \) and \( g_{NL} \) are present will be the focus of chapter 5.

Within the PBS approach, the first-order Gaussian part of the potential \( \varphi_G \) in eq. (4.40) is considered as a superposition of long and short modes,

\[
\varphi_G(q) = \varphi_{G,l}(q) + \varphi_{G,s}(q),
\]

which are statistically independent for a Gaussian random field. The local background is composed of long-wavelength modes \( \varphi_{G,l} \),

\[
\varphi_{G,l}(q) = \int_{k<l^{-1}} \frac{dk}{(2\pi)^3} \varphi_G(k) e^{-ik\cdot q},
\]

and acts as an approximately homogeneous background cosmology on comoving scale \( l \). On top of these the smaller scale peaks \( \varphi_{G,s} \),

\[
\varphi_{G,s}(q) = \int_{k \geq l^{-1}} \frac{dk}{(2\pi)^3} \varphi_G(k) e^{-ik\cdot q},
\]

61
lead to the collapse of dark matter into halos, on a scale $R \ll l$, when exceeding the threshold value $\delta_c$.

Substituting the peak-background split of eq. \(4.41\) into the non-Gaussian primordial potential of eq. \(4.40\), one obtains

$$
\Phi_{\text{in}}(q) = \varphi_{G,l} + f_{NL} \left( \varphi_{G,l}^2 - \langle \varphi_{G,l}^2 \rangle \right) + (1 + 2f_{NL}\varphi_{G,l}) \varphi_{G,s} + f_{NL} \left( \varphi_{G,s}^2 - [\varphi_{G,s}^2]_V \right),
$$

(4.44)

where the square brackets $[\varphi_{G,s}^2]_V$ account for a local expectation evaluated over the volume $V \sim l^3$ centred around $q$.

By Fourier transforming eq. \(4.44\) and defining the Gaussian long and short density mode respectively as $\delta_{G,l} = \alpha \varphi_{G,l}$ and $\delta_{G,s} = \alpha \varphi_{G,s}$, we can identify a "background" density perturbation

$$
\delta_{\text{lin},l}(k) = (1 + 2f_{NL}\varphi_{G,l})\delta_{G,s} + f_{NL} \left( \varphi_{G,l}^* \varphi_{G,l} - \langle \varphi_{G,l}^2 \rangle \right),
$$

(4.46)

which changes the effective collapse threshold for the small scale peaks

$$
\delta_c \rightarrow \delta_c - \delta_{\text{lin},l}.
$$

(4.47)

At the same time, local-type non-Gaussianity introduces a correlation between long and short wavelength modes in the density field. In particular the small scale density mode is affected by the long-wavelength potential, since

$$
\delta_{\text{lin},s}(k) = (1 + 2f_{NL}\varphi_{G,l})\delta_{G,s} + f_{NL} \left( \varphi_{G,l}^* \varphi_{G,l} - [\varphi_{G,s}^2]_V \right).
$$

(4.48)

Note that $\varphi_{G,l}$ is not convolved with $\delta_{G,s}$ because the long mode of $\varphi_G$ acts as an approximately homogeneous background mode on the small scale $s$. Equation \(4.48\) leads to the important conclusion that the small scale power is modulated by the long-modes of the primordial potential,

$$
\sigma_l = (1 + 2f_{NL}\varphi_{G,l})\sigma_G.
$$

(4.49)

The above discussion suggests that the number density of objects with mass $M$ at redshift $z$ in a volume $V \sim l^3$, i.e. the local mass function, depends not only on the mass and the redshift but also on the local density field and its moments $\langle \delta_{\text{lin},s}^n \rangle$,

$$
n_h = n_h(M, z, [\delta_{\text{lin},s}^n]_V),
$$

(4.50)

where
\[
[D_{\text{lin,s}}^n(V(q)] \equiv \int_{k \geq l-1} \frac{dk}{(2\pi)^3} \cdots \int_{k_n \geq l-1} \frac{dk_n}{(2\pi)^3} e^{-i(k+\cdots+k_n)\cdot q} \times \langle \delta_{\text{lin,s}}(k) \cdots \delta_{\text{lin,s}}(k_n) \rangle \, .
\] (4.51)

In the \( f_{\text{NL}}\)-only case, for a given halo mass \( M \) and redshift \( z \), eq. (4.50) simply reduces to \( n_h = n_h(\delta_{\text{lin,l}}, \sigma_l) \). The dimensionless variable \( \nu \) in the mass function thus needs to be replaced with the local effective value
\[
\nu = \frac{\delta_c}{\sigma_G} \rightarrow \nu(q) = \frac{\delta_c - \delta_{\text{lin,l}}(q)}{\sigma_l(q)} .
\] (4.52)

This result sets the basis for the Lagrangian bias model in the presence of PNG.

### 4.2.4 Bivariate model

The bias model provides an expression for the local halo overdensity in Lagrangian coordinates
\[
\delta_L^h(q) = \frac{n_h(q) - \langle n_h \rangle}{\langle n_h \rangle} .
\] (4.53)

Using the preceding arguments the local halo overdensity is expected to be a function of the large-scale linearly growing mode of the density field, \( \delta_{\text{lin,l}} \) (the local “background” density), and the small-scale variance of the linearly growing mode, \( \sigma_l \) (the “peaks”), averaged over the same large scale (\( l \) in the preceding section). Thus by Taylor expanding eq. (4.50) up to second order\(^3\) in terms of \( \delta_{\text{lin,l}} \) and \( \sigma_l \)
\[
\delta_L^h(q) = \beta_{10} \delta_{\text{lin,l}} + \beta_{01} \left( \frac{\sigma_l}{\sigma} - 1 \right) + \frac{1}{2} \left[ \beta_{20} (\delta_{\text{lin,l}})^2 + \beta_{02} \left( \frac{\sigma_l}{\sigma} - 1 \right)^2 + 2 \beta_{11} \delta_{\text{lin,l}} \left( \frac{\sigma_l}{\sigma} - 1 \right) \right] ,
\] (4.54)

where the bias coefficients are defined as
\[
\beta_{ij} \equiv \left[ \frac{(\sigma_l)^j n_h}{n_h} \left( \frac{\partial^i \delta_{\text{lin,l}}}{\partial n_h} \right)^i \left( \frac{\sigma_l}{\sigma} \right)^j \right] \bigg|_{\delta_{\text{lin,l}} = 0, \sigma_l = \sigma} .
\] (4.55)

Hereafter the subscript \( l \) will be dropped, and \( \delta_{\text{lin}} \) and \( \varphi_G \) indicate the long-wavelength modes of the linearly growing density contrast and the Gaussian primordial potential respectively.

The small-scale variance of eq. (4.49) depends on the local primordial potential \( \varphi_{G,l} \) and hence the Taylor expansion can be written in the bivariate form of [72,73],
\[
\delta_L^h(q) = b_{10}^h \delta_{\text{lin}} + b_{01}^h \varphi_G + b_{20}^h (\delta_{\text{lin}})^2 + b_{11}^h \delta_{\text{lin}} \varphi_G + b_{02}^h \varphi_G^2 ,
\] (4.56)

\(^3\)Note that to compute the tree-level bispectrum we need quantities up to \( \mathcal{O}(\delta^2) \).
where the Lagrangian bias coefficients are identified to be
\[
\begin{align*}
\beta^{L}_{10} &= \beta_{10}, \\
\beta^{L}_{01} &= 2f_{NL}\beta_{01}, \\
\beta^{L}_{20} &= \frac{\beta_{20}}{2}, \\
\beta^{L}_{11} &= 2f_{NL}\beta_{11}, \\
\beta^{L}_{02} &= 2f_{NL}^{2}\beta_{02}.
\end{align*}
\]

Assuming the Sheth-Tormen mass function corrected for non-Gaussian initial conditions [eq. (4.39)] allows us to get explicit formulas for the Lagrangian bias coefficients; only two of them are independent:
\[
\begin{align*}
\beta^{L}_{10} &= \frac{\nu^{2} - 1}{\delta_{c}} + \frac{2p}{1 + (\nu^{2})^{p}} \frac{\nu^{3} - \nu}{\delta_{c}} + \frac{d\kappa_{3}/dM}{d\ln \sigma^{-1}/dM} \frac{\nu + \nu^{-1}}{6\delta_{c}}, \\
\beta^{L}_{20} &= \frac{\nu^{2} \gamma^{2} - 3}{2 \delta_{c}^{2}} + \frac{p}{1 + (\nu^{2})^{p}} \frac{2 \gamma^{2} + 2p - 1}{\delta_{c}^{2}} - \frac{\kappa_{3}}{2} \left[ \frac{\gamma^{2} - (\gamma + 2)\nu^{3} + \nu}{\delta_{c}^{2}} + \frac{2p}{1 + (\nu^{2})^{p}} \frac{\nu^{3} - \nu}{\delta_{c}^{2}} \right] + \frac{1}{2} \frac{d\kappa_{3}/dM}{d\ln \sigma^{-1}/dM} \left[ \frac{\nu^{3} + (\gamma - 1)\nu}{3\delta_{c}^{2}} + \frac{2p}{1 + (\nu^{2})^{p}} \frac{\nu - \nu^{-1}}{3\delta_{c}^{2}} \right],
\end{align*}
\]
where \(\gamma \) and \(p\) were introduced in eq. (3.115). The other bias coefficients can be written in terms of these two by noting that from eq. (4.55) for a universal mass-function one finds \(\beta_{01} = \delta_{c}\beta_{10} \), etc. Thus the remaining coefficients are obtained from the following combinations
\[
\begin{align*}
\beta^{L}_{01} &= 2f_{NL}\delta_{c}b^{L}_{10}, \\
\beta^{L}_{11} &= 2f_{NL}(\delta_{c}b^{L}_{20} - b^{L}_{10}), \\
\beta^{L}_{02} &= 4f_{NL}^{2}\delta_{c}(\delta_{c}b^{L}_{20} - 2b^{L}_{10}).
\end{align*}
\]
In eq. (4.58) the first two terms are the usual Gaussian bias while the last two are a scale-independent correction introduced by PNG [92, 93], which has been shown to improve the comparison between theory and simulations [94]. The same structure is found for eq. (4.59).

### 4.2.5 Scale-dependent bias in the halo power spectrum

The Lagrangian halo overdensity can now be used to predict the tree-level halo power spectrum, once appropriately transformed into the Eulerian frame, as
explained in section 3.3.3:

\[ 1 + \delta_{Eh}(x) = (1 + \delta(x))(1 + \delta_{Lh}(x)) \]
\[ \simeq (1 + \delta_G(x))(1 + b_{E10}^L \delta_G(x) + b_{01}^L \varphi_G(x)) \quad (4.62) \]
\[ \simeq 1 + (1 + b_{E10}^L) \delta_G(x) + b_{01}^L \varphi_G(x). \]

Working in Fourier space, the Gaussian field \( \varphi_G \) can be replaced by using eq. (4.27), so that \( \varphi_G = \delta_G/\alpha \) and then

\[ \delta_{Eh}(k) \simeq \left( b_{E10}^L + \frac{b_{01}^L}{\alpha(k)} \right) \delta_G(k). \quad (4.63) \]

The tree-level halo-matter and halo-halo power spectrum immediately follow

\[ P_{hm}(k,z) = \left( b_{E10}^L + \Delta b_{E10}^L \right) P(k,z), \quad (4.64) \]
\[ P_{hh}(k,z) = \left( b_{E10}^L + \Delta b_{E10}^L \right)^2 P(k,z), \quad (4.65) \]

where

\[ b_{E10}^L \equiv 1 + b_{E10}^L, \quad (4.66) \]
\[ \Delta b_{E10}^L \equiv \frac{b_{01}^L}{\alpha(k)} = 3f_{NL}(b_{E10}^L - 1) \delta_c \frac{\Omega_m}{k^2 T(k) D(z)} \left( \frac{H_0}{c} \right)^2. \quad (4.67) \]

The analytic expression for \( \Delta b_{E10}^L \) was derived for the first time in [95] and later confirmed in [92] within the same framework: this is known as scale-dependent bias. The same result has been derived in [96], starting from the formula for the 2-point correlation function of regions above a high threshold [97], and considering only high peaks and large separation between halos. Note that in eq. (4.67) the bias \( b_{E10}^L \) is the sum of the Gaussian bias and a scale-independent correction due to PNG (see eq. (4.58) and the subsequent discussion), as pointed out in [86]. In all the previous works, the scale-independent correction to \( b_{E10}^L \) was mistakenly not considered in the definition of \( \Delta b_{E10}^L \), indeed it can affect the scale-dependence of the bias and lead to a small asymmetry between the predictions for \( f_{NL} > 0 \) and \( f_{NL} < 0 \).

Figure 4.2 shows the effect of the scale-dependent bias generated by the potential of eq. (4.40) on the power spectrum, at large scales (or small wavenumbers). The halo-matter power spectrum at \( k \simeq 0.01 \ h \text{Mpc}^{-1} \) more than doubles for \( f_{NL} = 500 \) and \( b_{E10}^L = 3.25 \) with respect to the Gaussian case \( (f_{NL} = 0) \), while negligible differences arise at \( k \simeq 0.1 \ h \text{Mpc}^{-1} \).

The pioneering result of Dalal et al [95] showed that the LSS power spectrum offers a way to constrain PNG, independently of the CMB. This result is due to the mode coupling in local-type models, as seen in section 4.2.3, the short
Figure 4.2: In the upper panel, the plot shows the halo-matter power spectrum $P_{hm}$ for objects of mass $1.6 \times 10^{13} M_\odot < M < 3.2 \times 10^{13} M_\odot$ at redshift $z = 1$ and various $f_{NL}$. The lines represent the theoretical prediction for $P_{hm}$ with $b_{10}^E = 3.25$ against $N$-body simulations (points). A strong scale-dependence arises at low wavenumbers (or large scales). The bottom panel shows the ratio $(b_{10}^E + \Delta b_{10}^E)/b_{10}^E$. Figure taken from [95].

Wavelength modes which drive the collapse of matter into halos are modulated by the long wavelength modes, effectively changing the short-mode variance from patch to patch of the sky. This introduces a scale-dependent correction in the bias model, which describes how the number of halos relates to the matter density. Since the correction is proportional to $f_{NL}/k^2$, the clustering of structures on large scales has the potential to discriminate among different models of inflation.

This new perspective on the subject has been studied in depth in a number of works [92, 94, 96, 98, 100]. However, constraining $f_{NL}$ from real data requires detailed understanding of the systematic errors; indeed, some of them mimic the PNG excess of power on large scales (see for instance [101,103]). Different techniques have been proposed to handle these errors [102,104,106], as well as methods to reduce the effect of cosmic variance [107,112] and the statistical uncertainties [113,114].

Constraints comparable with the WMAP bounds have already been achieved using the power spectrum of LSS [92,104,106,115,119] and even more stringent results will come thanks to the next generation of LSS experiments, such as Eu-
clid and SKA \cite{112,120,122}, despite the fact that no survey has been optimized for PNG so far \cite{123}. The novel technique introduced in \cite{124} may optimistically provide $\sigma_{f_{NL}} \sim 1$.

The effect of PNG on the LSS bispectrum is the topic of the following chapters.

### 4.3 Summary

In this chapter a concise description of how quantum fluctuations in the inflaton field (or in a field other than the inflaton) can generate the curvature perturbations responsible for both the CMB anisotropies and the matter density fluctuations have been introduced. No matter if the field fluctuations are Gaussian distributed or not, quantum fluctuations give rise to a model-dependent amount of non-Gaussianity in the primordial curvature perturbations, as shown by the $\delta N$ formalism.

The simple application considered at the beginning of the chapter is sufficient to justify the local-type expansion of the primordial Newtonian potential [see eq. (4.23)], which will be the ansatz for the initial conditions throughout the rest of this thesis.

Measuring the amount of PNG is of crucial importance in order to unveil the properties of the inflationary phase. Nowadays the best constraints come from the CMB and are compatible with a very small PNG as predicted by single-field slow-roll inflation. However, current bounds are not tight enough yet to rule out alternative models, like the curvaton and modulated reheating. Since the CMB constraining power has been almost fully exploited, other observable are needed.

Interestingly, the PBS argument makes evident a specific mode coupling in local-type models of PNG that affects structure formation by changing the local threshold for collapse and, in addition, modulating the small-scale variance from patch to patch of the sky. This introduces a large-scale correction proportional to $f_{NL}/k^2$ in the clustering of halos, which is manifest in the halo/galaxy power spectrum, as understood within the bivariate bias model.

This discovery provides a way to constrain PNG that is complementary to the CMB and many works indicate that future surveys will further decrease the error bars.

However there are reasons to consider also higher-order statistics as it will be fully explained in the following chapters.
Chapter 5

Primordial non-Gaussianity in the bispectra of LSS

As discussed in the preceding chapters, the statistics of large-scale structure in the Universe can be used to probe non-Gaussianity of the primordial density field, complementary to existing constraints from the cosmic microwave background.

In particular, the scale dependence of halo bias derived in chapter 4 affects the halo distribution at large scales, which represents a promising tool for analysing primordial non-Gaussianity of local form. Future observations, for example, may be able to constrain the trispectrum parameter $g_{NL}$ that is difficult to constrain using the CMB alone.

However it turns out to be challenging to disentangle the contributions from different non-Gaussian (nG) parameters using only halo and matter power spectra, since the characteristic scale dependence of the bias is primarily sensitive to a particular combination of the $f_{NL}$ and $g_{NL}$ parameters [125,126], as will be shown later. Mild corrections associated with the redshift dependence of the halo mass function have been used to set the first LSS constraints on $g_{NL}$ in [117], but it is difficult to convincingly distinguish the effect of $f_{NL}$ from that of $g_{NL}$ using only galaxy power spectra.

A promising method that may break the degeneracy between $f_{NL}$ and $g_{NL}$ is to study the bispectra of halo and matter densities, which are sensitive to a non-linear bias parameter that depends specifically on $g_{NL}$, and allows us to break the aforementioned degeneracy. Jeong and Komatsu [42] (see also [127]) were the first among various groups to include the scale dependence of halo bias when studying bispectra, by considering the non-linear evolution of the halo
overdensity in a local, Eulerian bias expansion, where the halo abundance is taken to be a function of the local matter density. They have shown that galaxy bispectra are sensitive to non-Gaussian parameters beyond $f_{NL}$, as confirmed by $N$-body simulations \textsuperscript{[128,129]}.

In this chapter, it will be shown that galaxy and matter bispectra have interesting qualitative features that may allow us to distinguish between the two non-Gaussian parameters $f_{NL}$ and $g_{NL}$. They are derived within the PBS approach and under the hypothesis that the number density of halos forming at a given position is a function of the local matter density contrast and of its local higher-order statistics.

In order to focus only on the consequences of primordial non-Gaussian initial conditions on the properties of bispectra, the effects of non-linear gravitational clustering will not be included in this chapter; that is, we work in Lagrangian space and linearly transform to Eulerian space. Non-linearity in the halo mass function are also neglected, i.e. non-linear local bias, which implies that any non-vanishing bispectra will be solely due to PNG.

The resulting bispectra contain several contributions scaling with different powers of the scale $k$, weighted by coefficients depending on primordial non-Gaussian parameters. In the limit of large scales, in which analytic expressions for the transfer functions can be derived, the qualitative features of bispectra that make it possible, at least in principle, to distinguish the effects of $f_{NL}$ and $g_{NL}$ will be discussed. For example, by studying the properties of the bispectra as a function of the scale, a connection between the sign of the halo bispectrum and nG parameters, in particular the value of $g_{NL}$, exists. Moreover, a combination of halo and matter bispectra that is sensitive to $f_{NL}$ only will be presented, as it allows us to probe $f_{NL}$ without contaminations from $g_{NL}$.

All these results, obtained in the case of a single source for the primordial gravitational potential, will be extended to a two-field example, allowing us to understand how multiple sources contribute to galaxy bispectra.

In \textsuperscript{[80,90]} it was pointed out that if multiple fields source the primordial gravitational potential, then halo overdensities need not be fully correlated with matter overdensities. In what follows, it will be shown that a combination of halo and matter power spectra is particularly convenient for studying the phenomenon of stochastic halo bias. This concept will be extended in the presence of PNG to the cases of both power spectra and bispectra, including the effect of $g_{NL}$.

The fact that this stochastic bias is non-vanishing can be interpreted in terms of inequalities satisfied by primordial non-Gaussian parameters, and these
provide information on the number of fundamental fields sourcing PNG. A new combination of halo and matter bispectra can be used to distinguish the effect of single and multiple sources on the primordial gravitational potential.

In the second part of the chapter, these theoretical findings will be applied to a systematic numerical analysis of bispectra in appropriate squeezed limits, by using LV halo mass function and considering scales and redshifts that can be probed in present or future surveys. By plotting the resulting halo bispectra, the different roles played by $f_{\text{NL}}$ and $g_{\text{NL}}$ will be explored. The numerical analysis confirms the analytical results: the qualitative features of the profiles of halo and matter bispectra as a function of the scale are sensitive to different nG parameters, possibly allowing to test them individually.

These results clearly demonstrate how galaxy bispectra have features that could distinguish between different non-Gaussian parameters, in a way that is not possible by studying only the scale dependence of galaxy bias in power spectra of halos and matter.

5.1 The power spectrum and the bispectrum: the single source case

Let’s start by considering the single source case, in which the local expansion of the primordial gravitational potential up to third order in terms of the Gaussian field $\varphi_G$ is (see the discussion in section 4.2.2)

$$\Phi_{\text{in}} = \varphi_G(q) + f_{\text{NL}} (\varphi_G^2(q) - \langle \varphi_G^2 \rangle) + g_{\text{NL}} (\varphi_G^3(q) - 3 \varphi_G(q) \langle \varphi_G^2 \rangle).$$  \hspace{1cm} (5.1)

As shown in section 4.2.3, the first-order Gaussian potential $\varphi_G$ can be considered as a superposition of long and short modes, $\varphi_G = \varphi_{G,l} + \varphi_{G,s}$, with respect to a fiducial scale $l$. Considering a sub-volume $V \sim l^3$ of the Universe large enough to contain many halos, over which the long mode is reasonably constant, then

$$\langle \varphi_G \rangle = 0,$$ \hspace{1cm} (5.2)

$$[\varphi_G]_V = \varphi_{G,l},$$ \hspace{1cm} (5.3)

where $[\cdot]_V$ denotes the spatial average over the subvolume and $\langle \cdot \rangle$ is the spatial average over the entire Universe.

After implementing the PBS in eq. (5.1), one finds in the Lagrangian space

$$\Phi_{\text{in}}(q) = \varphi_{G,l} + f_{\text{NL}} (\varphi_{G,l}^2 - \langle \varphi_{G,l}^2 \rangle) + g_{\text{NL}} (\varphi_{G,l}^3 - 3 \varphi_{G,l} \langle \varphi_{G,l}^2 \rangle) + [1 + 2f_{\text{NL}} \varphi_{G,l} + 3g_{\text{NL}} (\varphi_{G,l}^2 - \langle \varphi_{G,l}^2 \rangle)] \varphi_{G,s}$$
The previous expression demonstrates how the long wavelength mode modulates the gravitational potential. The coefficients of the different powers of \( \phi_{G,s} \), which control the statistics of this quantity within the subvolume \( V \), receive contributions depending on the long mode \( \phi_{G,l} \), and this acts as a background quantity from the point of view of the subvolume \( V \). In light of this, we can define the small scale effective power \( \sigma_l = [\Phi^2]^{1/2}_V \) and the effective non-Gaussian parameters \( f_{NL}^{l} \) and \( g_{NL}^{l} \) as

\[
\begin{align*}
\sigma_l &= [\phi_{G,s}^2]^{1/2}_V \left( 1 + 2f_{NL} \phi_{G,l} + 3g_{NL}(\phi_{G,l}^2 - \langle \phi_{G,l}^2 \rangle) \right), \\
f_{NL}^l &= f_{NL} + 3g_{NL} \phi_{G,l}, \\
g_{NL}^l &= g_{NL},
\end{align*}
\] (5.5)

so we learn that long wavelength modes and PNG affect the two point statistics of the short modes, as well as the non-Gaussian parameters inside the small box. The long mode dependence of eqs. (5.5) and (5.6) will be used to estimate the number of halos at a given position.

There are various approaches to the problem of quantifying how long modes contribute to the process of halo formation. The conceptually simplest one is to use a local, Eulerian bias model (discussed in section 3.3.2), in which the number of halos is expressed as a power series of \( \delta \), the matter density contrast (see for instance \[42\]). This method includes the effects of the long mode \( \phi_{G,l} \) of the field sourcing the gravitational potential in the halo distribution, as much as \( \phi_{G,l} \) modulates \( \delta \). A physical issue with this approach is that, strictly speaking, the number of halos does not depend only on the local value of the matter density contrast at a given position \( x \). Indeed, it also depends on how the matter density contrast is distributed around \( x \) – that is, on its correlation functions evaluated at \( x \).

In the presence of local non-Gaussianity, these correlation functions are affected by the long mode \( \phi_{G,l} \) of the primordial field that sources the gravitational potential, in a way that is not completely described by the dependence of \( \delta \) on \( \phi_{G,l} \). Indeed eqs. (5.5) and (5.6) suggest that the number of halos at a given position, \( n_h(x) \), does not depend on the matter density contrast only, but also on all its moments

\[
n_h = n_h (\delta(x), [\delta^n]_V(x)) ,
\] (5.8)

where \( [\delta^n]_V(x) \) denotes the \( n \)-th moment \( \langle \delta_1 \ldots \delta_n \rangle \) evaluated over a volume of size \( V \) around the point \( x \).
We now perform a Taylor expansion of the halo density contrast at first order in each of the arguments of \( n_h \) in eq. (5.8). Including higher-order terms in the expansion would induce further non-linearities on the long mode dependence of \( n_h \), and thus rely on the non-linear dependence of this function on the matter density contrast and its correlations. Since we intend to focus our attention on the specific non-linearities associated with PNG only, these contributions will be discarded, although it would be interesting to include them in a more complete analysis (see however section 5.4 for some discussion on this point). For the same reason, we do not include the effects of non-linear gravitational clustering in our analysis: that is, we work in Lagrangian space and linearly transform to Eulerian space. As will be shown, this procedure leads to manageable expressions for the bispectra, that can be used to derive interesting physical consequences.

Therefore, at very large scales, the halo overdensity is the Taylor expansion of the following function

\[
\delta_h(q) = \frac{n_h(\delta_{\text{lin},l}, \sigma_l, f_{NL}^l, g_{NL}^l) - \langle n_h \rangle}{\langle n_h \rangle},
\]

evaluated at \((\delta_{\text{lin},l} = 0, \sigma, f_{NL}, g_{NL})\). Its Fourier transform reads

\[
\delta_h = b_g \left[ \delta_{G,l} + \alpha f_{NL} (\varphi_{G,l} \ast \varphi_{G,l} - \langle \varphi_{G,l}^2 \rangle) + \alpha g_{NL} (\varphi_{G,l} \ast \varphi_{G,l} \ast \varphi_{G,l} - 3 \varphi_{G,l}^3) \right] + \frac{\beta_2}{2} \left( \frac{\sigma_l}{\langle \sigma^2 \rangle^{1/2}} - 1 \right) + \frac{\beta_3}{3} (f_{NL}^l - f_{NL}) + \frac{\beta_4}{4} (g_{NL}^l - g_{NL}),
\]

where the terms inside the square bracket in the first two lines give the linearly growing long mode of the smoothed density field [cf. eq. (4.24)],

\[
\delta_{\text{lin},l} = \alpha \Phi_{\text{lin}} = \delta_{G,l} + \alpha f_{NL} (\varphi_{G,l} \ast \varphi_{G,l} - \langle \varphi_{G,l}^2 \rangle) + \alpha g_{NL} (\varphi_{G,l} \ast \varphi_{G,l} \ast \varphi_{G,l} - 3 \varphi_{G,l}^3),
\]

and with the standard bias \( b_g \) evaluated at \((\delta_{\text{lin},l} = 0, \sigma, f_{NL}, g_{NL})\) by taking derivatives along the long wave-length mode \( \delta_{\text{lin},l} \),

\[
b_g = \frac{\partial \ln n_h}{\partial \delta_{\text{lin},l}}.
\]

Moreover, at the same point

\[
\beta_2 = 2 \frac{\partial \ln n_h}{\partial \ln \sigma_l}, \quad \beta_3 = 3 \frac{\partial \ln n_h}{\partial f_{NL}^l}, \quad \beta_4 = 4 \frac{\partial \ln n_h}{\partial g_{NL}^l}.
\]

The quantities of eqs. (5.12) and (5.13) depend on the specific halo mass function one considers: the theoretical expressions for the \( \beta_i \) coefficients will be discussed
In the following analysis, these parameters can be considered free. Note that, for convenience, the definitions of bias coefficients used in this chapter differ from those introduced in chapters 3 and 4.

Using formulae of eqs. (5.5) to (5.7), eq. (5.10) becomes

\[
\delta_h = \delta_{G,l} \left( b_g + \frac{\beta_2 f_{NL}}{\alpha} + \frac{\beta_3 g_{NL}}{\alpha} \right) + \left( \varphi_{G,l} \ast \varphi_{G,l} - \langle \varphi_{G,l}^2 \rangle \right) \left( b_g f_{NL} + \frac{3}{2} \beta_2 g_{NL} \right) \]

\[
+ \left( \varphi_{G,l} \ast \varphi_{G,l} \ast \varphi_{G,l} - 3 \varphi_{G,l} \langle \varphi_{G,l}^2 \rangle \right) \left( b_g g_{NL} \right). \tag{5.14}
\]

These results can be collected in the convenient expression

\[
\delta_h = b_1 \delta_{G,l} + b_2 \left( \varphi_{G,l} \ast \varphi_{G,l} - \langle \varphi_{G,l}^2 \rangle \right) + b_3 \left( \varphi_{G,l} \ast \varphi_{G,l} \ast \varphi_{G,l} - 3 \varphi_{G,l} \langle \varphi_{G,l}^2 \rangle \right), \tag{5.17}
\]

where the new bias parameters \(b_i\) are defined as

\[
b_1 \equiv b_g + \frac{\beta_2 f_{NL}}{\alpha} + \frac{\beta_3 g_{NL}}{\alpha}, \tag{5.18}
\]

\[
b_2 \equiv 2\alpha b_g f_{NL} + 3\beta_2 g_{NL}, \tag{5.19}
\]

\[
b_3 \equiv 6\alpha b_g g_{NL}. \tag{5.20}
\]

The quantity \(b_1\) corresponds to the linear bias. It receives a contribution due to PNG, that scales as \(1/\alpha \sim 1/k^2\) at large scales: this is the well-known scale dependence of halo bias due to PNG, introduced in section 4.2.5. In this large scale limit, the linear bias function \(b_1\) depends on a particular combination of \(f_{NL}\) and \(g_{NL}\), and cannot distinguish between these two quantities. On the other hand, the bias \(b_2\) depends specifically on \(g_{NL}\). In what follows, we study the bispectrum of halos and matter, which depend on \(b_2\): this can provide unambiguous information on \(g_{NL}\), allowing us to distinguish it from \(f_{NL}\).

5.1.1 The two point function of halo and matter densities

We adopt the following definitions for the power spectra associated with the two point functions of halo and matter densities:

\[
P_{mm}(2\pi)^3 \delta(k_1 + k_2) = \langle \delta_m \delta_m \rangle, \tag{5.21}
\]

\[
P_{hm}(2\pi)^3 \delta(k_1 + k_2) = \frac{\langle \delta_h \delta_m \rangle + \langle \delta_m \delta_h \rangle}{2}, \tag{5.22}
\]

\[
P_{hh}(2\pi)^3 \delta(k_1 + k_2) = \langle \delta_h \delta_h \rangle. \tag{5.23}
\]
We define the halo-halo power spectrum assuming that the shot-noise contribution $1/n_h$ has been subtracted, and analogously for the matter-halo and matter-matter power spectrum. Using the formula for the linearly-growing mode $\delta_{\text{lin}}$ [eq. (5.11)], the halo overdensity $\delta_h$ of eq. (5.17) and the linear, primordial power spectrum $\langle \phi_{G,l} \phi_{G,l} \rangle \equiv \Delta_0 / k^3$, we find

$$P_{mm} = \frac{\Delta_0 \alpha^2}{k^3}, \quad (5.24)$$

$$P_{hm} = \frac{b_1 \Delta_0 \alpha^2}{k^3}, \quad (5.25)$$

$$P_{hh} = \frac{b_2 \Delta_0 \alpha^2}{k^3}. \quad (5.26)$$

In the limit of very large scales, we can express the previous expressions for the power spectra as

$$P_{mm} = \Delta_0 \alpha_0^2 k, \quad (5.27)$$

$$P_{hm} = k \left( \alpha_0 b_g + \frac{\beta_2 f_{NL}}{k^2} + \frac{\beta_3 g_{NL}}{k^2} \right) \Delta_0 \alpha_0, \quad (5.28)$$

$$P_{hh} = k \left( \alpha_0 b_g + \frac{\beta_2 f_{NL}}{k^2} + \frac{\beta_3 g_{NL}}{k^2} \right)^2 \Delta_0, \quad (5.29)$$

neglecting the terms that are subleading in powers of $k$, and using

$$\alpha(k, z) = \frac{2k^2 T(k) D(z)}{3 \Omega_m H_0^2} \simeq \frac{2k^2 D(z)}{3 \Omega_m H_0^2} = \alpha_0(z) k^2 \quad (5.30)$$

where

$$\alpha_0(z) \simeq 2.16 \cdot 10^7 \left( \frac{0.277 D(z)}{\Omega_m 0} \right) \left( \frac{\text{Mpc}}{h} \right)^2, \quad (5.31)$$

which also holds specifically at large scales ($k \lesssim 0.01 h \text{ Mpc}^{-1}$). Notice the famous scale dependence of halo bias induced by PNG in the halo-halo power spectrum. On the other hand, the primordial non-Gaussian parameters $f_{NL}$ and $g_{NL}$ appear on the same footing in the previous expressions, rendering them difficult to disentangle. In order to overcome this degeneracy, in the following we will consider the statistics of the three-point function and study the bispectra of halo and matter density contrasts.

### 5.1.2 The three point functions of halo and matter densities

The three point functions are defined as

$$B_{mmm} (2\pi)^3 \delta(k_1 + k_2 + k_3) = \langle \delta_m \delta_m \delta_m \rangle, \quad (5.32)$$

1For simplicity we assume the scalar index $n_s = 1$.

2As for the power spectra, we assume that shot-noise contributions are removed.
obtaining at leading order (tree-level)

\[
B_{mmm} = 2f_{\text{NL}}\Delta_0^2\alpha^{(1)}\alpha^{(2)}\alpha^{(3)}\left(\frac{1}{k_1^2k_2^2} + \frac{1}{k_1^2k_3^2} + \frac{1}{k_2^2k_3^2}\right), \tag{5.36}
\]

\[
B_{hmm} = \frac{1}{3}\left\{\Delta_0^2\alpha^{(2)}\alpha^{(3)}\left[2f_{\text{NL}}\alpha^{(1)}b^{(1)}_1\left(\frac{1}{k_1^2} + \frac{1}{k_3^2}\right) + \frac{b^{(1)}_2}{k_2^3k_3^3}\right] + \text{perm.}\right\}, \tag{5.37}
\]

\[
B_{hhm} = \frac{1}{3}\left\{\Delta_0^2\alpha^{(3)}\left[2f_{\text{NL}}\alpha^{(1)}b^{(2)}_1\left(\frac{1}{k_1^2} + \frac{1}{k_2^2}\right) + \frac{b^{(1)}_2}{k_2^3k_3^3}\right] + \text{perm.}\right\}, \tag{5.38}
\]

\[
B_{hhh} = \Delta_0^2\left(b^{(1)}_2\alpha^{(2)}\frac{b^{(2)}_1}{k_2^2} + b^{(2)}_1\alpha^{(2)}\frac{b^{(1)}_2}{k_2^3} + b^{(2)}_2\alpha^{(2)}\frac{b^{(1)}_3}{k_2^3}\right) + \frac{b^{(3)}_2}{k_2^3k_3^3}, \tag{5.39}
\]

where \(\alpha^{(j)}\equiv \alpha(k_j, z)\) and \(b^{(j)}_i\equiv b_i(k_j, z)\). Notice that the scale dependence of \(b^{(j)}_i\) is due to PNG, and this induces a dependence on \(\alpha^{(j)}\) for these quantities [see eqs. (5.18) and (5.20)]. In the previous formulae, we focussed only on tree-level contributions to the bispectra, neglecting so-called “loop” effects. For this reason, the bias parameter \(b_3\) does not appear. Focussing on the halo bispectrum, eq. (5.39), recall that the bias parameters \(b_1\) and \(b_2\) depend on PNG [see eqs. (5.18) and (5.19)].

**Isosceles triangles and the squeezed limit in momentum space**

We focus on isosceles triangle configurations, considering a squeezed limit in which primordial local nG contributions yield the largest signal [42]. We now follow the convention of Jeong and Komatsu by setting \(k = k_1 = k_2 = \epsilon k_3\); in other words, we consider isosceles triangles in momentum space. The squeezed limit, then, corresponds to configurations in which \(\epsilon \gg 1\). The previous expressions [eqs. (5.36) and (5.39)] become

\[
B_{mmm} = \frac{2f_{\text{NL}}\Delta_0^2\alpha(k)^2\alpha(k/\epsilon)}{k^6} \left(1 + 2\epsilon^3\right), \tag{5.40}
\]
\[ B_{hmm} = \frac{\Delta^2_3 \alpha(k)}{3 k^6} \left\{ 4 f_{\text{NL}} \left[ \left( 1 + \epsilon^2 \right) \alpha \left( \frac{k}{\epsilon} \right) \alpha(k) b_1(k) + c^2 \alpha(k) \alpha \left( \frac{k}{\epsilon} \right) b_1 \left( \frac{k}{\epsilon} \right) \right] \\
+ 2 c^2 \alpha \left( \frac{k}{\epsilon} \right) b_2(k) + \alpha(k) b_2 \left( \frac{k}{\epsilon} \right) \right\}, \]  
(5.41)

\[ B_{hhm} = \frac{\Delta^2_3}{3 k^6} \left\{ 2 f_{\text{NL}} \left[ \alpha \left( \frac{k}{\epsilon} \right) \alpha(k) b_1(k)^2 + 2 c^2 \alpha(k)^2 \alpha \left( \frac{k}{\epsilon} \right) b_1(k) b_1 \left( \frac{k}{\epsilon} \right) \right] \\
+ 2 c^2 \alpha \left( \frac{k}{\epsilon} \right) \alpha(k) b_1(k) b_2(k) + \right. \\
+ 2 \alpha(k) \left( \epsilon^2 \alpha \left( \frac{k}{\epsilon} \right) b_1 \left( \frac{k}{\epsilon} \right) b_2(k) + b_2 \left( \frac{k}{\epsilon} \right) \alpha(k) b_1(k) \right\}, \]  
(5.42)

\[ B_{hhh} = \frac{\Delta^2_3 \alpha(k) b_1(k)}{k^6} \left\{ 2 c^2 \alpha \left( \frac{k}{\epsilon} \right) b_1 \left( \frac{k}{\epsilon} \right) b_2(k) + b_2 \left( \frac{k}{\epsilon} \right) \alpha(k) b_1(k) \right\}, \]  
(5.43)

where we keep all contributions. Expanding the previous functions, one finds at large scales \( k \ll 1 \) and in squeezed limit \( \epsilon \gg 1 \), using \( \alpha(k) \approx \alpha_0 k^2 \) and thus the approximation \( T(k/\epsilon) \approx T(k) \) for \( k \lesssim 0.01 h \text{ Mpc}^{-1} \),

\[ B_{mmm} \simeq 4 f_{\text{NL}} \Delta^2_3 \alpha_0^2 \epsilon, \]  
(5.44)

\[ B_{hmm} \simeq 4 f_{\text{NL}} \Delta^2_3 \alpha_0^2 b_0^2 \epsilon + \frac{2 \Delta^2_3 \alpha_0^2}{3 k^2} \left[ (\beta_2 (3 g_{\text{NL}} \epsilon + 2 f_{\text{NL}}^2 \epsilon^3) + 2 f_{\text{NL}} g_{\text{NL}} \beta_3 \epsilon^3) \right], \]  
(5.45)

\[ B_{hhm} \simeq 4 f_{\text{NL}} \Delta^2_3 \alpha_0^2 b_0^2 \epsilon + \frac{4 \Delta^2_3 \alpha_0^2}{3 k^2} \left[ \beta_2 (3 g_{\text{NL}} \epsilon + 2 f_{\text{NL}}^2 \epsilon^3) + 2 f_{\text{NL}} g_{\text{NL}} \beta_3 \epsilon^3 \right] + \\
+ \frac{2 \Delta^2_3 \alpha_0^2}{3 k^4} \left[ \beta_2^2 (2 f_{\text{NL}}^2 + 3 f_{\text{NL}} g_{\text{NL}}) + \beta_2 \beta_3 (4 f_{\text{NL}}^2 g_{\text{NL}} + 3 g_{\text{NL}}^2) + \\
+ 2 f_{\text{NL}} g_{\text{NL}} \beta_2^2 \right], \]  
(5.46)

\[ B_{hhh} \simeq 4 f_{\text{NL}} \Delta^2_3 \alpha_0^2 b_0^2 \epsilon + \frac{2 \Delta^2_3 \alpha_0^2 b_0^2}{k^2} \left[ \beta_2 (3 g_{\text{NL}} \epsilon + 2 f_{\text{NL}}^2 \epsilon^3) + 2 f_{\text{NL}} g_{\text{NL}} \beta_3 \epsilon^3 \right] + \\
+ \frac{2 \alpha_0 \Delta^2_3 \alpha_0^2 b_0^2}{k^4} \left[ \beta_2^2 (2 f_{\text{NL}}^2 + 3 f_{\text{NL}} g_{\text{NL}}) + \beta_2 \beta_3 (4 f_{\text{NL}}^2 g_{\text{NL}} + 3 g_{\text{NL}}^2) + \\
+ 2 f_{\text{NL}} g_{\text{NL}} \beta_2^2 \right] + \frac{6 g_{\text{NL}} \Delta^2_3 \beta_2^2 \epsilon^3}{k^6} (f_{\text{NL}} \beta_2 + g_{\text{NL}} \beta_3)^2. \]  
(5.47)

The expansion of the various expressions for the bispectra at large scales makes manifest how the non-Gaussian parameters \( f_{\text{NL}} \) and \( g_{\text{NL}} \) characterize different bispectra, and suggests that the study of bispectra allows us to distinguish the effects of each of them. Indeed, several contributions appear with different scale dependences and different coefficients: these can be used to distinguish the effects of \( f_{\text{NL}} \) and \( g_{\text{NL}} \).
5.1.3 Methods to disentangle $g_{\text{NL}}$ from $f_{\text{NL}}$

Here we discuss how our results allow one to break the degeneracy between $g_{\text{NL}}$ and $f_{\text{NL}}$ that affects the power spectra. We will discuss two possible methods to use bispectra to distinguish these two quantities.

The previous results are obtained in the limit of large scales. We use this limit in order to allow simple analytical approximations for the transfer function $T(k)$ and thus allow us to analytically understand physically interesting features of the bispectra as a function of the scale $k$, for configurations of squeezed isosceles triangles in momentum space $k = k_1 = k_2 = \epsilon k_3$.

As a specific example, let us study in more detail the properties of $B_{\text{hhh}}$, using the eq. (5.47) valid at large scales. The aim is to determine distinctive signatures of $g_{\text{NL}}$ in the profile of halo bispectra. While eq. (5.47) has been written in a form aimed to emphasize the different scale dependences of the various contributions, it can also be expressed in a more concise form as

$$B_{\text{hhh}} = \frac{2\Delta_{0\epsilon}}{k^6} (3\beta_2 g_{\text{NL}} + 2\alpha_0 b_g f_{\text{NL}} k^2) \times$$

$$\times \left[ \epsilon^2 (\beta_2 f_{\text{NL}} + \beta_3 g_{\text{NL}})^2 + \alpha_0 b_g \epsilon^2 (\beta_2 f_{\text{NL}} + \beta_3 g_{\text{NL}}) k^2 + \alpha_0^2 b_g^2 k^4 \right], \quad (5.48)$$

which makes the zeroes of $B_{\text{hhh}}$ clearly identifiable. Recall that the bispectrum eq. (5.48) is defined for isosceles configurations in momentum space, in which the equal sides of the triangle have length $k$, while the small side length is $k/\epsilon$.

Hence, we are studying $B_{\text{hhh}}$ as a function of the size of the long sides of an isosceles triangle in momentum space. In the limit of large $\epsilon$, eq. (5.48) has various roots: the ones that can be real are

$$k_{\text{root}}^{(1)} = \sqrt{-\frac{3\beta_2 g_{\text{NL}}}{2\alpha_0 b_g f_{\text{NL}}}}, \quad (5.49)$$

$$k_{\text{root}}^{(2)} = \sqrt{\frac{\epsilon^2 (\beta_2 f_{\text{NL}} + \beta_3 g_{\text{NL}})^2}{\alpha_0 |b_g|}}, \quad (5.50)$$

hence the existence and positions of the zeroes of galaxy bispectrum depend on the values of the non-Gaussian parameters $f_{\text{NL}}$ and $g_{\text{NL}}$. Working in regimes in which the parameters $\alpha_0$, $\beta_2$ and $b_g$ are positive (see the discussion in Section 5.4), the quantity $k_{\text{root}}^{(1)}$ is real only if $f_{\text{NL}}$ and $g_{\text{NL}}$ are both non-vanishing and have opposite signs. Hence, if the profile of $B_{\text{hhh}}$ changes sign as a function of the scale, it would be a tantalizing hint of the presence of non-vanishing $g_{\text{NL}}$ with an opposite sign to $f_{\text{NL}}$. While these roots have been derived in the limit of large scales where we can neglect the scale dependence of the transfer function, we will show that these features are accurately reproduced by a full numerical analysis in section 5.4.
The bispectrum $B_{hhh}$ is not the only quantity that allows to distinguish among different nG parameters. Combining different bispectra, indeed, one can more directly probe the individual effects of $f_{NL}$ with negligible contamination from $g_{NL}$. For example, let us consider the combination

$$C_{fnl} \equiv \frac{B_{hhh} P_{hh}^2}{P_{hh}} + 3 \frac{B_{hmm} P_{hm}^2}{P_{hm} P_{hh}} - 3 \frac{B_{hhm} P_{hm} P_{hh}}{P_{hm} P_{hh}}.$$  

(5.51)

Using eqs. (5.28), (5.29) and (5.41) to (5.43) without making any approximation, one finds that the previous quantity reads, for isosceles triangles,

$$C_{fnl} = \frac{2 f_{NL} (1 + 2 \epsilon^2) \alpha (k/\epsilon) \alpha(k) (b_g \alpha(k) + f_{NL} \beta_2 + g_{NL} \beta_3)}{\alpha(k) (b_g \alpha(k) + f_{NL} \beta_2 + g_{NL} \beta_3)}.$$  

(5.52)

Hence it depends specifically on $f_{NL}$ (with only a minor dependence on $g_{NL}$ in the denominator). Hence, a measurement of non-vanishing $C_{fnl}$ would provide a very clean probe of the quantity $f_{NL}$ that is (almost) independent on the size of $g_{NL}$. Nevertheless, the combination $C_{fnl}$ is challenging to probe observationally, since it is harder to observe bispectra and power spectra involving dark matter densities (although optimistically this might be realized in the future using gravitational lensing).

The bottom-line of this section is that the bispectra of halos and matter have distinctive qualitative features that allow one to distinguish the effects of different nG parameters, thereby providing observables that may allow one to break the degeneracies between $f_{NL}$ and $g_{NL}$ that are present in the study of power spectra only. Let us emphasize once more that the analytical results obtained so far include the effects of PNG only, without taking into account non-linear effects due to gravity. This approximation allowed us to analytically identify more directly the effects of different nG parameters in the halo distribution. See however section 5.4 for a numerical study of our analytic results, with some preliminary discussion on possible effects of non-linear gravitational clustering.

The next two sections are more theoretical, and investigate the implications of our findings when applied to the multiple source extension of our treatment.

### 5.2 The power spectrum and the bispectrum: the multiple source case

In this section, we discuss how to extend the previous analysis to the case of more than one source field for the primordial gravitational potential. We focus for definiteness on the case of two fields, that contribute to the gravitational
potential as in the following ansatz (see also [80])

$$\Phi_{\text{in}} = \varphi_G + \psi_G + f_{NL} (1 + \Pi)^{2} (\psi_G^2 - \langle \psi_G^2 \rangle) + g_{NL} (1 + \Pi)^{3} (\psi_G^3 - 3\langle \psi_G^2 \rangle \psi_G) ,$$

with

$$\Pi = \frac{P_{\phi_G}}{P_{\psi_G}}. \quad (5.54)$$

While $\varphi_G$ can be thought as the inflaton fluctuation, $\psi_G$ can be seen as a spectator field (like the curvaton) that is responsible for introducing PNG of local form in the gravitational potential. The two fields $\varphi_G$ and $\psi_G$ are by themselves Gaussian: we split them as long and short modes with respect to a fiducial scale $\ell$, as done in the previous section:

$$\varphi_G = \varphi_{G,s} + \varphi_{G,l} , \quad (5.55)$$

$$\psi_G = \psi_{G,s} + \psi_{G,l} . \quad (5.56)$$

This set-up can be seen as an extension of the curvaton-like model analysed in section 4.1 of [90], where we include the contribution of $g_{NL}$. From now on, $P_i = P_{\Phi_{\text{in}}}(k_i)$ and $P_{ij} = P_{\Phi_{\text{in}}}(|k_i + k_j|)$. Starting from ansatz eq. (5.53), the three- and four-point functions read

$$\langle \Phi_{\text{in}}(k_1)\Phi_{\text{in}}(k_2)\Phi_{\text{in}}(k_3) \rangle = f_{NL} [P_1 P_2 + 5 \text{ perms}] , \quad (5.57)$$

$$\langle \Phi_{\text{in}}(k_1)\Phi_{\text{in}}(k_2)\Phi_{\text{in}}(k_3)\Phi_{\text{in}}(k_4) \rangle = 2 \left( \frac{5}{6} \right)^2 \tau_{NL} [P_1 P_2 P_3 + 23 \text{ perms}] +
\left. + g_{NL} [P_1 P_2 P_3 + 11 \text{ perms}] , \quad (5.58)\right.$$

where

$$\tau_{NL} = \left( \frac{6}{5} f_{NL} \right)^2 (1 + \Pi) . \quad (5.59)$$

The usual equality $\tau_{NL} = \left( \frac{6}{5} f_{NL} \right)^2$ is recovered, as expected, in the single field limit $\Pi \rightarrow 0$. After implementing the long/short splitting, one gets in coordinate space

$$\Phi_{\text{in}} = \varphi_{G,l} + \psi_{G,l} + f_{NL} (1 + \Pi)^{2} (\psi_{G,l}^2 - \langle \psi_{G,l}^2 \rangle) +
\left. + 6g_{NL} (1 + \Pi)^{3} (\psi_{G,l}^3 - 3\langle \psi_{G,l}^2 \rangle \psi_{G,l}) + \varphi_{G,s} +
\left. + \left( 1 + 2f_{NL} (1 + \Pi)^{2} \psi_{G,l} + 3g_{NL} (1 + \Pi)^{3} (\psi_{G,l}^2 - \langle \psi_{G,l}^2 \rangle) \right) \psi_{G,s} +
\left. + (1 + \Pi)^{2} (f_{NL} + 3 (1 + \Pi) g_{NL} \psi_{G,l}) (\psi_{G,s} - [\psi_{G,s}]_V) \right)
\left. + g_{NL} (1 + \Pi)^{3} (\psi_{G,s}^3 - 3[\psi_{G,s}]_V \psi_{G,s}) . \quad (5.60)\right.$$

This expression demonstrates how long wavelength modes modulate the gravitational potential. The coefficients of the different powers of $\varphi_{G,s}$ and $\psi_{G,s}$,
which control the statistics of this quantity within the subvolume \( V \), receive contributions depending on the long mode \( \psi_{G,l} \), and thus acts as a background quantity from the point of view of the subvolume \( V \). In light of this, we can define the small scale effective power

\[
\sigma_l = \left[ \frac{\Phi^2}{V} \right]^{1/2} \left\{ 1 + 2 f_{NL}(1 + \Pi) \psi_{G,l} + (1 + \Pi)^2 (2 f_{NL}^2 \psi_{G,l}^2 + 3 g_{NL}(\psi_{G,l}^2 - \langle \psi_{G,l}^2 \rangle)) + \frac{1}{2} (1 + \Pi)^3 (-8 f_{NL}^3 \psi_{G,l}^3 + 12 f_{NL} g_{NL} \psi_{G,l} (\psi_{G,l}^2 - \langle \psi_{G,l}^2 \rangle)) \right\}.
\]

(5.61)

\[
f_{NL} = (f_{NL} + 3 (1 + \Pi) g_{NL} \psi_{G,l}),(5.62)
\]

\[
g_{NL} = g_{NL},(5.63)
\]

so we learn that long wavelength modes affect the two-point statistics of the short modes, as well as the non-Gaussian parameters inside the small box.

Following our treatment of the single source case, we can define the quantity \( \delta_{G,l} \) corresponding to the linearly evolved sum of primordial Gaussian perturbation as

\[
\delta_{G,l} = \alpha (\varphi_{G,l} + \psi_{G,l}).
\]

(5.64)

Using the Poisson equation, the linearly evolved long modes in the smoothed matter density contrast read

\[
\delta_{lin,l} = \delta_{G,l} + \alpha \left( (1 + \Pi)^2 f_{NL} (\psi_{G,l} * \psi_{G,l} - \langle \psi_{G,l}^2 \rangle) + (1 + \Pi)^3 g_{NL} (\psi_{G,l} * \psi_{G,l} * \psi_{G,l} - 3 \psi_{G,l} (\psi_{G,l}^2)) \right).
\]

(5.65)

We express the halo density contrast as the expansion

\[
\delta_h = b_g \left[ \delta_{G,l} + \alpha \left( (1 + \Pi)^2 f_{NL} (\psi_{G,l} * \psi_{G,l} - \langle \psi_{G,l}^2 \rangle) + (1 + \Pi)^3 g_{NL} (\psi_{G,l} * \psi_{G,l} * \psi_{G,l} - 3 \psi_{G,l} (\psi_{G,l}^2)) \right) \right] + \frac{\beta_2}{2} \left( \frac{\sigma_l}{\sigma_{l,1/2}^2} - 1 \right) + \frac{\beta_3}{3} (f_{NL} - f_{NL}) + \frac{\beta_4}{4} (g_{NL} - g_{NL}),
\]

(5.66)

with the standard bias evaluated at \((\delta_{lin,l} = 0, \sigma, f_{NL}, g_{NL})\),

\[
b_g = \frac{\partial \ln n_h}{\partial \delta_{lin,l}}.
\]

(5.67)

Moreover, we define at the same point

\[
\beta_2 = 2 \frac{\partial \ln n_h}{\partial \ln \sigma_l}, \quad \beta_3 = 3 \frac{\partial \ln n_h}{\partial f_{NL}^2}, \quad \beta_4 = 4 \frac{\partial \ln n_h}{\partial g_{NL}}.
\]

(5.68)

Substituting eqs. (5.61) to (5.63) into eq. (5.66), one finds

\[
\delta_h = b_g \delta_{G,l} + 2 \beta_2 (1 + \Pi) f_{NL} \psi_{G,l} + 6 \beta_3 g_{NL} (1 + \Pi) \psi_{G,l} +
\]

80
\[
+ \frac{(1 + \Pi)^2}{2} \left[ (\psi_{G,I} \ast \psi_{G,I} - (\psi_{G,I}^2)) \right. \\
\left. + \psi_{G,I} \ast \psi_{G,I} \beta_2 f_{NL}^2 \right] \\
+ \frac{(1 + \Pi)^3}{6} \left[ (\psi_{G,I} \ast \psi_{G,I} \ast \psi_{G,I} - 3\psi_{G,I}^2 (\psi_{G,I}^2)) \right. \\
\left. - 24 f_{NL}^2 \beta_2 \psi_{G,I} \ast \psi_{G,I} \ast \psi_{G,I} \right]
\]

\[
= b_2 \alpha \phi_L + b_1 \alpha \psi_{G,I} + \frac{1}{2} \left( b_2 \psi_{G,I} \ast \psi_{G,I} - b_3 \psi_{G,I} \right)
\]

\[
+ \frac{1}{6} \left( b_4 \psi_{G,I} \ast \psi_{G,I} \ast \psi_{G,I} - 3 b_5 \psi_{G,I} \right) .
\]

We define the bias \( b_i \) as

\[
b_1 \equiv b_2 + (1 + \Pi) \frac{f_{NL}}{\alpha} + \beta_3 (1 + \Pi) \frac{g_{NL}}{\alpha}, \quad (5.71)
\]

\[
b_2 \equiv (1 + \Pi)^2 \left[ 2 \alpha b g_{NL} + 3 \beta_2 g_{NL} + \Pi \beta_2 f_{NL}^2 \right], \quad (5.72)
\]

\[
b_3 \equiv (1 + \Pi)^2 \left[ 2 \alpha b g_{NL} + 3 \beta_2 g_{NL} \right], \quad (5.73)
\]

\[
b_4 \equiv (1 + \Pi)^3 \left[ 3 b_2 \alpha g_{NL} + 9 \Pi \beta_2 f_{NL} g_{NL} - 12 \Pi \beta_2 f_{NL}^2 \right], \quad (5.74)
\]

\[
b_5 \equiv (1 + \Pi)^3 \left[ 3 b_2 \alpha g_{NL} + 9 \Pi \beta_2 f_{NL} g_{NL} \right]. \quad (5.75)
\]

### 5.2.1 The two point functions of halo and matter densities

We adopt the definitions given in eqs. (5.21) to (5.23) for the two point functions, where from eqs. (5.54), (5.65) and (5.66) we obtain

\[
P_{mm} = \Delta_{mm} - \frac{\Delta_0}{k^3} (1 + \Pi) \alpha, \quad (5.76)
\]

\[
P_{hm} = \Delta_{hm} - \frac{\alpha b_1 + \alpha b g_{NL} \Pi}{k^3} \Delta_0 \alpha, \quad (5.77)
\]

\[
P_{hh} = \Delta_{hh} - \frac{\alpha^2 b_2^2 + \alpha^2 b_5^2 \Pi}{k^3} \Delta_0 \alpha, \quad (5.78)
\]

### 5.2.2 The three point functions of halo and matter densities

The three point functions are defined same as eqs. (5.32) to (5.35), which yields

\[
B_{mmm} = 2 \left( 1 + \Pi \right)^2 f_{NL} \Delta_0 \alpha \left[ \alpha(1) \alpha(2) \alpha(3) \left( \frac{1}{k_1^3 k_2^3} + \frac{1}{k_2^3 k_3^3} + \frac{1}{k_3^3 k_1^3} \right) \right], \quad (5.79)
\]

\[
B_{hmm} = \frac{1}{3} \left\{ \Delta_0^2 \alpha^2 \alpha \right\} \left[ 4 f_{NL} \left( 1 + \Pi \right)^2 \frac{\alpha(1)^2 \alpha(1)}{k_1^3 k_2^3} + \frac{1}{k_2^3 k_3^3} + \frac{1}{k_3^3 k_1^3} \right] + \text{perm.} \right\}, \quad (5.80)
\]

\[
B_{hhm} = \frac{1}{3} \left\{ \Delta_0^3 \alpha^3 \right\} \left[ 2 f_{NL} \left( 1 + \Pi \right)^2 \frac{\alpha(1)^2 b(1)}{k_1^3 k_2^3} + \frac{1}{k_2^3 k_3^3} + \frac{1}{k_3^3 k_1^3} \right], \quad (5.81)
\]

81
\begin{align*}
B_{hhh} &= \Delta_0^2 \left( b''_1 \frac{\alpha^{(2)} b''_1}{k_2^3} + b''_1 \frac{\alpha^{(2)} b''_1}{k_2^3} \right) + \text{perm.} \right) , \quad \text{(5.82)}
\end{align*}

where \( \alpha^{(j)} \equiv \alpha(k_j, z) \) and \( b^{(j)} \equiv b_j(k_j, z) \). In these formulae, we include only tree-level contributions, neglecting loop effects. Remarkably, only the bias parameter \( b_1 \) and \( b_2 \) appear.

**Isosceles triangles and the squeezed limit in momentum space**

We now set \( k = k_1 = k_2 = \epsilon k_3 \). That is, we consider isosceles triangles. The squeezed limit, then, corresponds to configurations in which \( \epsilon \gg 1 \). The previous expressions [eqs. (5.79) and (5.82)] become

\begin{align*}
B_{mmm} &= \frac{2 f_{NL} (1 + \Pi)^2}{k^6} \Delta_0^2 \alpha(k)^2 \alpha(k/\epsilon) \left( 1 + 2 \epsilon^3 \right) , \quad \text{(5.83)}
B_{hmm} &= \frac{\Delta_0^2}{3 k^6} \left\{ 4 f_{NL} (1 + \Pi)^2 \times \right.
&\left. \left[ (1 + \epsilon^3) \alpha \left( \frac{k}{\epsilon} \right) \alpha(k) b_1(k) + \epsilon^3 \alpha(k) \alpha \left( \frac{k}{\epsilon} \right) b_1 \left( \frac{k}{\epsilon} \right) \right] + \right.
&\left. \left[ 2 \epsilon^3 \alpha \left( \frac{k}{\epsilon} \right) b_2(k) + \alpha(k) b_2 \left( \frac{k}{\epsilon} \right) \right] \right\} , \quad \text{(5.84)}
B_{hhm} &= \frac{\Delta_0^2}{3 k^6} \left\{ 2 f_{NL} (1 + \Pi)^2 \times \right.
&\left. \left[ \alpha \left( \frac{k}{\epsilon} \right) \alpha(k) b_1(k) + 2 \epsilon^3 \alpha(k)^2 \alpha(k/\epsilon) b_1(k) b_1 \left( \frac{k}{\epsilon} \right) \right] + \right.
&\left. \left[ 2 \epsilon^3 \alpha \left( \frac{k}{\epsilon} \right) \alpha(k) b_1(k) b_2(k) \right. \right.
&\left. \left. + \right. \right.
&\left. \left. + 2 \alpha(k) \left( \epsilon^3 \alpha(k/\epsilon) b_1 \left( k/\epsilon \right) b_2(k) + b_2 \left( k/\epsilon \right) \alpha(k) b_1(k) \right) \right] \right\} , \quad \text{(5.85)}
\end{align*}

where we keep all contributions. Notice that the structure of the bispectra is very similar to the case of single source, apart from coefficients depending on \( \Pi \), the ratio of the power spectra of the Gaussian fields. This implies that when expanded using eqs. (5.71) and (5.72), one finds various contributions depending
both on bispectrum \((f_{ NL})\) and trispectrum \((\tau_{ NL} \text{ and } g_{ NL})\) parameters. The qualitative features of the bispectra as a function of the scale remain the same as the ones discussed in the single source case. For example, let us write the multi-source version of the quantity \(C_{ fnl} \) that we wrote in eq. (5.52): it reads

\[
C_{ fnl} = \frac{2 f_{ NL} (1 + \Pi) (1 + 2 \epsilon^3) \alpha (k/\epsilon)}{\alpha(k) (b_g \alpha(k) + f_{ NL} \beta_2 + g_{ NL} \beta_3)}. \quad (5.87)
\]

Also in the multiple source case, this quantity represents a clean probe of the nG parameter \(f_{ NL} \) (almost) independent from the value of \(g_{ NL} \).

In the next section, we investigate the contributions depending on \(\Pi\) that play an important role in defining the properties of the stochastic halo bias.

### 5.3 Stochastic halo bias, and combinations of power spectra and bispectra

Stochastic halo bias arises when halo overdensities are not fully correlated with matter overdensities [80,90]. An example is the inequality \(P_{hh} \geq b^2 P_{mm}\) where \(b = P_{mh}/P_{mm}\) is the halo-matter bias. As we have seen, halo and matter overdensities depend on PNG: neglecting the effect of shot-noise, the aforementioned stochasticity is associated with inequalities between non-Gaussian parameters. Such inequalities have been well studied in the analysis of PNG from inflation, and are typically (but not only [130, 131]) associated with the presence of multiple sources for the primordial gravitational potential. A famous inequality is the Suyama-Yamaguchi inequality \(\tau_{ NL} \geq \left(\frac{6}{5} f_{ NL}\right)^2\). In this section, we will investigate the role of \(g_{ NL}\) for characterizing the stochasticity of halo bias, and we will extend the notion of stochasticity to the bispectra. We will make use of some technical results on inequalities among primordial non-Gaussian parameters, which we relegate to appendix A. In this section, as in the previous ones, we will not include loop effects.

#### 5.3.1 Stochastic halo bias and power spectra

A convenient quantity to quantify the stochasticity of halo bias using power spectra is \(r_P\), defined as [90,90] \(3\)

\[
r_P = \frac{P_{hh}}{P_{mm}} - \left(\frac{P_{mh}}{P_{mm}}\right)^2. \quad (5.91)
\]

\(3\)Stochasticity was first discussed in [132] to study the scatter between the halo and matter density field in simulations. Stochasticity was defined there as

\[
r = \frac{P_{hm}}{\sqrt{P_{hh} P_{mm}}}, \quad (5.88)
\]
Using the results of section 5.2, we find at large scales $k \to 0$

$$r_P = \frac{4 \left( \beta_2 f_{NL} + 3 \beta_3 g_{NL} \right)^2 \Pi}{\alpha_0 k^4} \geq 0. \quad (5.92)$$

Hence $r_P \geq 0$, and the equality $r_P = 0$ can be obtained when PNG is absent, or in the single field limit $\Pi \to 0$. Notice that $f_{NL}$ and $g_{NL}$ appear in a combination that renders the identification of their individual effects difficult – a feature that we already discussed by studying power spectra in the previous sections.

Let us provide a heuristic understanding for this result, extending the arguments of [90] to include $g_{NL}$. In that paper, setting $g_{NL} = 0$, the inequality $r_P \geq 0$ was associated to the Suyama-Yamaguchi inequality $\tau_{NL} \geq (6/5 f_{NL})^2$.

On the other hand, when including $g_{NL}$, we learn that eq. (5.92) reads

$$\alpha_0 k^4 r_P = \left( \frac{5 \beta_2}{6} \right)^2 \tau_{NL} - \left( \frac{6}{5} f_{NL} \right)^2 + 2 \beta_2 \beta_3 f_{NL} g_{NL} \Pi + \beta_3 g_{NL}^2 \Pi \geq 0. \quad (5.93)$$

Hence we find two additional terms proportional to $g_{NL}$ besides the first corresponding to the Suyama-Yamaguchi inequality.

As we have seen in the previous sections, the halo density contrast can be schematically expressed as an expansion in terms of local correlation functions of the linearly evolved matter density field

$$\delta_h = b_2 \delta + \beta_2 [\delta^2] + \beta_3 [\delta^3] + \ldots. \quad (5.94)$$

Using this expansion, we can schematically express the quantity $r_P$ as

$$r_P = \frac{\langle \delta_h \delta_h \rangle - \langle \delta_h \delta \rangle^2}{\langle \delta \delta \rangle^2} = \beta_2^2 \frac{\langle [\delta^2] [\delta^2] \rangle}{\langle \delta \delta \rangle^2} + 2 \beta_2 \beta_3 \left( \frac{\langle [\delta^3] \delta \rangle}{\langle \delta \delta \rangle^2} - \frac{\langle [\delta^2] \delta \rangle \langle [\delta^3] \delta \rangle}{\langle \delta \delta \rangle^2} \right) + \beta_3^2 \left( \frac{\langle [\delta^3] [\delta^3] \rangle}{\langle \delta \delta \rangle^2} - \frac{\langle [\delta^3] \delta \rangle^2}{\langle \delta \delta \rangle^2} \right). \quad (5.95)$$

so that $r = \pm 1$ when $\delta_h$ is a deterministic linear function of $\delta$. The definitions of eq. (5.88) clearly differs from the one of eq. (5.91) and, indeed, one can show that

$$r_P = \frac{P_{hh}}{P_{mm}} \left( 1 - r^2 \right), \quad (5.89)$$

which implies $r_P = 0$ for $r = \pm 1$. Despite this difference, both definitions allow to study the case in which

$$\frac{P_{hh}}{P_{mm}} \neq \left( \frac{P_{mm}}{P_{mm}} \right)^2. \quad (5.90)$$

In particular, the definition of eq. (5.91) is well suited to study stochasticity from inflation in terms of squeezed and collapsed limits of n-point correlation functions (see the following discussion).
To write the previous formula, we used the schematic notation discussed in appendix A to express squeezed and collapsed limits of \( n \)-point functions. With \( \langle \delta[\delta^2] \rangle \) we denote the squeezed configuration of a 3-point function, in which one of the momenta is sent to zero: this notation clearly demonstrates that a long mode \( \delta \) modulates the 2-point function \( \delta[\delta^2] \). With \( \langle [\delta^2] [\delta^2] \rangle \) we denote the collapsed configuration of a 4-point function, in which the momentum connecting the two \( [\delta^2] \) is going to zero.

The combination in the first term in the rhs of eq. (5.95) relates the collapsed limit of a 4-point function with the squeezed limit of the 3-point function, and is associated with the Suyama-Yamaguchi inequality [see eq. (A.2)]. It corresponds to the first term in eq. (5.93). The remaining terms are instead associated to the inequalities eqs. (A.3) and (A.4), that involve collapsed limits of higher point functions, and are at the origin of the additional contributions in eq. (5.93) depending on \( g_{\text{NL}} \). The conclusion is that \( r_P \geq 0 \) does not measure just the Suyama-Yamaguchi inequality, but a combination of contributions associated with various inequalities: thus, it cannot clearly distinguish between \( f_{\text{NL}} \) and \( g_{\text{NL}} \).

5.3.2 Stochastic halo bias and the bispectra

Stochasticity can be defined also using bispectra: this allows one to break the degeneracy between \( f_{\text{NL}} \) and \( g_{\text{NL}} \) also at the level of stochastic halo bias. Consider the bispectra defined for isosceles triangles, both for the case of single and multiple sources as discussed in the previous sections. (Squeezed configurations are special cases in which one of the sides of the triangle has vanishing size, i.e. the parameter \( \epsilon \) defined in section 5.2.2 is large).

We define the following quantity \( r_B \) that uses bispectra to measure stochasticity,

\[
\tau_B = \frac{B_{hh}P_{hh}^2}{P_{hm}^2} + \frac{3 B_{hmm}}{P_{hm}P_{mm}P_{kh}} - \frac{3 B_{hhm}}{P_{hm}P_{mm}}. \tag{5.96}
\]

A non-vanishing \( r_B \) is associated with stochasticity: indeed, using the same arguments of the previous subsection, one can expand the previous quantity \( r_B \) in terms of \( \beta_i \), \( f_{\text{NL}} \) and \( g_{\text{NL}} \) finding a sum of several contributions that vanish in the single source limit. These contributions, among other things, depend on appropriated collapsed limits of 6-point functions; an example of contributions to \( r_B \) is the following combination, which appears in the expansion

\[\text{In [69] two independent parameters were introduced to generalise the definition of stochasticity in [132] to the case of the bispectrum. In this different context, stochasticity is studied in terms of effective bias parameters, which are built from ratios of various combinations of halo and crossed bispectra.}\]
for $r_B$ multiplied by suitable powers of the available parameters

$$
C \equiv \left[ \frac{\langle [\delta^2] [\delta^2] [\delta^2] \rangle - \langle \delta \delta \delta \rangle}{\langle \delta \delta \rangle \langle \delta \delta^2 \rangle} - 3 \frac{\langle \delta [\delta^2] [\delta^2] \rangle}{\langle \delta \delta^2 \rangle \langle \delta^2 \rangle^2} + 3 \frac{\langle \delta \delta \delta^2 \rangle}{\langle \delta \delta \rangle \langle \delta^2 \rangle^2} \right]. \tag{5.97}
$$

Using the inequalities among non-Gaussian parameters discussed in appendix A, one can check that $C$ vanishes in the single source case (neglecting loop effects), while it is generally non-vanishing for multiple sources.

For the specific two-field model of section 5.2, neglecting loop corrections, one can straightforwardly check that $r_B$ is non-vanishing only when $\Pi \neq 0$, i.e. in presence of multiple sources. We find the following value for $r_B$ at large scales and in the limit of squeezed configurations $\epsilon \gg 1$

$$
r_B = \frac{\epsilon^2 f_{NL}^2}{2(2f_{NL}^2 + 3g_{NL}^2)} - \frac{3 \epsilon^2 f_{NL} b_y a_0 k^2}{4(2f_{NL}^2 + 3g_{NL}^2)^2} + O(k^4). \tag{5.98}
$$

Hence, at very large scales $k \to 0$, $r_B$ is sensitive to $f_{NL}$ and its sign: if $f_{NL}$ vanishes, then $r_B$ vanishes no matter of the size of $g_{NL}$. Hence $r_B$ can be an useful complementary observable besides $r_P$ to study the stochasticity of halo bias.

### 5.4 Distinguishing $g_{NL}$ with large-scale structure

The analytical results of the previous sections indicate that the effects of $f_{NL}$ and $g_{NL}$ can be distinguished by measuring bispectra of halo and matter densities, in a way that is not possible by studying power spectra only. Indeed, in the scale-dependent bias of the halo power spectrum the parameters $f_{NL}$ and $g_{NL}$ are weighted by the same power of the scale $k$, while the bispectra contain different contributions depending on non-Gaussian parameters that scale with different powers of $k$, implying it is possible to overcome the degeneracy between $f_{NL}$ and $g_{NL}$ without having to study galaxy 4-point functions.

We theoretically analysed this subject in the previous sections, making simplifying assumptions in order to concentrate on the effects of PNG on the halo bias. While in our theoretical discussions, in order to obtain analytic expressions for our quantities, we focussed on the limit of large scales where simple approximations for the transfer functions hold, in this section we numerically investigate the features of the bispectrum also for smaller scales, in order to determine properties of this quantity that allow us to distinguish the effects of $g_{NL}$ from those of $f_{NL}$. In particular, we investigate two properties of halo and matter bispectra that we found when analytically studying them in the previous theoretical sections: 1) The dependence on $g_{NL}$ of the sign of halo bispectra as a function of the scale. and 2) The fact that a particular combination of halo...
and matter bispectra, \( C_{fnl} \) [see eq. (5.52)] depends on \( f_{nl} \) only, allowing one to cleanly distinguish \( f_{nl} \) from \( g_{nl} \).

To start with, we focus on the single source case and investigate the slope of the halo bispectrum \( B_{hhh} \) associated with squeezed isosceles triangles in momentum space. We consider highly biased halos for all the possible combinations of \( f_{nl} = 0, \pm 1 \) and \( g_{nl} = 0, \pm 10^4, \pm 10^3 \), in the squeezed configuration \( \epsilon = 100 \) (that is, the ratio between the long and short sides of an isosceles triangle in momentum space is 100). The aim of our analysis is to confirm the analytical results we determined in the previous sections, and to investigate at which scales, redshifts and masses a distinct signature of the effects of \( g_{nl} \) might be observed by future surveys, like Euclid [133]. On the other hand, our analysis intends to concentrate on the effects of PNG, without including the non-linear evolution of gravity, nor non-linearities associated with the particular dependence of the mass function on the matter density contrast.

The starting point of our arguments is eq. (5.39) for \( B_{hhh} \). In order to obtain the quantities \( b_1, b_2 \) defined in eqs. (5.18) and (5.19), we compute \( \alpha(k,z) \) in eq. (4.22) by using the matter transfer function \( T(k,z) \) from CAMB [39] with the state parameters consistent with the WMAP5+BAO+SN cosmology [134] (the reason of this choice will be clear later). Then, to evaluate the bias coefficients \( b_g, \beta_2, \beta_3 \) defined in eqs. (5.12) and (5.13), a specific mass function must be chosen at the expense of making additional assumptions.

As anticipated in chapter 4, we assume the LV mass function [eq. (4.38)]. Within the validity of the Edgeworth expansion and under the approximations that recover universality, the bias coefficients are obtained by taking the derivatives of eq. (4.33). Decomposing

\[
 b_g = b_{g0} + f_{nl} b_{f_{nl}0} + g_{nl} b_{g_{nl}0},
\]

we have

\[
 b_{g0} = 1 + \frac{\nu^2 - 1}{\delta_c},
\]

\[
 b_{f_{nl}0} = -\kappa^{(1)}_3(M) \left( \frac{\nu^3 - \nu}{2\delta_c} \right) - \frac{d\kappa^{(1)}_3/M}{d\ln(\sigma_M^3)/dM} \left( \frac{\nu + \nu^{-1}}{6\delta_c} \right),
\]

\[
 b_{g_{nl}0} = -\kappa^{(1)}_3(M) \left( \frac{\nu^4 - 3\nu^2}{6\delta_c} \right) - \frac{d\kappa^{(1)}_3/M}{d\ln(\sigma_M^3)/dM} \left( \frac{\nu^2}{12\delta_c} \right).
\]

Moreover, from eq. (5.13)

\[
 \beta_2 = 2\nu^2 - 2,
\]

\[
 \beta_3 = \kappa^{(1)}_3(M) \left( \frac{\nu^3 - \nu}{2} \right) - \frac{d\kappa^{(1)}_3/M}{d\ln(\sigma_M^3)/dM} \left( \frac{\nu - \nu^{-1}}{2} \right).
\]
The physical interpretation of the results above is the following: $b_{g0}$ is the usual Gaussian bias in Eulerian coordinates, while the terms $b_{fNL,0}$ and $b_{gNL,0}$ describe the scale-independent shift to $b_{g0}$ due to PNG. Unfortunately they cannot be used to constrain PNG, since the bias of a real tracer population is not known a priori. Then, $\beta_2$ is the well-known scale-dependent bias for an $f_{NL}$ cosmology.

By neglecting the explicit mass dependence of the Edgeworth expansion, we have forced eq. (5.103) to satisfy the relation $\beta_2 = 2h_c(b_{g0} - 1)$ found in [92]; this is valid for a universal mass function. Finally, $\beta_3$ describes the scale-dependent bias introduced by the $g_{NL}$ parameter.

Recall that the mass function here is initially computed in Lagrangian coordinates, therefore the +1 in eq. (5.100) takes into account the dynamical effects produced by linear gravity and bring us to Eulerian coordinates [61]. Let us emphasize again that here we are only including the linear effects of gravity. As discussed in subsequent chapters non-linear effects would add further non-Gaussian features to bispectra – not specifically due uniquely to PNG, the focus of this chapter.

Smith et al [135] compared the Edgeworth prediction for $\beta_3$ with N-body simulations, finding that it breaks down at lower halo mass, $M \lesssim 10^{14} h^{-1} M_\odot$. To fix this problem, they first noted that eq. (5.104) can be written as

$$\beta_3 = \kappa_3 \left[ -1 + \frac{3}{2} (\nu - 1)^2 + \frac{1}{2} (\nu - 1)^3 \right] - \frac{d\kappa_3}{d\ln \sigma_{NL}} \left( \frac{\nu - \nu^{-1}}{2} \right), \quad (5.105)$$

then, they found a good agreement with simulations by changing the coefficients of the polynomial in the brackets as follows

$$\beta_3^* = \kappa_3 \left[ -0.7 + 1.4 (\nu - 1)^2 + 0.6 (\nu - 1)^3 \right] - \frac{d\kappa_3}{d\ln \sigma_{NL}} \left( \frac{\nu - \nu^{-1}}{2} \right). \quad (5.106)$$

Our numerical analysis can be easily performed by using the following fitting functions [135]

$$\kappa_3 = 0.000329 (1 + 0.09z) b_{g0}^{-0.09}, \quad (5.107)$$

$$\frac{d\kappa_3}{d\ln \sigma_{NL}} = -0.000061 (1 + 0.22z) b_{g0}^{-0.25}. \quad (5.108)$$

The quantities of eqs. (5.100), (5.103) and (5.106) provide the necessary terms to describe highly biased tracers. All the results provided by Smith et al [135] have been tested against 4 simulations with Gaussian initial conditions, 5 simulations with $f_{NL} = \pm 250$ and 3 simulations with $g_{NL} = \pm 2 \times 10^6$, for a total of 20 simulations in the WMAP5+BAO+SN cosmology. Given that these results have been well tested on several simulations, we assume the same dataset, and we do not expect to see any radical change in our order-of-magnitude analysis
for a slightly different ΛCDM cosmology. However, further simulations should be carried out to test the validity of $\beta^*_3$ for the low values of $f_{\text{NL}}$ and $g_{\text{NL}}$ we are interested in.

Let us now present our numerical results and their interpretation. Figures 5.1 to 5.3 display the absolute value of the halo bispectrum (using a log-scaling), for halos of mass $M = 10^{13} M_\odot$, at redshift $z = 1$, and $k$-space triangles with squeezing parameter $\epsilon = 100$. Each of the three plots shows the results for a different value of $f_{\text{NL}}$: $f_{\text{NL}} = -1$ for fig. 5.1, $f_{\text{NL}} = 0$ for fig. 5.2 and $f_{\text{NL}} = 1$ for fig. 5.3. The lines on each plot correspond to different values of $g_{\text{NL}}$. Each point on the $k$-axis actually involves three different values of the wavenumber specifying the three sides of a squeezed triangle in momentum space: two at $k_1 = k_2 = k$ (the long equal sides) and one at $k_3 = k/\epsilon$ (the small side). The plots span an interval of $k$ between $10^{-2}$ and $10^{-1} h/\text{Mpc}$, corresponding to scales within ranges that will be probed by future observations made by Euclid, under the hypothesis that this survey will gain one order of magnitude in scale with respect to BOSS [108, 136]. Since negative values of the bispectrum are possible, we plot the absolute value of $B_{hhh}$. The presence of cusps indicate the changing from positive to negative values (or vice-versa). For the sake of clarity, we use dashed lines for negative values of the bispectrum and solid lines for positive ones.

An inspection of these plots suggests an important qualitative feature of these results. If the parameters are contained within appropriate intervals, the halo bispectrum as a function of $k$ changes sign more times if $g_{\text{NL}}$ has opposite sign with respect to $f_{\text{NL}}$. For example, in fig. 5.1 corresponding to $f_{\text{NL}} = -1$, we see that the green curve, corresponding to the logarithm of the bispectrum for $g_{\text{NL}} = 10^3$, changes sign twice instead of once as the other curves. Moreover, the positions of the zero at smaller scales, that exists for all the curves, depends on the value of $g_{\text{NL}}$. In fig. 5.2 with $f_{\text{NL}} = 0$, the bispectra for different values of $g_{\text{NL}}$ change sign in the interval of scales we consider only for $g_{\text{NL}} = -10^4$ (the magenta curve). Finally, in fig. 5.3 with $f_{\text{NL}} = +1$, the red curve with $g_{\text{NL}} = -10^3$ changes sign while all the other curves do not.

The qualitative features in the bispectra profiles as a function of scale become even more pronounced when considering higher redshifts or higher mass objects. To show this behaviour, we present plots displaying the halo bispectrum for two additional cases. The first set, displayed in figs. 5.4 to 5.6 is for $M = 10^{14} M_\odot$, $z = 1$, $\epsilon = 100$, while the second one is for $M = 10^{13} M_\odot$, $z = 2$, $\epsilon = 100$ and is displayed in figs. 5.7 to 5.9. Each plot uses the same conventions as figs. 5.1 to 5.3. The qualitative considerations we made above are still
Figure 5.1: The absolute value of the halo bispectrum $B_{hhh}$ in the squeezed configuration $k_1 = k_2 = 100k_3$, for halo mass $M = 10^{13}M_\odot$, redshift $z = 1$, $f_{NL} = -1$. Different lines correspond to the values $g_{NL} = 0, \pm 10^3, \pm 10^4$. We use solid lines to describe positive values and dotted lines for negative values; the presence of a cusp indicates a change of sign. The pattern of zeros in the plotted range of wavenumbers can be used to distinguish $f_{NL}$ from $g_{NL}$: see the text for further details.

Figure 5.2: Same as fig. 5.1 but for $f_{NL} = 0$. For $g_{NL} = -10^4$ a zero is present.
valid. In particular, we see, at higher mass or redshift than our fiducial choice ($M = 10^{13} M_\odot$, $z = 1$, $\epsilon = 100$), an increase in the absolute value of the halo bispectrum. Interestingly, the various contributions to the bispectrum weighted by different powers of $k$ make the magenta curve for $g_{NL} = -10^4$ in the plots with $f_{NL} = +1$ change sign, in both cases presented here (see figs. 5.6 and 5.9). This new feature is due to the dependence of the bias parameters $b_1$, $b_2$ on mass and redshift.

Hence, the previous plots show that the presence or absence of $g_{NL}$ affects the bispectra profiles: in particular it governs how many times the halo bispectrum changes sign as a function of the scale. We already discussed this qualitative behavior in Section 5.1.2, when analytically studying the roots of the bispectrum in the large scale limit. The numerical results of this section confirm those analytical findings, indicating that the qualitative profile of the halo bispectrum for isosceles triangles as a function of the scale can provide an interesting signature of the presence of $g_{NL}$. Hence, we learn that the fact that the bispectrum changes sign in multiple locations as a function of the scale is a distinctive signal of the presence of $g_{NL}$. Optimistically, this can be used to constraint values of $g_{NL}$ smaller than the ones that can be tested by CMB. These features might be observed in cases in which large values of $b_g$ are realized, since in this case the bispectrum generated by PNG is more significant. Thus, the bispectrum is greater for larger halo masses $M$ and, at constant $M$, greater at higher redshift. For example, in fig. 5.10 we provide an example of halos at redshift $z = 2$, for $f_{NL} = -1$. In this case, we see that the green curve
Figure 5.4: Same as fig. 5.1 but for $M = 10^{14}M_\odot$. The amplitude of the bispectra have increased considerably over the case with $M = 10^{13}M_\odot$, the positions of the zeros have changed for $f_{NL} = -1$.

Figure 5.5: Same as fig. 5.2 but for $M = 10^{14}M_\odot$. The amplitude of the bispectra have increased considerably over the case with $M = 10^{13}M_\odot$ and the number of zeros has increased for $f_{NL} = 0$. 
Figure 5.6: Same as fig. 5.3 but for $M = 10^{14} M_\odot$. The amplitude of the bispectra have increased considerably over the case with $M = 10^{13} M_\odot$ (shown in Fig. 1) and the number of zeros has increased for $f_{NL} = +1$.

Figure 5.7: Same as fig. 5.1 but for $z = 2$. The amplitude of the bispectra have increased significantly over the case of $z = 1$ and the positions of the zeros differ from those observed both in fig. 5.1 and fig. 5.4.
Figure 5.8: Same as fig. 5.2, but for $z = 2$. The amplitude of the bispectra have increased significantly over the case of $z = 1$ and the positions of the zeros differ from those observed both in fig. 5.2 and fig. 5.5.

Figure 5.9: Same as fig. 5.3, but for $z = 2$. The amplitude of the bispectra have increased significantly over the case of $z = 1$ and the positions of the zeros differ from those observed both in fig. 5.3 and fig. 5.6.
corresponding to the halo bispectrum changes sign twice for $g_{NL} = 400$. An observation of this phenomenon can then probe quite small values for $g_{NL}$. We can also compare the results of fig. 5.10 with the analytical formulae for the zeros of the bispectrum, eqs. (5.49) and (5.50). For the halo mass, redshift and $\epsilon$ parameter we are considering, we find $b_g \simeq 4.3$, $\beta_2 \simeq 9.5$, $\beta_3 \simeq 1.1 \times 10^{-3}$. Computing the positions of the zeros of the bispectrum for the case $g_{NL} = 400$ with the analytical formulae of eqs. (5.49) and (5.50), one finds the zeros at $k_{(1)\text{root}} \simeq 1.2 \times 10^{-2}$ and $k_{(2)\text{root}} \simeq 4.9 \times 10^{-2}$ h/Mpc, in agreement with the numerical results within one order of magnitude. This implies that our analytical findings, although obtained in the approximation in which we neglect the scale dependence of the transfer function, provide reasonably accurate predictions for the positions of the zeroes.

Figure 5.10: The plot shows the absolute value of the halo bispectrum $B_{hh}h$ in the squeezed configuration $k_1 = k_2 = 100k_3$, for halo mass $M = 10^{13} M_{\odot}$ and redshift $z = 2$. $f_{NL}$ is kept constant and equal to $-1$, while different lines correspond to the values $g_{NL} = 0, 400$. We use solid lines to describe positive values and dotted lines for negative values; the presence of a cusp indicate a change of sign. $f_{NL} = -1$ produces a specific pattern of zeros in the line with $g_{NL} = 400$. If this feature can be observed in the halo bispectrum, it will improve the constrain on $g_{NL}$ to few hundreds.

We now present a plot that represents the second theoretical observation we made in the previous sections: if it is possible to probe and observe bispectra correlating halo and matter densities (for example using gravitational lensing) then by analyzing combinations such as $C_{fnl}$ of eq. (5.52) we can have clean measurements of $f_{NL}$ only, with a negligible contamination of $g_{NL}$. In fig. 5.11
we represent this quantity for different values of $f_{NL}$, halo masses and redshifts; the lines are practically insensitive on the values of $g_{NL}$ compatible with current constraints [81], and depend only on $f_{NL}$ and on the halo mass. This feature, as we discussed around eq. (5.52), is valid at all scales (although at smaller scales one should take into more proper account the non-linear effects of gravity). This concretely shows that an observation of a non-vanishing value for the quantity $C_{fnl}$ would be a clean indication of a non-vanishing $f_{NL}$. Combined with other measurements (for example associated with power spectra) this fact could then be used to obtain independent constraints on $g_{NL}$. We conclude that the scale dependence of bispectra of isosceles configurations in momentum space have qualitative features that might allow us to distinguish the effects of different nG parameters, in a way that is not possible by studying power spectra only.

While in the previous discussion we neglected the effects of non-linear gravity in the results for the bispectra, let us end this section with a preliminary analysis of its contributions. Focussing on squeezed configurations, the next plot (fig. 5.12) shows a comparison between the amplitude of bispectra associated with PNG only (color lines) and bispectra due uniquely to the non-linear effects of gravity (black lines). We divide the bispectrum due to gravity in two contributions. The black dotted line corresponds to the amplitude of bispectra
associated only with linear bias $b_{g0}$, and is controlled by the kernel $F_2$ (term A in eq. (3.145)). The black continuous line, instead, is the contribution associated with the quadratic coefficient $B_2$ of the local non-linear biasing model of [65] (term L in eq. (3.145)). We make the choice $B_2 = 2b_{g0}$ in order to be more conservative respect to [42] (See the review [28] for more details.)

Figure 5.12: The plot shows the amplitude of halo bispectra as a function of scale. The coloured lines are bispectra associated with PNG only; the black lines are contributions to bispectra due to non-linear gravitational clustering. The black dotted line corresponds to the gravitational clustering contributions to $B_{hhh}$ depending on the linear bias $b_{g0}$ only. The continuous line is the gravity bispectrum associated with non-linear bias $b_2$ (we take $b_2 = 2b_{g0}$).

We notice that the amplitude of the bispectrum due to PNG tends to dominate at large scales (small $k$). However, if the primordial nG parameters are chosen to be small, the bispectrum induced by gravity is more important at smaller scales (see green line). For somewhat larger values of the nG parameters (with the magnitude of $g_{NL}$ still below CMB constraints) the amplitude of the primordial bispectrum can dominate over the gravity contributions also at relatively small scales (see violet line). The primordial bispectrum represented in the violet line has non-trivial profile due to competing effects of primordial $f_{NL}$ and $g_{NL}$, in particular it has zeroes associated with the presence of $g_{NL}$: if such features can be detected in the bispectrum profile, they can be used to help distinguishing among the effects of $f_{NL}$ and $g_{NL}$.

The plot of fig. 5.12 is only indicative, in the sense that one should study in

\[ 8 \] To avoid confusion with eq. (5.19) we call it here $B_2$, although in the literature it is usually referred to as $b_2$. 

97
more detail the combined effects of non-linear gravity and PNG in shaping the bispectra, and not only comparing their relative amplitude (see the following chapter). On the other hand, it gives the idea that for interesting values of nG parameters the primordial contribution can lead to a rich profile for the bispectrum, controlled by the primordial nG parameters, with an amplitude larger or comparable to the one due to gravitational clustering only.

5.5 Summary

PNG characterizes the interactions of the fields sourcing the density fluctuations that seed the large-scale structure of our universe. In this chapter, we analytically and numerically investigated how the statistics of LSS, in particular the study of bispectra of halos and matter, allow one to probe PNG. Our aim was to find ways to determine distinctive features of the local nG parameters $f_{NL}$ and $g_{NL}$, controlling respectively skewness and kurtosis of the probability distribution function, and whose effects in the halo and matter power spectra are nearly degenerate. This investigation is important since although CMB constraints on $f_{NL}$ have greatly improved with Planck, it is not clear whether CMB measurements only can set stringent constraints on $g_{NL}$.

In the first part of this chapter, we showed analytically that the profiles of halo and matter bispectra have qualitative features that if measurable would clearly distinguish the effects of each nG parameter. We exploited the scale dependence of the halo bias induced by PNG. We worked in a simplified situation in which only linear bias and linear effects of gravitational clustering are taken into account, to single out the effects of PNG in the statistics of LSS. We have shown that the profile of the halo bispectrum as a function of the scale, for isosceles configurations of triangles in momentum space, has properties that provide simple, qualitative information on each nG parameter. For example, the number and the position of the zeros of this function depend on the size of $g_{NL}$, and can be analytically computed with our formulae. If $g_{NL}$ is present, and its sign is opposite with respect to the sign of $f_{NL}$, the number of zeros of the bispectra increases as a function of the scale. Then, we determined a particular combination of halo and matter bispectra and power spectra, denoted $C_{fnl}$, that is proportional to $f_{NL}$ only and is (nearly) independent on the value of $g_{NL}$. A detection of this combination would be a clean probe of $f_{NL}$ with no contaminations associated with $g_{NL}$. We then generalized our findings to the case in which multiple sources contribute to form the primordial density fluctuations, and studied how stochastic halo bias is affected by the presence
of $g_{NL}$. We also provided new ways to study stochastic halo bias in terms of combinations of bispectra and power spectra.

In the second part of this chapter, we numerically confirmed and further developed our analytical results, focussing on a Press-Schechter approach to halo formation supplemented by appropriate $nG$ initial conditions. This analysis allowed us to apply our theoretical formulae for the bispectra focussing on scales and redshifts that might be probed by future surveys. We confirmed our theoretical considerations, showing plots representing how the qualitative profiles of halo bispectra in suitable configurations depend on $nG$ parameters, and might be used to set constraints on them. We ended our discussion with a preliminary analysis of the contributions of non-linear gravitational clustering, indicating that non-linear gravity tends to contaminate the bispectrum profiles, but that the qualitative features we investigated survive for sufficiently large primordial $nG$ parameters.

In the following chapters we will consider the contribution of non-linear clustering to the halo bispectra and how this affects our ability to constrain PNG.
Chapter 6

Non-local bias in the halo bispectrumbispectrum with primordial non-Gaussianity

As discussed in the preceding chapter the 2-point function is not the natural statistic in which to look for PNG, as the full shape information is available only in higher-order statistics, and moreover local-type models including higher-order non-Gaussian parameters like $g_{\text{NL}}$ are approximately degenerate with $f_{\text{NL}}$ in the power spectrum [126]. These issues naturally drive us towards studying higher-order $n$-points correlation functions, in particular the 3-point function and its Fourier transform, the *bispectrum*\(^1\). Many more triangle configurations compared to 2-point statistics are available in $k$-space, suggesting potentially a stronger constraining power with respect to the power spectrum. Indeed, $\sigma_{f_{\text{NL}}} \sim 1$ is expected from the bispectrum [73, 138, 139], although this forecast is based on idealized surveys and ignores redshift space distortions (RSD). In addition, the degeneracy between $f_{\text{NL}}$ and $g_{\text{NL}}$ is broken by combining both the power spectrum and bispectrum constraints [42, 91].

There are a number of complications, which explains why few measurements of the 3-point function or bispectrum from real data have been obtained so far [138, 140, 148]. For example, redshift-space distortions and the complicated mask geometry intrinsic to any survey are hard to model. Further, non-linearities in halo populations and the non-linear nature of Einstein’s field

\(^1\)In [137] the possibility to use the higher-order moments of LSS to constraint PNG has been explored. Although these avoid the complexities of the full $n$-point statistics, they may not be able to detect small PNG.
equations produce non-Gaussian features in the evolved matter field that may be hard to disentangle from the PNG signal.

The ability to go beyond current constraints on PNG, crucially depends on our understanding of all possible effects that can influence LSS measurements and how these depend on the primordial fluctuations. In this chapter we make improvements to the bias model which relates the halo density to the underlying matter density, particularly focusing on an accurate description of the second-order, non-local and non-Gaussian effects.

In the presence of PNG of local type,

$$\Phi_{\text{in}}(\mathbf{q}) = \varphi_G(\mathbf{q}) + f_{\text{NL}} (\varphi_G^2(\mathbf{q}) - \langle \varphi_G^2 \rangle),$$

(6.1)

the local Lagrangian bias model needs to be extended to account for the correlations in the initial density field. As shown in sections 4.2.3 and 4.2.4, the PBS argument naturally leads to the bivariate model of [72], in which the Lagrangian halo overdensity is written as a double expansion in terms of the initial, linear density field and the potential \( \varphi_G \). The bivariate model shows good fits against simulations even when applied to the bispectrum [38, 73].

Non-linear evolution of the matter density introduces a non-local tidal term in the halo bias model (see section 3.3.3), which we here consider for the first time in combination with the effect of local-type non-Gaussianity. Furthermore, the presence of local-type non-Gaussianity in the Lagrangian frame leads to a novel non-local convective term in the Eulerian frame, that is proportional to the displacement field when going beyond the spherical collapse approximation and contributes to the number density of halos.

By using an extended Press-Schechter approach to evaluate the halo mass function and thus the halo bispectrum, we will quantitatively investigate how much the tidal term, and our new non-local contribution, affect the tree-level halo bispectrum in comparison with the reference model of [73] (hereafter BSS). We will show that including these non-local terms in the halo bispectra can lead to corrections of up to 25% for some configurations, on large scales or at high redshift.

### 6.1 Non-local Eulerian bias

As shown in section 3.3.3 in a local Lagrangian biasing scheme \((f_{\text{NL}} = 0)\),

$$\delta_L^h(\mathbf{q}) = b_{10} \delta_{\text{lin}} + b_{20} (\delta_{\text{lin}})^2 + \ldots,$$

(6.2)

the formation sites of halos is identified from the initial density field. The halo density is expanded in terms of density at the initial spatial coordinate \( \mathbf{q} \), which
is related to the evolved Eulerian coordinate $x$ through the displacement field $\Psi$:

$$x(q, \tau) = q + \Psi(q, \tau). \quad (6.3)$$

Hence the transformation captures the dynamics of halos and the number density in Eulerian coordinates is described by (see section 3.3.3)

$$1 + \delta_{\text{E}}^E(x, z) = [1 + \delta_{\text{E}}^E(x, z)] \left[ 1 + \delta_{\text{B}}^E(q, z) \right]. \quad (6.4)$$

At first-order the density perturbation $\delta_{\text{lin}}^E$ has the same form in either frame but, at second-order, they differ because of a convective term proportional to the displacement $\Psi$. For this reason at second-order in the evolved density field in Eulerian coordinates one has (see section 3.1.3)

$$\delta_{\text{nonlin}}^E(x, z) \simeq \frac{17}{21} (\delta_{\text{lin}}^E(x, z))^2 + \frac{2}{7} s^2(x, z) - \Psi(x, z) \cdot \nabla \delta(x, z), \quad (6.5)$$

and the halo density in the Eulerian frame is

$$\delta_{\text{E}}^E(x) = b_{E1}^E \delta + b_{E2}^E \delta^2 - \frac{2}{7} b_1^E s^2, \quad (6.6)$$

which is non-local because of the tidal term $s^2$. As usual throughout the thesis, $\delta = \delta_{\text{lin}} + \delta_{\text{nonlin}} \simeq \delta_{\text{lin}} + \delta^{(2)}$ (see section 3.1).

With local-type non-Gaussianity of the form of eq. (6.1), the PBS argument shows that a local Lagrangian bias model should give the bivariate form (see section 4.2.4)

$$\delta_{\text{E}}^L(q) = b_{L0}^E \delta_{\text{lin}} + b_{L1}^E \varphi^G + b_{L2}^E (\delta_{\text{lin}})^2 + b_{L3}^E \delta_{\text{lin}} \varphi^G + b_{L4}^E \varphi^G, \quad (6.7)$$

In analogy with the density field, the Newtonian potential $\varphi^G$ has the same form in either frame to first order, but we must include a convective term when writing the primordial potential eq. (6.1) up to second order in Euclidean coordinates,

$$\varphi^G(x) \simeq \varphi^G(q) + \Psi(q, z) \cdot \nabla \varphi^G(q), \quad (6.8)$$

and thus

$$\Phi_{\text{lin}} \simeq \varphi^G(x) - \Psi(x, z) \cdot \nabla \varphi^G(x) + f_{\text{NL}} \left( \varphi^G_2(x) - \langle \varphi^G_2 \rangle \right). \quad (6.9)$$

Hence we learn that, even though the primordial potential is by definition a local function of an initial Gaussian random field $\varphi^G(q)$ at each initial spatial coordinate $q$ and at a fixed initial time, the primordial potential at a fixed Eulerian position $x$ becomes time-dependent at second-order, due to the time-dependent relation between Eulerian and Lagrangian coordinates [eq. (6.3)]. In appendix B we show that even a Gaussian initial potential in the Lagrangian
frame becomes a non-Gaussian field at second-order in the Eulerian frame due to the first-order displacement. We shall see that this gives rise to an additional non-local term in the distribution of collapsed halos in Eulerian space.

Using the transformation law of eq. (6.4), we thus obtain our expression for the Eulerian halo overdensity up to second order in terms of the density contrast and the Gaussian potential in Eulerian coordinates,

\[ \delta^E_h(x) = b^E_{10} \delta + b^E_{01} \varphi + b^E_{20} \delta^2 + b^E_{11} \varphi_G \delta + b^E_{02} \varphi_G^2 - \frac{2}{7} b^L_{10} s^2 - b^L_{01} \Psi(x, z) \cdot \nabla \varphi_G, \]

(6.10)

where we define the standard Eulerian bias coefficients

\[
\begin{align*}
    b^E_{10} &= 1 + b^L_{10}, \\
    b^E_{01} &= b^L_{01}, \\
    b^E_{20} &= \frac{8}{21} b^L_{10} + b^L_{20}, \\
    b^E_{11} &= b^L_{01} + b^L_{11}, \\
    b^E_{02} &= b^L_{02}.
\end{align*}
\]

(6.11)

Following BSS, we estimate them assuming the ST mass function corrected for PNG in light of LV mass function, \( f = f_{ST} f_{LV} / f_{PS} \) [see eq. (4.39)]. Figure 6.1 shows the dependence of the bias coefficients on halo mass and redshift.

Equation (6.10) generalises the result of BSS, obtained under the spherical collapse approximation (\( \Psi = 0 \)). Indeed, the last two terms of eq. (6.10) are the non-local, non-linear terms of our bias model. While \( s^2 \) is an already known tidal term, we have derived here for the first time the convective contribution \( \Psi(x, z) \cdot \nabla \varphi_G \). We decide to keep the corresponding bias coefficients written as \( b^L_{10} \) and \( b^L_{01} \), instead of replacing them with \( b^E_{10} - 1 \) and \( b^E_{01} \) respectively. In this way we will be able to recognise more easily the differences they introduce with respect to the reference model of BSS, especially in the discussion in section 6.3.

Finally, if we perform a Fourier transform with respect to Euclidean coordinates, we obtain

\[ \delta^E_h(k) = b^E_{10} \delta + b^E_{01} \varphi_G + b^E_{20} \delta^2 + b^E_{11} \varphi_G \delta + b^E_{02} \varphi_G^2 - \frac{2}{7} b^L_{10} s^2 - b^L_{01} n^2, \]

(6.12)

where we define

\[
\delta(k) = \delta_G(k) + f_{NL} \alpha(k) \int \frac{dq}{(2\pi)^3} \frac{\delta_G(q) \delta_G(k - q)}{\alpha(q) \alpha(|k - q|)} +
\]

\[
+ \int \frac{dq}{(2\pi)^3} F_2(q, k - q) \delta_G(q) \delta_G(k - q),
\]

(6.13)
Figure 6.1: The Eulerian bias coefficients as a function of mass and redshift, assuming $f_{NL} = 1$. 
\[ s^2(k) = \int \frac{dq}{(2\pi)^3} S_2(q, k - q) \delta_G(q) \delta_G(k - q), \quad (6.14) \]
\[ n^2(k) = 2 \int \frac{dq}{(2\pi)^3} N_2(q, k - q) \delta_G(q) \delta_G(k - q) \frac{\delta_G(k - q)}{\alpha(|k - q|)}, \quad (6.15) \]

and the kernels are given by
\[ F_2(k_1, k_2) = \frac{5}{7} + \frac{1}{2} \frac{k_1 \cdot k_2}{k_1 k_2} \left( \frac{k_1^2}{k_2} + \frac{k_2^2}{k_1} \right) + \frac{2}{7} \frac{(k_1 \cdot k_2)^2}{k_1^2 k_2^2}, \]
\[ S_2(k_1, k_2) = \frac{(k_1 \cdot k_2)^2}{k_1^2 k_2^2} - \frac{1}{3}, \]
\[ N_2(k_1, k_2) = \frac{k_1 \cdot k_2}{2k_1^2}. \]

The standard second-order Newtonian kernel \( F_2 \) is generated by the non-linear gravitational evolution. \( S_2 \) by the tidal term and the new kernel, \( N_2 \), is generated by the convective term \( \Psi \cdot \nabla \varphi_G \).

### 6.2 Three-point functions of halo and matter overdensities

In sections 6.2.1 and 6.2.2 we will present the tree-level bispectra for the halo and matter overdensities. We adopt the definition
\[ \langle \delta_E^H(k_1) \delta_E^H(k_2) \delta_E^H(k_3) \rangle = (2\pi)^3 \delta^D(k_1 + k_2 + k_3) B_{\alpha \beta \gamma}(k_1, k_2, k_3), \]
where \( \alpha, \beta, \gamma = h, m \) with the labels \( h \) and \( m \) standing for halo and matter respectively. To simplify the notation, we omit hereafter to write the explicit dependence on the redshift \( z \) of the power spectrum and bispectrum (see section 3.2.1). The crossed halo-matter bispectra are given in appendix D.

In sections 6.2.2 and 6.3 we will make use of the graphical representation introduced by Jeong and Komatsu [42] to show the shape dependence of the bispectrum. This was introduced in section 3.3.6.

#### 6.2.1 Matter bispectrum

The matter bispectrum is given from eqs. (6.13) and (6.19)
\[ B_{mmm}(k_1, k_2, k_3) = \left( 2P(k_1)P(k_2)F_2(k_1, k_2) + 2P(k_1)P(k_2)\alpha(k_3) \right. \]
\[ + 2 f_{NL} P(k_1)P(k_2) \alpha(k_1) \alpha(k_2) \) + 2 cyc., \]
where we have dropped the redshift \( z \) dependence from the function \( \alpha(k, z) \) and the matter power spectrum \( P(k, z) \) in order to simplify the notation; we will do
the same in the following sections. The matter bispectrum is thus generated by both primordial non-Gaussianity, \( f_{\text{NL}} \), and non-linear gravitational evolution, \( \mathcal{F}_2 \).

The tree-level approximation based on perturbation theory well describes the simulation results at scales up to \( k \simeq 0.05 - 0.1 \text{hMpc}^{-1} \), depending on the redshift. Including one-loop corrections can significantly extend the validity for bispectrum to \( k \simeq 0.3 \text{hMpc}^{-1} \) at redshift \( z \gtrsim 1 \) [38]. Alternatively, it is possible to use an effective kernel \( \mathcal{F}_2^{\text{eff}} \) calibrated against simulations [149]. This phenomenological approach gives simpler expressions in the non-linear regime, and accurate predictions for the bispectrum, up to \( k \simeq 0.4 \text{hMpc}^{-1} \) in the redshift range \( 0 \leq z \leq 1.5 \), when Gaussian initial conditions are assumed [150].

### 6.2.2 Halo bispectrum

In the presence of primordial non-Gaussianity, many more terms contributes to the halo bispectrum\(^2\), respect to the case with Gaussian (\( f_{\text{NL}} = 0 \)) initial conditions [see eq. (3.145)]:

\[
B_{\text{hhh}}(k_1, k_2, k_3) = B_{\text{hhh}}^{(\Lambda^\rightarrow \bar{L})}(k_1, k_2, k_3)
\]

\[
\quad - \frac{2}{7} b_{10}^2 b_{10}^L \left( 2P(k_1)P(k_2)S_2(k_1, k_2) + 2 \text{cyc.} \right)_M
\]

\[
\quad - \frac{2}{7} b_{10} b_{10}^L \left( 2P(k_1)P(k_2) \left( \frac{1}{\alpha(k_1)} + \frac{1}{\alpha(k_2)} \right) S_2(k_1, k_2) + 2 \text{cyc.} \right)_N
\]

\[
\quad - \frac{2}{7} b_{10}^2 b_{10}^L \left( \frac{2}{\alpha(k_1)\alpha(k_2)} S_2(k_1, k_2) + 2 \text{cyc.} \right)_O
\]

\[
\quad - b_{10} b_{01} b_{10}^L \left( 2P(k_1)P(k_2) \left( \frac{N_2(k_1, k_2)}{\alpha(k_2)} + \frac{N_2(k_2, k_1)}{\alpha(k_1)} \right) + 2 \text{cyc.} \right)_P
\]

\[
\quad \times \left( \frac{1}{\alpha(k_1)} + \frac{1}{\alpha(k_2)} \right) + 2 \text{cyc.}_Q
\]

\[
\quad - b_{01} b_{01} b_{01}^L \left( 2P(k_1)P(k_2) \left( \frac{N_2(k_1, k_2)}{\alpha(k_2)} + \frac{N_2(k_2, k_1)}{\alpha(k_1)} \right) + 2 \text{cyc.} \right)_R.
\]

\(^2\)It is interesting to note that a local Eulerian bias model which includes the effects of PNG leads to a different result, even at the Lagrangian time, i.e. \( \Psi = 0 \). This is the model used for instance in [22], it correctly recognises that the scale dependence of the bias in the bispectrum case is stronger than the one in the power spectrum, but fails to capture all the physics, as showed in [128], where the mismatch with N-body simulations is evident. On the other hand, the bivariate expansion formulated in the Eulerian frame showed in general good fits against simulations [73, 129]. However, as noted in [70], it does not mean that one approach is superior to the other, but rather that the local approximation is not a good one in the Eulerian biasing scheme.

106
Figure 6.2: The term $A$ is given by the sum of two pieces, which we label $A_1$ and $A_2$. The former is sourced by non-linear gravitational evolution while the latter by PNG. We show them separately, normalized to the maximum value $A$ can take and assuming $z = 0$ and $f_{NL} = 10$. The normalization does not completely cancel the redshift dependence, which is present in $A_2$ through the factor $D(z)^{-1}$. The effect of $A_2$ is visible in the squeezed configuration, where $A_1$ is vanishing and $A_2$ is at its maximum (the logarithmic scale might hide this aspect but a quick look back to fig. 3.5 should clarify this point).

Note that hereafter we suppress the $E$ superscript in the bias factors to simplify the notation but not the $L$ superscript. This allows to keep track of the effects generated by the non-local terms $s^2$ and $n^2$. Note that each contribution to the halo bispectrum of eq. (6.21) is labelled with a letter, running from $A$ to $R$, in accordance with the labelling of BSS; this will make easier in the following to identify them. Also, the schematic source of each term is presented in table 6.1, which naturally extends table 3.1 to the case of PNG.

The quantity $B_{hhh}^{(A \rightarrow L)}$ accounts for the terms with label going from $A$ to $L$; it matches exactly the halo bispectrum model of BSS, that we reproduce in appendix C.

The first line in eq. (C.1), term $A$, comes from linear bias acting on the matter bispectrum. We can split this into two terms

$$A_1(k_1, k_2, k_3) = 2P(k_1)P(k_2)F_2(k_1, k_2) + 2 \text{ cyc.}, \quad (6.22)$$

$$A_2(k_1, k_2, k_3) = 2f_{NL} \frac{P(k_1)P(k_2)\alpha(k_3)}{\alpha(k_1)\alpha(k_2)} + 2 \text{ cyc.}, \quad (6.23)$$

identifying an additional term, $A_2$, with respect to the Gaussian initial conditions case of eq. (3.145), that is proportional to $f_{NL}$. We study the shape dependence of $A_1$ and $A_2$ in fig. 6.2. Again, we plot them separately but normalize to the maximum value taken by $A(k_1, k_2, k_3) = A_1 + A_2$ in the $k_2/k_1, k_3/k_1$-space, thus the relative values of the two plots can be compared. Note that $A_2$ has...
Table 6.1: Schematic sources of the terms, from A to R, contributing to the halo bispectrum with non-Gaussian initial conditions [cf. eq. (6.21)]. Permutations of the components are subtended in the middle column. In the right column it is indicated whether a term is present for $f_{NL} = 0$ (i.e. Gaussian initial conditions) or only when $f_{NL} \neq 0$. The former is abbreviated with G, while the latter with PNG. The abbreviation G + PNG means that a term has contributions coming from both cases.

<table>
<thead>
<tr>
<th>Term</th>
<th>Source</th>
<th>Origin</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$\langle \delta^{(2)} \delta^{(1)} \delta^{(1)} \rangle$</td>
<td>G + PNG</td>
</tr>
<tr>
<td>B</td>
<td>$\langle \delta^{(2)} \varphi \delta^{(1)} \rangle$</td>
<td>PNG</td>
</tr>
<tr>
<td>C</td>
<td>$\langle \delta^{(2)} \varphi \varphi \rangle$</td>
<td>PNG</td>
</tr>
<tr>
<td>D</td>
<td>$\langle \varphi \varphi \varphi \varphi \rangle$</td>
<td>PNG</td>
</tr>
<tr>
<td>E</td>
<td>$\langle \varphi \varphi \delta^{(1)} \delta^{(1)} \rangle$</td>
<td>PNG</td>
</tr>
<tr>
<td>F</td>
<td>$\langle \varphi \varphi \delta^{(1)} \delta^{(1)} \rangle$</td>
<td>PNG</td>
</tr>
<tr>
<td>G</td>
<td>$\langle \varphi \delta^{(1)} \varphi \delta^{(1)} \rangle$</td>
<td>PNG</td>
</tr>
<tr>
<td>H</td>
<td>$\langle \varphi \delta^{(1)} \varphi \delta^{(1)} \rangle$</td>
<td>PNG</td>
</tr>
<tr>
<td>I</td>
<td>$\langle \varphi \delta^{(1)} \delta^{(1)} \delta^{(1)} \rangle$</td>
<td>PNG</td>
</tr>
<tr>
<td>J</td>
<td>$\langle \delta^{(1)} \varphi \delta^{(1)} \varphi \rangle$</td>
<td>PNG</td>
</tr>
<tr>
<td>K</td>
<td>$\langle \delta^{(1)} \varphi \delta^{(1)} \delta^{(1)} \rangle$</td>
<td>PNG</td>
</tr>
<tr>
<td>L</td>
<td>$\langle \delta^{(1)} \varphi \delta^{(1)} \delta^{(1)} \rangle$</td>
<td>G</td>
</tr>
<tr>
<td>M</td>
<td>$\langle s^2 \delta^{(1)} \delta^{(1)} \rangle$</td>
<td>G</td>
</tr>
<tr>
<td>N</td>
<td>$\langle s^2 \delta^{(1)} \varphi \rangle$</td>
<td>PNG</td>
</tr>
<tr>
<td>O</td>
<td>$\langle s^2 \varphi \varphi \rangle$</td>
<td>PNG</td>
</tr>
<tr>
<td>P</td>
<td>$\langle n^2 \delta^{(1)} \delta^{(1)} \rangle$</td>
<td>PNG</td>
</tr>
<tr>
<td>Q</td>
<td>$\langle n^2 \delta^{(1)} \varphi \rangle$</td>
<td>PNG</td>
</tr>
<tr>
<td>R</td>
<td>$\langle n^2 \varphi \varphi \rangle$</td>
<td>PNG</td>
</tr>
</tbody>
</table>
Figure 6.3: Shape dependence of the terms, from B to F, contributing to the halo bispectrum. These are generated by non-Gaussian initial conditions. A value $f_{\text{NL}} = 10$ and redshift $z = 0$ are assumed. Each term is normalized to the maximum it can take. This choice does not completely cancel the redshift dependence in the terms B and C, where it is present as a factor $D(z)^{-1}$, and does not allow a comparison of the amplitude between different plots. We clearly see that the effect of PNG is prominent in the squeezed configuration.

An extra $1/\alpha \propto 1/D(z)$ factor compared to $A_1$. Hence the relative amplitude of the primordial nG contribution $A_2$ grows with redshift relative to the term $A_1$ generated by non-linear evolution even without PNG. Figure 6.2 highlights the interesting shape dependence of $A_2$; it peaks in the extremely squeezed configuration (top left), exactly where the $A_1$ term vanishes (see also fig. 3.5).

We emphasize that $A_2$ and the terms from B to K are generated by primordial non-Gaussianity. We plot the shape dependence of the terms B to F in fig. 6.3 and the term G to K in fig. 6.4. They confirm what we have previously stated: the PNG terms are greatest in the squeezed configuration.

As a result of the presence of $s^2$ and our new term $n^2$ in the expression for $\delta_h^2$ (see eq. (6.12)), additional terms appear with respect to BSS. M, N and O that are generated by the tidal term $s^2$; schematically, they are sourced by
Figure 6.4: The terms contributing to the halo bispectrum, with label going from G to K. These are generated by non-Gaussian initial conditions. A value $f_{\text{NL}} = 10$ is assumed. Each term is normalized to the maximum it can take. This choice completely cancels the redshift dependence and does not allow a comparison of the amplitude between different terms. We clearly see that the effect of PNG is prominent in the squeezed configuration.
Figure 6.5: Shape dependence of the terms with label going from N to R contributing to the halo bispectrum. These are the additional terms generated by the presence of $s^2$ and $n^2$ in $\delta_h^\rho$. A value $f_{NL} = 10$ is assumed. Each term is normalized to maximum value it can take so that the redshift dependence is completely dropped; note however that the different scaling does not allow one to compare the amplitude between plots. Since these terms can take negative values, we show their absolute value. Actually, the blue line indicates where they change sign, with the top right part of the plots being positive.

$\langle s^2 \delta^{(1)} \delta^{(1)} \rangle$, $\langle s^2 \delta^{(1)} \phi \rangle$ and $\langle s^2 \phi \phi \rangle$ respectively. We have seen in eq. (3.145) that $M$ is present regardless of the presence of PNG, but $N$ and $O$ come from the coupling between the tidal term and $\phi_G$ for $f_{NL} \neq 0$. On the other hand, $P$, $Q$ and $R$ are generated by $n^2$. They account for the non-local effect of the potential $\phi_G$ and are therefore due to the presence of PNG. Schematically, they are generated by $\langle n^2 \delta^{(1)} \delta^{(1)} \rangle$, $\langle n^2 \delta^{(1)} \phi \rangle$, $\langle n^2 \phi \phi \rangle$ respectively. In fig. 6.5 we show the terms from N to R. Since they can take negative values, we plot their absolute value. Interestingly, in all of the plots we can identify a violet strip, indicating a change of sign and, hence, where the kernels $S_2$ and $N_2$ make each term vanish.
Figure 6.6: The relative difference in absolute value [eq. (6.24)] between our bispectrum of eq. (6.21) and the one by Baldauf et al ($B_{\text{BSS}} = B_{hhh}^{(A \rightarrow L)}$), assuming $M = 10^{13} h^{-1} M_\odot$ and $f_{\text{NL}} = 10$. Note that we saturate differences above 10% to the same red colour of the palette; as the plots range between different values, this choice allows us to present them in a compact way, highlighting where the most relevant differences are expected. However we do not observe discrepancies above 25%. Interestingly, the squeezed limit is not affected while the other configurations show differences from only a few percent. A no-difference region going approximately from the squeezed to the isosceles configurations is present, following the shape dependence that we showed in figs. 3.5 and 6.5.

### 6.3 Analytic estimates

In this section we further investigate our result for the halo bispectrum and, in particular, we compare it to our reference model BSS [73]. Since the model of BSS has shown a good fit against simulations [73, 129], we want to understand if and where differences between these two models arise.

In fig. 6.6 we plot the absolute value of the relative difference between our halo bispectrum $B_{hhh}$ of eq. (6.21) and that of BSS ($B_{\text{BSS}} = B_{hhh}^{(A \rightarrow L)}$),

$$\text{Diff}(k_1, k_2, k_3) = \left| \frac{B_{hhh}(k_1, k_2, k_3) - B_{\text{BSS}}(k_1, k_2, k_3)}{B_{\text{BSS}}(k_1, k_2, k_3)} \right|,$$

for values of $k_1 = 0.01, 0.05, 0.1h\text{Mpc}^{-1}$ and redshift $z = 0, 0.5, 1$. We consider halos of mass $M = 10^{13} h^{-1} M_\odot$ and primordial non-Gaussianity with $f_{\text{NL}} = 10$. We choose to saturate differences above 10% to the same red colour of the palette for the purpose of presenting many different plots with different ranges.
of values in a compact way. However we do not observe differences above 25%.

For the value of the halo mass and $f_{\text{NL}}$ considered, we find that the terms $M$ and $P$ are the main sources of the differences, with all the other terms contributing very little or having a negligible effect. The most relevant differences appear for $k_1 = 0.01 \, h \text{Mpc}^{-1}$: up to 25% in the elongated, folded and equilateral regions for all the redshifts considered. These discrepancies drop to a few percent for $k_1 = 0.05 \, h \text{Mpc}^{-1}$ at $z = 0$, while for $z = 0.5, 1$ they are reduced to about 5% when approaching the equilateral configuration and to order 10% in the elongated and folded regions. For $k_1 = 0.1 \, h \text{Mpc}^{-1}$ we observe a similar pattern, but at $z = 0.5$ the approximately 10% difference area that was present in the elongated and folded regions for $k_1 = 0.05 \, h \text{Mpc}^{-1}$ is almost completely washed out, decreased to about 5%. However, that area is still present for $z = 1$, although reduced in size. We also recognise an area where the difference is close to zero [the violet region going approximately from the squeezed (top left) to the isosceles (bottom right) configuration] corresponding to a vanishing contribution from the terms $M$ to $R$, as shown in figs. 3.5 and 6.5. Interestingly, in all plots the squeezed configuration is unaffected.

We can understand the features that we have just described by studying the analytic solutions of the halo bispectrum in three simple configurations:

- In the equilateral configuration, where $k_1 = k_2 = k_3 = k$, the halo bispectrum becomes

$$B_{hhh} = \left\{ \begin{array}{l}
\frac{b_{01}^2}{7} (12b_{10} + b_{10}^L + 42b_{20}) \\
+ \frac{1}{\alpha(k)} \left[ \frac{b_{01}^2}{7} b_{10} b_{10} (12b_{10} + b_{10}^L + 42b_{20}) + 3b_{10}^2 \left( b_{01}^L + 2( f_{\text{NL}} b_{10} + b_{11}) \right) \right] \\
+ \frac{1}{\alpha(k)^2} \left[ b_{01}^2 (12b_{10} + b_{10}^L + 42b_{20}) + 6b_{01} b_{10} ( b_{01}^L + 2( f_{\text{NL}} b_{10} + b_{11}) ) + 6b_{01}^2 b_{10}^2 \right] \\
+ \frac{3}{\alpha(k)^3} \left[ b_{01}^2 \left( b_{01}^2 + 2( f_{\text{NL}} b_{10} + b_{11}) \right) + 84b_{01} b_{02} b_{10} \right] + \frac{6b_{01} b_{02}}{\alpha(k)^4} \right\} P(k)^2
\end{array} \right. \tag{6.25}$$

We remind the reader that the contributions of $b_{01}^2$ and $b_{01}^L$ indicate where the non-local, non-linear terms of our model ($s^2$ and $n^2$) are introducing differences respect to the BSS model. By looking at eq. (6.21) we notice that the terms $M$, $N$, $O$, $P$, $Q$, $R$ are respectively linked to the bias com-

\footnote{An exception is the top left plot of fig. 6.6, where there is a small curve close to the squeezed configuration in which the discrepancy is actually bigger than that. This is because in that area the BSS halo bispectrum crosses zero for these values of the bias coefficients.}
bimensions $b_{10}^2 b_{10}^L$, $b_{10} b_{10}^L$, $b_{10}^3 b_{10}^L$, $b_{10}^2 b_{10}^L$, $b_{10} b_{10}^L$, $b_{10}^2 b_{10}^L$, so that we can actually recognize each of them in eq. (6.25); M appears in the first line of eq. (6.25), while N and P appear in the second line, O and Q in the third and R in the first term on the fourth line.

The presence of a larger difference at high redshift and on large scales can be explained by noting that the function $\alpha(k)$ takes smaller values in those regimes and enhances the terms inside the square brackets [see eq. (4.21) and fig. 4.1] and, therefore, accentuates the effect of $b_{10}^L$ and $b_{10}^L$. 

- In the folded configuration, $k_2 = k_3 = k$ and $k_1 = 2k$, the halo bispectrum can be written as

$$B_{\text{hhh}} = \left\{ 2b_{10}^3 (2 - \Pi^2) + b_{10}^2 \left( 2b_{20} - \frac{8}{21} b_{10}^L \right) (1 + 4\Pi^2) + \frac{1}{\alpha(k)} \left[ 10b_{10} f_{\text{NL}} \Pi^2 + b_{10} \left( 8 - \frac{\Pi}{2} - 2\Pi^2 \right) + 2b_{10}^L \left( -1 + \Pi + \Pi^2 \right) + b_{10} \left( 2b_{20} - \frac{8}{21} b_{10}^L \right) (2 + \Pi + 4\Pi^2) \right] ight\} P(k)^2,$$

(6.26)

where we define $\Pi = T(2k)/T(k)$ to make the notation more compact. The full expression is long so we have given just the first two terms. These are enough to explain what we observe in fig. 6.6. Again, $b_{10}^L$ and $b_{10}^L$ are present. As for the equilateral configuration, the effect of M appears in the first term, while N and P in the second one, O and Q in the third and R in the fourth one. We see that the effect of $b_{10}^L$ and $b_{10}^L$ can be enhanced depending on the value of $\Pi$. When $k_1 = 0.01h \text{Mpc}$ the ratio $\Pi \approx 1$, while for the other two values of $k_1$ the ratio $\Pi < 1$. This suggests why bigger differences should be expected in the case with $k_1 = 0.01h \text{Mpc}$.

The same considerations apply to the function $\alpha(k)$ as in the equilateral case, so that the differences are larger at high redshift and on large scales.

- A simple expression for the halo bispectrum in the squeezed limit can be found by setting $k_1 = k_2 = \epsilon k_3 = k$ with $\epsilon \gg 1$, corresponding to an isosceles triangle, whose degree of squeezing is controlled by the parameter $\epsilon$. For squeezed triangles (large values of $\epsilon$), the leading term in the
The absence of terms involving bias coefficients $b_{L10}$ and $b_{L01}$ explains why our predictions do not differ from the BSS model in extremely squeezed configurations.

### 6.4 Summary

An important application of measurements of large-scale structure in our Universe is to determine the distribution of primordial perturbations, in particular possible non-Gaussian signatures of scenarios for the origin of structure in the very early universe. For example, the shape information contained in the primordial bispectrum is a valuable tool to discriminate between different inflationary models. The matter bispectrum at later times is due to a combination of both primordial non-Gaussianity and non-linear evolution under gravity. However, the way in which gravitationally collapsed halos trace the density field, the bias model, can enhance the effect of primordial non-Gaussianity in the galaxy distribution. In particular, local-type non-Gaussianity leads to a scale-dependent bias which can have a dramatic effect on very large scales in both the halo power spectrum \cite{95,96} and the halo bispectrum \cite{91}.

In this chapter we applied a local Lagrangian biasing scheme to a general set-up with local-type primordial non-Gaussianity, focussing on second-order, non-local and non-Gaussian effects. For an $f_{\text{NL}}$ cosmology, this biasing scheme is in the bivariate form of \cite{72}, i.e. the halo overdensity is expanded in terms of the linear matter overdensity in the Lagrangian frame and the primordial Gaussian potential.

Non-linear evolution of the matter field in general gives rise to second-order terms in the matter density which include non-local, tidal terms [see eq. (3.90)], while transforming from the Lagrangian to the Eulerian frame introduces a non-local convective term at second order [see eq. (3.92)]. Non-local here implies terms derived from derivatives of the potential, not directly from the local density or its derivatives. Both these terms are included in the usual kernel, $\mathcal{F}_2$, for the second-order density in Eulerian space. But since the halo density is determined by the linear matter overdensity in the Lagrangian frame, one must
account separately for these non-local terms to reconstruct the halo density at late times. These terms are absent at early times, or if we restrict ourselves to the spherical collapse approximation.

We have shown in this chapter that in the bivariate expansion we must also account at second order for the convective term relating the primordial potential in the Lagrangian frame to that in Eulerian space at later times [see eq. (6.8)]. This gives rise to a new term in the halo bispectrum in the presence of local-type primordial non-Gaussianity which has not previously been studied as far as we are aware.

Setting $f_{\text{NL}} = 0$, we are able to recover the halo bispectrum model of eq. (3.145) [74], when a local Lagrangian biasing scheme is applied. Three terms appear in the halo bispectrum (A, L and M) that are sourced by (A) the non-linear matter density encoded in the kernel $F_2$, (L) the non-linear bias $b_{20}$ and (M) the tidal term $s^2$.

Generalising to $f_{\text{NL}} \neq 0$, we found 12 terms in the halo bispectrum (labelled A to L) matching the BSS model [73]. In this case, the non-linear matter density term, A, includes a correction due to PNG, while the non-linear bias term, L, is left unchanged. The other contributions (B to K) come from a mixture of bivariate terms, involving the bias coefficients $b_{01}$, $b_{11}$, $b_{02}$. We also found a contribution, M, sourced by the tidal term, $s^2$, which also couples with terms in the halo overdensity that are specifically due to PNG and, hence, generate new contributions, N and O, in the halo bispectrum. The new convective term, $n^2$, also generates contributions in the presence of primordial non-Gaussianity; in the halo bispectrum we have found three new contributions, P, Q and R, due to this term.

In order to investigate the magnitude and shape of the various contributions to the bispectrum, we have implemented a version of the Sheth-Tormen mass function corrected for PNG in light of the Lo Verde et al mass function, following BSS [73]. This allowed us to numerically calculate the bias coefficients for our model and predict the halo bispectrum in different configurations for sample values of $f_{\text{NL}}$ and at various scales and redshifts.

We investigated the halo bispectrum by comparing it to the fiducial model of BSS. Assuming halos of mass $M = 10^{13} h^{-1} M_{\odot}$ and $f_{\text{NL}} = 10$, we found:

- at redshift $z = 0$ differences up to 25% in the halo bispectrum for $k_1 = 0.01 h \text{Mpc}^{-1}$ in the elongated and folded configurations and approximately 7–8% when approaching the equilateral configuration, while these drop to a few percent when $k_1 = 0.05$ or $0.1 h \text{Mpc}^{-1}$.

- At redshift $z = 0.5$ differences up to 25% for $k_1 = 0.01 h \text{Mpc}^{-1}$ in the
elongated, folded and equilateral configurations. When $k_1 = 0.05 h \text{Mpc}^{-1}$
differences of order 10\% are still visible in the elongated and folded shapes, while they decrease to approximately 5\% towards the equilateral configuration. For $k_1 = 0.1 h \text{Mpc}^{-1}$ these differences reduce to about 5\%.

- At redshift $z = 1$ we find results similar to those for $z = 0.5$, except that differences of order 10\% are still visible in the elongated and folded regions for $k_1 = 0.1 h \text{Mpc}^{-1}$.

In general, we observe that the non-local terms have a negligible effect in extremely squeezed configurations.

Our results indicate that the non-local terms in the halo overdensity could have a significant contribution to the galaxy bispectrum, especially on large scales and at high redshift.
Chapter 7

Galaxy bispectrum, primordial non-Gaussianity and redshift space distortions

In this chapter we consider the primordial potential $\Phi_{\text{in}}(x)$ with local non-Gaussianity of the form

$$
\Phi_{\text{in}}(x) = \varphi_G(x) + f_{\text{NL}} \left( \langle \varphi_G^2 \rangle - \langle \varphi_G \rangle \right). 
$$

(7.1)

Studies of the galaxy bispectrum indicate that an accuracy in determining local $f_{\text{NL}}$ of order $\sigma_{f_{\text{NL}}}$ ~ few is achievable \[73, 138, 139\]. The accuracy achievable could even be less than one if the survey is optimised for detecting primordial non-Gaussianity \[151\]. Considering such higher-order statistics gives access to the full shape information of the non-Gaussian signal, with the primordial one having a scale dependence even stronger than $k^{-2}$ \[42, 73, 91\] (see also section 6.3). Additionally, information contained in the bispectrum potentially allows us to break the degeneracy in the power spectrum between $f_{\text{NL}}$ and the next-order non-Gaussian parameter $g_{\text{NL}}$ \[126\], as discussed in chapter 5 (or equivalently \[91\]).

On the other hand, there are significant challenges in measuring $f_{\text{NL}}$ with LSS that are both theoretical and observational. At the theoretical level, as the Universe evolves density perturbations undergo non-linear evolution through gravitational collapse, and therefore we require an accurate modelling of the
density evolution, capable of separating the primordial non-Gaussian signal from the one generated by clustering, as discussed in chapters 4 and 6. Moreover, a precise description of how dark matter halos form starting from the primordial density field is necessary. Further, the high accuracy required for measuring a signal of $f_{NL} \sim 1$ implies that GR effects cannot be neglected. Although in the simplest halo models they do not give rise to a scale-dependent bias [152–154], GR effects do source contributions to the squeezed limit of the matter bispectrum [24, 26, 155] and generate secondary non-Gaussianities along the path of the photons from the emitting galaxy and the observer, in analogy with the CMB (see for instance [122, 156] and references therein).

At the observational level, several issues should be taken into account, such as mask geometry and systematic effects, which can mimic the scale dependence of PNG [101–103]. Moreover, redshift space distortions (RSD) are an additional source of complexity [157]: since the redshift measurements used to infer the distances of galaxies are contaminated by peculiar velocities, distortions appear along the line of sight. They can either be due to the in-fall of galaxies into clusters or due to the velocity dispersion inside a cluster, when its non-linear structure is resolved. The former leads to an apparent squashing of the clustering along the line of sight on large scales (say $k \lesssim 0.1 \, h/\text{Mpc}$), modelled at linear level through the Kaiser factor [158], while the latter is responsible for elongation on small scales (say $k \gtrsim 0.1 \, h/\text{Mpc}$), usually referred to as Fingers of God (FoG).

In this chapter we address the problem of computing the galaxy bispectrum in redshift space with primordial non-Gaussianity of local type. For the purpose of obtaining an analytic result, we focus mainly on large-scale regimes. We point out new potentially significant effects, induced by primordial non-Gaussianity, associated with large-scale amplifications of RSD. By decomposing the line of sight dependence of the bispectrum into spherical harmonics, we also make a prediction for the galaxy monopole, motivated by the recent measurement of [147].

7.1 Galaxy overdensity in the Eulerian frame

The bivariate model describes the statistics of the objects in the Lagrangian frame, while their dynamics are obtained by the transformation to Eulerian coordinates. As we explained previously (see section 3.1.3), the two frames are linked by $\mathbf{x}(\mathbf{q}, \tau) = \mathbf{q} + \mathbf{\Psi}(\mathbf{q}, \tau)$, where the displacement field $\mathbf{\Psi}$ is the dynamical quantity in the Lagrangian picture.
In [72, 73], the halo/galaxy overdensity in the Eulerian frame is obtained by assuming spherical collapse. By dropping this assumption, we got a more general (non-local) result in chapter 6 (or equivalently [159]):

$$\delta^E_g(k) = b^E_{10}\delta + b^E_{01}\varphi_G + b^E_{20}\delta \ast \delta + b^E_{02}\varphi_G \ast \varphi_G - \frac{2}{7}b^L_{10}s^2 - b_{01}n^2,$$

(7.2)

where the Eulerian bias coefficients are related to Lagrangian ones through the relations previously given in eq. (6.11). As usual, we will drop the superscript E to indicate the Eulerian bias coefficients, in order to simplify the notation. However, we continue to write the superscript L for Lagrangian where needed for avoiding confusion.

The Fourier transform of the density field in Eulerian coordinates follows from the discussion of section 4.2.1,

$$\delta(k) = \delta_G(k) + \int \frac{dq}{(2\pi)^3} \left[ \mathcal{F}_2(q, k - q) + f_{NL} \frac{\alpha(k)}{\alpha(q) \alpha(|k - q|)} \right] \delta_G(q) \delta_G(k - q),$$

(7.3)

while the tidal term $s^2$ reads [36, 74]

$$s^2(k) = \int \frac{dq}{(2\pi)^3} S_2(q, k - q) \delta_G(q) \delta_G(k - q),$$

(7.4)

with the kernel defined as follows

$$S_2(k_1, k_2) = \frac{(k_1 \cdot k_2)^2}{k_1^2 k_2^2} - \frac{1}{3}.$$  

(7.5)

The non-Gaussian shift term $n^2$ in eq. (7.2) is a consequence of the displacement of halos/galaxies respect to their initial positions $q$ in the Lagrangian frame, which affects the field $\varphi_G(x)$:

$$n^2(k) = 2 \int \frac{dq}{(2\pi)^3} N_2(q, k - q) \frac{\delta_G(q) \delta_G(k - q)}{\alpha(|k - q|)},$$

(7.6)

where the kernel is

$$N_2(k_1, k_2) = \frac{k_1 \cdot k_2}{2k_1^2}.$$  

(7.7)

We use the standard definitions for the galaxy power spectrum $P_{gg}$ and bispectrum $B_{ggg}$ given in eqs. [3.94] and [3.95] but we omit to write the explicit dependence on the redshift $z$ and the mass scale $M$, through the filter function, of the power spectrum and bispectrum to make the notation less cumbersome (see section 3.2.1). At tree-level, they can be conveniently written as

$$P_{gg}(k_1) = E^2_1(k_1)P(k_1)$$

(7.8)

$$B_{ggg}(k_1, k_2, k_3) = 2E_1(k_1)E_1(k_2)E_2(k_1, k_2)P(k_1)P(k_2) + 2 \text{ cyc.},$$

(7.9)
where $P(k)$ is the linear matter power spectrum for the Gaussian source field $\varphi_G$, while the kernels $E_i$ are defined as

$$E_{1}(k) = b_{10} + \frac{b_{01}}{\alpha(k)}, \quad (7.10)$$

$$E_{2}(k_1, k_2) = b_{10} \left[ F_{2}(k_1, k_2) + f_{\text{NL}} \frac{\alpha([k_1 + k_2])}{\alpha(k_1)\alpha(k_2)} \right] + \left[ b_{20} - \frac{2}{7} b_{10} \alpha(k_1) \alpha(k_2) \right]$$

$$+ \frac{b_{11}}{2} \left[ \frac{1}{\alpha(k_1)} + \frac{1}{\alpha(k_2)} \right] + \frac{b_{02}}{\alpha(k_1)\alpha(k_2)} - b_{01} \left[ \frac{N_2(k_1, k_2)}{\alpha(k_2)} + \frac{N_2(k_2, k_1)}{\alpha(k_1)} \right], \quad (7.11)$$

The term $b_{01}/\alpha(k_1) \propto f_{\text{NL}}/k_1^{2}$ in $E_1$ is the so-called scale-dependent bias: it is responsible for deviations on the large-scale clustering, with respect to the scale-independent bias $b_{10}$. The first and second term that appears in $E_2$ account for non-linear clustering and non-linear biasing respectively, while the terms in the last two lines describe the non-linear effects due to PNG. A detailed analysis of the results for the bispectrum is presented in chapter 6 (or equivalently [159]).

To conclude this section, let us point out that for avoiding the complexities of a full bispectrum measurement, a new observable is proposed in [160] in terms of position dependent power spectrum. We explore this possibility in appendix E, in light of the result of eq. (7.9).

### 7.2 Redshift space distortions

The peculiar velocities of galaxies contaminate the redshift measurements of surveys, resulting in distortions along the line of sight. The in-fall of galaxies into clusters is responsible for large-scale distortions, while the velocity dispersion inside a cluster leads to the FoG, usually a small-scale effect. In this chapter, for the first time, we study how PNG affects the bivariate halo distribution when formulated in redshift space, finding new and potentially sizeable large scale effects. We model RSD within a perturbative approach and focus mainly on the large scale effects, for the purpose of obtaining analytic results.

An object at some position $x$ and redshift $z$ appears in redshift space at position [161]:

$$x_s(z) = x + (1 + z) \frac{\mathbf{v}(x)}{H(z)} \hat{x} = x + \frac{v_s(x)}{H(z)} \hat{z} \approx x + f u_s(x) \hat{z}. \quad (7.12)$$

In the second line we use $f \equiv d\ln D/d\ln a$, and we introduce the reduced
component along the line of sight, \( u_z \), given by

\[
 u_z(k) = u(k) \cdot \frac{\hat{z}}{k} = \frac{i\mu \theta(k)}{k} = \frac{i\mu}{k} \eta(k),
\]  

(7.13)

where \( \mu \equiv \hat{k} \cdot \hat{z} = k \cdot \hat{z}/k \). The second line of eq. (7.12) holds under the plane parallel (or distant observer) approximation. If the distances between galaxies are much smaller than the distance between the observer and the galaxies (resulting therefore in a small transverse component with respect to the radial direction) the line of sight \( \hat{x} \) can be assumed to be fixed along \( \hat{z} \), pointing towards the centre of the galaxies of interest. When used to compute the galaxy power spectrum, this approximation has been shown to be valid for pairs separated by an angle less than \( 10^\circ \) [162]. If one has to use the full data from surveys such as BOSS and Euclid, where distances between galaxies can be comparable with the observer distance, then wide angle effects must be considered as well. This problem has been addressed in several works with various approaches (see for instance [162,170]), investigated in numerical simulations (for example in [171]) and the impact on measurements considered [172,173]. However, in this first work about the effect of PNG on the bispectrum in redshift space, as predicted by the bivariate model, we will neglect for simplicity wide angle effects. Since we will show that PNG could be enhanced by RSD on large scales, in regimes where wide angle effects may not be negligible, the results of section 7.2 should be considered as a zero order approximation of a more general framework which combines PNG, RSD and wide angle effects. The development of this framework is left for future work.

Equation (7.13) approximates the redshift space mapping of eq. (7.12) as a power series. By comparing eq. (7.13) with eq. (3.38), it follows that

\[
 \eta(k) = \sum_{n=1}^{\infty} \int \frac{dk_1}{(2\pi)^3} \cdots \int \frac{dk_{n-1}}{(2\pi)^3} \int \frac{dk_n}{(2\pi)^3} \delta^D(k - k_1 - \cdots - k_n) \times \\
 \times G_n(k_1, \ldots, k_n) \delta_{\text{lin}}(k_1) \cdots \delta_{\text{lin}}(k_n).
\]

(7.14)

7.2.1 Galaxy overdensity in redshift space

The transformation from real to redshift space is obtained by requiring the conservation of the number density of objects,

\[
 [1 + \delta^s_g(x_s)] dx_s = [1 + \delta^F_g(x)] dx,
\]

(7.15)

so that in Fourier space we have [174]

\[
 \delta^s_g(k) = \int d\mathbf{x}_s \delta^s_g(x_s) e^{-i\mathbf{k} \cdot \mathbf{x}_s}.
\]

(7.16)
where eq. (7.19) holds under the assumption of a small velocity component along the line of sight, $u_z \to 0$.

In general, the redshift-space galaxy overdensity can be written as
\[
\delta_g^E(k) = \sum_{n=1}^{\infty} \int \frac{dk_1}{(2\pi)^3} \cdots \int \frac{dk_{n-1}}{(2\pi)^3} \int \frac{dk_n}{(2\pi)^3} \int d\mathbf{k}_n \delta^E(k - \mathbf{k}_1 - \cdots - \mathbf{k}_n) \times
\]
\[
Z_n(k_1, \ldots, k_n) \delta_G(k_1) \cdots \delta_G(k_n),
\]
(7.21)
where the redshift space kernels $Z_n(k_1, \ldots, k_n)$ are, up to second order,
\[
Z_1 = b_{10} \left(1 + \beta \mu^2\right) + \frac{b_{01}}{nG1},
\]
(7.22)
\[
Z_2 = b_{10} \left[F_2(k_1, k_2) + f_{NL} \frac{\alpha(k)}{\alpha(k_1)\alpha(k_2)}\right] + \left[b_{20} - \frac{2}{7} b_{10} S_2(k_1, k_2)\right]
\]
\[
+ \frac{b_{11}}{2} \left[\frac{1}{\alpha(k_1)} + \frac{1}{\alpha(k_2)}\right] + \frac{b_{02}}{\alpha(k_1)\alpha(k_2)} + \frac{b_{01}}{\alpha(k)} \left[N_2(k_1, k_2) + N_2(k_2, k_1)\right]
\]
\[
+ f\mu^2 \left[G_2(k_1, k_2) + f_{NL} \frac{\alpha(k)}{\alpha(k_1)\alpha(k_2)}\right] + \frac{f^2 k^2 \mu^2}{2} \frac{\mu_1 \mu_2}{k_1 k_2} + b_{10} \frac{f\mu k}{2} \left[\frac{\mu_1}{k_1} + \frac{\mu_2}{k_2}\right]
\]
\[
+ b_{01} \frac{f\mu k}{2} \left[\frac{\mu_1}{k_1 \alpha(k_2)} + \frac{\mu_2}{k_2 \alpha(k_1)}\right],
\]
(7.23)
with $\mu_i \equiv \hat{k}_i \cdot \hat{z} = k_i \cdot z/k_i$. These kernels $Z_i$ are important since they allow us to express the tree-level galaxy power spectrum and bispectrum in redshift space, by replacing $E_{1(2)} \to Z_{1(2)}$ in eqs. (7.8) and (7.9):
\[
P^s_{gg}(k_1) = Z_{1g}^2(k_1)P(k_1),
\]
(7.24)
\[
B^s_{ggg}(k_1, k_2, k_3) = 2Z_1(k_1)Z_1(k_2)Z_2(k_1, k_2)P(k_1)P(k_2) + 2 \text{ cyc}.,
\]
(7.25)
including the effects of PNG. The resulting expression for $B^s_{ggg}$ — whose physics we will discuss more in detail in the next section — represents one of the main
results of this chapter. At this stage we can make some considerations with respect to the various contributions to the $Z_i$, which we will then develop in what comes next. Clearly, the angular dependence $\mu_j$ in the kernels $Z_i$ breaks the isotropy of the clustering along the line of sight.

Considering $Z_1$, it contains the bias $b_{10}$ and the scale-dependent correction ($nG1$). At linear level, RSD introduce the quantity $\beta \mu^2 \geq 0$, explaining why objects are more clustered in redshift space (compared to real space). The term $(1 + \beta \mu^2)$ is often referred to as the Kaiser factor \cite{158}, where $\beta = f/b_{10}$ and is regularly accounted for in studies of galaxy clustering (e.g. \cite{176}).

For $Z_2$, we notice that the contributions that we label with SQ1 (linear squashing), NLB (non-linear bias), nG2 (second order non Gaussian effects) are already present in the expressions for $E_2$, eq. (7.11), controlling the bispectrum in real space. On the other hand, the remaining three contributions are generated by redshift space distortions. The quantities SQ2 (second order squashing) and FOG are already well studied in the literature.

Interestingly, we notice the presence of a qualitatively new term (FoGnG), induced by PNG. It mimics the FOG contribution, but with an amplification of $1/\alpha(k)$. In a sense, it is an analogue for RSD of the scale-dependent bias of the galaxy power spectrum induced by PNG. The FoGnG term is sourced by the coupling between $u_z$ and $\phi_G$ (first integrand in eq. (7.20)), potentially affecting large-scale measurements. Therefore, neglecting it would introduce a systematic error, resulting in a biased $f_{NL}$ measurement from the tree-level of the bispectrum.

As pointed out in \cite{28}, the galaxy overdensity of eq. (7.21) is the result of two approximations: one is the power series expansion [eq. (7.13)] of the redshift space mapping [eq. (7.12)], the other one is the perturbative expansion of $\delta(k)$ and $\theta(k)$ [eqs. (3.37) and (3.38)]. Therefore, the perturbation theory in redshift space is expected to break down on larger scales than in real space. However, replacing the kernels with effective kernels calibrated against simulations can extend the validity of the results based on eq. (7.21), as shown in \cite{177}. Although we will not implement these techniques here, the replacement is straightforward.

1 A common factor $D_{\Phi G}^B(k_1,k_2,k_3,\sigma_{\Phi G}[z])$ is usually included in the r.h.s of eq. (7.23), accounting for the FOG damping due to intra-cluster velocity dispersion, beyond linear level \cite{179}. This phenomenological extension, which describes N-body data, will not be considered here for the purpose of getting an analytic result in the next section.

2 At the power spectrum level, the FoGnG term enters as a loop correction. However, we do not consider its consequences in this chapter, since we only focus on large scale effects.
7.2.2 Galaxy bispectrum monopole

In this section, we investigate the galaxy bispectrum monopole, i.e. the angle averaged bispectrum along the line of sight direction. The reason is to extend the monopole model used in the recent measurement by Gil-Marín et al. [147, 148, 177] to the case of PNG. The result of this section can thus be applied in a similar analysis, aiming to measure $f_{\text{NL}}$.

The galaxy bispectrum in redshift space is a function of five variables: three of them (say $k_1$, $k_2$ and $\hat{k}_1 \cdot \hat{k}_2 = \cos \theta_{12}$) fully define the shape of the triangle, while the polar angle $\omega = \arccos \mu_1$ and the azimuthal angle $\phi$ about $\hat{k}_1$ describe how it is oriented with respect to the line of sight. All the angles between the vectors $k_1$, $k_2$, $k_3$ and the line of sight $\hat{z}$ can be written in terms of $\mu_1$ and $\phi$ [175]:

\begin{align}
\mu_1 &= \cos \omega = \hat{k}_1 \cdot \hat{z}, \\
\mu_2 &= \mu_1 \cos \theta_{12} - \sqrt{(1 - \mu_1^2) \sin \theta_{12} \cos \phi}, \\
\mu_3 &= -\frac{k_1}{k_3} \mu_1 - \frac{k_2}{k_3} \mu_2.
\end{align}

The ($\mu_1, \phi$)-dependence introduced by redshift space distortions can be conveniently decomposed into spherical harmonics,

\[ B_{ggg}(k_1, k_2, \omega, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} B_{ggg}^{(l,m)}(k_1, k_2) Y_{lm}(\omega, \phi). \]  

As we mentioned, for simplicity we focus only on the monopole ($l = 0, m = 0$), i.e. the average over all the possible orientations of the bispectrum with respect to the line of sight,

\[ B_{ggg}^{(0,0)}(k_1, k_2) = \frac{1}{4\pi} \int_{-1}^{+1} d\mu_1 \int_{0}^{2\pi} d\phi B_{ggg}^{(0,0)}(k_1, k_2, \omega, \phi), \]

although the large-scale enhancement of PNG in redshift space, associated with the term called FoGnG (see eq. (7.23)), is found also in higher multipoles, since it is not cancelled by angular integrations.

We start by quoting the result for Gaussian initial conditions ($f_{\text{NL}} = 0$) [175, 177]:

\[ B_{ggg}^{G(0,0)}(k_1, k_2) = b_{10}^4 \left\{ \frac{1}{b_{10}^2} F_2(k_1, k_2) D_{S1} + \frac{1}{b_{10}^2} G_2(k_1, k_2) D_{S2} + \frac{b_{20}^2}{b_{10}^2} S_2(k_1, k_2) \right\} P(k_1)P(k_2) + 2 \text{ cyc.} \]  

\[ + \left\{ \frac{b_{20}^2}{b_{10}^2} - \frac{2}{7} \frac{b_{10}^4}{b_{20}^2} \right\} P(k_1)P(k_2) + 2 \text{ cyc.}. \]
The terms $D_{SQ1}$ and $D_{SQ2}$ represent the linear and non-linear contributions to the large-scale squashing, $D_{NLB}$ the non-linear bias contribution and, finally, $D_{FOG}$ accounts for the linear part of FoG, i.e. the damping effect due to velocity dispersion. The labelling that we introduced in eqs. (7.22) and (7.23) helps to understand where these factors come from: schematically $D_{SQ1}$ is the result of the angular average of the Kaiser factor squared times the term $SQ1$, $D_{SQ2}$ of the Kaiser factor squared times $SQ2$ and so on. Appendix F further clarifies these points, with explicit expressions for the $D$-factors. Here and after we omit their explicit dependence $D(k_i, k_j, \cos \theta_{ij}, y_{ij}, \beta)$ to simplify the notation, but they are among the quantities to be permuted.

We now generalise the previous result to the case of local-type PNG; it can be written as

$$B_{999}^{(0,0)}(k_1, k_2) = b_{10} b_{01} b_{01} b_{01} \left\{ \begin{array}{l}
F_2(k_1, k_2) + f_{NL} \frac{\alpha(k_1)}{\alpha(k_1) \alpha(k_2)} D_{SQ1} R_{SQ1} + \\
+ \frac{1}{b_{10}} \left[ g_2(k_1, k_2) + f_{NL} \frac{\alpha(k_1)}{\alpha(k_1) \alpha(k_2)} \right] D_{SQ2} R_{SQ2} + \\
+ \left[ \frac{b_{10}}{b_{01} b_{01}} - \frac{2 b_{10}}{7 b_{01}^2} S_2(k_1, k_2) \right] D_{NLB} R_{NLB} + D_{FOG} R_{FOG} \\
+ \frac{1}{b_{01}} \left[ \frac{1}{\alpha(k_1)} + \frac{1}{\alpha(k_2)} \right] - b_{01} \left[ \frac{N_2(k_1, k_2)}{\alpha(k_2)} + \frac{N_2(k_2, k_1)}{\alpha(k_1)} \right] + \\
+ \frac{b_{02}}{b_{10}^2 \alpha(k_1) \alpha(k_2)} \right\} P(k_1) P(k_2) + 2 \text{ cyc.},
\end{array} \right. \tag{7.33}$$

where we have introduced the correction factors

$$R_{SQ1} = 1 + \frac{1}{D_{SQ1}} \left( \frac{b_{01}^2}{b_{10}^2 \alpha(k_1) \alpha(k_2)} + \frac{b_{01} b_{01}}{b_{10} D_{NLB}} \right), \tag{7.34}$$

$$R_{SQ2} = 1 + \frac{1}{D_{SQ2}} \left( \frac{b_{01}^2}{b_{10} \alpha(k_1) \alpha(k_2)} + \frac{b_{01} b_{01}}{b_{10} D_{NLB}} \right), \tag{7.35}$$

$$R_{NLB} = 1 + \frac{1}{D_{NLB}} \left( \frac{b_{10}^2}{b_{10}^2 \alpha(k_1) \alpha(k_2)} + \frac{b_{01} b_{01}}{b_{10} D_{NLB}} \right), \tag{7.36}$$

$$R_{FOG} = 1 + \frac{1}{D_{FOG}} \left( \frac{b_{01}^2}{b_{10} \alpha(k_1) \alpha(k_2)} + \frac{b_{01} b_{01}}{b_{10} D_{NLB}} \right), \tag{7.37}$$

$$R_{nG2} = 1 + \frac{1}{D_{nG2}} \left( \frac{b_{01}^2}{b_{10}^2 \alpha(k_1) \alpha(k_2)} + \frac{b_{01} b_{01}}{b_{10} D_{nG2}} \right), \tag{7.38}$$

$$R_{FGnG} = 1 + \frac{1}{D_{FGnG}} \left( \frac{b_{01}^2}{b_{10} \alpha(k_1) \alpha(k_2)} + \frac{b_{01} b_{01}}{b_{10} D_{FGnG}} \right). \tag{7.39}$$

We see that PNG enters into the expression (7.33) in four different ways:

- The kernels $F_2$ and $G_2$ acquire a correction proportional to $f_{NL}$, as seen in section 4.2.1.
The linear (SQ1) and non-linear (SQ2) squashing, non-linear biasing (NLB) and linear part of FoG (FOG) are modified by the correction factors $\mathcal{R}_{SQ1}$, $\mathcal{R}_{SQ2}$, $\mathcal{R}_{NLB}$ and $\mathcal{R}_{FOG}$, respectively. In these, for instance, $\mathcal{D}_{nG1}^{SQ1}$ comes from the angular average of the Kaiser factor times SQ1 times nG1 (linear effect of PNG), $\mathcal{D}_{nG1}^{NLB}$ from the Kaiser factor times NLB time nG1, $\mathcal{D}_{SQ2}^{nG12}$ from the SQ2 term times nG1 squared, and so on.

Non-Gaussianity distortions appear in the non-linear effect of PNG through the term $\mathcal{D}_{nG2} \mathcal{R}_{nG2}$. In particular, $\mathcal{D}_{nG2}^{nl}$ is generated by the integration of the Kaiser factor squared times the non-linear effect of PNG (nG2) and $\mathcal{D}_{nG1}^{nG1}$ by the angular average of the Kaiser factor times nG1 times nG2.

Importantly, a new set of terms appear, potentially relevant at large scales, related to the quantity called FoGnG in eq. (7.23). $\mathcal{D}_{FoGnG}$ is the result of the integration of the Kaiser factor squared times FoGnG, $\mathcal{D}_{FoGnG}^{nG1}$ of the Kaiser factor times FoGnG times nG1 and, finally, $\mathcal{D}_{FoGnG}^{nG12}$ by the angular average of the FoGnG term times nG1 squared.

Appendix F further discusses in detail all the $\mathcal{D}$-factors that we schematically described here.

In order to provide an illustration of the role played by PNG in redshift space, we plot in fig. 7.1 the absolute value of the relative difference between the non-Gaussian and the Gaussian monopole,

\[
\text{Diff}(k_1, k_2, k_3) = \left| \frac{B_{gG}^{s}(0,0) - B_{gG}^{sG}(0,0)}{B_{gG}^{s}(0,0)} \right|, \tag{7.40}
\]

assuming $f_{NL} = 10$ and objects with mass $M = 10^{13}h^{-1}M_\odot$. The plots are based on the graphical representation introduced in section 3.2.1 i.e. the amplitude of the signal is presented in a colour map as a function of $k_2/k_1$ and $k_3/k_1$, under the condition $k_3 \leq k_2 \leq k_1$, which avoids multiple visualizations of the same triangle/configuration.

The primordial non-Gaussian signal clearly peaks in the squeezed limit (top left corners) of the galaxy monopole, mainly on large scales and/or high redshift. On the other hand, interestingly, a non-negligible signal propagates also into other configurations, with decreasing amplitude as it approaches the equilateral configuration (top right corners). We interpret this effect, at least partly, as due to the new contributions FoGnG discussed above; primordial non-Gaussianity in redshift space induces distortions that can affect large scale measurements. Failing to include all the non-Gaussian effects together with RSD would result into biased measurements of $f_{NL}$.
Figure 7.1: The plots show the absolute value of the relative difference between the galaxy bispectrum monopole with PNG and the one without, for objects with mass $M = 10^{13} h^{-1} M_{\odot}$ and $f_{\text{NL}} = 10$. The colour maps display the amplitude of the signal as a function of $k_2/k_1$ and $k_3/k_1$, under the condition $k_3 \leq k_2 \leq k_1$. Differences above 100% are saturated to the same red colour of the palette.

### 7.3 Conclusions

A future target sensitivity for measurements of $f_{\text{NL}}$ to values $f_{\text{NL}} \sim 1$ sets a new challenge in searches for primordial non-Gaussianity. While future CMB experiments may not be able to achieve this goal, large-scale structure observations might allow us to reach this level of sensitivity, by exploiting the characteristic scale-dependence introduced by local-type models in the bias relation between collapsed objects and the density field, and the very large amount of data available with future redshift surveys.

In this chapter we addressed the problem of modelling redshift space distortions in the tree-level galaxy bispectrum with primordial non-Gaussianity of local-type. We examined how redshift space distortions can affect large-scale measurements, and therefore potentially lead to a biased measurement of $f_{\text{NL}}$ if not properly described. In particular we identified new contributions to the galaxy bispectrum, which physically correspond to large-scale amplifications – induced by primordial non-Gaussianity – of redshift space distortion effects. Moreover, we proposed an analytic prediction for the monopole which can be used to fit against data, in the large scale regimes where the non-linear part of FoG can be neglected. We analysed the physical consequences of our findings, providing a graphical method for comparing our results for bispectra with the
case in which primordial non-Gaussianity is not included.
Chapter 8

Conclusions

The origin of structures is nowadays understood as arising via a mechanism generating initial density perturbations in the very early Universe. Among different possibilities, inflation has been proven to be consistent with CMB data and is now a well established paradigm, often considered a building block of modern cosmology.

Remarkably, different realizations of this process can leave an imprint in the web of galaxies surrounding us, potentially affecting the large-scale clustering. Thus galaxy survey data can be used to measure the parameters controlling the inflationary set-up; from our ability to make accurate and unbiased measurements we have the possibility of discriminating among different scenarios and, hence, improving our understanding of inflation.

The present thesis explores this line of research. In particular we reviewed a possible approach to the evolution of initial density perturbations into structures, deriving the scale-dependent bias term that has attracted a lot of interest in recent years, as it can potentially surpass the CMB constraints on primordial non-Gaussian parameters.

We showed that the galaxy power spectrum might not be sufficient to unveil all the properties of inflation. In particular, we found that the information available in the next-order statistic, i.e. the bispectrum, potentially allows us to break the degeneracy between different non-Gaussian parameters.

Among the many challenges that are intrinsic to this observable, the bispectrum involves non-linear evolution of matter density even at leading order. We improved the existing bivariate model for the halo density by recognising the presence of a novel convective term in the Eulerian frame, when the spherical collapse assumption is dropped: it contributes in a non-negligible way to the tree-level bispectrum [see eq. (6.21)].
Moreover, we extended our analytic prediction for the bispectrum by taking into account redshift space distortions. Galaxy survey data are contaminated by peculiar velocities when the measured redshift is used to infer the distance and, thus, line-of-sight distortions can lead to biased measurements if not properly accounted for.

In this chapter we briefly discuss the results that we have obtained, indicating further possible developments of this work and considering the possibility of measuring primordial non-Gaussianity with the bispectrum of LSS in current and future galaxy surveys.

8.1 Breaking the degeneracy between $g_{\text{NL}}$ and $f_{\text{NL}}$

In chapter 5 we investigated how galaxy and matter bispectra can distinguish between the two nG parameters $f_{\text{NL}}$ and $g_{\text{NL}}$, respectively controlling the skewness and kurtosis in the PDF of the initial density perturbations. We found a connection between the sign of the halo bispectrum on large scales and the sign of the parameter $g_{\text{NL}}$ and constructed a combination of halo and matter bispectra that is sensitive to $f_{\text{NL}}$, with little contamination from $g_{\text{NL}}$. These results were then generalised to the case of multiple sources for the primordial density perturbations, leading to a model of stochastic bias.

These results can be extended in various directions. From the theoretical side, the non-linear effects associated with features of the mass function, bias and gravitational clustering (the non-linear transformation from Lagrangian to Eulerian coordinates), as well as loop effects, should be properly included in the context of perturbation theory, in order to understand whether and how PNG controls the qualitative features of bispectra also after taking these non-linearities into account. Having a good control of gravitational effects would also allow one to reliably study the bispectra for smaller scales and for configurations that are less squeezed than the ones we considered in chapter 5 and extend the validity of our results into these regimes.

Another direction of investigation is a more systematic study of observational applications of our results, trying to quantify whether the methods we propose are sufficient to set constraints on $g_{\text{NL}}$ that are more stringent than those available from the CMB. An important extension of our work is thus to forecast the signal-to-noise of galaxy bispectra as a function of scale, for future surveys such as the ESA Euclid mission, and thereby test whether the features we predict are observable and can further shrink the already existing $f_{\text{NL}}$ and $
$g_{\text{NL}}$ constraints from the galaxy power spectrum \cite{104,117}. Such a study would exploit our theoretical inputs to develop optimised methods to distinguish the effects of $f_{\text{NL}}$ and $g_{\text{NL}}$ using galaxy bispectra and thus quantify their size and sign independently.

8.2 Non-local bias in the halo bispectrum

Many pieces of supporting evidence show that a local bias model in the Eulerian frame is not an accurate description of the halo density \cite{36,66,69,70}, resulting in missing terms in the leading order of the bispectrum \cite{36,70}. This motivates dropping spherical collapse approximation, often used to transform the local Lagrangian bias model into Eulerian coordinates, which introduces the tidal term \cite{36,74}.

When a $f_{\text{NL}}$ cosmology is considered, the local Lagrangian bias model has to be replaced with a local bivariate expansion. The Lagrangian overdensity is thus modelled in terms of the initial density field and a Gaussian gravitational potential, accounting for the specific mode coupling in local-type models. Within this picture, we considered for the first time the transformation of the potential into Eulerian coordinates when the spherical collapse approximation is dropped and found a non-Gaussian, non-local shift term.

The presence of the tidal and shift terms in the halo overdensity sources new contributions in the tree-level halo bispectrum. Our numerical analysis suggests that they lead to non-negligible corrections on large scales and high redshift in the presence of primordial non-Gaussianity, when compared to the model of Baldauf \textit{et al.} \cite{73} which assumes spherical collapse.

Building upon the findings of chapter 6, the next challenge is to test these theoretical predictions against N-body simulations with non-Gaussian initial conditions. As the effect of the novel convective term is controlled by the parameter $f_{\text{NL}}$, one could use values of PNG that are now unrealistic to augment the significance of the new terms contributing to the bispectrum. This would make easier to test the presence of the new effects that we are predicting. However, as they contribute the most on very large scales, it is computationally challenging to create a sufficiently large set of simulations to reduce the error bars and, thus, have a clear detection of the signal. This could be feasible in principle by using novel techniques for the fast creation of mock catalogues, like for instance \cite{178}.

Ultimately we would wish to be able to estimate the signal-to-noise for upcoming surveys, like \textit{Euclid}, which probes large scales and high redshifts, in order
to explore the observability of these non-local effects in the bispectrum. A full discussion of the observability of these effects must include a halo occupation model, to describe how galaxies populate halos and many other effects including redshift space distortions and lensing along the line of sight, from galaxies to the observer, in order to translate our theoretical model for the halo bispectrum into predictions for the observed galaxy angular bispectrum in redshift space.

8.3 Galaxy bispectrum in redshift space

Our understanding of the physics of inflation would improve considerably if we could increase by at least one order of magnitude the current accuracy of measures of non-Gaussianity of the primordial density field. In the future, large-scale galaxy redshift surveys might allow us to reach this goal, by exploiting the scale-dependent bias induced by primordial non-Gaussianity.

The analytic bispectrum developed in chapter 6 for local-type inflationary models with $f_{NL}$ only is a starting point to build more refined predictions, which would include further theoretical and observational effects.

For instance, at the observational level, issues such as mask geometry and systematic effects, which can mimic the scale dependence of PNG (see for instance [101, 103]), should be taken into account. Theoretically, one should also worry about GR effects. It has been proven that GR does source contributions to the squeezed limit of the matter bispectrum [24, 26, 155] and generate secondary non-Gaussianities along the path of the photons from the emitting galaxy and the observer, in analogy with the CMB (see for instance [122, 156] and references therein). These may well introduce a non-primordial $f_{NL} \sim 1$ and thus bias measurements of primordial non-Gaussianity.

Among other effects, redshift space distortions are an additional source of complexity. Indeed, the redshift measurements used to infer the distances of galaxies are contaminated by peculiar velocities and thus produce distortions along the line of sight.

In chapter 7 we addressed the problem of computing the galaxy bispectrum in redshift space with primordial non-Gaussianity of local type. An analytic result can be achieved by focussing on large-scale regimes, i.e. ignoring the non-linear part of the Fingers-of-God term. However, we showed that primordial non-Gaussianity can potentially source large-scale amplifications in redshift space and thus lead to biased measurements if not properly accounted for. Moreover, we proposed a prediction for the galaxy monopole, by decomposing the line of sight dependence of the bispectrum into spherical harmonics.
8.4 Measuring $f_{\text{NL}}$ with redshift surveys: forecasts

As the recent measurement of Gil-Marín et al [147, 148] shows, the bispectrum is a valuable tool for cosmology. In particular, the larger amount of available configurations in a wavelength range between $k_{\text{min}}$ and $k_{\text{max}}$ compared to the power spectrum may well shrink the observational bounds on $f_{\text{NL}}$ even further, as many works show [73, 129, 138, 139], potentially below the Planck constraint [179]. The monopole derived in section 7.2.2 naturally extends the model considered by Gil-Marín et al [177] to the case of local-type PNG and can be used to measure $f_{\text{NL}}$.

However, it may not be the only way to improve our constraints on inflation: the multi-tracer technique is a promising tool [107, 111, 112, 180–182]. Even the novel position-dependent power spectrum could be an interesting alternative, although a detailed analysis in the case of PNG is still missing (see the discussion in appendix E). As both involve measurements of the power spectrum, they require less effort than a full bispectrum analysis.

In this section we present forecasts of the accuracy in determining $f_{\text{NL}}$ – a quantity that we call $\sigma_{f_{\text{NL}}}$ – based on a Fisher analysis for the bispectrum. Our aim is to give an illustration of the best possible improvement one could get with respect to power spectrum forecasts, when a single tracer is considered [183, 184]. On the other hand, important effects (like the covariance between different triangles) will be neglected and the quoted results are by no means intended to be fully realistic; rather, they are meant to motivate future work. For this reason we will consider only the bispectrum in real space, which is computationally easier to handle than the redshift space result. Although forecasts based on the redshift-space bispectrum can have some small quantitative differences respect to our results, we expect the following qualitative discussion to hold anyway.

¹Technically, studies on the cumulative signal-to-noise, i.e. summed over all the configurations, show the critical dependence of the halo bispectrum signal on some kind of triangle configurations and the maximum wavenumber, $k_{\text{max}}$, considered [73, 129]. At large scales ($k_{\text{max}} < 0.05h\text{Mpc}^{-1}$), the signal is strongly suppressed because only few configurations are available, and with a large variance. By increasing $k_{\text{max}}$, the number of triangles considerably grows ($N_{\text{Tr}} \sim k_{\text{max}}^3$) and, consequently, the signal. As fig. 7.1 suggests, a large fraction of it is in squeezed configurations. Among these, the FoGnG term can play a role on those that have their smallest $k$ on sufficiently large scales. However, one should bear in mind that these large-scale, squeezed triangles are highly correlated, i.e. the covariance cannot be neglected (see also the discussion in section 8.4.1). Thus, we expect that forecasts based on the redshift-space bispectrum will have small differences respect to our results, but it is clear that failing to include RSD in the bispectrum model could bias a $f_{\text{NL}}$ measurement at the level of accuracy.
The Fisher formalism is a tool for setting a lower limit on the statistical uncertainties that future surveys will have in the measurements of cosmological parameters of interest (see [185–187] for an introduction). The Fisher matrix information is defined as

\[ F_{\alpha\beta} \equiv -\left\langle \frac{\partial^2 \ln L(x; p)}{\partial p_\alpha \partial p_\beta} \right\rangle, \quad (8.1) \]

where \( L(x; p) \) is the likelihood function, i.e. the probability of the data \( x \) given the parameters \( p \), and \( p_\alpha \) is the \( \alpha \)-th unknown parameter. If all the parameters are fixed except one (say \( p_\alpha \)), then the lower limit on the 1\( \sigma \) error bar in the \( p_\alpha \) measurement is \( \sigma_{p_\alpha} = 1/\sqrt{F_{\alpha\alpha}} \). Otherwise, if we marginalise over the parameters, the lower bound becomes \( \sigma_{p_\alpha} = \sqrt{F_{\alpha\alpha}} \).

The Fisher matrix for the bispectrum is [138,179]

\[ F_{\alpha\beta} \equiv \sum_z \sum_{k_1, k_2, k_3 \geq k_{\text{min}}} \frac{1}{\Delta^2(k_1, k_2, k_3)} \frac{\partial B(k_1, k_2, k_3)}{\partial p_\alpha} \frac{\partial B(k_1, k_2, k_3)}{\partial p_\beta}, \quad (8.2) \]

where \( B \) and \( \Delta^2 \) are the bispectrum and covariance estimator respectively and we assume the minimum value of \( k \) to be fixed by the survey volume \( V \), \( k_{\text{min}} = 2\pi/V^{1/3} \), while the maximum is \( k_{\text{max}} = 0.1D(0)/D(z) \) – a reasonable limit for the validity of the non-linear analytic model [184].

8.4.1 Methodology

To compute the Fisher matrix we need to define \( B, \Delta^2 \) and the set of unknown parameters \( p \); our assumptions are described below.

We assume the bispectrum model of eq. (7.9), while the covariance for a survey of volume \( V \) is given by [188]

\[ (\Delta B)^2 = s_{123} \frac{V_f}{V_{123}} \left( P_{gg}(k_1) + \frac{1}{\bar{n}} \right) \left( P_{gg}(k_2) + \frac{1}{\bar{n}} \right) \left( P_{gg}(k_3) + \frac{1}{\bar{n}} \right), \quad (8.3) \]

where the volume of the fundamental cell is \( V_f = (2\pi)^3/V \) and

\[ V_{123} \approx 8\pi^2 k_1 k_2 k_3 \delta k^3, \quad (8.4) \]

with \( \delta k \) being the bin size. \( \bar{n} \) is the number density of objects accounting for the shot noise, while \( s_{123} \) is the symmetry factor, respectively \( s_{123} = 6, 2, 1 \) for equilateral, isosceles and general configurations. In the noise estimator, the power spectrum is approximated by the leading contribution \( P_{gg}(k) = (b_{10} + b_{01}/\alpha(k))^2 P(k) \).

that is now required.
For simplicity and as in other studies \[73, 129, 138, 139, 189\], we neglect the covariance between different configurations of the bispectrum which is expected to be non-negligible for triangles sharing one or two sides, in particular on large scales \[190\]. The induced covariance arising from survey selection effects (i.e. complicated survey geometry and mask) is neglected as well.

As explained in \[139\], the results of \[138\] suggest that ignoring covariance can over-estimate the constraining power of a given sample by a factor of two for \(k < 0.1 h \text{Mpc}^{-1}\) and up to a factor of eight for \(k < 0.3 h \text{Mpc}^{-1}\) at redshift zero. At higher redshift the contribution from a connected 6-point function generated by non-linear gravitational evolution is expected to be less important and thus the (theoretical) covariance reduced. The largest \(k\) values used in our forecasts lie between 0.15\(h\)Mpc\(^{-1}\) and 0.25\(h\)Mpc\(^{-1}\) in the redshift range \(0 < z < 2.2\); suggesting covariance could make our forecasts optimistic by a factor of \(\sim 5\).

However, we will show that the constraining power in current and future surveys would provide competitive \(f_{NL}\) constraints even if the covariance degrades our idealised forecasts by a factor of 5, although a more realistic analysis is needed to fully explore this.

We assume all the cosmological parameters to be fixed to Planck’s central values \[6\], except for the linear and non-linear bias and \(f_{NL}\), thus \(p = \{b_{10}, b_{20}, f_{NL}\}\). The fiducial model that maximizes the likelihood is assumed to be \(p = \{b_{10}^{\text{fid}}, b_{20}^{\text{fid}}, f_{NL}^{\text{fid}} = 0\}\), where \(b_{10}^{\text{fid}}\) is calibrated against real data (either from the particular survey or characteristic of the type of galaxy expected to be observed by the particular survey) and will be quoted in the following paragraphs, depending on the survey and the tracer. \(f_{NL}^{\text{fid}}\) is assumed to be vanishing, this being compatible with current data: the final results will give an idea of the significance level at which a non-null primordial signal can be detected.

Since the non-linear bias is not a well constrained parameter, the choice of the fiducial value is very important. We take it to be the analytic prediction \(b_{20}^{\text{fid}}(\nu)\), based on eqs. \((3.122), (3.123)\) and \((3.136)\), where the variable \(\nu\) is estimated under the assumption \(b_{10}(\nu) = b_{10}^{\text{fid}}\). We will then show how much \(\sigma_{f_{NL}}\) is affected by the choice of \(b_{20}^{\text{fid}}\) by allowing for a \(\pm 1\) range around this value. Therefore, the results are presented in table 8.1 in the following form:

\[
\sigma_{f_{NL}, b_{20}^{\text{fid}}} = \min\left(\frac{\sigma_{f_{NL}, b_{20}^{\text{fid}} + 1}}{\sigma_{f_{NL}, b_{20}^{\text{fid}} - 1}}\right),
\]

In our analysis we consider four redshift surveys: BOSS \[191\], eBOSS \[192\], DESI \[193\], Euclid \[133\], briefly presented below. The data used for each of them can be found in the tables in appendix \[3\].
BOSS

SDSS-III’s Baryon Oscillation Spectroscopic Survey [191] is a galaxy redshift survey, which finished observations in 2014. It mapped the spatial distribution of about 1.5 million luminous red galaxies (LRGs) covering 10,000 deg$^2$ in the redshift range $0 < z < 0.8$, with the primary goal of detecting the characteristic scale imprinted by sound waves in the early universe, i.e. the Baryon Acoustic Oscillations (BAO). Also, about 160,000 quasars (QSOs) were observed in the redshift range $2.2 < z < 3$, so that correlations can be measured in the Lyman-α forest, which we will not consider here. Table G.1 in appendix G shows the basic numbers for BOSS, with the linear bias assumed to be $b_{10}^{\text{LRG}} = 1.7/D(z)$ (see [183] and references therein).

eBOSS

The extended Baryon Oscillation Spectroscopic Survey [192] is part of the SDSS-IV project and started observations in 2014. It will extend the BAO measurements to $0.6 < z < 2.2$ by observing LRGs, Emission Line Galaxies (ELGs) and QSOs.

The eBOSS numbers we use match those presented in [184]. LRGs will be observed in the redshift range $0.6 < z < 1$ over 7,000 deg$^2$, with a linear bias assumed to be $b_{10}^{\text{LRG}} = 1.7/D(z)$, while QSOs will fall in the range $0.6 < z < 2.2$ over 7,500 deg$^2$, with bias $b_{10}^{\text{QSO}} = 0.53 + 0.29(1+z)^2$. The ELG target selection definitions have not been finalised, but each of the three proposals considered in [184] result in samples that have a significant overlap in volume with the LRG sample. We have tested each potential ELG sample and found that even if they are treated independently, they do not add substantial constraining power. Therefore, we omit them from the forecast constraints we present. Tables G.2 to G.4 show the basic numbers for eBOSS LRGs, QSOs and ELGs respectively. Refer to [184] and references therein for further details.

DESI

Dark Energy Spectroscopic Instrument [193] is a redshift survey with the primary target of measuring the effect of dark energy on the expansion of the Universe. It is expected to run between 2018 and 2022 and will map the universe from low to high redshift over 14,000 deg$^2$, measuring the optical spectra for tens of million objects, including LRGs, ELGs and QSOs.

---

2Strictly speaking, BOSS also contains a sample of luminous galaxies with more star formation and greater disk morphology than typical LRGs.
The LRGs will fall in the redshift range \(0.1 < z < 1.1\) with a linear bias assumed to be \(b_{10}^{\text{LRG}} = 1.7/D(z)\), ELGs in \(0.1 < z < 1.8\) with \(b_{10}^{\text{ELG}} = 0.84/D(z)\) and QSOs will be considered in redshift range \(0.1 < z < 1.9\), with bias \(b_{10}^{\text{QSO}} = 1.2/D(z)\). Table G.5 shows the basic numbers for DESI (see [183] and references therein).

Euclid

Euclid [133] is a space mission developed to study the imprints of dark energy and gravity. The expansion rate of the Universe and the growth of structures will be tracked by using two complementary observables: weak gravitational lensing and galaxy clustering. Its launch is planned for 2020.

We focus on the redshift survey part of the mission, which is expected to detect about 50 million galaxies in the redshift range \(0.6 < z < 2.1\), over 15,000 deg\(^2\). The fiducial value for the bias is assumed to be \(b_{10} = 0.76/D(z)\). Table G.6 shows the basic numbers for Euclid (see [183] and references therein).

8.4.2 Results

Given the assumptions listed above, table 8.1 shows the lower limit on \(\sigma_{f_{\text{NL}}}\) that could be expected from the bispectrum of BOSS, eBOSS, DESI and Euclid. We also present the forecast results for the power spectrum, in order to provide a comparison. The results combine all the tracers available for each survey (LRGs, ELGs, QSOs), which are treated as independent, except that we omit any ELG sample for eBOSS, as previously noted.

If we focus first on the results labelled ‘bias float’ (marginalising over bias) of the bispectrum set of columns, our analysis suggests that BOSS and eBOSS will both be able to reach \(\sigma_{f_{\text{NL}}} \simeq 1\). eBOSS appears to be penalised compared to BOSS because of the lower number densities. Interestingly, DESI and Euclid may give \(\sigma_{f_{\text{NL}}} < 1\), regardless of the chosen fiducial value for \(b_{20}\), within \(\pm 1\) range. The DESI result is more stringent than the Euclid one because DESI is assumed to observe more biased objects; however, combining the BOSS and Euclid data tighten the constraint towards the DESI result. Clearly, the \(f_{\text{NL}}\) measurements improve by fixing the linear and non-linear bias: the results on the last column (bias fixed) of table 8.1 indicate an improvement factor between 1.4 and 2 on \(\sigma_{f_{\text{NL}}}\).

A comparison between \(\sigma_{f_{\text{NL}}}\) expected from the bispectrum and those from the power spectrum of a single tracer (first two columns of table 8.1 but see also [183][184]) seems to indicate about an order of magnitude improvement.
Table 8.1: Forecasts for $\sigma_{f_{NL}}$ from the bispectrum of BOSS, eBOSS, DESI and Euclid, assuming the fiducial values $p = \{b_{10}^{\text{fid}}, b_{20}^{\text{fid}}, f_{NL}^{\text{fid}} = 0\}$, as described in section 8.4.1. Forecasts from the power spectrum are obtained considering only the tree-level, with the fiducial model $p = \{b_{10}^{\text{fid}}, f_{NL}^{\text{fid}} = 0\}$. The results with marginalisation over the bias factors are shown on the left columns (bias float), while those without on the right (bias fixed). The numbers inside the parenthesis in the superscripts are the predictions for $\sigma_{f_{NL}}$ considering the fiducial value for the non-linear bias to be $b_{20}^{\text{fid}} + 1$, while those in the subscripts assume $b_{20}^{\text{fid}} - 1$.

<table>
<thead>
<tr>
<th>Sample</th>
<th>Power Spectrum</th>
<th>Bispectrum</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\sigma_{f_{NL}}$</td>
<td>$\sigma_{f_{NL}}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>bias float</td>
<td>bias fixed</td>
<td></td>
</tr>
<tr>
<td>BOSS</td>
<td>21.30</td>
<td>13.28</td>
<td>1.04^{(0.65)}_{(2.47)}</td>
</tr>
<tr>
<td>eBOSS</td>
<td>14.21</td>
<td>11.12</td>
<td>1.18^{(0.82)}_{(2.02)}</td>
</tr>
<tr>
<td>Euclid</td>
<td>6.00</td>
<td>4.71</td>
<td>0.45^{(0.18)}_{(0.71)}</td>
</tr>
<tr>
<td>DESI</td>
<td>5.43</td>
<td>4.37</td>
<td>0.33^{(0.17)}_{(0.48)}</td>
</tr>
<tr>
<td>BOSS + Euclid</td>
<td>5.64</td>
<td>4.44</td>
<td>0.39^{(0.17)}_{(0.59)}</td>
</tr>
</tbody>
</table>

Even allowing for a factor of five dilution in constraining power potentially caused by covariance between triangle configurations, our forecasts remain impressive. Indeed, we forecast that current BOSS data should allow $f_{NL}$ constraints competitive with those obtained from Planck [81].

8.4.3 Conclusions

We performed idealised forecasts of $\sigma_{f_{NL}}$, the accuracy of the determination of local $f_{NL}$, that could be obtained from measurements of the galaxy bispectrum using data from surveys like BOSS, eBOSS, DESI and Euclid.

These results are approximate, as for instance we ignored the covariance between different triangles, which is known to degrade the signal, in particular on large scales. Nonetheless, our findings suggest that the constraining power in current and future surveys would provide competitive $f_{NL}$ constraints even if the covariance between triangle configurations degrades our idealised forecasts by a factor of 5. In particular, current BOSS data should allow for Planck-like constraints on $f_{NL}$, while future surveys like Euclid and DESI may shrink the bound even further, potentially by a factor of three.

We leave as a challenge for future work to obtain improved predictions for $\sigma_{f_{NL}}$ fully accounting for the covariance: this will be necessary if we are to
completely understand the power of bispectrum measurements to constrain $f_{NL}$ compared to alternative approaches, such as the multi-tracer technique or the position-dependent power spectrum.

However, our preliminary forecasts are encouraging and we would like to end this thesis with a positive perspective for the future: extracting information about the primordial density perturbations from the galaxy bispectrum could be a rich source of insight into the very early Universe.
Appendix A

Inequalities among non-Gaussian parameters

In this appendix we collect some technical results on inequalities among primordial non-Gaussian parameters, that we use in Section 5.3. The most famous of these inequalities is dubbed Suyama-Yamaguchi inequality, and relates the collapsed limit of the 4-point function with the square of the squeezed limit of a 3-point function of a given fluctuation $\delta$ in Fourier space [79]

$$\lim_{q \to 0} \langle \delta_{k_1} \delta_{-k_1-q} \delta_{k_2} \delta_{-k_2+q} \rangle' \geq \lim_{q \to 0} \frac{\langle \delta_{k_1} \delta_{-k_1-q} \delta_q \rangle'^2}{\langle \delta_{k_1} \delta_q \rangle}.$$ (A.1)

Using the definitions for $\tau_{NL}$ and $f_{NL}$ and specializing to the case of primordial curvature fluctuations, this inequality corresponds to the Suyama-Yamaguchi inequality. It is convenient to simplify the notation, expressing the squeezed limit of the 3-point function in eq. (A.1) in a synthetic way as $\langle \delta_2 \rangle$, meaning that the momentum $q$ connecting $\delta$ and $[\delta^2]$ is sent to zero: this notation emphasizes that the long mode $\delta_q$ with $q \to 0$ modulates the 2-point function $[\delta^2]$. Analogously, the collapsed limit of the 4-point function can be expressed as $\langle [\delta^2][\delta^2] \rangle$. With the help of this notation, inequality eq. (A.1) succinctly reads

$$\langle [\delta^2][\delta^2] \rangle \geq \frac{\langle \delta_2 \rangle^2}{\langle \delta \delta \rangle}.$$ (A.2)

In [194] a simple proof of this inequality has been provided, by inserting a complete set of normalized momentum eigenstates $|n_{\vec{k}}\rangle \equiv |\delta_{\vec{k}}\rangle / \langle \delta \delta \rangle^{1/2}$ into the quantity $\langle [\delta^2][\delta^2] \rangle$ of the left hand side of eq. (A.2). The zero-momentum, long wavelength eigenstate provides the square of the squeezed 3-point function in the right hand side, to which one has to add the additional (positive definite)
contributions that lead to the inequality above. In the collapsed limit and single source case, the additional contributions vanish and one saturates the inequality.

Although less explored in the literature, it is also possible to build further inequalities involving collapsed and squeezed limits of higher point functions (see for example [131,195]): the virtue of the method of [194] is that the proofs of such inequalities can be straightforwardly generalized to those cases. As specific examples, we can write the following inequalities satisfied by five and six point functions, in appropriate collapsed limits

\[ \langle [\delta^3][\delta^2] \rangle \geq \frac{\langle [\delta^3][\delta] \rangle \langle [\delta^2][\delta] \rangle}{\langle \delta \delta \rangle} , \]  \hspace{1cm} (A.3)

\[ \langle [\delta^3][\delta^3] \rangle \geq \frac{\langle [\delta^3][\delta]^2 \rangle}{\langle \delta \delta \rangle} . \]  \hspace{1cm} (A.4)

Using the methods of [194], it is relatively straightforward to prove that the inequalities saturate to equalities in the single source case. These inequalities have been used in section 5.3.1 for physically interpreting the notion of stochastic halo bias applied to power spectra of halos and matter. When considering stochastic halo bias for the bispectra in section 5.3.2 other inequalities are needed, that involve subtler collapsed limits among higher order point functions. We collect them here

\[ \langle [\delta^2][\delta^2][\delta^2] \rangle \geq \frac{\langle \delta \delta \delta^2 \rangle \langle [\delta^2][\delta^2] \rangle}{\langle \delta \delta \rangle} \]  \hspace{1cm} (A.5)

\[ \langle [\delta^2][\delta^2][\delta^2] \rangle \geq \frac{\langle \delta \delta \delta^2 \rangle \langle [\delta^2][\delta^2] \rangle}{\langle \delta \delta \rangle} \]  \hspace{1cm} (A.6)

\[ \langle [\delta^2][\delta^2][\delta^2] \rangle \geq \frac{\langle \delta \delta \delta \delta \rangle \langle [\delta^2][\delta^2] \rangle}{\langle \delta \delta \rangle} \]  \hspace{1cm} (A.7)

They can be analyzed and proved again with the methods of [194].
Appendix B

Displaced Gaussian random fields

In deriving the halo abundance in Eulerian space in the presence of PNG we need to map the initial gravitational potential in Lagrangian coordinates, eq. (6.1), into Eulerian coordinates under the coordinate displacement of eq. (6.3), which is itself determined by the gravitational potential. In this appendix we shall demonstrate how even an initial Gaussian field in Lagrangian coordinates may be transformed into a non-Gaussian field by this displacement to Eulerian coordinates.

We start from a random field, \( \hat{\phi}_G(q) \), defined with respect to the Lagrangian coordinate chart, \( q \). Let

\[
\hat{\phi}_G(q) = \epsilon f(q) \hat{a}, \tag{B.1}
\]

where \( \hat{a} \) denotes a Gaussian random variable and \( \epsilon \) is a small perturbative parameter. Thus \( \hat{\phi}_G(q) \) is a Gaussian random field (first order with respect to \( \epsilon \)) with, for example, vanishing 3-point function

\[
\langle \hat{\phi}_G(q_1) \hat{\phi}_G(q_2) \hat{\phi}_G(q_3) \rangle = \epsilon^3 f(q_1) f(q_2) f(q_3) \langle \hat{a}^3 \rangle = 0 , \tag{B.2}
\]

where angle-brackets denote the ensemble average.

Let the Eulerian coordinate \( x \) be related to \( q \) by a first-order displacement field \( \hat{x}(q) \), correlated with the field \( \hat{\phi}_G \) such that

\[
\hat{x}(q) = q + \epsilon \psi(q) \hat{a} . \tag{B.3}
\]

If we consider a fixed coordinate \( q \) then \( \hat{x}(q) \) is itself a random variable, correlated with \( \hat{\phi}_G(q) \). We can then construct the field

\[
\hat{\phi}_G(\hat{x}(q)) = \hat{\phi}_G(q) + \epsilon^2 \psi(q) f'(q) \hat{a}^2 + O(\epsilon^3) , \tag{B.4}
\]
which is Gaussian at first order in $\epsilon$, but non-Gaussian at second order. For example, the 3-point function of $\hat{\phi}(\hat{x}(q))$ with respect to the coordinate chart $q$ is non-vanishing at fourth order

$$\langle \hat{\phi}_G(\hat{x}(q_1))\hat{\phi}_G(\hat{x}(q_2))\hat{\phi}_G(\hat{x}(q_3)) \rangle_q = \epsilon^4 [\psi(q_1)f'(q_1)f(q_2)f(q_3) + \text{perms}] \langle \hat{a}^4 \rangle + O(\epsilon^6) \neq 0 .$$

Conversely, if we work with respect to the Eulerian coordinate chart $x$, it is the Lagrangian coordinate that becomes a random field at fixed coordinate $x$:

$$\hat{q}(x) = x - \epsilon \psi(x) \hat{a} + O(\epsilon^2) ,$$

and $\hat{\phi}_G(\hat{q}(x))$ becomes a non-Gaussian field at second order

$$\hat{\phi}_G(\hat{q}(x)) = \hat{\phi}_G(x) - \epsilon^2 \psi(x)f'(x)\hat{a}^2 + O(\epsilon^3) ,$$

where $\hat{\phi}_G(x) = \epsilon f(x) \hat{a}$ is a first-order Gaussian random field in Eulerian space.
Appendix C

BSS halo bispectrum

The halo bispectrum model of BSS [73] predicts

\[ B_{\text{BSS}}(k_1, k_2, k_3) = B^{(A-L)}_{\text{hhh}}(k_1, k_2, k_3) = \]

\[ b_{10}^3 \left( 2P(k_1)P(k_2)\mathcal{F}_2(k_1, k_2) + 2f_{\text{NL}} \frac{P(k_1)P(k_2)\alpha(k_3)}{\alpha(k_1)\alpha(k_2)} + 2 \text{ cyc.} \right)_A \]

\[ + b_{10}^3 b_{01} \left( 2P(k_1)P(k_2) \left( \frac{1}{\alpha(k_1)} + \frac{1}{\alpha(k_2)} \right) \mathcal{F}_2(k_1, k_2) \right) \]

\[ + 2f_{\text{NL}} \frac{P(k_1)P(k_2)\alpha(k_3)}{\alpha(k_1)\alpha(k_2)} \left( \frac{1}{\alpha(k_1)} + \frac{1}{\alpha(k_2)} \right) + 2 \text{ cyc.} \] \quad \text{B} \]

\[ + b_{10} b_{01}^2 \left( 2P(k_1)P(k_2) \frac{\alpha(k_1)\alpha(k_2)}{\alpha(k_1)\alpha(k_2)} \mathcal{F}_2(k_1, k_2) + 2f_{\text{NL}} \frac{P(k_1)P(k_2)\alpha(k_3)}{\alpha^2(k_1)\alpha^2(k_2)} + 2 \text{ cyc.} \right) \]

\[ + b_{01}^2 b_{02} \left( 2P(k_1)P(k_2) \frac{1}{\alpha(k_1)\alpha^2(k_2)} + 2 \text{ cyc.} \right) \quad \text{D} \]

\[ + b_{01} b_{10} b_{02} \left( \frac{2P(k_1)P(k_2)}{\alpha(k_1)\alpha(k_2)} \left( \frac{1}{\alpha(k_1)} + \frac{1}{\alpha(k_2)} \right) + 2 \text{ cyc.} \right) \quad \text{E} \]

\[ + b_{10}^2 b_{02} \left( 2P(k_1)P(k_2) \frac{1}{\alpha(k_1)\alpha(k_2)} + 2 \text{ cyc.} \right) \quad \text{F} \]

\[ + b_{10}^2 b_{11} \left( \frac{P(k_1)P(k_2)}{\alpha(k_1)\alpha(k_2)} \left( \frac{1}{\alpha(k_1)} + \frac{1}{\alpha(k_2)} \right) + 2 \text{ cyc.} \right) \quad \text{G} \]

\[ + b_{01} b_{10} b_{11} \left( P(k_1)P(k_2) \left( \frac{1}{\alpha^2(k_1)} + \frac{2}{\alpha(k_1)\alpha(k_2)} + \frac{1}{\alpha^2(k_2)} \right) + 2 \text{ cyc.} \right) \quad \text{H} \]

\[ + b_{10}^2 b_{11} \left( \frac{P(k_1)P(k_2)}{\alpha(k_1)\alpha(k_2)} \left( \frac{1}{\alpha(k_1)} + \frac{1}{\alpha(k_2)} \right) + 2 \text{ cyc.} \right) \quad \text{I} \]

\[ + b_{01}^2 b_{20} \left( 2P(k_1)P(k_2) \frac{1}{\alpha(k_1)\alpha(k_2)} + 2 \text{ cyc.} \right) \quad \text{J} \]

\[ + b_{01} b_{10} b_{20} \left( 2P(k_1)P(k_2) \left( \frac{1}{\alpha(k_1)} + \frac{1}{\alpha(k_2)} \right) + 2 \text{ cyc.} \right) \quad \text{K} \]

\[ + b_{02}^2 b_{20} \left( 2P(k_1)P(k_2) + 2 \text{ cyc.} \right)_L. \]
Equation (C.1) is incorporated in our prediction for the halo bispectrum but new terms appear (see eq. (6.21)).
Appendix D

Halo-matter bispectra

Here we consider the case of crossed bispectra between halos and matter. Potentially, weak lensing measurements will allow to cross correlate the dark matter density field with galaxies in the future. Also, these results provide additional predictions of our model (see chapter 6) that can be tested against simulations.

We start by quoting the halo-halo-matter bispectrum in presence of Gaussian initial conditions

$$B_{\text{hhm}}(k_1, k_2, k_3) = b_{10}^2 \left( 6 F_2(k_1, k_2) P(k_1) P(k_2) + 2 \text{ cyc.} \right) \text{A}$$

$$+ b_{10} b_{20} \left( 4 P(k_1) P(k_2) + 2 \text{ cyc.} \right) \text{D}$$

$$- \frac{2}{7} b_{10} b_{L10} \left( 2 S_2(k_1, k_2) P(k_1) P(k_2) + 2 \text{ cyc.} \right) \text{J}. \quad (D.1)$$

Again, we recognise that it is sourced by non-linear gravitational evolution and non-linear bias, while the last term is generated by $s^2$ in eq. (3.139).

Assuming PNG, the halo-halo-matter bispectrum\(^1\) reads

$$B_{\text{hhm}}(k_1, k_2, k_3) =$$

$$b_{10}^2 \left( 6 F_2(k_1, k_2) P(k_1) P(k_2) + 6 f_{\text{NL}} \alpha(k_3) \frac{P(k_1) P(k_2)}{\alpha(k_1) \alpha(k_2)} + 2 \text{ cyc.} \right) \text{A}$$

$$+ b_{01} b_{10} \left( 4 F_2(k_1, k_2) P(k_1) P(k_2) \left( \frac{1}{\alpha(k_1)} + \frac{1}{\alpha(k_2)} \right) \right)$$

$$+ 4 f_{\text{NL}} \alpha(k_3) P(k_1) P(k_2) \left( \frac{\alpha^2(k_1) \alpha(k_2)}{\alpha(k_1) \alpha^2(k_2)} + \frac{1}{\alpha(k_1) \alpha^2(k_2)} \right) + 2 \text{ cyc.} \text{B}$$

$$+ b_{01}^2 \left( 2 F_2(k_1, k_2) \frac{P(k_1) P(k_2)}{\alpha(k_1) \alpha(k_2)} + 2 f_{\text{NL}} \alpha(k_3) \frac{P(k_1) P(k_2)}{\alpha^2(k_1) \alpha^2(k_2)} + 2 \text{ cyc.} \right) \text{C}$$

\(^1\)Note that the term H corrects a typo present in Eq.(5.6) of BSS.
\begin{equation}
+ b_{10}b_{20}\left(4P(k_1)P(k_2) + 2 \text{ cyc.}\right)_D
+ b_{20}b_{01}\left(2P(k_1)P(k_2)\left(\frac{1}{\alpha(k_1)} + \frac{1}{\alpha(k_2)}\right) + 2 \text{ cyc.}\right)_E
+ b_{11}b_{10}\left(2P(k_1)P(k_2)\left(\frac{1}{\alpha(k_1)} + \frac{1}{\alpha(k_2)}\right) + 2 \text{ cyc.}\right)_F
+ b_{11}b_{01}\left(P(k_1)P(k_2)\left(1 + \frac{2}{\alpha(k_1)\alpha(k_2)} + \frac{1}{\alpha^2(k_2)}\right) + 2 \text{ cyc.}\right)_G
+ b_{10}b_{02}\left(\frac{4P(k_1)P(k_2)}{\alpha(k_1)\alpha(k_2)} + 2 \text{ cyc.}\right)_H
+ b_{02}b_{01}\left(2P(k_1)P(k_2)\left(\frac{1}{\alpha^2(k_1)\alpha(k_2)} + \frac{1}{\alpha(k_1)\alpha^2(k_2)}\right) + 2 \text{ cyc.}\right)_I
- \frac{2}{7}b_{10}b_{10}\left(2\delta^2(k_1,k_2)P(k_1)P(k_2) + 2 \text{ cyc.}\right)_J
- \frac{2}{7}b_{01}b_{10}\left(2\delta^2(k_1,k_2)P(k_1)P(k_2)\left(\frac{1}{\alpha(k_1)} + \frac{1}{\alpha(k_2)}\right) + 2 \text{ cyc.}\right)_K
- b_{10}b_{10}\left(4P(k_1)P(k_2)\left(\frac{N_2(k_1,k_2)}{\alpha(k_2)} + \frac{N_2(k_2,k_1)}{\alpha(k_1)}\right) + 2 \text{ cyc.}\right)_L
- b_{01}b_{10}\left(2P(k_1)P(k_2)\left(\frac{N_2(k_1,k_2)}{\alpha(k_2)} + \frac{N_2(k_2,k_1)}{\alpha(k_1)}\right)\times\frac{1}{\alpha(k_1)} + \frac{1}{\alpha(k_2)} + 2 \text{ cyc.}\right)_L,
\end{equation}

The terms with label going from A to I match exactly the result of Eq. (5.6) in BSS but, as for the halo bispectrum, new terms appear. J, K are due to the tidal term $s^2$; schematically, they generated by $\langle s^2 \delta^{(1)} \delta^{(1)} \rangle$, $\langle s^2 \delta^{(1)} \varphi \rangle$ respectively. We see by comparison with eq. (D.1) that J is present regardless of PNG, while K comes from the coupling between the tidal term and $\varphi$ which is specifically introduced by PNG. L and M are present because of $n^2$ and, therefore, depend on the presence of PNG. These are schematically generated by $\langle n^2 \delta^{(1)} \delta^{(1)} \rangle$, $\langle n^2 \delta^{(1)} \varphi \rangle$, respectively.

Finally, for the halo-matter-matter bispectrum when Gaussian initial conditions are assumed we find

\begin{equation}
B_{\text{hmm}}(k_1,k_2,k_3) = b_{10}\left(6\mathcal{F}_2(k_1,k_2)P(k_1)P(k_2) + 2 \text{ cyc.}\right)_A
+ b_{20}\left(2P(k_1)P(k_2) + 2 \text{ cyc.}\right)_C
\end{equation}

\begin{equation}
- \frac{2}{7}b_{10}\left(2\delta^2(k_1,k_2)P(k_1)P(k_2) + 2 \text{ cyc.}\right)_F,
\end{equation}

where the same considerations for eq. (D.2) apply. Then, assuming PNG, the
halo-matter-matter bispectrum reads

\[ B_{\text{hm}}(k_1, k_2, k_3) = \]

\[ b_{10} \left( 6F_2(k_1, k_2)P(k_1)P(k_2) + 6f_{\text{NL}}\alpha(k_3)\frac{P(k_1)P(k_2)}{\alpha(k_1)\alpha(k_2)} + 2 \text{ cyc.} \right)_A \]

\[ + b_{01} \left( 2F_2(k_1, k_2)P(k_1)P(k_2) \left( \frac{1}{\alpha(k_1)} + \frac{1}{\alpha(k_2)} \right) \right)_B \]

\[ + 2f_{\text{NL}}\alpha(k_3)P(k_1)P(k_2) \left( \frac{1}{\alpha^2(k_1)\alpha(k_2)} + \frac{1}{\alpha(k_1)\alpha^2(k_2)} \right) + 2 \text{ cyc.} \right]_B \]

\[ + b_{20} \left( 2P(k_1)P(k_2) + 2 \text{ cyc.} \right)_C \]

\[ + b_{11} \left( P(k_1)P(k_2) \left( \frac{1}{\alpha(k_1)} + \frac{1}{\alpha(k_2)} \right) + 2 \text{ cyc.} \right)_D \]

\[ + b_{02} \left( 2P(k_1)P(k_2) \frac{\alpha(k_1)\alpha(k_2)}{\alpha(k_1)\alpha(k_2)} + 2 \text{ cyc.} \right)_E \]

\[ - \frac{2}{7} b_{10} \left( 2S_2(k_1, k_2)P(k_1)P(k_2) + 2 \text{ cyc.} \right)_F \]

\[ - \frac{1}{7} b_{01} \left( 2P(k_1)P(k_2) \left( \frac{N_2(k_1, k_2)}{\alpha(k_2)} + \frac{N_2(k_2, k_1)}{\alpha(k_1)} \right) + 2 \text{ cyc.} \right)_G, \]

with the terms going from A to E matching exactly Eq.(5.8) of BSS. As above, new terms appear: F is due to tidal term \( s^2 \) and it is generated by \( \langle s^2 \delta^{(1)} \rangle \), regardless of the presence of PNG, while G is due to \( n^2 \), sourced by \( \langle n^2 \delta^{(1)} \rangle \).
Appendix E

Position-dependent power spectrum

Although the bispectrum contains more information than power spectrum, it is more challenging to measure and, indeed, only few measurements have been reported so far [138, 141–148]. To overcome this issue, a new observable has been proposed in [160], which measures an integral of the squeezed configuration of the bispectrum. In [196], this has been applied to measure the non-linear bias $b_{20}$ from the BOSS data release 10.

This new observable, called integrated bispectrum, correlates the power spectrum in a subvolume of the survey volume (i.e. the position-dependent power spectrum) to the mean overdensity of the subvolume itself: basically it measures the response of the spectra of short density modes to a large-scale fluctuation. We briefly describe below how the position-dependent power spectrum and integrated bispectrum are built, referring the reader to [160, 196] for further details.

Given a density field $\delta(x)$ in a cubic survey volume $V$ with length side $L_B$, suppose to split it into $N$ subvolumes, with side $L = L_B/N$. If we now focus on the subvolume centred at $x_L$, we can measure the local mean overdensity as

$$\bar{\delta}(x_L) = \frac{1}{V_L} \int d^3x \, \delta(x) W_L(x - x_L), \quad (E.1)$$

where the volume of the subvolume is $V_L = L^3$ and the window function is assumed to be

$$W_L(x) = \prod_{i=1}^{3} \theta(x_i), \quad \theta(x_i) = \begin{cases} 1, & |x_i| \leq L/2, \\ 0, & \text{otherwise}. \end{cases} \quad (E.2)$$

The Fourier transform of the window is $W_L(k) = L^3 \prod_{i=1}^{3} j_0(k_i L/2)$, where the 0-th spherical Bessel function is $j_0(x) = \sin(x)/x$. The position-dependent
power spectrum is defined as
\[ P(k, x_L) = \frac{1}{V_L} |\delta(k, x_L)|^2 , \]
where \( \delta(k, x_L) \equiv \int_{V_L} d^3x \, \delta(x)e^{-ik \cdot x} \) is the Fourier transformation of the density field with integral ranging over the subvolume centred at \( x_L \).

If we now correlate the mean overdensity to the position-dependent power spectrum in the corresponding subvolume, it can be shown that
\[
\langle P(k, x_L)\delta(x_L) \rangle = \frac{1}{V_L^2} \int \frac{d^3p_1}{(2\pi)^3} \int \frac{d^3p_2}{(2\pi)^3} W_L(p_1)W_L(-p_1 - p_3)W_L(p_3) \times B(k - p_1, -k + p_1 + p_3, -p_3) \equiv iB(k) ,
\]
where \( iB(k) \) is the integrated bispectrum and \( B(k_1, k_2, k_3) \) can be the matter bispectrum, the galaxy bispectrum or cross-correlations between matter and galaxies. An angular average over \( iB(k) \) removes the remaining \( \hat{k} \)-dependence due to the choice of a cubic window function and one finally gets
\[
iB(k) \equiv \int \frac{d^2\Omega_k}{4\pi} iB(k) = \frac{1}{V_L^2} \int \frac{d^2\Omega_k}{4\pi} \int \frac{d^3p_1}{(2\pi)^3} \int \frac{d^3p_2}{(2\pi)^3} \times W_L(p_1)W_L(-p_1 - p_3)W_L(p_3)B(k - p_1, -k + p_1 + p_3, -p_3) .
\]

From the behaviour of the 0-th spherical Bessel function, the dominant contribution to the integrated bispectrum comes from wavenumbers \( k \) that are larger than \( 1/L \), i.e. from the squeezed configuration of the bispectrum \( B(k, -k + p_1 + p_3, -p_3) \to B(k, -k, -p_3) \) with \( p_1 \ll k \) and \( p_3 \ll k \).

If we consider the tree-level matter bispectrum with Gaussian initial conditions \( \langle f_{NL} \rangle = 0 \) [see eq. (3.100)],
\[
B_{mmn}^G(k_1, k_2, k_3) = 2[P(k_1)P(k_2)F_2(k_1, k_2) + 2 \text{perm}], \]

it can be shown that the integrated bispectrum is
\[
iB_{mmn}^G(k) \quad \text{as} \quad \frac{68}{21} - \frac{1}{3} \frac{d\ln k^3 P(k)}{d\ln k} P(k)\sigma_L^2 ,
\]
where \( \sigma_L^2 \) is the variance of the density field on the subvolume scale,
\[
\sigma_L^2 \equiv \frac{1}{V_L^2} \int \frac{d^3p_2}{(2\pi)^3} W_L^2(p_3)P(p_3) .
\]
By including local type PNG in the form of eq. (7.1), the tree-level matter bispectrum reads [see eq. (6.20)]
\[
B_{mmm}(k_1, k_2, k_3) = 2 \left[ F_2(k_1, k_2) + f_{NL}\frac{\alpha(k_3)}{\alpha(k_1)\alpha(k_2)} \right] P(k_1)P(k_2) + 2 \text{cyc.}
\]

151
and the linear response of the small-scale matter power spectrum to large-scale density perturbation is now

\[ iB_{mnm}(k) \xrightarrow{kL \to \infty} \left[ \frac{68}{21} - \frac{1}{3} \frac{d \ln k^3 P(k)}{d \ln k} \right] P(k)\sigma_L^2 + \]

\[ + 4f_{\text{NL}}\sigma_{\text{NG},i}^2 P(k) + 2f_{\text{NL}}\sigma_{\alpha}^2 \frac{P^2(k)}{\alpha^2(k)}, \quad (E.10) \]

where we have introduced the new quantities

\[ \sigma_{\text{NG},i}^2 = \frac{1}{V^2} \int \frac{d^3p}{(2\pi)^3} W_L^2(p_i) \frac{P(p_i)}{\alpha^3(p_i)} , \quad (E.11) \]

\[ \sigma_{\alpha}^2 = \frac{1}{V^2} \int \frac{d^3p}{(2\pi)^3} W_L^2(p_i) \alpha(p_i) . \quad (E.12) \]

If we now consider the galaxy bispectrum of eq. (7.9), the integrated bispectrum is

\[ iB_{agg}(k) \xrightarrow{kL \to \infty} P(k)\sigma_L^2 \left\{ \left[ b_{10}^2 \left( \frac{68}{21} - \frac{1}{3} \frac{d \ln k^3 P(k)}{d \ln k} \right) + 4b_{10}^2 b_{20} \right] + \right. \]

\[ + \frac{1}{\alpha(k)} \left\{ b_{10}^2 b_{11} + 2b_{10} b_{01} b_{20} + b_{10}^2 b_{01} \left( \frac{68}{21} - \frac{1}{3} \frac{d \ln k^3 P(k)}{d \ln k} \right) \right\} + \frac{2}{\alpha(k)^2} b_{10} b_{01} b_{11} \left( b_{01}^2 b_{11} + 2b_{01} b_{10} b_{20} \right) \]

\[ + P(k)\sigma_{\text{NG},1}^2 \left\{ b_{10}^2 b_{01} \left( \frac{68}{21} - \frac{1}{3} \frac{d \ln k^3 P(k)}{d \ln k} \right) + 2b_{10}^2 b_{11} + 2b_{10} b_{01} b_{20} + 4f_{\text{NL}} b_{10}^2 \right\} \]

\[ + \frac{1}{\alpha(k)} \left\{ b_{10}^2 b_{01} \left( \frac{68}{21} - \frac{1}{3} \frac{d \ln k^3 P(k)}{d \ln k} \right) + 4b_{10}^2 b_{02} + 2b_{10} b_{01} b_{11} + 4b_{01}^2 b_{20} \right\} + \]

\[ + \frac{2}{\alpha(k)^2} \left( b_{01}^2 b_{11} + 2b_{01} b_{10} b_{20} \right) \]

\[ + P(k)\sigma_{\text{NG},2}^2 \left\{ 2 \left( b_{10} b_{01} b_{11} + 2f_{\text{NL}} b_{10}^2 b_{01} \right) + \right. \]

\[ + \frac{2}{\alpha(k)} \left( b_{01}^2 b_{11} + 2b_{10} b_{01} b_{02} + 2f_{\text{NL}} b_{10} b_{02}^2 \right) + \frac{4}{\alpha(k)^2} \left( b_{01}^2 b_{02} \right) \right\} \]

\[ + P^2(k)\sigma_{\alpha}^2 \left\{ 2f_{\text{NL}} \frac{1}{\alpha(k)} \left( b_{10}^3 + \frac{1}{\alpha^3(k)} \left( b_{10}^2 b_{01} + b_{10} b_{01}^2 \right) \right) \right\} + \]

\[ + \frac{P^2(k)}{V_L} \left\{ 2 \left( b_{10}^2 b_{20} - \frac{4}{21} b_{10}^2 b_{10}^L \right) + \right. \]

\[ + \frac{2}{\alpha(k)} \left( b_{10}^2 b_{11} + b_{10} b_{01} b_{20} - \frac{8}{21} b_{10} b_{01} b_{10}^L + b_{10}^2 b_{01} \right) + \]

\[ + \frac{2}{\alpha^2(k)} \left( b_{10}^2 b_{02} + 2b_{10} b_{01} b_{11} + b_{01}^2 b_{20} - \frac{4}{21} b_{01}^2 b_{10}^L + 2b_{10} b_{02}^2 + 2b_{01}^2 \right) + \]

\[ + \frac{2}{\alpha^3(k)} \left( b_{01}^2 b_{02} + 2b_{10} b_{01} b_{02} \right) \right\} . \]

152
Equations \((E.10)\) and \((E.13)\) are our prediction for \(iB(k)\) coming from the matter and galaxy bispectrum with local-type non-Gaussianity, respectively. In [196], an analysis on \(iB_{ggg}\) shows poor constraints on \(f_{NL}\) compared to those from the power spectrum. However, since the scale-dependent bias due to local-type non-Gaussianity was ignored there, it would be interesting to see how much the constraint on \(f_{NL}\) from the position-dependent power spectrum would improve when the result of eq. \((E.13)\) is used. We leave this for future work.
Appendix F

\( \mathcal{D} \) factors

The factors \( \mathcal{D}(k_i, k_j, \cos \theta_{ij}, y_{ij}, \beta) \) introduced in section 7.2.2 are defined as the integrals below

\[
\mathcal{D}_{SQ1} = \frac{1}{4\pi} \int_{-1}^{+1} d\mu_1 \int_{0}^{2\pi} d\phi \ 2 \left( 1 + \beta \mu_i^2 \right) \left( 1 + \beta \mu_j^2 \right), \quad (F.1)
\]

\[
\mathcal{D}_{NLB} = \mathcal{D}_{SQ1}, \quad (F.2)
\]

\[
\mathcal{D}_{SQ2} = \frac{1}{4\pi} \int_{-1}^{+1} d\mu_1 \int_{0}^{2\pi} d\phi \ 2 \beta \mu_k^2 \left( 1 + \beta \mu_i^2 \right) \left( 1 + \beta \mu_j^2 \right), \quad (F.3)
\]

\[
\mathcal{D}_{FOG} = \frac{1}{4\pi} \int_{-1}^{+1} d\mu_1 \int_{0}^{2\pi} d\phi \ \beta \mu_k k_k \left( 1 + \beta \mu_i^2 \right) \left( 1 + \beta \mu_j^2 \right) \times \\
\times \left[ \beta \mu_k k_k \left\{ \frac{\mu_i}{k_i} \frac{\mu_j}{k_j} - \left( \frac{\mu_i}{k_i} + \frac{\mu_j}{k_j} \right) \right\} \right], \quad (F.4)
\]

\[
\mathcal{D}_{nG2} = \mathcal{D}_{SQ1}, \quad (F.5)
\]

\[
\mathcal{D}_{FOGnG} = -\frac{1}{4\pi} \int_{-1}^{+1} d\mu_1 \int_{0}^{2\pi} d\phi \ \beta \mu_k k_k \left( 1 + \beta \mu_i^2 \right) \left( 1 + \beta \mu_j^2 \right) \times \\
\times \left( \frac{\mu_i}{k_i} \alpha(k_i) + \frac{\mu_j}{k_j} \alpha(k_j) \right), \quad (F.6)
\]

\[
\mathcal{D}_{nG1}^{SQ1} = \frac{1}{4\pi} \int_{-1}^{+1} d\mu_1 \int_{0}^{2\pi} d\phi \ 2 \left( \frac{1 + \beta \mu_i^2}{\alpha(k_i)} + \frac{1 + \beta \mu_j^2}{\alpha(k_j)} \right), \quad (F.7)
\]

\[
\mathcal{D}_{nG1}^{NLB} = \mathcal{D}_{nG1}^{SQ1}, \quad (F.8)
\]

\[
\mathcal{D}_{nG1}^{SQ2} = \frac{1}{4\pi} \int_{-1}^{+1} d\mu_1 \int_{0}^{2\pi} d\phi \ 2 \beta \mu_k^2 \left( \frac{1 + \beta \mu_i^2}{\alpha(k_i)} + \frac{1 + \beta \mu_j^2}{\alpha(k_j)} \right), \quad (F.9)
\]

\[
\mathcal{D}_{nG1}^{FOG} = \frac{1}{4\pi} \int_{-1}^{+1} d\mu_1 \int_{0}^{2\pi} d\phi \ \beta \mu_k k_k \left( \frac{1 + \beta \mu_i^2}{\alpha(k_i)} + \frac{1 + \beta \mu_j^2}{\alpha(k_j)} \right) \times \\
\times \left[ \beta \mu_k k_k \left\{ \frac{\mu_i}{k_i} \frac{\mu_j}{k_j} - \left( \frac{\mu_i}{k_i} + \frac{\mu_j}{k_j} \right) \right\} \right], \quad (F.10)
\]
yielding the following results

\[ D_{SQ1} = \frac{2}{15} \left[ 15 + 10\beta + \beta^2 \left( 2x_{ij}^2 + 1 \right) \right] , \quad \text{(F.16)} \]

\[ D_{NLB} = D_{SQ1} , \quad \text{(F.17)} \]

\[ D_{SQ2} = \frac{2\beta}{105 \left( 2x_{ij} y_{ij} + y_{ij}^2 + 1 \right)} \left[ 12\beta^2 y_{ij} x_{ij}^3 + 2\beta x_{ij}^2 \left( 6\beta + 7 \right) (y_{ij}^2 + 1) + 2x_{ij} y_{ij} \left( 9\beta^2 + 42\beta + 35 \right) \right. \]

\[ \left. + \left( 3\beta^2 + 28\beta + 35 \right) (y_{ij}^2 + 1) \right] , \quad \text{(F.18)} \]

\[ D_{FOG} = \frac{\beta}{315 y_{ij}} \left[ 16\beta^3 y_{ij} x_{ij}^4 + 4\beta^2 x_{ij}^3 (5\beta + 9) (y_{ij}^2 + 1) + 24\beta x_{ij}^2 (2\beta^2 + 9\beta + 7) + 63\beta + 33\beta^2 + 63\beta + 35 \right. \]

\[ \times \left. \left( y_{ij}^2 + 1 \right) + 6y_{ij} \left( \beta^2 + 9\beta^2 + 35\beta + 35 \right) \right] , \quad \text{(F.19)} \]

\[ D_{nG2} = D_{SQ1} , \quad \text{(F.20)} \]

\[ D_{FoGnG} = \frac{\beta}{105 \alpha (k_i) \alpha (k_j) y_{ij}} \left[ 6\beta^2 x_{ij}^3 \left( \alpha (k_i) + \alpha (k_j) \right) y_{ij}^2 \right. \]

\[ + 2\beta (6\beta + 7) x_{ij}^2 y_{ij} \left( \alpha (k_i) + \alpha (k_j) \right) + (9\beta^2 + 42\beta + 35) x_{ij} \left( \alpha (k_i) + \alpha (k_j) \right) y_{ij}^2 \]

\[ + \left( 3\beta^2 + 28\beta + 35 \right) y_{ij} \left( \alpha (k_i) + \alpha (k_j) \right) \right] , \quad \text{(F.21)} \]

\[ D_{SQ1} = \frac{2}{3} \left( 3 + \beta \right) \left( \frac{1}{\alpha (k_i)} + \frac{1}{\alpha (k_j)} \right) , \quad \text{(F.22)} \]

\[ D_{NLB} = D_{SQ1} , \quad \text{(F.23)} \]

\[ D_{SQ2} = \frac{2\beta}{15 \alpha (k_i) \alpha (k_j) \left( 2x_{ij} y_{ij} + y_{ij}^2 + 1 \right} \times \]
\begin{align*}
\times \left[ \alpha(k_i) \left( \beta + 2 \beta x_{ij}^2 + 2(3 \beta + 5)x_{ij}y_{ij} + (3 \beta + 5)y_{ij}^2 + 5 \right) + \\
+ \alpha(k_j) \left( 3 \beta + 2 \beta x_{ij}^2 y_{ij}^2 + 2(3 \beta + 5)x_{ij}y_{ij} + (\beta + 5)y_{ij}^2 + 5 \right) \right],
\end{align*}

(D.24)

\begin{align*}
D_{nG1} = \frac{\beta}{105 \alpha(k_i)\alpha(k_j)y_{ij}} \times \\
\times \left[ 6 \beta^2 x_{ij}^3 (\alpha(k_i) + \alpha(k_j)y_{ij}^2) + 6 \beta(4 \beta + 7)x_{ij}y_{ij}(\alpha(k_i) + \alpha(k_j)) + \\
+ x_{ij} \left( (\alpha(k_i) (9 \beta^2 + 42 \beta + (15 \beta^2 + 42 \beta + 35)y_{ij}^2 + 35) + \\
+ \alpha(k_j) (15 \beta^2 + 42 \beta + (9 \beta^2 + 42 \beta + 35)y_{ij}^2 + 35) \right) + \\
+ 2 (3 \beta^2 + 21 \beta + 35) y_{ij}(\alpha(k_i) + \alpha(k_j)) \right],
\end{align*}

(D.25)

\begin{align*}
D_{nG2} &= D_{SQ1},
\end{align*}

(F.26)

\begin{align*}
D_{nG1}^2 &= \frac{\beta}{15 \alpha(k_i)^2\alpha(k_j)^2y_{ij}} \left[ 4 \alpha(k_i)\alpha(k_j)y_{ij}^2 + \\
+ (3 \beta + 5)x_{ij} (\alpha(k_i)^2 + \alpha(k_i)\alpha(k_j) (y_{ij}^2 + 1) + \alpha(k_j)^2y_{ij}^2) + \\
+ y_{ij} (\alpha(k_i)^2(3 \beta + 5) + 2\alpha(k_i)\alpha(k_j)(\beta + 5) + \alpha(k_j)^2(3 \beta + 5)) \right],
\end{align*}

(F.27)

\begin{align*}
D_{SQ2}^2 &= \frac{2}{3} \frac{\beta}{\alpha(k_i)\alpha(k_j)},
\end{align*}

(F.28)

\begin{align*}
D_{FOG}^2 &= \frac{\beta}{15 \alpha(k_i)\alpha(k_j)y_{ij}} \left[ 4 \beta^2 y_{ij}^2 + (3 \beta + 5)x_{ij} (y_{ij}^2 + 1) + 2(\beta + 5)y_{ij} \right],
\end{align*}

(F.29)

\begin{align*}
D_{FOG}^2 &= \frac{\beta}{3 \alpha(k_i)^2\alpha(k_j)^2y_{ij}} \left[ x_{ij} (\alpha(k_i) + \alpha(k_j)y_{ij}^2) + y_{ij}(\alpha(k_i) + \alpha(k_j)) \right],
\end{align*}

(F.30)

where \(x_{ij} = (k_i \cdot k_j)/k_i k_j\) and \(y_{ij} = k_i/k_j\).
Appendix G

Basic numbers for BOSS, eBOSS, DESI, Euclid

Here we present tables with the numbers describing the BOSS, DESI, Euclid [183] and eBOSS [184] surveys, which we use to forecast constraints on primordial non-Gaussianity in section 8.4.

Table G.1: Basic numbers for BOSS LRGs. The shell volume $V$ is in units of $(\text{Gpc}/h)^3$, while the number density $N_{\text{LRG}}$ in $10^{-4} (h/\text{Mpc})^3$. The fiducial value for $b_{10}^{\text{LRG}}$ and the estimates of $\nu_{\text{LRG}}$ and the non-linear bias $b_{20}^{\text{LRG}}$ are also presented.

<table>
<thead>
<tr>
<th>$z$</th>
<th>$V$</th>
<th>$N_{\text{LRG}}$</th>
<th>$b_{10}^{\text{LRG}}$</th>
<th>$\nu_{\text{LRG}}$</th>
<th>$b_{20}^{\text{LRG}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.03</td>
<td>3.14</td>
<td>1.74</td>
<td>1.68</td>
<td>-0.04</td>
</tr>
<tr>
<td>0.15</td>
<td>0.16</td>
<td>3.06</td>
<td>1.84</td>
<td>1.74</td>
<td>0.02</td>
</tr>
<tr>
<td>0.25</td>
<td>0.40</td>
<td>3.12</td>
<td>1.94</td>
<td>1.81</td>
<td>0.09</td>
</tr>
<tr>
<td>0.35</td>
<td>0.70</td>
<td>3.17</td>
<td>2.04</td>
<td>1.88</td>
<td>0.18</td>
</tr>
<tr>
<td>0.45</td>
<td>1.03</td>
<td>3.21</td>
<td>2.15</td>
<td>1.95</td>
<td>0.29</td>
</tr>
<tr>
<td>0.55</td>
<td>1.38</td>
<td>3.25</td>
<td>2.26</td>
<td>2.01</td>
<td>0.41</td>
</tr>
<tr>
<td>0.65</td>
<td>1.71</td>
<td>1.22</td>
<td>2.37</td>
<td>2.08</td>
<td>0.55</td>
</tr>
<tr>
<td>0.75</td>
<td>2.03</td>
<td>0.15</td>
<td>2.49</td>
<td>2.15</td>
<td>0.70</td>
</tr>
</tbody>
</table>
Table G.2: Basic numbers for eBOSS LRGs. The shell volume $V$ is in units of $(\text{Gpc}/h)^3$, while the number density $N_{\text{LRG}}$ in $10^{-4} (h/\text{Mpc})^3$. The fiducial value for $b_{10}^{\text{LRG}}$ and the estimates of $\nu_{\text{LRG}}$ and the non-linear bias $b_{20}^{\text{LRG}}$ are also presented.

<table>
<thead>
<tr>
<th>$z$</th>
<th>$V$</th>
<th>$N_{\text{LRG}}$</th>
<th>$b_{10}^{\text{LRG}}$</th>
<th>$\nu_{\text{LRG}}$</th>
<th>$b_{20}^{\text{LRG}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.65</td>
<td>1.20</td>
<td>0.810</td>
<td>2.37</td>
<td>2.08</td>
<td>0.55</td>
</tr>
<tr>
<td>0.75</td>
<td>1.42</td>
<td>0.678</td>
<td>2.49</td>
<td>2.15</td>
<td>0.70</td>
</tr>
<tr>
<td>0.85</td>
<td>1.63</td>
<td>0.350</td>
<td>2.61</td>
<td>2.21</td>
<td>0.87</td>
</tr>
<tr>
<td>0.95</td>
<td>1.82</td>
<td>0.097</td>
<td>2.73</td>
<td>2.28</td>
<td>1.06</td>
</tr>
</tbody>
</table>

Table G.3: Basic numbers for eBOSS QSOs. The shell volume $V$ is in units of $(\text{Gpc}/h)^3$, while the number density $N_{\text{QSO}}$ in $10^{-4} (h/\text{Mpc})^3$. The fiducial value for $b_{10}^{\text{QSO}}$ and the estimates of $\nu_{\text{QSO}}$ and the non-linear bias $b_{20}^{\text{QSO}}$ are also presented.

<table>
<thead>
<tr>
<th>$z$</th>
<th>$V$</th>
<th>$N_{\text{QSO}}$</th>
<th>$b_{10}^{\text{QSO}}$</th>
<th>$\nu_{\text{QSO}}$</th>
<th>$b_{20}^{\text{QSO}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.65</td>
<td>1.28</td>
<td>0.119</td>
<td>1.32</td>
<td>1.33</td>
<td>-0.22</td>
</tr>
<tr>
<td>0.75</td>
<td>1.52</td>
<td>0.130</td>
<td>1.42</td>
<td>1.42</td>
<td>-0.19</td>
</tr>
<tr>
<td>0.85</td>
<td>1.74</td>
<td>0.154</td>
<td>1.52</td>
<td>1.51</td>
<td>-0.16</td>
</tr>
<tr>
<td>0.95</td>
<td>1.95</td>
<td>0.171</td>
<td>1.63</td>
<td>1.59</td>
<td>-0.11</td>
</tr>
<tr>
<td>1.05</td>
<td>2.12</td>
<td>0.163</td>
<td>1.75</td>
<td>1.68</td>
<td>-0.04</td>
</tr>
<tr>
<td>1.15</td>
<td>2.28</td>
<td>0.170</td>
<td>1.87</td>
<td>1.77</td>
<td>0.04</td>
</tr>
<tr>
<td>1.30</td>
<td>4.96</td>
<td>0.175</td>
<td>2.06</td>
<td>1.89</td>
<td>0.21</td>
</tr>
<tr>
<td>1.50</td>
<td>5.36</td>
<td>0.166</td>
<td>2.34</td>
<td>2.06</td>
<td>0.51</td>
</tr>
<tr>
<td>1.70</td>
<td>5.65</td>
<td>0.151</td>
<td>2.64</td>
<td>2.23</td>
<td>0.93</td>
</tr>
<tr>
<td>1.90</td>
<td>5.84</td>
<td>0.137</td>
<td>2.97</td>
<td>2.40</td>
<td>1.48</td>
</tr>
<tr>
<td>2.05</td>
<td>2.96</td>
<td>0.122</td>
<td>3.23</td>
<td>2.53</td>
<td>1.99</td>
</tr>
<tr>
<td>2.15</td>
<td>2.98</td>
<td>0.093</td>
<td>3.41</td>
<td>2.61</td>
<td>2.39</td>
</tr>
</tbody>
</table>

158
Table G.4: Basic numbers for eBOSS ELGs. The labels Fisher, LD, HD stand respectively for Fisher Discriminant, Low Density DECam and High Density DECam selected objects. The shell volume $V$ is in units of $(\text{Gpc}/h)^3$, while the expected number density $N_X$ based on target selection definition $X$ is in $10^{-4} (\text{h/Mpc})^3$. The fiducial value for $b^{\text{ELG}}_{10}$ and the estimates of $\nu_{\text{ELG}}$ and the non-linear bias $b^{\text{ELG}}_{20}$ are also presented.

<table>
<thead>
<tr>
<th>$z$</th>
<th>$V$</th>
<th>$N_{\text{Fisher}}$</th>
<th>$N_{\text{LD}}$</th>
<th>$N_{\text{HD}}$</th>
<th>$b^{\text{ELG}}_{10}$</th>
<th>$\nu_{\text{ELG}}$</th>
<th>$b^{\text{ELG}}_{20}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.65</td>
<td>0.26</td>
<td>1.41</td>
<td>0.183</td>
<td>0.205</td>
<td>1.40</td>
<td>1.40</td>
<td>-0.20</td>
</tr>
<tr>
<td>0.75</td>
<td>0.30</td>
<td>2.17</td>
<td>1.91</td>
<td>2.07</td>
<td>1.46</td>
<td>1.46</td>
<td>-0.18</td>
</tr>
<tr>
<td>0.85</td>
<td>0.35</td>
<td>1.65</td>
<td>2.67</td>
<td>3.03</td>
<td>1.53</td>
<td>1.51</td>
<td>-0.15</td>
</tr>
<tr>
<td>0.95</td>
<td>0.39</td>
<td>0.624</td>
<td>1.14</td>
<td>1.61</td>
<td>1.60</td>
<td>1.57</td>
<td>-0.12</td>
</tr>
<tr>
<td>1.05</td>
<td>0.42</td>
<td>0.218</td>
<td>0.373</td>
<td>0.568</td>
<td>1.68</td>
<td>1.63</td>
<td>-0.08</td>
</tr>
<tr>
<td>1.15</td>
<td>0.46</td>
<td>0.081</td>
<td>0.159</td>
<td>0.241</td>
<td>1.75</td>
<td>1.68</td>
<td>-0.04</td>
</tr>
</tbody>
</table>

Table G.5: Basic numbers for DESI. The shell volume $V$ is in units of $(\text{Gpc}/h)^3$, while the number density $N_X$ for the tracers $X$ (LRGs, ELGs, QSOs) is in $10^{-4} (\text{h/Mpc})^3$. The corresponding fiducial value for $b^{X}_{10}$ and the estimates of $\nu_{X}$ and the non-linear bias $b^{X}_{20}$ are also presented.

<table>
<thead>
<tr>
<th>$z$</th>
<th>$V$</th>
<th>$N_{\text{ELG}}$</th>
<th>$b^{\text{ELG}}_{10}$</th>
<th>$\nu_{\text{ELG}}$</th>
<th>$b^{\text{ELG}}_{20}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.15</td>
<td>0.23</td>
<td>23.0</td>
<td>0.91</td>
<td>0.85</td>
<td>3.06</td>
</tr>
<tr>
<td>0.25</td>
<td>0.56</td>
<td>8.65</td>
<td>0.96</td>
<td>0.92</td>
<td>3.12</td>
</tr>
<tr>
<td>0.35</td>
<td>0.98</td>
<td>4.15</td>
<td>1.01</td>
<td>0.99</td>
<td>3.17</td>
</tr>
<tr>
<td>0.45</td>
<td>1.45</td>
<td>2.76</td>
<td>1.06</td>
<td>1.06</td>
<td>3.21</td>
</tr>
<tr>
<td>0.55</td>
<td>1.93</td>
<td>1.33</td>
<td>1.12</td>
<td>1.12</td>
<td>3.26</td>
</tr>
<tr>
<td>0.65</td>
<td>2.40</td>
<td>1.17</td>
<td>1.18</td>
<td>1.03</td>
<td>3.29</td>
</tr>
<tr>
<td>0.75</td>
<td>2.84</td>
<td>1.23</td>
<td>1.24</td>
<td>1.03</td>
<td>3.32</td>
</tr>
<tr>
<td>0.85</td>
<td>3.26</td>
<td>1.29</td>
<td>1.30</td>
<td>1.02</td>
<td>2.03</td>
</tr>
<tr>
<td>0.95</td>
<td>3.63</td>
<td>1.40</td>
<td>1.35</td>
<td>1.01</td>
<td>0.35</td>
</tr>
<tr>
<td>1.05</td>
<td>3.97</td>
<td>1.59</td>
<td>1.41</td>
<td>1.01</td>
<td>0.04</td>
</tr>
<tr>
<td>1.15</td>
<td>4.26</td>
<td>1.47</td>
<td>1.46</td>
<td>-0.18</td>
<td>0.00</td>
</tr>
<tr>
<td>1.25</td>
<td>4.52</td>
<td>1.40</td>
<td>1.51</td>
<td>-0.15</td>
<td>0.00</td>
</tr>
<tr>
<td>1.35</td>
<td>4.74</td>
<td>1.31</td>
<td>1.59</td>
<td>1.56</td>
<td>-0.13</td>
</tr>
<tr>
<td>1.45</td>
<td>4.93</td>
<td>1.20</td>
<td>1.65</td>
<td>1.61</td>
<td>-0.10</td>
</tr>
<tr>
<td>1.55</td>
<td>5.09</td>
<td>1.27</td>
<td>1.72</td>
<td>1.66</td>
<td>-0.06</td>
</tr>
<tr>
<td>1.65</td>
<td>5.22</td>
<td>0.480</td>
<td>1.78</td>
<td>1.70</td>
<td>-0.02</td>
</tr>
<tr>
<td>1.75</td>
<td>5.33</td>
<td>0.129</td>
<td>1.84</td>
<td>1.75</td>
<td>0.02</td>
</tr>
<tr>
<td>1.85</td>
<td>5.41</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>
Table G.6: Basic numbers for Euclid. The shell volume $V$ is in units of $(\text{Gpc}/h)^3$, while the number density $N$ in $10^{-4} (h/\text{Mpc})^3$. The fiducial value for $b_{10}$ and the estimates of $\nu$ and the non-linear bias $b_{20}$ are also presented.

<table>
<thead>
<tr>
<th>$z$</th>
<th>$V$</th>
<th>$N$</th>
<th>$b_{10}$</th>
<th>$\nu$</th>
<th>$b_{20}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.65</td>
<td>2.57</td>
<td>6.42</td>
<td>1.06</td>
<td>1.06</td>
<td>-0.23</td>
</tr>
<tr>
<td>0.75</td>
<td>3.05</td>
<td>14.5</td>
<td>1.11</td>
<td>1.12</td>
<td>-0.23</td>
</tr>
<tr>
<td>0.85</td>
<td>3.49</td>
<td>16.3</td>
<td>1.17</td>
<td>1.18</td>
<td>-0.23</td>
</tr>
<tr>
<td>0.95</td>
<td>3.89</td>
<td>15.0</td>
<td>1.22</td>
<td>1.23</td>
<td>-0.23</td>
</tr>
<tr>
<td>1.05</td>
<td>4.25</td>
<td>13.3</td>
<td>1.27</td>
<td>1.29</td>
<td>-0.22</td>
</tr>
<tr>
<td>1.15</td>
<td>4.57</td>
<td>11.6</td>
<td>1.33</td>
<td>1.34</td>
<td>-0.21</td>
</tr>
<tr>
<td>1.25</td>
<td>4.84</td>
<td>10.1</td>
<td>1.38</td>
<td>1.39</td>
<td>-0.20</td>
</tr>
<tr>
<td>1.35</td>
<td>5.08</td>
<td>8.42</td>
<td>1.44</td>
<td>1.44</td>
<td>-0.19</td>
</tr>
<tr>
<td>1.45</td>
<td>5.28</td>
<td>6.68</td>
<td>1.50</td>
<td>1.48</td>
<td>-0.17</td>
</tr>
<tr>
<td>1.55</td>
<td>5.45</td>
<td>5.09</td>
<td>1.55</td>
<td>1.53</td>
<td>-0.14</td>
</tr>
<tr>
<td>1.65</td>
<td>5.59</td>
<td>3.69</td>
<td>1.61</td>
<td>1.58</td>
<td>-0.12</td>
</tr>
<tr>
<td>1.75</td>
<td>5.71</td>
<td>2.56</td>
<td>1.67</td>
<td>1.62</td>
<td>-0.09</td>
</tr>
<tr>
<td>1.85</td>
<td>5.80</td>
<td>1.68</td>
<td>1.73</td>
<td>1.66</td>
<td>-0.05</td>
</tr>
<tr>
<td>1.95</td>
<td>5.87</td>
<td>1.02</td>
<td>1.78</td>
<td>1.70</td>
<td>-0.02</td>
</tr>
<tr>
<td>2.05</td>
<td>5.93</td>
<td>0.380</td>
<td>1.84</td>
<td>1.74</td>
<td>0.02</td>
</tr>
</tbody>
</table>

160
Bibliography


[39] [http://camb.info/](http://camb.info/)


[78] K. Koyama, “Non-Gaussianity of quantum fields during inflation,”


[80] D. Tseliakhovich, C. Hirata and A. Slosar, “Non-Gaussianity and large-scale structure in a two-field inflationary model,”

[81] Planck Collaboration, P. Ade et al, “Planck 2015 results. XVII. Constraints on primordial non-Gaussianity,”


[85] T. Suyama, T. Takahashi, M. Yamaguchi and S. Yokoyama, “Implications of Planck results for models with local type non-Gaussianity,”


[88] M. LoVerde and K. M. Smith, “The non-Gaussian halo mass function with f_{NL}, g_{NL} and \tau_{NL},”


[139] E. Sefusatti and E. Komatsu, “The bispectrum of galaxies from high-redshift galaxy surveys: Primordial non-Gaussianity and non-linear


**Research Ethics Review Checklist**

Please include this completed form as an appendix to your thesis (see the Postgraduate Research Student Handbook for more information).

<table>
<thead>
<tr>
<th>Postgraduate Research Student (PGRS) Information</th>
<th>Student ID: 673707</th>
</tr>
</thead>
<tbody>
<tr>
<td>PGRS Name: Matteo Tellarini</td>
<td></td>
</tr>
<tr>
<td>Department: ICG</td>
<td></td>
</tr>
<tr>
<td>First Supervisor: Prof. David Wands</td>
<td></td>
</tr>
<tr>
<td>Start Date: 1st October 2012</td>
<td></td>
</tr>
<tr>
<td>Study Mode and Route: Full-time</td>
<td></td>
</tr>
<tr>
<td>Title of Thesis: Primordial non-Gaussianity in the large-scale structure of the Universe</td>
<td></td>
</tr>
<tr>
<td>Thesis Word Count: 31,152</td>
<td></td>
</tr>
</tbody>
</table>

If you are unsure about any of the following, please contact the local representative on your Faculty Ethics Committee for advice. Please note that it is your responsibility to follow the University's Ethics Policy and any relevant University, academic or professional guidelines in the conduct of your study.

Although the Ethics Committee may have given your study a favourable opinion, the final responsibility for the ethical conduct of this work lies with the researcher(s).

**UKRIO Finished Research Checklist:**

(a) Have all of your research and findings been reported accurately, honestly and within a reasonable time frame? YES ☒ NO ☐

(b) Have all contributions to knowledge been acknowledged? YES ☒ NO ☐

(c) Have you complied with all agreements relating to intellectual property, publication and authorship? YES ☒ NO ☐

(d) Has your research data been retained in a secure and accessible form and will it remain so for the required duration? YES ☒ NO ☐

(e) Does your research comply with all legal, ethical, and contractual requirements? YES ☒ NO ☐

**Candidate Statement:**

I have considered the ethical dimensions of the above named research project, and have successfully obtained the necessary ethical approval(s).

Ethical review number(s) from Faculty Ethics Committee (or from NRES/SCREC): 100F-488B-4C3D-48E8-AAAA-554D-D318-C16F

Signed (PGRS): [Signature] Date: 30th March 2016

UPR16 – August 2015