NON-LINEAR VECTOR INTERACTIONS AND COSMOLOGICAL SELF-ACCELERATION

by

Matthew Dean Hull

This thesis is submitted in partial fulfilment of the requirements for the award of the degree of Doctor of Philosophy of the University of Portsmouth.

May 15, 2017
Abstract

Observations of the local Universe indicate that we are currently in an accelerated phase of cosmic expansion. This behaviour is compatible with our best theory of gravity, General Relativity, with the addition of a non-luminous form of energy that exerts a negative pressure known as ‘dark energy’. Alternatively, the observed acceleration could be an indication of the presence of new degrees of freedom active on cosmological scales. These degrees of freedom could be in the form of a dynamical field called ‘quintessence’, which is typically isolated from the rest of energy-momentum; although models which allow for interactions with ‘dark matter’ have recently been explored. This thesis explores a more radical idea; that the acceleration is a result of additional gravitational degrees of freedom acting on the largest scales.

Any modification to our description of gravity on cosmological scales must take into account that we have very accurate tests of General Relativity within the solar system. Therefore the theories we explore rely on non-linearities that allow them to evade solar system tests via a ‘screening mechanism’.

In particular, we focus on the use of special, non-linear, derivative structures which dominate on scales shorter than a characteristic length; suppressing the coupling to energy-momentum and thus screening the effect of the additional force. On the other hand, these interactions are negligible over cosmic scales and we recover a linear theory that communicates a fifth force. A typical theory of this type and the first to be discovered is the ‘Galileon’. We introduce this theory and discuss its behaviour around a static a spherically symmetric source which provides an example of ‘Vainshtein screening’.
We introduce Ostrogradsky’s construction which proves that non-degenerate theories with higher order time derivatives always have instabilities. However, by being degenerate, the special structure of Galileons ensures that they evade this result. The same structure is used to construct vector theories with non-linear derivative self-interactions. These theories, named ‘vector Galileons’, break gauge symmetries and have been shown to have interesting cosmological applications. We introduce a way to spontaneously break the gauge symmetry and construct these theories via a Higgs mechanism. In addition to the purely gauge field interactions, our method generates new ghost-free scalar-vector interactions between the Higgs field and the gauge boson. We show how these additional terms are found to reduce, in a suitable decoupling limit, to scalar bi-Galileon interactions between the Higgs field and Goldstone bosons. Our formalism is first developed in the context of abelian symmetry, which allows us to connect with earlier work on the extension of the Proca action. We then show how this formalism is straightforwardly generalised to generate theories with non-abelian symmetry.

Using an Arnowitt-Deser-Misner approach, we carefully reconsider the coupling with gravity of vector Galileons, with the aim of studying the necessary conditions to avoid the propagation of ghosts. We develop arguments that put on a more solid footing the results previously obtained in the literature. Moreover, working in analogy with the scalar counterpart, we find indications for the existence of a ‘beyond Horndeski’ theory involving vector degrees of freedom.

After identifying the decoupled longitudinal mode of the vector Galileon with the scalar Galileon, we investigate the number of degrees of freedom present in the theory. We discuss how to construct the theory from the extrinsic curvature of the constant scalar field hypersurface, and find a simple expression for the action which guarantees the existence of the primary constraint necessary to avoid the Ostrogradsky instability.

We then return to the ‘Galileonic Higgs mechanism’ and consider the effect of interactions between the higher order operators and a dynamical metric. We find a consistent covariantisation through the use of gravitational counter-terms that serve to also restrict the parameter space of the theory.
After a brief introduction to cosmological perturbation theory, we explore the cosmological applications of the Galileonic Higgs. We find self-accelerating background solutions, associated with a non-trivial profile of the vector. We then expand the action to quadratic order in linear perturbations, diagonalise and discover that one of the modes is a ghost. This is in contrast with the positive results of related scenarios where an instability on Minkowski space is removed by gravitational interactions.
# Table of Contents

Abstract i

Declaration vii

Acknowledgements ix

1 Cosmological Expansion 2
   1.1 Gravitation ................................. 7
   1.2 The expansion history of the Universe .......... 10
      1.2.1 Friedmann-Lemaître-Robertson-Walker metric .... 11
      1.2.2 Cosmic distances .......................... 13
      1.2.3 Observational evidence of cosmic acceleration .... 17
   1.3 Geometrical modifications of Einstein’s theory .... 18
      1.3.1 Galileons .................................. 22
      1.3.2 Vainshtein screening ......................... 24
      1.3.3 Extending the Proca action ................. 28
      1.3.4 The vector Galileon System .................. 30
   1.4 Ostrogradsky’s construction ..................... 31

2 A Higgs Mechanism for Vector Galileons 34
   2.1 The Higgs mechanism ............................ 36
   2.2 G-Higgs ........................................ 38
      2.2.1 Coupling to matter ......................... 45
   2.3 Higgs mechanism and generalized non-abelian symmetry breaking 46
Declaration

Whilst registered as a candidate for the above degree, I have not been registered for any other research award. The results and conclusions embodied in this thesis are the work of the named candidate and have not been submitted for any other academic award.

Word count: 30,000 words.
To Marjorie Dadson, for being such a fantastic Nanna.
Acknowledgements

I would like to use this opportunity to express my deepest gratitude to my family. Kanae and Percy, you are brilliant. Thank you for bringing so much joy to my life. To Dr. Tsuneo and Mrs. Yasuko Hanajima, your support has been truly fantastic, I couldn’t ask for a better father and mother-in-law. To Mum and Andrew; to Martin, Emma and Evie; to Michelle, Dan, Amabelle and ‘bump’; to Michael and Kate (and Buster); to Andrew, Aline, Amelie, Hugo and Esther; thank you all for you love, kindness and friendship.

I would also like to express my enormous debt to my supervisors, Professor Kazuya Koyama and Dr. Gianmassimo Tasinato. Without their dedication, patience and instruction this thesis would never have been possible.

To my fellows at the Institute of Cosmology and Gravitation; thank you for making my time at the institute so lively and interesting. In particular, I thank Marco Surace for being ace and reading through the manuscript and Dan Goddard, Robert Hardwick and Ben Mawdsley, for all the company at the pub and the great banter that came with it. Similarly to former fellows: Cullan Howlett, Bridget Falk, Kyle Westfall, James Etherington and Taichi Kidani, I thank you for all your conversations. To David Wilkinson, thank you for being kind and generous and introducing me to ‘Bar Billiards’. To Jeremy Sakstein, a big thank you for all your advice and for organising ‘Chicken Club’.

Last but not least, I would like to thank Julie and Holly for helping me out in so many ways and for all their patience with my last minute reimbursement forms.
Dissemination

This doctoral thesis covers a detailed presentation of the scientific results published in the following articles. The results are not presented in chronological order of their publication but rather in the logically most comprehensible way. The thesis also contains unpublished results, notably in chapter 6 and the appendices.

- Regan, D., Anderson, G., Hull, M., Seery, D.,
  *Constraining Galileon Inflation*,
  JCAP 1502 (2015) 015,
  arXiv:1411.4501

- Hull, M., Koyama, K., Tasinato, G.,
  *A Higgs Mechanism for Vector Galileons*,
  JHEP 1503 (2015) 154,
  arXiv:1408.6871

- Hull, M., Koyama, K., Tasinato, G.,
  *Covariantised Vector Galileons*,
  Phys.Rev. D93 (2016) no.6, 064012,
  arXiv: 1510.07029.

- Crisostomi, M., Hull, M., Koyama, K., Tasinato, G.,
  *Horndeski: beyond or not beyond*,
  JCAP03 (2016) 038,
Notation and conventions

Throughout this thesis we work with the *mostly plus* signature \((-,+,+,+\)} for the metric. Furthermore, we make extensive use of the antisymmetric properties of the *Levi-Civita epsilon tensor*. In particular, we make use of the following property,

\[
\epsilon_{\gamma_1...\gamma_{D-n}a_1...a_n} \epsilon^{\gamma_1...\gamma_{D-n}\beta_1...\beta_n} = -(D-n)! n! \delta^{[\beta_1...\beta_n]}_{a_1...a_n}.
\]  

(1)

We also find it convenient to define, \(\pi_{\mu_1...\mu_n} \equiv \partial_{\mu_n}...\partial_{\mu_1}\pi\). Four dimensional indices are written with greek lower case letters: \(\mu, \nu, \ldots\) whereas for the three dimensional indices we use lower case latin: \(i, j, k, \ldots\). The three-metric is written \(g_{ij}\) and raises and lowers three dimensional objects like the extrinsic curvature, \(K_{ij}\), or the three dimensional Riemann tensor \(\tilde{R}_{ijk}\). The four dimensional covariant derivative is written as \(\nabla_{\mu}\) and the three dimensional covariant derivative, (which is compatible with \(g_{ij}\)) is written as \(D_i\). The corresponding four dimensional connection is written as \(\Gamma^{\mu}_{\nu\rho}\) and the three dimensional one as \(\tilde{\Gamma}^{i}_{jk}\).
Chapter 1

Cosmological Expansion

Introduction

Cosmology informs our world view and has been a part of our identity long before the work of the ancient Greeks from whom we inherited its name. After thousands of years of philosophical debate, scientific endeavour and technical advancement we have arrived at a time of ‘precision cosmology’. Armed with a set of observations of unprecedented accuracy, we have come to view the Universe as being described by a ‘Standard Model of Cosmology’. This is a model based on the gravitational field equations of Einstein’s ‘General Relativity’ (GR) and describes the evolution of the Universe from a fraction of a second to the present day whilst spanning distances from the very deep sub-nuclear scale to tens of billions of light years. Such is the extent of gravitation described by Einstein’s theory.

According to our current understanding of gravitation, our picture of the Universe is one that started in a very hot and dense state which then led to a period of immense accelerated expansion called ‘inflation’. Inflation then ended and the decay of the ‘inflaton’ reheated the Universe. Radiation came to dominate the evolution of an extremely hot and dense Universe. Nuclei were formed in a process known as ‘big bang nucleosynthesis’ (BBN) and the expansion of the Universe continued so that eventually matter began to dominate and photons were able to decouple from the hot primordial plasma. Observational evidence for this early epoch is
imprinted in the distribution of the decoupled photons which now form the ‘Cosmic Microwave Background Radiation’ (CMB). This radiation contains a wealth of information about our Universe and has been measured to exquisite detail by experiments such as BICEP2 [1] and the Planck collaboration [2]. Tiny anisotropies in the temperature measured in the CMB are directly related to fluctuations in the density field. These anisotropies form the initial seed from which all structure in the known Universe find its origin.

After expanding and cooling, structures such as stars, galaxies and clusters of galaxies formed. With this expansion, the energy density of radiation and matter in the Universe continued to decrease. Without any other source of gravity, the Universe would either continue to expand indefinitely, expand but asymptote to zero expansion, or start to contract. Note that alternative cosmologies with a cosmological constant were certainly being seriously considered in light of the Flatness problem, where the spatial curvature of the Universe appears to be fine-tuned to some value very close to zero [3]. Moreover, in an attempt to reconcile observations of large scale structure with that of the cold dark matter model, [4] suggested the presence of a cosmological constant dominating the recent expansion of the Universe. However, it was not until the late nineties when two collaborations of astronomers measuring high red-shift supernovae found strong evidence for late time accelerated cosmological expansion. Indeed, observational data of Supernovae Type Ia (SN Ia) independently collected by the High-redshift Supernova Search Team and the Supernova Cosmology Project Team showed that the light from distant supernovae was actually dimmer than they would be under a constant or decelerating expansion [5, 6].

The motivation and focus of this work is to try to understand the cause of the acceleration from the point of view of gravity. The use of General Relativity to describe the acceleration requires that we add a homogeneous and isotropic source of near constant energy-momentum with negative pressure. This new and unknown form of energy is often called ‘dark energy’ but it can also be exactly constant, in this case, it is called the ‘cosmological constant’ and was originally introduced by Einstein to ensure his field equations would have static (albeit unsta-
ble) solutions [7]. Our current formulation of particle physics in terms of ‘quantum field theory’, also predicts the existence of a cosmological constant in the form of ‘vacuum energy density’. Unfortunately, the expected value of this vacuum energy, (assuming supersymmetry removes the contributions above the electroweak scale), is calculated to be at least \(10^{12}\text{eV}^4\) which is enormous when compared to the value of this constant extrapolated from the supernovae observations \((2.3 \times 10^{-3}\text{eV})^4[5, 6, 8, 9]\).

In this work, we focus on an alternative scenario where rather than inferring an exotic form of non-luminous energy, we instead speculate on whether GR needs modifying when considering gravitation over the largest scales. As a universal force, there is every reason to expect that gravity acts over all these scales. What is still unknown however, is whether there is any limit to the applicability of General Relativity. Recent laboratory tests of gravity confirm that it follows an inverse square law down to at least the sub-millimetre scales whilst laser ranging and observations of the solar system confirm General Relativity’s predictions in the weak field approximation (see [10] for an excellent review). Moreover, with the recent detection of gravitational waves from binary black hole mergers by the ‘Advanced Laser Interferometer Gravitational-Wave Observatory’ (aLIGO), it is now possible to probe the accuracy of GR in the strong field regime [11, 12].

In the course of this thesis we will discuss the feasibility of an alternative description for the Universe. This is because despite the success of General Relativity, we have yet to truly test its accuracy on the galactic or cosmological scales. Furthermore, the standard model of cosmology is an extrapolation of our knowledge of gravity from local measurements that is kept consistent with current observations only once a predominantly ‘dark’ component is included in the energy content of the Universe.

In the first half of this chapter we introduce the formalism of General Relativity and motivate expressions for describing the background evolution of the Universe. We also discuss the distinct forms of matter and how they affect the expansion of the Universe. We then define contrasting measures of cosmic distances and give an overview of the observations that are used to constrain them.
In the latter half of this chapter we move on to discuss how the results of these observations can be used to motivate exploring possible extensions and modifications of General Relativity. We introduce Galileons as an interesting example of a modified gravity theory. We then discuss the important properties of this class of theory such as Vainshtein screening and superluminal sound speeds. The special structure of the Galileons can be used to construct a ghost free extension of the Proca action. The Proca action describes the dynamics of a massive spin one field, such as a massive photon. We introduce this extended theory, named ‘vector Galileons’, at the end of this chapter.

In chapter 2 we discuss one of our main results; a Higgs mechanism for generating vector Galileons. This is a consistent extension of the Higgs mechanism that generates the vector Galileon system when the Higgs field acquires a vacuum expectation value (vev). Furthermore, we discover that by taking a suitable decoupling limit the Higgs-vector interactions are rendered into a ‘bi-Galileon’ system formed out of the Higgs and the, would-be, Goldstone boson. We end the chapter by discussing the generalisation to non-abelian gauge theories.

Vector Galileons have been shown to have interesting cosmological applications. In chapter 3 we analyse how the inclusion of interactions with a dynamical metric affects the stability of the system. Using the Arnowitt-Deser-Misner (ADM) formalism and working with perturbations around a special anstaz consistent with a homogeneous and isotropic background, we find that for the quartic and quintic operators, the inclusion of non-minimal couplings remove terms associated with unhealthy dynamics. Interestingly, these same terms are also removed in the minimal covariantisation of the fully antisymmetric version of the theory.

We continue our analysis in chapter 4 for general backgrounds but concentrate on the interactions of the longitudinal mode only. We again work with the ADM formalism but unlike previous analysis in the literature which used the unitary gauge we use a foliation formed of constant scalar field hypersurfaces. These are hypersurfaces on which the scalar (longitudinal mode) takes a constant value. This construction allows us to show how the presence of a primary constraint removes the unwanted ghost degrees of freedom.
Having already discussed an extension to the Higgs mechanism which is able to generate the vector Galileons spontaneously, we now address issues related to the consistency of including interactions with a dynamical metric. Having a consistent covariant theory on a general space-time will allow us to connect this theory with the interesting cosmological applications of the vector Galileons. In chapter 5 we follow the procedure developed for scalar Galileons in [13] and derive non-minimal counter terms for the quartic and the quintic level operators. Interestingly, when considering the abelian symmetry of the action, we find an obstruction to constructing a counter term for the quintic.

In chapter 6 we investigate the cosmological applications of the cubic Galileonic Higgs. After setting up our notation and reviewing general cosmological perturbation theory, we discuss two examples from the literature, both with self-accelerating background solutions. Furthermore, both of these theories possess cubic order non-linear interactions and instabilities on a flat, Minkowski space-time.

Motivated by these examples, we consider a ‘Ghost Galileonic Higgs’ model. This is a model formed of the Einstein-Hilbert action, Maxwell action and a Galileonic Higgs action with a ghost signature for the kinetic term and G-Higgs operators up to third order. The ghost signature for the kinetic term allows us to find consistent non-trivial background solutions; including a self-accelerating scenario. We then examine the linear (scalar) perturbations around this background and expand the action to second order. We find a system of two scalars that after diagonalising reveals that one of them is a ghost. Therefore, although the background generated by the non-trivial profile of the vector is able to ‘cure’ the perturbations associated with the longitudinal mode, the Higgs perturbations remain unhealthy. We end the chapter by discussing how including a conformal coupling between the Higgs and the Ricci curvature scalar can generate extra dynamics for the Higgs such that its kinetic term is rendered healthy.

Finally in chapter 7 we review our results and discuss the future outlook for the Galileonic Higgs and its relevance to cosmology.
1.1 Gravitation

Our modern understanding of gravity was for the most part developed by Einstein at the beginning of the 20th century. Relying on his theory of special relativity and the geometrical picture developed by Minkowski that space-time is a four dimensional hyperbolic space, Einstein was able to utilise his intuition concerning the equivalence of acceleration and gravitation to create an entirely new theory, ‘General Relativity’.

Gravitation as conceptualised in General Relativity is a geometrical property of space-time. In the mathematical language of differential geometry, space-time is modelled as a four dimensional manifold with a metric of Lorentzian signature. Space-time is no longer considered a fixed background for physics to work on but is actually a physical system with its own dynamics. Mathematical concepts such as displacement, covariant differentiation, curvature and torsion are used to describe the effect of gravitational interactions with itself and on the properties of physical systems.

Einstein famously relied on his physical intuition to reach this description of gravitation. However, there exists an alternative path that paints a picture of gravitation being the exchange of massless particles, called gravitons. This path is based on quantum mechanics and special relativity and fits in with the description of the rest of fundamental physics in which forces are modelled as the exchange of particles.

In the mathematical language of special relativity, we understand nature to be Poincaré invariant. That is, we characterise particles by the way they transform under translations, rotations and boosts. Particles can be either massive or massless and are distinguished by their spin.

Weinberg, following on from earlier work by Feynman, considered the quantum mechanical properties of massless spin two particles and found that the equivalence principle is actually a consequence of unitarity [14]. Furthermore, the leading contribution in the low energy, ‘infra-red’ limit results uniquely in a linearised
form of the Einstein-Hilbert action\(^1\).

The linearised Einstein-Hilbert action is a gauge theory of linearised diffeomorphisms, (general coordinate transformations). However, although the free massless spin-2 field equations of the linear theory are perfectly consistent, this is not true when they are sourced by a dynamical system’s energy-momentum. This is because the free field equations are divergenceless, whereas the energy-momentum of the source is not conserved. Conservation of energy-momentum at the linear level can be restored by including the stress tensor of the quadratic action that led to the linear field equations. However, the obstruction then reappears at the next order. When summed over, this series of contributions leads to the full non-linear Einstein equations, (see [16]). Also, in [16], by utilising the ‘Palantini formalism’ and perturbing about the inverse metric, Deser showed that this action can be nonlinearly completed in a single step to obtain the full non-linear theory of General Relativity. This result therefore completed a picture of General Relativity as being the unique infra-red theory of a massless spin two field.

Moreover, this non-geometric path to General Relativity indicates its true nature as ‘... the theory of a non-trivially interacting massless helicity 2 particle’ [17]. The geometrical properties of the fully non-linear theory emerge as a consequence the coupling of this field to the total energy-momentum tensor [16].

In what follows we work with the modern formulation of General Relativity where space-time is modelled in terms of Riemannian Geometry. Given a metric, \( g_{\mu\nu} \), we assume there is no torsion and construct the Cristoffel symbol related to the Levi-Civita connection,

\[
\Gamma_{\nu\rho}^{\mu} = \frac{1}{2} g^{\mu\sigma} \left\{ g_{\nu\sigma,\rho} + g_{\rho\sigma,\nu} - g_{\nu\rho,\sigma} \right\}, \tag{1.1}
\]

where \( g_{\mu\nu,\lambda} \equiv \partial g_{\mu\nu}/\partial x^{\lambda} \).

The covariant derivative on a tensor \( A_{\mu_1 \ldots}^{V_1 \ldots} \) compatible with this metric is given by,

\[
\nabla_{\rho} A_{\mu_1 \ldots}^{V_1 \ldots} = A_{\mu_1 \ldots}^{V_1 \ldots,\rho} = A_{\mu_1 \ldots,\rho}^{V_1 \ldots} - \Gamma_{\mu_1,\rho}^{\sigma} A_{\sigma \mu_2 \ldots}^{V_1 \ldots} \ldots + \Gamma_{\sigma,\rho}^{\sigma_{\mu_2 \ldots} \mu_1 \ldots} \ldots \tag{1.2}
\]

We make use of the Ricci identity to define the Riemann curvature tensor,

\[
[\nabla_{\mu}, \nabla_{\nu}] A_{\sigma} = R_{\sigma \mu \nu}^{\rho} A_{\rho}, \tag{1.3}
\]

\(^1\)There is an alternative class of interacting spin two theory but this turns out to be acasual [15].
where $[,]$ denotes commutation.

The components of the Riemann curvature tensor are given by,

$$ R^\mu_{\nu\alpha\beta} = \Gamma^\mu_{\beta\nu,\alpha} - \Gamma^\mu_{\alpha\nu,\beta} + \Gamma^\mu_{\alpha\rho} \Gamma^\rho_{\beta\nu} - \Gamma^\mu_{\beta\rho} \Gamma^\rho_{\alpha\nu}, \quad (1.4) $$

and has the following symmetry properties:

$$ R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu} = R_{\mu\alpha\beta\nu}. \quad (1.5) $$

The Riemann curvature tensor obeys two Bianchi identities\(^2\),

$$ R_{\alpha[\beta\mu\nu]} \equiv 0, \quad (1.6) $$
$$ R_{\alpha\beta[\mu\nu;\rho]} \equiv 0, \quad (1.7) $$

where the square brackets on the indices denotes weighted anti-symmetrisation, e.g. $A_{[\mu\nu]} := \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu})$. Contracting the second identity gives,

$$ 0 \equiv g^{\alpha\mu} g^{\beta\nu} R_{\alpha\beta[\mu\nu;\rho]} = g^{\alpha\mu} g^{\beta\nu} (R_{\alpha\beta\mu\nu;\rho} + R_{\alpha\beta\nu\rho;\mu} + R_{\alpha\beta\rho\mu;\nu}), $$
$$ = R_{\rho\mu} - g^{\alpha\mu} R_{\alpha\rho;\mu} - g^{\beta\nu} R_{\beta\rho;\nu} = R_{\rho\mu} - 2R^\mu_{\rho;\mu}, \quad (1.8) $$

where $R_{\rho\mu} \equiv R^\alpha_{\mu\alpha\nu}$ is the Ricci tensor and $R \equiv R^\mu_{\mu}$ is the Ricci scalar. On the second line, the last term on the right hand side is just the divergence of the Einstein tensor, which is defined to be, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$, and therefore we have,

$$ \nabla_\mu G^{\mu\nu} \equiv 0. \quad (1.9) $$

Note that it was this property that led Einstein to chose the specific form of his field equations as it ensures local energy-momentum conservation (see equation (1.11)).

The action for General Relativity is given by the Einstein-Hilbert action, $S_{EH}$, together with an action for matter,

$$ S = \frac{c^4}{16\pi G} \int d^4x \sqrt{-g} R + \int d^4x \sqrt{-g} \mathcal{L}_m. \quad (1.10) $$

---

\(^2\)Note that this second identity is actually a consequence of the Jacobi identity.
Note that we have chosen not to include the cosmological constant as we include a dark energy term in the matter sector. Varying this action with respect to the metric leads to Einstein's field equations for gravity,

$$G_{\mu \nu} = \frac{8\pi G}{c^4} T_{\mu \nu}, \quad (1.11)$$

where $T_{\mu \nu}$ is the energy-momentum tensor,

$$T_{\mu \nu} \equiv \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}.L_m)}{\delta g^{\mu \nu}}. \quad (1.12)$$

We therefore see that equation (1.9) ensures that we have local energy-momentum conservation.

In the rest of the text we will often set the speed of light, $c$, and Planck’s constant, $\hbar$ to one and replace Newton’s constant, $G$, with the reduced Planck mass,

$$M_{Pl} \equiv \sqrt{\frac{\hbar c}{8\pi G}} = 2.4 \times 10^{18} \text{GeV}. \quad (1.13)$$

## 1.2 The expansion history of the Universe

Observations of the cosmos on the largest scales indicate that the Universe is statistically isotropic. Furthermore, if we apply the *Copernican principle*, which states that we should not consider ourselves to be located at a special position in the Universe, then we come to the conclusion that the Universe must also be statistically homogeneous. Therefore the observed statistical isotropy of the Universe suggests the accuracy of the *Cosmological Principle* that hypothesises that the Universe should be both isotropic and homogeneous. In addition, we also observe that the Universe has been expanding for over ten billion years. Therefore, four-dimensionally, our Universe is not flat but actually warped in the direction of ‘time’.

In this section we discuss how the formalism of General Relativity is used to model the Universe. In particular we introduce the ‘Friedmann-Lemaître-Robertson-Walker metric’ (FLRW), which models the largest scales in the cosmos, i.e. the ‘background’. We then discuss the different measurements of cosmic distance
used to put observational constraints on the Universe’s expansion. Finally, we present some of the evidence for our current understanding of its expansion history.

1.2.1 Friedmann-Lemaître-Robertson-Walker metric

As mentioned earlier, the FLRW metric models an expanding, spatially homogeneous and isotropic Universe. It can be derived from a metric where the hypersurfaces of constant cosmic time are maximally symmetric subspaces of the whole of space-time, (see [18]), and the line element is given by (\(c := 1\)),

\[
ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)\gamma_{ij}dx^idx^j,
\]

(1.14)

where \(a(t)\) is the scale factor, \(t\) is the cosmic time and \(\gamma_{ij}\) is the time independent, three dimensional spatial metric with constant scalar curvature, \(K\). In polar coordinates \((x^1, x^2, x^3) = (r, \theta, \phi)\) we have,

\[
\gamma_{ij} = \begin{bmatrix}
\frac{1}{1-Kr^2} & 0 & 0 \\
0 & r^2 & 0 \\
0 & 0 & r^2\sin^2\theta
\end{bmatrix}.
\]

(1.15)

The constant scalar curvature, \(K\), has units of inverse length squared and can be either positive, negative or zero which correspond to closed, open and flat Universes respectively\(^3\).

In a FLRW space-time the energy-momentum is modelled by a perfect fluid, therefore the energy-momentum tensor takes the following form,

\[
T^\mu_\nu = (\rho + P)u^\mu u_\nu + P\delta^\mu_\nu,
\]

(1.16)

where we use comoving coordinates and \(u^\mu = (-1, 0, 0, 0)\) is the four-velocity of the fluid (with \(c = 1\)), \(\rho\) is the energy density and \(P\) is the pressure.

The FLRW metric is a solution to the Einstein field equations given by equation

\(^3\text{We could choose to use dimensionless coordinates such that the constant curvature } K \text{ takes values of } \pm 1, 0. \text{ However, we prefer to keep the scale factor, } a(t) \text{ dimensionless.}\)
Substituting both the metric and the previous expression for the energy-momentum tensor into the field equations gives for the $(00)$ and $(ii)$ components of the equation,

$$H^2 = \frac{8\pi G}{3} - \rho - \frac{K}{a^2},$$  \hspace{1cm} (1.17)

$$H^2 + 2\dot{H} = -8\pi Gp - \frac{K}{a^2}.$$  \hspace{1cm} (1.18)

Equation (1.17) is known as the ‘Friedmann equation’. It is actually a constraint for the time evolution of the Universe. We can combine the above two equations and eliminate the terms dependent on the scalar curvature to find the ‘acceleration equation’,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p).$$  \hspace{1cm} (1.19)

Substituting this relation into the time derivative of the Friedmann equation results in the ‘energy conservation equation’,

$$\dot{\rho} + 3H(\rho + p) = 0.$$  \hspace{1cm} (1.20)

It is often useful to define a ‘Critical energy density’, $\rho_c$ which is defined as the minimal amount of energy-momentum to ensure that the Universe is spatially flat ($K = 0$). In order to find this we set $K := 0$ in equation (1.17) and rearrange to find, $\rho_c$,

$$H^2 = \frac{8\pi G}{3} - \rho_c \quad \Rightarrow \quad \rho_c = \frac{3H^2}{8\pi G}.$$  \hspace{1cm} (1.21)

This allows us to define the cosmological density parameters, $\Omega_M$ and $\Omega_K$ such that the Friedmann equation can be re-expressed as,

$$\Omega_M + \Omega_K = 1,$$  \hspace{1cm} (1.22)

where $\Omega_K \equiv -K/(aH)^2$. The matter density parameter, $\Omega_M$, is further split into the respective contributions from radiation, matter, and dark energy (cosmological constant), $\sum \Omega_M = \Omega_{\text{rad}} + \Omega_m + \Omega_{\text{DE}}$. Note that for models with a cosmological constant we define, $\Omega_{\Lambda} \equiv \Lambda/3H^2$.

Considering the expression for $\Omega_K$, we see that if our universe were to decelerate, $\ddot{a} < 0$, then $aH$ decreases and any amount of primordial curvature would increase.
However, we measure our Universe to be extremely close to being flat. Therefore unless the curvature was exactly zero, we require a period of cosmic acceleration in the early Universe [19]. This is referred as the ‘flatness problem’ and is one of the original motivations for theories of inflation.

The Universe has undergone different phases in its expansion. These phases have been dominated by the different components of energy density in equation (1.22). In these periods of dominance, we can model the dominant component as having an ‘equation of state’ given by,

\[ P = w \rho . \quad (1.23) \]

Assuming \( w \) is constant we solve the Friedmann and energy conservation equations (equations (1.17) and (1.20)) to obtain the behaviour of the energy density and scale factor,

\[ \rho \propto a^{-(1+w)}, \quad a(t) = (t/t_0)^{2/(1+w)}, \quad (1.24) \]

where the present value of the scale factor is set to unity \( a(t_0) = a_0 := 1 \) and \( t_0 \) denotes the present value of cosmic time. We therefore find for the following values of \( w \), corresponding expressions for \( \rho \) and \( a \):

- **Radiation** \((w = 1/3)\): \( \rho_{\text{rad}} \propto a^{-4} \) and \( a = (t/t_0)^{1/2} \),
- **Non-relativistic matter** \((w \approx 0)\): \( \rho_{m} \propto a^{-3} \) and \( a = (t/t_0)^{2/3} \).
- **Cosmological constant** \((w = -1)\): \( \rho = \text{const.} \) and \( a = \exp[H(t-t_0)] \) where \( H = \text{constant.} \)

### 1.2.2 Cosmic distances

As we mentioned in the introduction, the accelerated expansion of the Universe was discovered by measuring the ‘dimness’ of distant supernovae versus what was expected from their ‘redshift’,

\[ z = \frac{\lambda_0}{\lambda} - 1 = \frac{a_0}{a} - 1, \quad (1.25) \]
where in an expanding Universe, the observed wavelength $\lambda_0$ of absorption lines in distant supernovae is larger than the wavelength observed in the rest frame, $\lambda$. The physical distance from an observer to an object in an expanding space-time in ‘comoving coordinates’ is given by, $\bar{r} = a(t)\bar{x}$. Taking the time derivative gives us,

$$\dot{\bar{r}} = H\bar{r} + a\dot{\bar{x}}. \quad (1.26)$$

This velocity is decomposed into the ‘recessional velocity’, $v_H \equiv H\bar{r}$, the velocity associated with the expansion and the ‘peculiar velocity’, $v_p \equiv a\dot{\bar{x}}$, which is associated with the motion within the local environment of the object. At large enough distances where the contribution from the peculiar motion is negligible, we recover a relationship between redshift and distance which is known as ‘Hubble’s law’,

$$v \simeq H_0 \bar{r}, \quad (1.27)$$

where $H_0$ is the present value of $H$.

In this subsection, we define the different concepts of cosmic distance that are directly related to observations in the FLRW Universe.

Consider the FLRW metric from equation (1.14). We set $r = r_0 \sin(\chi/r_0)$, $r = \chi$ and $r = r_0 \sinh(\chi/r_0)$ for $k \equiv K r_0^2 = 1, 0, -1$, respectively. The three dimensional line element can be written as,

$$dx^2 = d\chi^2 + (f_K(\chi))^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (1.28)$$

where,

$$f_K(\chi) = \begin{cases} 
    r_0 \sin(\chi/r_0) & \text{if } k = 1, \\
    \chi & \text{if } k = 0, \\
    r_0 \sinh(\chi/r_0) & \text{if } k = -1.
\end{cases} \quad (1.29)$$

**Comoving distance**

Consider the case where light is emitted at time $t = t_1$ with $\chi = \chi'$ (corresponding to redshift $z$) and reaches an observer (at $z = 0$) at time $t = t_0$ with $\chi = 0$ integrating
equation (1.28) gives,
\[ d_c \equiv \chi' = \int_0^{\chi'} d\chi = - \int_{t_0}^{t_1} \frac{c}{a(t)} \, dt. \] (1.30)
This is the ‘comoving distance’. Using the definition of redshift in equation (1.25) we have that the comoving distance is given by,
\[ d_c = \frac{c}{a_0 H_0} \int_0^z \frac{H_0}{H(\tilde{z})} \, d\tilde{z}. \] (1.31)

**Luminosity distance**

The ‘luminosity distance’ is used with observations of SN Ia supernovae to ascertain the expansion rate of the Universe. The ‘luminosity’ of a supernovae, or other astronomical object is the total amount of energy emitted per unit time. For an observed flux \( \mathcal{F} \) it is given by,
\[ d_L^2 \equiv \frac{L_s}{4\pi \mathcal{F}}, \] (1.32)
where \( L_s \) is the absolute luminosity of a source. The flux depends on the observed luminosity, \( L_0 \) at \( z = 0 \) and the area of a sphere; \( S = 4\pi (a_0 f_K(\chi))^2 \) and is given by \( \mathcal{F} = L_0 / S \). Substituting these relations into equation (1.32) we find that the luminosity distance is dependent on the ratio of the luminosity at the source and the observed luminosity,
\[ d_L^2 = (a_0 f_K(\chi))^2 \frac{L_s}{L_0}. \] (1.33)
The ratio of the luminosities can be calculated by taking in to account both the effect expansion has on the wavelength of light as well as on the rate of change of time.

Consider the energy of light \( \Delta E_1 \) being emitted by a source within a period of time \( \Delta t_1 \) such that \( L_s = \Delta E_1 / \Delta t_1 \). This light is then observed with a luminosity given by, \( L_0 = \Delta E_0 / \Delta t_0 \) where \( \Delta E_0 \) is the total energy of light detected within a \( \Delta t_0 \) period of time. The energy of a photon is proportional to the inverse of its wavelength, \( \lambda \). Therefore we use equation (1.25) to relate the ratio of the two energies to redshift, \( \Delta E_1 / \Delta E_0 = \lambda_0 / \lambda_1 = 1 + z \).
Furthermore, the rate of particles emitted is higher than the rate at which they
are absorbed. We can see the reason for this by examining the comoving distance. Equation (1.30) provides an expression for the comoving distance in terms of time. Light travelling from the source from time \( t_1 \) reaches the observer at time \( t_0 \) then light travelling from \( t_1 + \Delta t_1 \) reaches the observer at \( t_0 + \Delta t_0 \). Since the comoving distance for both signals is the same we have the following relation,

\[
d_c = - \int_{t_0}^{t_1} \frac{c}{a(t)} dt = - \int_{t_0 + \Delta t_0}^{t_1 + \Delta t_1} \frac{c}{a(t)} dt ,
\]

\[
\Rightarrow \quad \int_{t_0}^{t_0 + \Delta t_0} \frac{dt}{a(t)} = \int_{t_0 + \Delta t_0}^{t_1 + \Delta t_1} \frac{dt}{a(t)} ,
\]

\[
\Rightarrow \quad \frac{\Delta t_0}{a_0} = \frac{\Delta t_1}{a_1} .
\]

Using equation (1.25) we find that,

\[
\frac{\Delta t_0}{\Delta t_1} = \frac{a_0}{a_1} = \frac{\lambda_0}{\lambda_1} = (1 + z) .
\]

Therefore, in terms of redshift, the ratio of luminosities is given by,

\[
\frac{L_s}{L_0} = \frac{\Delta E_1}{\Delta E_0} \frac{\Delta t_0}{\Delta t_1} = (1 + z)^2 ,
\]

and the luminosity distance can be written as,

\[
d_L = a_0 f_K(\chi)(1 + z) .
\]

**Angular diameter distance**

Given an observation of an extended object, we define the ‘angular diameter distance’, \( d_A \) to be the ratio of the size of the object versus the angle it subtends in the sky, formally it is given by,

\[
d_A \equiv \frac{\Delta x}{\Delta \theta} ,
\]

where \( \Delta \theta \) is the angle that subtends the object of physical size \( \Delta x \) in the orthogonal direction to the line of sight.

Visually, the object lies on the surface of a sphere with radius \( \chi \). With the FLRW metric we find that at time \( t = t_1 \) its size is given by,

\[
\Delta x = a(t_1) f_K(\chi) \Delta \theta .
\]
Equations (1.40) and (1.25) give the angular diameter distance as,

\[ d_A = a(t_1) f_K(\chi) = \frac{a_0 f_K(\chi)}{1 + z}. \]  

(1.42)

Comparing this expression with our result for the luminosity distance in equation (1.39), we find the ‘Etherington relation’ [20].

\[ d_A = \frac{d_L}{(1 + z)^2}. \]  

(1.43)

### 1.2.3 Observational evidence of cosmic acceleration

We have discussed how we model an expanding universe using the FLRW metric. In this subsection we will briefly review the observational evidence that shows our Universe is not only expanding but that the rate of this expansion is actually increasing.

The evidence for cosmic acceleration is derived from multiple observations that together remove the degeneracy between the way dark energy and negative spatial curvature affect the luminosity. Here we mention three major observations that have been used to provide evidence for cosmic acceleration:

1. **Type Ia supernovae**: These type of supernovae are thought to have their origin from binary systems where one member is a ‘white dwarf’ which accretes mass from its partner. The supernova occurs when the mass of the white dwarf exceeds the ‘Chandrasekhar limit’. In the nineties an analysis of a high-quality sample of local \((z < 1)\) observations found that brighter supernovae have a broader light curve. This correlation led them to be fashioned as ‘standard candles’ [21]. In 1998 the High-redshift Supernovae Search Team (HSST) [5] and the Supernovae Cosmology Project (SCP) [6] independently announced a measurement of the accelerated expansion of the Universe.

2. **CMB**: Measurements of anisotropies in the temperature in the cosmic microwave background radiation provide another independent test of cosmic...
acceleration. The anisotropies are an artefact of acoustic oscillations in the photon-baryon plasma. The location of the first peak in the angular power spectrum indicates that the geometry of the universe is very close to being flat. Furthermore measurements of the anisotropies also indicate that the total amount of matter (including ‘dark matter’) and radiation in the Universe was insufficient to ensure that it is flat, indicating the need for dark energy. [22].

3. Baryon acoustic oscillations: In the primordial plasma baryons were strongly coupled with photons. This ensured that their distributions closely resembled each other. Once the photons decoupled and their density had sufficiently diluted enough, the baryons remained distributed according to the last oscillation of the sound waves. The baryon acoustic oscillations provide a characteristic angular scale determined by the physical size of the sound horizon at decoupling. This physical size is well understood from the physics of plasmas and the expansion history of the universe up to decoupling. Therefore we can use this physical scale as a standard rule for $\Delta x$ and observe $\Delta \theta$ to determine the angular diameter distance from equation (1.40) [23, 24].

The measurements from these three major sources help to pin down a measurement of the relative energy density of matter versus curvature and dark energy (see figure 1.1). Other observational evidence can be found in measurements of the age of the Universe [22] and also of the matter power spectrum of the large scale structure which is formed of galaxy clusters, voids and filaments [25].

1.3 Geometrical modifications of Einstein’s theory

We have discussed the observational evidence that shows the Universe is undergoing a phase of accelerated expansion and how this can be modelled via including a dark energy component or cosmological constant in the Friedmann equation. In
Figure 1.1: Left (from [26]): Constraints on $\Omega_L$ and $\Omega_m$ from Supernovae (SNe), Cosmic Microwave Background (CMB) and Baryon Acoustic Oscillations (BAO) [27]. Right (from [28]): The relative amounts of the different constituents of the Universe [29]. From [28].

In this section we will discuss the possibility that extra degrees of freedom active on cosmological scales can mimic, in the background, the effect of having a small cosmological constant. These degrees of freedom can come in the form of exotic matter, coined ‘quintessence’, or as modifications to the field equations of General Relativity, commonly referred to as ‘modified gravity’. Here we will concentrate on models of modified gravity.

It is important that we clarify what we consider to be a theory of gravity. Although we wish to investigate the cosmological applications of alternative theories of gravity, we keep the notion, inherited from General Relativity, that gravity is principally a metric theory. That is, the effect of gravity on matter is mediated by a rank two tensor. Moreover, we follow [30] and consider a gravitational theory to be the set of field equations obeyed by the rank two tensor and those of any non-matter fields it interacts with.

Note that in some circumstances, there exist special transformations that are able
to switch between the different interpretations and therefore the two categories are not mutually exclusive [31]. For the simplest models these are ‘conformal transformations’ but for the models we discuss in this thesis, the more general ‘disformal transformations’ are involved [32].

We follow [34] and discuss screening in the context of a scalar field conformally coupled to matter. The general Lagrangian for a modified gravity model based on a Lorentz invariant scalar field theory can be written schematically as,

\[ \mathcal{L} = -\mathcal{D}^{\mu\nu}(\phi, \partial \phi, \partial^2 \phi) \partial_\mu \phi \partial_\nu \phi - V(\phi) + \tilde{\beta}(\phi) T^{\mu}_{\mu}, \]  

(1.44)

where \( \mathcal{D}^{\mu\nu}(\phi, \partial \phi, \partial^2 \phi) \) is schematic for a non-linear function of the derivatives of \( \phi \) and \( g_{\mu\nu}T^{\mu\nu} = T^\mu_{\mu} \) is the trace of the energy-momentum tensor (see equation (1.12)). For a non-relativistic source we can write \( T^\mu_{\mu} = -\rho \), where \( \rho \) is the energy density. The dynamics of the scalar field depends on the local density. For a point source, \( \rho = M \delta^3(\vec{x}) \), where \( \delta^3(\vec{x}) \) represents the three dimensional Dirac function. We then expand the action to quadratic order in fluctuations \( \phi \) around a background solution, \( \bar{\phi} \), with \( \phi = \bar{\phi} + \phi \),

\[ -\mathcal{D}^{00}(\bar{\phi}) \partial_t \phi \partial_t \phi + \mathcal{D}^{ij}(\bar{\phi}) \nabla_i \phi \nabla_j \phi - \frac{\partial^2 V}{\partial \phi^2}(\bar{\phi}) \phi^2 - \beta(\bar{\phi}) M \delta^3(\vec{x}), \]  

(1.45)

where \( \mathcal{D}^{00}(\bar{\phi}), \mathcal{D}^{ij}(\bar{\phi}) \) and \( \beta(\bar{\phi}) \) are schematic background dependent factors. We vary with respect to \( \phi \) to find the equation of motion,

\[ \mathcal{D}(\bar{\phi}) \left( \partial_t^2 \phi - c_s^2 \nabla^2 \phi \right) + m(\bar{\phi})^2 \phi = \beta(\bar{\phi}) M \delta^3(\vec{x}). \]  

(1.46)

where \( m(\bar{\phi})^2 = 2 \partial^2 V / \partial \phi^2 \) and we have factored out \( \mathcal{D}^{00}(\bar{\phi}) \) such that \( c_s^2 \) is the effective speed at which the fluctuations propagate. That is, we focus on the homogeneous equation,

\[ \partial_t^2 \phi - \frac{\mathcal{D}^{ij}(\bar{\phi})}{\mathcal{D}^{00}(\bar{\phi})} \nabla_i \nabla_j \phi = 0, \]  

(1.47)

and find a solution in the form of plane wave, \( \phi(t, \vec{x}) \sim e^{i(\omega t - \vec{\kappa} \cdot \vec{x})} \) and use the wave dispersion relation, \( \omega^2 = c_s^2 k^2 \), to define the sound speed \( c_s \).

\(^5\)See [33] for a recent investigation into the viability of using even more general transformations to build novel theories.
We find that the dynamics of the fluctuations is determined by three parameters: the mass $m(\phi)$, the coupling $\beta(\phi)$ and the kinetic function $\mathcal{Z}^{\mu\nu}(\phi)$. Furthermore, from the geodesic equation, the acceleration of a test particle is given by,

$$\ddot{a} = -\nabla \Phi - \nabla U = -\nabla \Phi - \beta(\phi) \nabla \varphi,$$  \hspace{1cm} (1.48)

where $\Phi$ is the Newtonian potential. Therefore, we find an extra factor of $\beta(\phi)$ and the local potential, $U(r)$, for this scalar field around a spherically symmetric and static source is then given by,

$$U(r) = -\frac{\beta(\phi)^2}{\mathcal{Z}(\phi)c_s^2(\phi)} e^{-\frac{m(\phi)^2 r}{\sqrt{\mathcal{Z}(\phi)c_s(\phi)}}} M.$$  \hspace{1cm} (1.49)

Expression (1.49) allows us to identify two typical classes of modification:

- The Chameleon field has a potential and coupling to energy density such that its effective mass is high within regions of high energy density. A typical choice is to set $\beta := \frac{d \ln \lambda}{d \phi}|_{\phi = \bar{\phi}}$:

$$A(\phi) = 1 + \xi \frac{\phi}{M_{\text{Pl}}} \quad V(\phi) = \frac{M^{4+n}}{\phi^n}$$  \hspace{1cm} (1.50)

- The Galileon field realises Vainshtein screening by the presence of derivative self-interactions which dominate at short distances. Schematically, $\mathcal{Z}^{\mu\nu}$ for Galileons takes the following form:

$$\mathcal{Z}^{\mu\nu}(\phi, \partial \phi, \partial^2 \phi) \equiv \eta^{\mu\nu} \left\{ 1 + \frac{\Box \phi}{\Lambda^3} + \frac{((\Box \phi)^2 - (\partial \phi \partial \phi)^2)}{\Lambda^6} + \ldots \right\}$$  \hspace{1cm} (1.51)

In this section we will discuss in more detail the structure and properties of those theories with non-canonical kinetic terms. In general, higher derivative theories are plagued with pathologies related to having an Ostrogradsky instability in the system. We introduce the Ostrogradsky instability in section 1.4. There exist examples of theories with higher order derivatives that are able to evade the Ostrogradsky instability by being degenerate. The most famous example of this type of theory is the Galileon which we introduce in section 1.3.1. We then go on to discuss in section 1.3.3 a related vector theory, called ‘vector Galileons’, which can be considered as a generalisation of the Proca equation for the massive photon.
1.3.1 Galileons

Galileons were first introduced to cosmology in [35] yet their special properties have been analysed ever since their discovery in the context of Horndeski’s work on the most general scalar-tensor theory in four space-time dimensions that has field equations which are up to second order in their derivatives [36].

In the context of cosmology, Galileons have attracted a lot of attention due to their ability to generate an accelerated expansion of the Universe whilst evading local tests of gravity via a particularly robust screening mechanism. They were discovered in this modern context within a braneworld scenario created to explain the observed acceleration in the cosmic expansion. Indeed, in [39] it was found that the special properties exhibited by the Dvali-Gabadadze-Porrati (DGP) braneworld model [40] were controlled, in a suitable decoupling limit, by a scalar field with self derivative interactions. This scalar, also called the brane-bending mode, is the Goldstone boson generated from the breaking of five dimensional Poincaré invariance of the bulk by the presence of the brane. The special properties of the scalar field were then abstracted and generalised by [35].

The term ‘Galileon’ was used to highlight that the scalar field $\pi$ has an action that is invariant under ‘Galilean’ shifts in field space, $\pi(x) \to \tilde{\pi}(x) = \pi(x) + b_\mu x^\mu + c$. Where by ‘shift in field space’ we mean an $x^\mu$ dependent field redefinition. Galileons have the additional restricting property that their equations of motion are strictly of second order. Furthermore, under Galileon shifts, their Lagrangian transforms and so they are not strictly invariant. This makes them an example of a Wess-Zumino term as, although their Lagrangian density doesn’t respect the symmetry, the effect of a transform only produces a total derivative. In this respect they have interesting topological applications and represent the existence of non-trivial elements of a cohomology [42].

Moreover, Galileons have been shown to possess interesting cosmological applications in both the late and early Universe. Galileons are mostly used as an

---

6They have also been of significant mathematical interest [37]. See also [38] for an overview of their mathematical properties.

7Thus allowing them to avoid the consequences of Ostrogradsky’s theorem [41].
alternative model to dark energy for the accelerated expansion of the Universe. However, their ability to drive self-acceleration makes them attractive as an alternative to standard slow-roll inflation. We discuss this scenario in appendix A. We can make use of the Levi-Civita epsilon tensor to write the Lagrangian for the Galileons in a compact form [43]. Using the following property:

\[ \varepsilon_{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_n} \varepsilon^{\nu_1 \cdots \nu_n \alpha_1 \cdots \alpha_n} \delta_{\alpha_1 \cdots \alpha_n} = -(D - n)! \delta^{[\beta_1 \cdots \beta_n]}_{\alpha_1 \cdots \alpha_n} \]

(1.52)

where the square brackets represent normalised anti-symmetric permutations, we can write the Galileon Lagrangians as:

\[ \mathcal{L}_1 = \pi \]

(1.53)

\[ \mathcal{L}_2 = \frac{1}{3!} \varepsilon^{\mu_1 \nu_1 \gamma} \varepsilon^{\mu_2 \nu_2 \lambda} \pi_{\mu_1} \pi_{\mu_2} := \varepsilon^{(2)}_{\mu_1} \pi_1 \pi_2 \]

(1.54)

\[ \mathcal{L}_3 = \frac{1}{2!} \varepsilon^{\mu_1 \mu_2 \nu_1 \nu_2 \lambda} \pi_{\mu_1} \pi_{\mu_2} (\pi_{\mu_3} \pi_{\mu_4}) := \varepsilon^{(4)}_{\mu_1 \mu_2} \pi_1 \pi_2 (\pi_{\mu_3} \pi_{\mu_4}) \]

(1.55)

\[ \mathcal{L}_4 = \varepsilon^{\mu_1 \mu_2 \mu_3 \nu_1 \nu_2 \nu_3 \lambda} \pi_{\mu_1} \pi_{\mu_2} (\pi_{\mu_3} \pi_{\mu_4} \pi_{\mu_5} \pi_{\mu_6}) := \varepsilon^{(6)}_{\mu_1 \mu_2 \mu_3} \pi_1 \pi_2 (\pi_{\mu_3} \pi_{\mu_4} \pi_{\mu_5} \pi_{\mu_6}) \]

(1.56)

\[ \mathcal{L}_5 = \varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6 \lambda} \pi_{\mu_1} \pi_{\mu_2} (\pi_{\mu_3} \pi_{\mu_4} \pi_{\mu_5} \pi_{\mu_6} \pi_{\mu_7} \pi_{\mu_8}) := \varepsilon^{(8)}_{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \mu_6 \mu_7 \mu_8} \pi_1 \pi_2 (\pi_{\mu_3} \pi_{\mu_4} \pi_{\mu_5} \pi_{\mu_6} \pi_{\mu_7} \pi_{\mu_8}) \]

(1.57)

where we have defined \( \varepsilon^{1234 \cdots}_{2n} := \frac{1}{(D - n)!} \varepsilon^{135 \cdots \nu_1 \nu_2 \cdots \nu_{D-n}} \varepsilon^{246 \cdots \nu_1 \nu_2 \cdots \nu_{D-n}} \) which has been written in shorthand as \( \varepsilon^{(2n)} \) and the numbers are shorthand for labelled indices: \{ \mu_1 \mu_2 \cdots \}. Furthermore, we have that \( \pi_{\mu_1 \cdots \mu_n} = \partial_{\mu_1} \cdots \partial_{\mu_n} \pi \) and use the terminology such that \( \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4 \) and \( \mathcal{L}_5 \) are described as quadratic, cubic, quartic and quintic Galileons respectively.

With this notation it is very easy to see that the variation of these Lagrangians would never have higher than two derivatives. For instance, taking the variation of \( \mathcal{L}_5 \) gives us:

\[
0 = \delta \mathcal{S}_5 = \int d^4 x \delta \mathcal{L}_5 = \int d^4 x \varepsilon^{(8)} [2 \delta \pi_1 \pi_2 (\pi_{34} \pi_{56} \pi_{78}) + 3 \pi_1 \pi_2 (\delta \pi_{34} \pi_{56} \pi_{78})]
= \int d^4 x \varepsilon^{(8)} [-2 \partial_1 (\pi_2 \pi_{34} \pi_{56} \pi_{78}) - 3 \partial_3 \partial_4 (\pi_{12} \pi_{34} \pi_{56} \pi_{78})] \delta \pi
= -5 \int d^4 x \varepsilon^{(8)} (\pi_{12} \pi_{34} \pi_{56} \pi_{78})
\]

(1.58)
where we have integrated by parts and found that the only term to survive the summation with the totally antisymmetric tensor $\varepsilon_{(8)}$ has, indeed, only derivatives of second order.

### 1.3.2 Vainshtein screening

A fascinating consequence of the non-linearity of Galileons is that they are able to realise the Vainshtein mechanism. This is a screening mechanism, first proposed in [44], for evading the vDVZ discontinuity [45, 46] which is simply the fact that the zero mass limit of Fierz-Pauli massive gravity does not recover GR (see [47] for a modern review). The Vainshtein mechanism relies on the theory becoming non-linear at scales $r \ll R_V$, where $R_V$ is the Vainshtein radius. Deep inside the Vainshtein radius the non-linear self interactions cause the extra (scalar) degrees of freedom to decouple from the rest of the theory (including matter) and GR is recovered. To summarise, the theory has distinct regimes parametrised by the scale $R_V$: for $r \gg R_V$, the theory is linear and the massive graviton experiences Yukawa suppression but for $r \ll R_V$ the theory is non-linear and the longitudinal mode of massive gravity is screened.

**An example for a spherically symmetric and static source.**

We follow the example given in [48] where the boundary effective action of DGP is studied in the decoupling limit. This is a consistent truncation of the theory and is essentially the cubic Galileon coupled to matter via the trace of the energy-momentum tensor, $T \equiv \text{tr}[T_{\mu \nu}] = T_{\mu}^{\mu}$. Thus we start with the following action:

$$S = \int d^4x \sqrt{-g} \left[ 3(\partial \pi)^2 + \frac{1}{\Lambda^3} (\partial \pi)^2 \Box \pi + \frac{1}{2 M_{Pl}} \pi T \right].$$

(1.59)

Note the sign of the kinetic term appears to be unhealthy, we address this further on when considering the expression for the effective metric. The Euler-Lagrange
equation leads to,

\[-6\Box \pi + \frac{1}{\Lambda^3} (\Box \partial \pi)^2 - \frac{2}{\Lambda^3} \partial_{\mu} (\partial^\mu \pi \Box \pi) + \frac{1}{2M_{Pl}} T = 0 \quad (1.60)\]

To illustrate the Vainshtein mechanism we choose the special case of a spherically symmetric and static mass, \( T = \delta^3(r) M \), where \( \delta^3(r) \) is the three dimensional Dirac function. Therefore, qualitatively we can say,

- \( \partial_0 \pi = 0 \),
- \( \nabla^2 \pi = \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\pi}{dr} \right) \)

and the equation of motion becomes,

\[\nabla \cdot \left[ 6\nabla \pi - \frac{1}{\Lambda^3} \nabla (\nabla \pi)^2 + \frac{2}{\Lambda^3} (\nabla \pi \nabla^2 \pi) \right] = \frac{M}{2M_{Pl}} \delta^3(r) \quad (1.61)\]

Notice that,

- \( (\nabla \pi)^2 = (\pi'(r))^2 \Rightarrow \nabla (\nabla \pi)^2 = 2\pi'(r)\pi''(r)\hat{r} \)
- \( \nabla^2 \pi = \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\pi}{dr} \right) = \frac{2}{r} \frac{d\pi}{dr} + \frac{d^2\pi}{dr^2} \)
- \( \Rightarrow -\nabla (\nabla \pi)^2 + \nabla^2 \pi \nabla \pi = \frac{2}{r} \pi'(r)\pi'(r)\hat{r} \)

where, \( \hat{r} \) is the unit 3-vector along the direction of \( r \). This allows us to write the equation of motion as,

\[\nabla \cdot \hat{r} \left[ 6\pi' + \frac{4}{r\Lambda^3} \pi'\pi' \right] = \frac{M}{2M_{Pl}} \delta^3(r) \quad (1.62)\]

Rewriting \( \pi'(r) = u(r) \) and integrating \( \int dV \) gives us,

\[4\pi r^2 \left[ 6u + \frac{4u^2}{r\Lambda^3} \right] = \frac{M}{2M_{Pl}} \quad (1.63)\]

This is a quadratic equation in \( u \). We immediately solve it to get,

\[u = \frac{\Lambda^3}{4r} \left[ \pm \sqrt{9r^4 + \frac{1}{2\pi^2} \frac{M}{\Lambda^3 M_{Pl}}} r - 3r^2 \right] \quad (1.64)\]

We define the Vainshtein radius as \( R_V \equiv (M/\Lambda^3 M_{Pl})^{1/3} \) and finally write our solution as,

\[\pi'(r) = \frac{\Lambda^3}{4r} \left[ \pm \sqrt{9r^4 + \frac{1}{2\pi^2} R_V^2 r - 3r^2} \right] \quad (1.65)\]

If we concentrate on the positive solution and look at two limits:
• For $r \gg R_V$: we use that for $0 < \epsilon \ll 1$, $\sqrt{1 + \epsilon} \approx 1 + \frac{1}{2}\epsilon$ and we find $\pi' \sim O(1/r^2)$.

• For $r \ll R_V$: we have that $R_V^3/r^3 \gg 1$ which gives us,

$$\pi'(r) \sim \frac{3}{4} \left[ \frac{1}{2\pi M_{Pl}^4} \frac{\Lambda}{L^4} \right]^{1/2}$$

and we see that the force is weaker at short distances.

There are two branches for the DGP model. Here we have focussed on the ‘normal’ branch which we see in equation (1.59) has the correct sign for the kinetic term. The alternative, ‘self-accelerating’ branch has the opposite signature for its kinetic term. This is a self-accelerating solution which describes a quasi-deSitter space-time without the need of a cosmological constant. However, in the full DGP theory, this turns out to be related to the existence of a ghost [49, 50, 51].

Other properties of Vainshtein screening

1. Interestingly, the effectiveness of the screening depends on the shape of the distribution of energy-momentum sourcing the field. For example, in [52] it was found that for an infinitely large flat mass sheet, a Galileon isn’t screened at all (see [53] for a recent investigation and [54] for results from cosmological simulations).

2. The size of $R_V$ is governed by the ratio of the strong coupling scale, $\Lambda$ to the four dimensional Planck mass, $M_{Pl}$. For DGP this is related to a large hierarchy between the relative strength of 5D gravity to 4D gravity, i.e. $M_5 \ll M_{Pl}$.

3. The theory becomes non-linear at $R_V$ where the higher order operators begin to dominate. A theorem by Leray\(^9\) tells us that causality in our theory will be governed by an effective metric which appears to render the theory

superluminal. To see how this happens we perturb our solution $\pi = \hat{\pi} + \phi$ and expand the action, given by equation (1.59), to quadratic order,

$$ S_\phi = \int d^4x \sqrt{-g} \left[ 3(\partial \phi)^2 + \frac{1}{\Lambda^3} [\partial^2 (\partial \phi)^2 + 2\partial_\mu \hat{\pi} \partial^\mu \phi \square \phi] \right]. \quad (1.67) $$

Using the identity, $\partial^\mu \phi \square \phi \equiv \partial_\nu [\partial^\nu \phi \partial^\mu \phi - \frac{1}{2} \eta^{\mu \nu} (\partial \phi)^2]$ gives us,

$$ S_\phi = \int d^4x \sqrt{-g} \left[ 3(\partial \phi)^2 + \frac{2}{\Lambda^3} (\partial_\mu \phi \partial_\nu \phi) [\square \hat{\pi} \eta^{\mu \nu} - \partial^\mu \partial^\nu \hat{\pi}] \right]. \quad (1.68) $$

Therefore we see we have an effective metric given by,

$$ Z^{\mu \nu} = 3 + \frac{2}{\Lambda^3} [\square \hat{\pi} \eta^{\mu \nu} - \partial^\mu \partial^\nu \hat{\pi}]. \quad (1.69) $$

Notice that the sign for the perturbations can now be healthy. Furthermore, issues to do with superluminality and causality of our theory will now be determined by this effective metric.

Perturbing around a spherical background, $\pi(r,t) = \hat{\pi}(r) + \phi(r,t)$, gives,

$$ S_\phi = \int d^4x \sqrt{-g} \left[ - (3 + \frac{2}{\Lambda^3} \nabla^2 \hat{\pi}) (-\phi^2 + g^{ij} \partial_i \phi \partial_j \phi) + \frac{2}{\Lambda^3} \nabla_k \nabla_i \hat{\pi} g^{ik} g^{jl} \partial_i \phi \partial_j \phi \right], \quad (1.70) $$

where $g_{ij} = \text{diag}(1, r^2, r^2 \sin^2 \theta)$ is the spacial metric tensor and $\nabla^2 \hat{\pi} = u' + \frac{2u}{r}$.

Concentrating on the radial components only, we find,

$$ S_\phi = \int d^4x \sqrt{-g} \left\{ 3 + \frac{2}{\Lambda^3} (u' + \frac{2u}{r}) \right\} \phi^2 - \left[ 3 + \frac{4u}{\Lambda^3 r} \right] (\partial_r \phi)^2 \right\}. \quad (1.71) $$

Since both terms in square brackets are positive, we have a theory without any instabilities. However, by examining these terms in the limit as $r/R_V \to 0$, we find that the sound speed of the theory, $c_s^2$ can be greater than one,

$$ c_s^2 = \frac{3 + \frac{2}{\Lambda^3} (u' + \frac{2u}{r})}{3 + \frac{4u}{\Lambda^3 r}} \approx 1 + \frac{2u' r}{3\Lambda^3 r + 4u} \approx 1 + \frac{\sqrt{2}}{2} > 1. \quad (1.72) $$

Although this brings up issues, that have yet to be settled, to do with whether or not there exists a standard Lorentz invariant UV completion for the theory [55], superluminality does not imply that the theory is acausal as for physically interesting solutions it can be shown that closed time-like curves do not appear [56, 57, 58].
1.3.3 Extending the Proca action

The dynamics of a massive vector boson can be described by adding a mass term to the Maxwell Lagrangian. This is usually called the Proca action and is given by,

\[ \mathcal{L}_{\text{Proca}} \equiv \mathcal{L}_{\text{EM}} - m^2 A_\mu A^\mu = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - m^2 A_\mu A^\mu \quad (1.73) \]

where \( A_\mu \) is the four-vector potential and \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the Faraday tensor.

The presence of the mass term explicitly breaks the gauge invariance. Furthermore, since the boson is massive, we can go to the rest frame and use ordinary quantum mechanics to decompose the field into its \((2s+1) = 3\) spin degrees of freedom. However, unlike in Fierz-Pauli massive gravity, we can recover the correct massless theory (with helicity \( h = \pm 1 \)) in the zero mass limit\(^\text{10}\). This is due to the fact that in the infinite momentum frame the scalar (longitudinal mode) decouples from the vectors such that they are parametrised by \( m \). Thus taking the limit \( m \to 0 \) effectively removes any interaction between the scalar and the transverse vector modes.

In [59, 60] the Proca action was extended with additional self derivative interactions that nonetheless ensure the theory remains ghost free,

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_{i=1}^{4} \mathcal{L}_{(i)} \quad (1.74) \]

where, as with the Galileons in section 1.3.1, we use the epsilon tensor to write the extended non-linear interactions,

\[ \mathcal{L}_{(i)} \propto \varepsilon_{(2i)^k} A_{\mu_1 A_{\nu_2}} \left( \alpha_{(2)} A_{\mu_3 v_4} + \beta_{(2)} F_{\mu_3 v_4} \right) \cdots \]

\[ \left( \alpha_{(i)} A_{(2-1)\nu_{(2)}(2)} + \beta_{(i)} F_{(2-1)\nu_{(2)}(2)} \right). \quad (1.75) \]

Notice that we have decomposed the derivatives into their symmetric, \( A_{\mu\nu} \equiv \partial_{(\mu} A_{\nu)} \) and antisymmetric, \( F_{\mu\nu} \) parts because, as was discussed in [60], extra parameters appear in the theory due to the fact that the indices on the vectors cannot

\(^{10}\)See [17] for a review of massive gravity.
be swapped with those on the partial derivatives (see appendix C.1.2 for a more complete discussion).

The use of the epsilon tensor to construct the theory guarantees that under varying $A_0$, any terms like $\partial_{a_i} \partial_{b_j} A_0$ in the equation of motion cannot have $a_i = 0$ or $b_j = 0$ since one index on each epsilon tensor has been chosen to be zero and the rest have to take other values. Thus no time derivatives act on $A_0$. This ensures that the Euler equation for $A_0$ is a constraint equation which removes this additional ghost degree of freedom. The remaining three degrees of freedom propagate as physical modes. Two of these degrees of freedom are transversal and therefore can be related to the helicities of the (would be massless) vector boson and the remaining scalar degree of freedom is associated with the longitudinal mode (Goldstone boson). Therefore we have a classically stable theory.

**Stückelberg formalism**

We can use the Stückelberg mechanism to substantiate that our vector theory does indeed only propagate three degrees of freedom. The Stückelberg trick involves introducing a non-physical degree of freedom that allows us to recover gauge invariance. We then find that, in a suitable decoupling limit, the theory resembles a gauge invariant theory for a massless vector boson plus a separate scalar Galileon theory.

For simplicity we focus only up to the cubic level where $i = 2$ and in this case the term factored by $\beta_2$ does not contribute. Making the substitution [59]: $A_\mu := A_\mu + 1/\sqrt{2m} \partial_\mu \chi$ we find the Lagrangian is now locally $U(1)$ invariant under $A_\mu \rightarrow A_\mu - 1/m \partial_\mu \xi$ and $\chi \rightarrow \chi + \sqrt{2} \xi$. Here $\chi$ is known as the Stückelberg field. Moreover, in this formalism it is clear that physically $\partial_\mu \chi$ represents the longitudinal mode for $A_\mu$:

$$\mathcal{L}_\phi = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \left( \sqrt{2m} A_\mu + \partial_\mu \chi \right) \left( \sqrt{2m} A^\mu + \partial^\mu \chi \right)$$

$$+ \mathcal{E}_{(4)} \frac{\alpha_2}{\sqrt{8m^3}} \left( \sqrt{2m} A_1 + \partial_1 \chi \right) \left( \sqrt{2m} A_2 + \partial_2 \chi \right) \left( \sqrt{2m} A_3 + \partial_3 \partial_4 \chi \right)$$

(1.76)
Taking the following decoupling limit [59],

\[ m \to 0, \quad \alpha_2 \to 0, \quad \text{with} \quad \frac{\alpha_2}{m^3} = \frac{\sqrt{8}}{3\Lambda_G^3} = \text{fixed} \quad (1.77) \]

We find as suggested earlier, a theory with two separate symmetries:

\[ \mathcal{L}_{\Lambda} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial \chi)^2 - \frac{1}{2\Lambda_G^3} (\partial \chi)^2 \nabla^2 \chi \quad (1.78) \]

where \( \Lambda_G \) is the strong coupling scale for the cubic Galileon operator.

### 1.3.4 The vector Galileon System

We have discussed how the gauged abelian symmetry found in electromagnetism can be broken by a mass term for the photon controlled by a scale \( m_A \). In addition to this, we showed how we can add derivative interactions for the vector field \( A_\mu \), the simplest of which is a dimension-4 operator weighted by a dimensionless coupling, (denoted as \( \beta \)):

\[ \mathcal{L}_A = -m_A^2 A_\mu A^\mu - \beta A_\mu A^\mu \partial_\rho A^\rho . \quad (1.79) \]

One can consider a handful of higher-dimensional operators with a similar structure as above. These operators break abelian gauge invariance, but are nevertheless consistent since the \( A_0 \)-component of the gauge field remains a constraint: Its action does not contain time derivatives. These systems are interesting for their cosmological applications as they can have self-accelerating background solutions, with stable dynamics, realise Vainshtein screening and even alleviate the tension in the rate of structure growth, versus what we would extrapolate from the \( \Lambda \)CDM model [61, 62]. We discuss the cosmological applications of the cubic model in chapter 6.
The vector Galileon system

We use the antisymmetric properties of the \textit{Levi-Civita epsilon tensor} to write the vector Galileons on Minkowski space-time as,

\begin{align}
\mathcal{L}_F &= -\frac{1}{4} F_{\mu \nu} F^{\mu \nu}, \\
\mathcal{L}_v^{(2)} &= A_\mu A^\mu, \\
\mathcal{L}_v^{(3)} &= \frac{1}{2} \epsilon^{\mu_1 \mu_2 \lambda_3} A_{\mu_1} A_{\mu_2} A_{\mu_3}, \\
\mathcal{L}_v^{(4)} &= \epsilon^{\mu_1 \mu_2 \mu_3} A_{\mu_1} A_{\mu_2} \left( A_{\mu_3} F_{\nu_1} F_{\nu_2} + c_2 F_{\mu_1} F_{\nu_3} F_{\nu_4} \right), \\
\mathcal{L}_v^{(5)} &= \epsilon^{\mu_1 \mu_2 \mu_3 \mu_4} A_{\mu_1} A_{\mu_2} \left( A_{\mu_3} A_{\mu_4} A_{\mu_5} F_{\nu_1} F_{\nu_2} + d_3 A_{\mu_3} A_{\mu_4} F_{\mu_5} F_{\nu_6} \right),
\end{align}

where $A_{\mu \nu} \equiv \partial_{(\mu} A_{\nu)}$ and $F_{\mu \nu} \equiv \partial_{[\mu} A_{\nu]}$.

As in the case for scalar Galileons, on a Minkowski background, there are different forms for the vector Galileons which are related by a total derivative. In addition to this, the vectors also have two extra free parameters, $(c_2, d_3)$, due to the ability to generate ghost free terms of the form, $f_2(A^2, A \cdot F, F^2, FF^* )$ [60, 63].

1.4 Ostrogradsky’s construction

In this section we reproduce the arguments made in [64] concerning a result due to Ostrogradsky [41] on the dynamical stability of theories with equations of motion higher than second order derivatives in time. Ostrogradsky very generally states that there is a linear instability in the Hamiltonian associated with a Lagrangian with time derivatives higher than order two. The result implies that all non-degenerate higher-derivative theories contain ghost degrees of freedom.

\footnote{We follow [60] and use $A \cdot F$ to denote all possible contractions of $A_\mu$ with $F_{\mu \nu}$ and $(c_2, d_3)$ to denote the two extra parameters. There is some degeneracy here, for example, starting with the quartic, $\mathcal{L}_v^{(4)}$, we can use integrations by parts to find expressions like $A^2 F^2$ and $A_\mu A_\nu F^{\mu \rho} F^{\nu \rho}$.}
Single variable non-degenerate instability

Here we present the argument, based on Ostrogradsky’s construction and found in [64], that in the context of a single, one-dimensional point particle, a Lagrangian which involves second order derivatives contains a linear instability.

We denote the position of the particle by \( q(t) \) and consider a system described by a Lagrangian \( L(q, \dot{q}, \ddot{q}) \) that depends non-degenerately on \( \ddot{q}(t) \). In this case the equation of motion is given by,

\[
\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}} = 0. \tag{1.85}
\]

The assumption of non-degeneracy means that we cannot remove \( \ddot{q}(t) \) by partial integration. This implies that \( \frac{\partial L}{\partial \ddot{q}} \) depends on \( \ddot{q}(t) \) which allows us to re-write equation (1.85) as,

\[
\ddot{q} = F(q, \dot{q}, \ddot{q}, \ldots) \quad \Rightarrow \quad q(t) = Q(t, q_0, \dot{q}_0, \ddot{q}_0, \ldots). \tag{1.86}
\]

so that the solutions depend upon four pieces of initial value data: \( q_0 = q(0) \), \( \dot{q}_0 = \dot{q}(0) \), \( \ddot{q}_0 = \ddot{q}(0) \) and \( \ldots \). Since the solutions require four pieces of initial value data there must be four canonical coordinates. Ostrogradsky used the following construction,

\[
Q_1 \equiv q, \quad P_1 = \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}}; \tag{1.87}
\]

\[
Q_2 \equiv \dot{q}, \quad P_2 = \frac{\partial L}{\partial \ddot{q}}. \tag{1.88}
\]

Since we have a non-degenerate system, we are able to invert the phase space transformation to solve for \( \ddot{q} \) in terms of \( Q_1, Q_2 \) and \( P_2 \). That is, we have a function \( A(Q_1, Q_2, P_2) \) such that,

\[
\frac{\partial L}{\partial \ddot{q}} \bigg|_{\begin{array}{l} q = Q_1 \\ \dot{q} = Q_2 \\ \ddot{q} = A \end{array}} = P_2. \tag{1.89}
\]

We follow the same procedure that is taken with Lagrangians dependent on first derivatives and Legendre transform to obtain the Hamiltonian. However, in this
case we transform along $\dot{q} = q^{(1)}$ and $\ddot{q} = q^{(2)}$, and find,

\begin{equation}
H(Q_1, Q_2, P_1, P_2) \equiv \sum_{i=1}^{2} P_i q^{(i)} - L, \quad (1.90)
\end{equation}

\begin{equation}
= P_1 Q_1 + P_2 A(Q_1, Q_2, P_2) - L(Q_1, Q_2, A(Q_1, Q_2, P_2)). \quad (1.91)
\end{equation}

Thus the Hamiltonian is linear in the canonical momentum $P_1$ indicating that this system can reduce its energy indefinitely and without bound. Therefore this system and any theory of this form is unstable.
Chapter 2

A Higgs Mechanism for Vector Galileons

The content of this chapter is based on my paper [65], ‘A Higgs Mechanism for Vector Galileons’, published in the Journal of High Energy Physics (JHEP).

In this chapter I will introduce the phenomenology of spontaneous symmetry breaking and discuss its application to the vector theories introduced in chapter 1. The physics of spontaneous symmetry breaking is well understood and can be attributed to the phenomenology of various systems. When associated with the breaking of a gauge degree of freedom, we find two well-known examples in the phenomenology of superconductors and the standard model of particle physics. In both cases, the physics is effectively encoded in how the vector degrees of freedom acquire a mass via their interactions with a scalar. At low temperatures (long wavelengths), these interactions are dominated by the presence of a vacuum expectation value ($\vev$) for the scalar and it is through this background that the vector perturbations propagate with a mass. In superconductors, this leads to spontaneous de-electrification whereas in nuclear physics we find short-range forces.

We saw in chapter 1 that massive vectors are described by the Proca action (equation (1.73)). Furthermore, by utilising the structure of the Galileon, we found that
this system can be extended with higher non-linear self-interactions. Given that the Higgs mechanism is able to generate the original Proca system spontaneously, it seems natural to use the connection between this system and the vector Galilean theories to find a mechanism to spontaneously generate the entire class.

We discussed in chapter 1 how in order to find interesting infra-red behaviour for vector systems we must add interactions which break the gauge symmetry. Furthermore, we found that such systems necessarily include extra degrees of freedom. For the Proca system, the standard picture is that the Goldstone bosons, which are usually associated with a broken *global* symmetry, are in this case, ‘eaten’ by the longitudinal modes of the vector, or rather more precisely, a unitary gauge can be selected that sets them to zero. Moreover, when this symmetry breaking is governed by an abelian Higgs mechanism, we recover interactions between the longitudinal modes and the Higgs which completely decouple at high energies.

In the extension of this scenario for the vector Galileon system, we found in chapter 1 that in an appropriate decoupling limit, the dynamics of the vector longitudinal modes corresponding to one of the would-be Goldstone bosons, is controlled by Galileon interactions. Furthermore, we will show in section 2.2 that the interactions of the scalar Higgs field itself also enjoys Galileonic symmetries, and the combined Higgs-Goldstone boson system assembles into a specific bi-Galileon combination.

In Section 2.3 we straightforwardly extend our constructions to the case of non-abelian symmetry, and discuss some of its physical consequences.

As far as we are aware, this is the first example of a consistent realization of a Higgs mechanism in theories with a spontaneously broken symmetry, that lead to Galileonic theories in the remaining degrees of freedom.

---

1That is, unless we break Lorentz invariance.
2.1 The Higgs mechanism

In this section I will discuss how, with the Higgs mechanism, the Proca equation follows naturally from the presence of a vacuum expectation value for the scalar field.

A Higgs mechanism, by construction, adds some new degrees of freedom to the theory, gauged under the symmetry being considered together with a non-trivial potential that spontaneously breaks this symmetry. We start by discussing the case of abelian interactions. We consider as a fundamental degree of freedom a complex Higgs scalar field charged under the \( U(1) \) abelian gauge symmetry, with a classical ‘Mexican hat’ potential. We work in four dimensional Minkowski space with a Lagrangian given by,

\[
L_H = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} - (\partial_{\mu} \phi)(\partial^{\mu} \phi)^* - V(\phi) \tag{2.1}
\]

The covariant derivative acting on the Higgs field contains the gauge field \( A_{\mu} \), and is defined as

\[
\partial_{\mu} = \partial_{\mu} - igA_{\mu}, \tag{2.2}
\]

with \( g \) a coupling constant. The Higgs potential has the traditional ‘Mexican hat’ form

\[
V(\phi) = -\mu^2 \phi \phi^* + \frac{\lambda}{2} (\phi \phi^*)^2, \tag{2.3}
\]

and has a minimum at

\[
\langle \phi \rangle \equiv v = \left( \frac{\mu^2}{\lambda} \right)^{1/2}. \tag{2.4}
\]

We demand that the Lagrangian, \( L_H \), is invariant under a \( U(1) \) gauge symmetry, acting on the scalar and on the vector as

\[
\phi \rightarrow \phi e^{i\xi}, \quad A_{\mu} \rightarrow A_{\mu} + \frac{1}{g} \partial_{\mu} \xi, \tag{2.5, 2.6}
\]

for an arbitrary function \( \xi \). Under a \( U(1) \) transformation, the covariant derivative transforms as

\[
\partial_{\mu} \phi \rightarrow e^{i\xi} \partial_{\mu} \phi \tag{2.7}
\]
In order to make the physics more transparent, it is convenient to decompose the complex scalar into its norm and phase:

\[ \phi = \varphi e^{i\pi}, \tag{2.8} \]

where \( \varphi, \pi \) are two real fields. \( \varphi \) does not transform under \( U(1) \) gauge symmetry, while the field \( \pi \) transforms non-linearly as \( \pi \to \pi + \frac{\xi}{\lambda} \); the phase \( \pi \) behaves as the would-be Goldstone boson for the broken abelian symmetry. Hence defining the gauge invariant combination

\[ \hat{A}_\mu \equiv A_\mu - \partial_\mu \pi, \tag{2.9} \]

we can express the covariant derivatives as

\[ \mathcal{D}_\mu \phi = [\partial_\mu \varphi - i g \varphi \hat{A}_\mu] e^{i\pi} \tag{2.10} \]

It is important to stress that, using this Higgs construction, the would-be Goldstone fields combine automatically with the vectors and appear in the action only in the gauge invariant combination in equation (2.9).

The phenomenon of spontaneous symmetry breaking is associated with the Higgs developing a vacuum expectation value \( v \) as in equation (2.4), and acquiring non-trivial dynamics when fluctuating around the minimum of its potential. In order to study the dynamics of Higgs fluctuations, it is convenient to expand the norm of the Higgs around the minimum \( v \) of the potential, and write

\[ \varphi = \left( v + \frac{h}{\sqrt{2}} \right) \tag{2.11} \]

which allows us to canonically normalize the Higgs fluctuations \( h \). Applying this expansion to \( \mathcal{L}_H \) we find

\[ \mathcal{L}_H = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - m_A^2 \hat{A}^2 - \sqrt{2} g m_A h \hat{A}_\mu \hat{A}^\mu - \frac{g^2}{2} h^2 \hat{A}_\mu \hat{A}^\mu \]

\[ -\frac{1}{2} (\partial h)^2 - \frac{1}{2} m_h^2 h^2 - \frac{\sqrt{\lambda} m_h}{2} h^3 - \frac{\lambda}{8} h^4 \tag{2.12} \]

with

\[ m_A = gv, \tag{2.13} \]

\[ m_h = \sqrt{2\lambda} v, \tag{2.14} \]
where we have neglected the field-independent part of the potential, that contributes to the cosmological constant.

The previous Lagrangian is fully gauge invariant, being expressed in terms of the gauge invariant combination given in equation (2.9), and describes the dynamics of four degrees of freedom, two scalars and a massless vector. Choosing the ‘unitary gauge’ \( \pi = 0 \) enables us to analyse the dynamics of the physical degrees of freedom: the Higgs scalar \( h \) and a massive gauge boson \( A_\mu \) (again, with a total of four degrees of freedom).

Working in the physically transparent unitary gauge, one finds that the previous Lagrangian in equation (2.12) leads to renormalizable interactions described by up-to dimension-4 operators. Moreover, we find, as promised, the Higgs’ vev, \( v \), gives a mass to the gauge field, \( m_A = gv \). Hence, the phenomenon of spontaneous symmetry breaking automatically generates the desired Proca equation.

### 2.2 G-Higgs

In this section I use the connection between the Proca system and the vector theories with broken gauge symmetries discussed in chapter 1 to motivate the construction of a ‘Galileonic’ Higgs mechanism. We will find that the gauge symmetry can be spontaneously broken by a Higgs scalar field acquiring a vacuum expectation value and that the theory after symmetry breaking coincides with the broken abelian gauge theory of [59].

The new Higgs interactions that we consider correspond to higher dimensional non-renormalizable operators, involving gauge invariant derivative self-couplings of the Higgs field. When the Higgs field sits at the minimum of its potential and acquires a vacuum expectation value \( v \), the resulting theory corresponds to the vector self-interacting theory discussed in chapter 1, with parameters depending on \( v \), the gauge coupling constant \( g \), as well as on the parameters characterizing the higher derivative Higgs self-interactions. Moreover, when considering Higgs excitations around its minimum, one finds new scalar-vector derivative interac-
tions – absent in the original theory that involved vector self-interactions only – appearing in consistent combinations built in such a way as to avoid the appearance of ghosts.

This is a stringent requirement that constrains the structure of the Higgs self-interaction\(^2\). We determine various examples of higher dimensional derivative self-interactions for the Higgs boson, that once expanded around the minimum of the Higgs potential lead to ghost-free derivative interactions between the vector and scalar, that generalize multi-Galileon constructions to the scalar-vector case.

We show that, in a suitable decoupling limit, the theory reduces to a scalar bi-Galileon theory that describes, with Galileon invariant interactions, the coupling of the Higgs boson with the would-be Goldstone modes of the broken symmetry.

Let us consider an extension of the gauge invariant action for a complex scalar Higgs field, \(\mathcal{L}_H\), by higher order derivative couplings,

\[
\mathcal{L}_{\text{tot}} = -\frac{1}{4} F^{\mu \nu} F_{\mu \nu} - (\partial_{\mu} \phi)(\partial^{\mu} \phi)^* - V(\phi) + \mathcal{L}_{(8)} + \mathcal{L}_{(12)} + \mathcal{L}_{(16)}.
\] (2.15)

The first line contains the usual kinetic terms for scalar and vector \((F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu)\) and the Higgs potential. The second line contains new dimension 8, 12, 16 gauge invariant operators, that are suppressed by a mass scale \(\Lambda\), and describe the Higgs derivative self-interactions associated with the pattern of spontaneous symmetry breaking that we are interested in.

As with, \(\mathcal{L}_H\), we demand that the Lagrangian \(\mathcal{L}_{\text{tot}}\) is invariant under a \(U(1)\) gauge symmetry, acting on the scalar and on the vector as equations (2.5 & 2.6).

Due to the transformation property of the covariant derivative, \(\partial_{\mu}\), given by equation (2.7), we have that under a \(U(1)\) transformation, the covariant double derivative transforms as

\[
\partial_{\mu} \partial_{\nu} \phi \rightarrow e^{i \xi} \partial_{\mu} \partial_{\nu} \phi.
\] (2.16)

\(^2\)Although recent progress in degenerate scalar-tensor theories point towards a multi-field evasion of the Ostrogradsky instability [66].
Using these transformation properties under gauge transformations, it is straightforward to check that the following tensors are gauge invariant:

\[
L_{\mu \nu} \equiv \frac{1}{2} \left[ (\partial_\mu \phi)^* (\partial_\nu \phi) + (\partial_\nu \phi)^* (\partial_\mu \phi) \right],
\]

(2.17)

\[
P_{\mu \nu} \equiv \frac{1}{2} \left[ \phi^* \partial_\mu \partial_\nu \phi + \phi \left( \partial_\mu \partial_\nu \phi \right)^* \right],
\]

(2.18)

\[
Q_{\mu \nu} \equiv \frac{i}{2} \left[ \phi \left( \partial_\mu \partial_\nu \phi \right)^* - \phi^* \partial_\mu \partial_\nu \phi \right].
\]

(2.19)

Notice that \(P_{\mu \nu}\) and \(Q_{\mu \nu}\) are formed by second covariant derivatives: these contain derivatives of the vectors, that are needed to build derivative vector self-interactions as in equation (1.79). Together with the totally antisymmetric \(\varepsilon\)-tensor in four dimensions (with \(\varepsilon_{0123} = 1\)), the previous tensors are the ingredients we use to define the operators \(L_{(8)}, L_{(12)}, L_{(16)}\) introduced in the second line of equation (2.15) as

\[
L_{(8)} = \frac{1}{2! \Lambda^4} \varepsilon^{\alpha \beta \mu_1 \mu_2} \varepsilon_{\alpha \beta \nu_1 \nu_2} \left[ \alpha_{(8)} L^{\nu_1}_{\mu_1} p^{\nu_2}_{\mu_2} + \beta_{(8)} L^{\nu_1}_{\mu_1} Q^{\nu_2}_{\mu_2} \right]
\]

(2.20)

\[
L_{(12)} = \frac{1}{\Lambda^8} \varepsilon^{\alpha \beta \mu_1 \mu_2 \mu_3} \varepsilon_{\alpha \nu_1 \nu_2 \nu_3} \left[ \alpha_{(12)} L^{\nu_1}_{\mu_1} p^{\nu_2}_{\mu_2} p^{\nu_3}_{\mu_3} + \beta_{(12)} L^{\nu_1}_{\mu_1} Q^{\nu_2}_{\mu_2} Q^{\nu_3}_{\mu_3} \right]
\]

(2.21)

\[
L_{(16)} = \frac{1}{\Lambda^{12}} \varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \varepsilon_{\nu_1 \nu_2 \nu_3 \nu_4} \left[ \alpha_{(16)} L^{\nu_1}_{\mu_1} p^{\nu_2}_{\mu_2} p^{\nu_3}_{\mu_3} p^{\nu_4}_{\mu_4} + \beta_{(16)} L^{\nu_1}_{\mu_1} Q^{\nu_2}_{\mu_2} Q^{\nu_3}_{\mu_3} Q^{\nu_4}_{\mu_4} \right]
\]

(2.22)

that are weighted by dimensionless parameters \(\alpha_{(i)}, \beta_{(i)}\), and suppressed by an energy scale \(\Lambda\) to the appropriate powers. We presented in appendix ?? arguments that show that these operators lead to equations of motion with at most two space-time derivatives, analogously to what happens for standard Galileons [35]. Indeed, the \(\varepsilon\)-tensors present in the above definitions have been introduced to automatically avoid the emergence of ghost degrees of freedom.

Similar gauge invariant Higgs Lagrangians were also studied in [67, 68]. Notice that all these operators are higher-dimensional and hence apparently non-renormalizable: we will return to this point at the very end of this section.

As in section 2.1 we express our complex scalar field in terms of a complex exponential and use equation (2.9) to expand the second order covariant derivatives
\[ \mathcal{D}_\mu \mathcal{D}_\nu \phi = [\partial_\mu \partial_\nu \phi - ig \phi \partial_\mu \hat{A}_\nu - ig \hat{A}_\mu \partial_\nu \phi - ig \hat{A}_\nu \partial_\mu \phi - g^2 \phi \hat{A}_\mu \hat{A}_\nu] e^{ig \pi}, \tag{2.23} \]

with the piece inside the square parenthesis invariant under the gauge transformation.

Using these relations, the operators defined in equations (2.20)-(2.22) can be expressed as,

\[ L_{\mu \nu} = \partial_\mu \phi \partial_\nu \phi + g^2 \phi^2 \hat{A}_\mu \hat{A}_\nu, \tag{2.24} \]
\[ P_{\mu \nu} = \phi \partial_\mu \partial_\nu \phi - g^2 \phi^2 \hat{A}_\mu \hat{A}_\nu, \tag{2.25} \]
\[ Q_{\mu \nu} = \frac{g}{2} [\partial_\mu (\phi^2 \hat{A}_\nu) + \partial_\nu (\phi^2 \hat{A}_\mu)], \tag{2.26} \]

which shows that they are symmetric in their two indexes. It is straightforward to plug these expressions into equations (2.20)-(2.22) to derive explicit forms for the Lagrangians \( \mathcal{L}(8), (12), (16) \), by also using the following identity involving contractions of the \( \varepsilon \)-tensors:

\[ \varepsilon_{\alpha_1 \ldots \alpha_n \beta_1 \ldots \beta_n} = - (4 - n)! n! \delta_{\alpha_1 \ldots \alpha_n}^{[\beta_1 \ldots \beta_n]}. \tag{2.27} \]

where \([\ldots]\) denotes weighted index anti-symmetrization.

For example, let us focus on the lower dimensional interaction contained in \( \mathcal{L}(8) \), proportional to the dimensionless coefficient \( \beta(8) \). We get

\[ \mathcal{L}(8) = - \frac{\beta(8)}{\Lambda^4} \left( L_\rho \mathcal{Q}^\rho - L_\mu \mathcal{Q}^\mu \right), \tag{2.28} \]
\[ = - \frac{g \beta(8)}{\Lambda^4} \left( \partial_\mu \phi \partial_\nu \phi + g^2 \phi^2 \hat{A}_\mu \hat{A}_\nu \right) \partial_\rho (\phi^2 \hat{A}_\sigma) \left( \delta_\nu^\rho \delta_\sigma^\mu - \delta_\nu^\mu \delta_\sigma^\rho \right). \tag{2.29} \]

This expression is manifestly gauge invariant, and describes the interactions between the norm \( \phi \) of the Higgs field and the gauge-invariant combination of vector and would-be Goldstone bosons. Additional dimension-8 operators proportional to \( \alpha(8) \) could be included, that lead to other interactions between gauge fields and first derivatives of the scalar \( \phi \); these are of less interest in the present context, so we ignore them here. Analogous expressions can be straightforwardly
obtained for $\mathcal{L}_{(12)}, \mathcal{L}_{(16)}$: the resulting formulae are however cumbersome so we include them in appendix (C). We instead move on to discuss some phenomenological aspects of the Higgs interactions associated with $\mathcal{L}_{(8)}$.

As we explained, our main motivation is to generate, by the phenomenon of spontaneous symmetry breaking, the vector self-interactions of equation (1.79) and their generalizations discussed in [59, 60]. As in section 2.1 we expand the norm of the Higgs around the minimum $v$ of the potential, and write

$$\varphi = \left( v + \frac{h}{\sqrt{2}} \right)$$ (2.30)

By applying this expansion, the initial Lagrangian $\mathcal{L}_{tot}$ \(=\) including only the $\beta_{(8)}$ contribution to $\mathcal{L}_{(8)}$ written in equation (2.29) \(\rightarrow\) results

$$\mathcal{L}_{tot} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} - m_A^2 A^2 - \sqrt{2} g m_A h \hat{A}_\mu \hat{A}^\mu - \frac{g^2}{2} h^2 \hat{A}_\mu \hat{A}^\mu$$

$$- \frac{1}{2} (\partial h)^2 - \frac{1}{2} m_h^2 h^2 - \sqrt{2} \frac{\lambda m_h}{2} h^3 - \frac{\lambda}{8} h^4 - \bar{\beta} \hat{A}_\mu \hat{A}^\mu \partial_\rho \hat{A}^\rho$$

$$+ \frac{4 g \bar{\beta}}{3 m_A} \left( \sqrt{2} h + \frac{3 g}{2 m_A} h^2 + \frac{g^2}{\sqrt{2} m_A^2} h^3 + \frac{g^3}{8 m_A^3} h^4 \right) \times$$

$$\left( \hat{A}_\mu \hat{A}^\nu \partial_\nu \hat{A}^\mu - \hat{A}_\mu \hat{A}^\mu \partial_\rho \hat{A}^\rho \right) ,$$ (2.31)

where, in addition to $m_A$ and $m_h$, we also have the parameter, $\bar{\beta}$ given by,

$$\bar{\beta} = \frac{3 g^3 \beta_{(8)} v^4}{2 \Lambda^4}$$ (2.32)

We again neglect the field-independent part of the potential that contributes to the cosmological constant.

Similarly to what we found for $\mathcal{L}_H$, the previous Lagrangian is fully gauge invariant. However, working in the physically transparent unitary gauge, one finds that the previous Lagrangian equation (2.31) leads to several interesting interactions. Indeed, in addition to the renormalizable interactions found in section 2.1 the vev of the Higgs, $v$, now provides the simplest example of a derivative vector self-interaction: that of equation (1.79), which was discussed in chapter 1. Hence,
the phenomenon of spontaneous symmetry breaking automatically generates the desired vector derivative self-interactions; the dimensionless coupling constant $\beta$ in front of this derivative operator depends on the ratio of the Higgs vev $v$ and the scale $\Lambda$, (see equation (2.32)).

On the other hand, we discover that in addition to these renormalizable derivative vector self-interactions, this Lagrangian contains new higher dimensional operators between the physical Higgs field $h$ and the gauge field, contained in the last two lines of equation (2.31). The couplings that govern those interactions are fixed by the mechanism of symmetry breaking and gauge invariance, and are suppressed by a mass scale corresponding to the vector mass $m_A$ to appropriate powers. Notice that all these new higher dimensional interactions are derived from our initial Lagrangian, and consequently are ghost-free since the associated equations of motion contain at most two space-time derivatives. It is indeed straightforward to show that for all these interactions the $A_0$ component of the gauge field remains a constraint, and the equations of motion for all the fields contain at most two space-time derivatives (including the new vector-scalar interactions in the last line of equation (2.31)). One can further generalize these results by including the Lagrangians $L_{12}$ and $L_{16}$, that lead to the complete set of derivative vector interactions discussed in [59] in addition to new scalar-vector interactions that generalize the last line of equation (2.31). These are a subset of the general scalar-vector interactions considered in appendix C and are therefore ghost free by construction. These new operators could have very interesting observational effects. Indeed, since they are suppressed by powers of $m_A$, they can lead to sizeable effects if $m_A$ is not large. However, screening mechanisms might occur, similar to what happens with the Vainshtein effect and Galileon interactions in gravitational set-ups. In fact, vector Galileons have been shown to possess a Vainshtein effect around dynamical space-times [61]. That such mechanisms exist for both the scalar and vector sectors and, in particular, for bi-Galileons [69], it does not seem presumptuous to suggest that such a mechanism would play a role in the phenomenology of the previous system\textsuperscript{3}. However, the existence of such a mechanism would strongly

\textsuperscript{3}See [70] for a discussion on the effect that vector-scalar interactions can have on Vainshtein
depend on how the theory couples to matter. See section 2.2.1 for a discussion of issues along these lines. For now, let us develop some intriguing relations between the previous system and Galileons.

We return to the fully gauge invariant Lagrangian (2.31) before choosing any gauge, with the aim to study the dynamics of would-be Goldstone bosons. In chapter 1 we presented the argument, first made in [59], which establishes that a decoupling limit exists in which the dynamics of the Goldstone bosons $\pi$ is described by Galileonic derivative self-interactions.

This is a regime where some kind of equivalence theorem should hold, with the physics of the Goldstone bosons being equivalent to that of the longitudinal polarization of the vectors (see for example [71]). In our Higgs set-up, we can go one step further: we show that in this decoupling limit, not only do the Goldstone self-interactions preserve Galileon invariance by themselves, but in addition they acquire new derivative couplings with the Higgs field $h$. These automatically preserve the Galileon symmetry by assembling into the bi-Galileon combinations considered in chapter 1.

To exhibit these features, the limit we have to consider is,

$$g \to 0 \ , \ \lambda \to 0 \ , \ \beta_{(8)} \to 0 \ , \ v \to \infty , \quad (2.33)$$

such that,

$$m_A \to 0 \ , \ m_h \to 0 \ , \ \bar{\beta} \to 0 \ , \ \frac{\bar{\beta}}{m_A^3} = \text{fixed} \equiv \frac{1}{\Lambda_g^3} , \quad (2.34)$$

where $\Lambda_g$ is a mass scale that, as we will see in a moment, is associated with the strength of the Galileon interactions.

Notice that the previous limits imply that $g/m_A = 1/v \to 0$. In order to have a correctly normalized kinetic term for the Goldstone boson $\pi$ we have to rescale this field, and define $\hat{\pi} = \pi/(\sqrt{2}m_A)$. Indeed the second term in the first line of screening.
(2.31) becomes, in the limit (2.33),

$$-m_A^2 (A_\mu - \partial_\mu \pi)^2 = - \left( m_A A_\mu - \frac{1}{\sqrt{2}} \partial_\mu \hat{\pi} \right)^2$$

$$\rightarrow - \frac{1}{2} (\partial_\mu \hat{\pi})^2,$$  \hspace{1cm} (2.35)

so the Goldstone boson acquires a standard kinetic term. In the limits (2.33, 2.34), when expressed in terms of the canonically normalized Goldstone field $\hat{\pi}$, the total Lagrangian $\mathcal{L}_{\text{tot}}$ reduces to

$$\mathcal{L}_{\text{tot}} = - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial_\mu h)^2 - \frac{1}{2} (\partial_\mu \hat{\pi})^2$$

$$- \frac{1}{\Lambda^3} (\partial_\mu \hat{\pi} \partial^\mu \hat{\pi}) \Box \hat{\pi} - \frac{1}{3 \Lambda^3} \left( \partial_\mu h \partial^\mu h \Box \hat{\pi} - \partial_\mu h \partial^\nu h \partial_\nu \partial^\mu \hat{\pi} \right).$$  \hspace{1cm} (2.36)

Hence, as announced, in this decoupling limit the Lagrangian acquires a bi-Galileon structure, and the physical Higgs itself acquires bi-Galileon couplings\(^4\) [72, 73] with the Goldstone boson describing the dynamics of the longitudinal vector polarization.

The connection that we pointed out with Galileons can help to render the structure of the theory stable under radiative corrections. Galileon Lagrangians are known to enjoy powerful non-renormalization theorems [39, 48] that might be applied in the present context to protect the size of the higher dimensional operators $\mathcal{L}^{(8),(12),(16)}$ that we introduced in this section.

### 2.2.1 Coupling to matter

We can think of two different ways in which the Higgs field can couple to matter, that would allow us to exploit the bi-Galileon interactions. The first is a direct coupling of the Higgs $\phi$ to the trace of the energy momentum tensor $T$ via operators that respect gauge invariance such as for example $\phi^* \phi T$. In the case in which the Higgs scalar of our model is very light – as might be required for cosmological applications – such couplings could be associated with a long range force that

\(^4\)The above bi-Galileon interaction corresponds to equation (C.9) in appendix C but with $h$ and $\pi$ exchanged.
needs to be screened. In our set-up we have shown that, in an appropriate regime, the Higgs scalar combines with the longitudinal polarization of the vector to form bi-Galileon derivative combinations. These non-linear operators can then lead to a Vainshtein mechanism that is able to suppress the aforementioned long range force.

Other possible couplings involve derivative operators. An example among others is a gauge invariant coupling of the form $(\mathcal{D}_\mu \phi)^\ast (\mathcal{D}^\mu \phi) T$, where the $\mathcal{D}_\mu$ is a covariant derivative containing gauge fields (see equation (2.10)). Once the covariant derivatives are expanded, such a combination leads among others to operators of the form $A_\mu A^\mu T$, that couple vectors to the energy momentum tensor. More generally, one could generalize the derivative disformal couplings of scalars to matter proposed by Bekenstein [32], by promoting the standard derivative to covariant derivatives.

These arguments of course only scratch the surface of the possible couplings of our Higgs field to matter and their phenomenological consequences.

### 2.3 Higgs mechanism and generalized non-abelian symmetry breaking

The Higgs construction that we developed in the abelian case can be directly extended to theories with a non-abelian gauge symmetry. This is interesting because, applying the Stückelberg approach in this case would be more laborious than in the abelian set-up.

Again we focus on theories that contain dimension-8 operators with derivative self-interactions of the Higgs field. We investigate theories that spontaneously break non-abelian symmetries, leading to consistent derivative self-interactions for gauge vectors and generalizing the abelian symmetry breaking case discussed in the previous section. The mathematical consistency and cosmological applications of a subclass of these theories has recently been considered in [74, 75]. Instead of providing a fully general treatment, we concentrate on a representative example to make clear our arguments.
We consider an $SU(2)$ theory with a doublet of complex scalars $\phi = \{\phi^\alpha\}$, with $\alpha = 1, 2$ transforming in the fundamental representation. The construction of a Higgs model for this theory, which spontaneously breaks the $SU(2)$ symmetry, is a standard textbook example (see [76]). Here we consider additional derivative self-interactions of the Higgs field, that lead to derivative self-interactions of the gauge vectors.

The Lagrangian we are interested in, is invariant under the non-abelian $SU(2)$ symmetry, and is written,

$$\mathcal{L}_{SU(2)} = - (D^\mu \phi)^\dagger D_\mu \phi - V(\phi) - \frac{1}{2} \text{tr} [F_{\mu\nu} F^{\mu\nu}] + \mathcal{L}_{(8)}^{SU(2)}. \quad (2.37)$$

The field $\phi$ is our Higgs, that as stated above is a doublet under the $SU(2)$ symmetry; the covariant derivative acts on its components as

$$(D_\mu \phi)^\alpha = \partial_\mu \phi^\alpha - ig A_\mu^a (T^a)^\alpha_\beta \phi^\beta,$$ \quad (2.38)

where $T^a$ are the generators in the fundamental representation, that for $SU(2)$ are proportional to the Pauli matrices, $T^a = \sigma^a/2$. The non-abelian transformation acts as,

$$\phi \rightarrow U \phi,$$ \quad (2.39)

$$A_\mu \rightarrow UA_\mu U^\dagger - \frac{i}{g} (\partial_\mu U) U^\dagger,$$ \quad (2.40)

with $A_\mu \equiv A_\mu^a T^a$, and the transformation group element is $U \equiv \exp [ig \theta^a(x) T^a]$. The covariant derivative (equation (2.38)) transforms as expected

$$(D_\mu \phi) \rightarrow U (D_\mu \phi). \quad (2.41)$$

The field strength associated with the vector potential is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu, A_\nu], \quad (2.42)$$

and transforms as

$$F_{\mu\nu} \rightarrow U F_{\mu\nu} U^\dagger,$$ \quad (2.43)
the corresponding gauge invariant vector kinetic term is,
\[ -\frac{1}{2} \text{tr} \left[ F_{\mu \nu} F^{\mu \nu} \right] = -\frac{1}{4} F_{\mu \nu}^a F^{a \mu \nu}, \]  
(2.44)
where we used the following identity valid for fundamental representations of the
gauge group \( \text{tr} \left[ T^a T^b \right] = \frac{1}{2} \delta^{ab} \).

The Higgs potential is written as,
\[ V(\phi) = \lambda \left( \phi \phi^\dagger - v^2 \right)^2, \]
and is invariant under the unitary transformations that we are considering. It is
characterized by a family of degenerate vacua, with \( \phi \phi^\dagger = v^2 \), that spontaneously
break the gauge symmetry.

The dimension-8 Lagrangian \( \mathcal{L}_{SU(2)}^{(8)} \) in the second line of equation (2.37),
responsible for breaking the non-abelian symmetry in such a way as to generate
consistent derivative vector self-interactions, is constructed similarly to what was
done for the case of abelian symmetry in the previous section.

We define the gauge invariant tensor combinations,
\[ L_{\mu \nu} \equiv \frac{1}{2} \left[ (\mathcal{D}_\mu \phi)^\dagger (\mathcal{D}_\nu \phi) + (\mathcal{D}_\nu \phi)^\dagger (\mathcal{D}_\mu \phi) \right], \]
(2.46)
\[ Q_{\mu \nu} \equiv \frac{i}{2} \left[ \phi (\mathcal{D}_\mu \mathcal{D}_\nu \phi)^\dagger - \phi^\dagger \mathcal{D}_\mu \mathcal{D}_\nu \phi \right], \]
(2.47)
built in terms of the Higgs doublet \( \phi \). Then,
\[ \mathcal{L}_{SU(2)}^{(8)} \equiv -\frac{\beta}{\Lambda^4} \left[ L_{\mu \nu}^\rho Q_{\sigma}^{\rho} - L_{\mu}^{\mu} Q_{\nu}^{\nu} \right], \]
(2.48)
with \( \beta \) a dimensionless coupling constant, and \( \Lambda \) a scale.

For the very same arguments discussed in the abelian case, this dimension-8 op-
erator is gauge invariant, and consistent since it does not introduce ghost degrees
of freedom.

To proceed, we recall that \( SU(2) \) transformations are characterized by three free
parameters, while our Higgs field has four independent real components. At this
stage, we can use the gauge freedom to fix a unitary gauge and eliminate three of
the Higgs four components. We write,
\[ \phi = \begin{pmatrix} 0 \\ v + \frac{1}{\sqrt{2}} h \end{pmatrix} \]
(2.49)
with $h$ a real scalar field.

The covariant derivative acting on the Higgs becomes

$$D_\mu \phi = \frac{1}{\sqrt{2}} \left( \partial_\mu h - i \frac{g}{2} \left( v + \frac{1}{\sqrt{2}} h \right) \left( A_\mu^1 - i A_\mu^2 \right) \right).$$

(2.50)

On the other hand, the second covariant derivative on the complex scalar $f$ acts as,

$$D_\nu D_\mu f = \partial_\nu \partial_\mu f - i g \left( \partial_\nu A_\mu^c \right) \left( T^c \right)_\nu^\gamma \phi^\gamma - i g A_\mu^c \left( T^c \right)_\nu^\gamma \partial_\nu \phi^\gamma$$

$$- i g A_\nu^c \left( T^c \right)_\mu^\gamma \partial_\nu \phi^\gamma - g^2 A_\mu^a A_\mu^b \left( T^a \right)_\nu^\gamma \left( T^b \right)_\mu^\gamma \phi^\gamma.$$ (2.51)

Plugging these ingredients in the expression (2.37) for $\mathcal{L}_{SU(2)}$ and expanding, we find the following Lagrangian for the Higgs field $h$, the vectors $A_\mu^a$, and their couplings (sum over repeated indexes),

$$\mathcal{L}_{SU(2)} = -\frac{1}{4} F_\mu^a F^{a \mu v} - \frac{g^2 v^2}{4} \left( A_\mu^a A^a_\mu \right)$$

$$- \frac{\beta g^3}{8 A^4} \left[ \left( A_\mu^a A^a_\mu \right) \partial_\nu A^3_\nu - \left( A_\mu^a A^a_\nu \right) \partial_\mu A^3_\nu \right]$$

$$- \frac{1}{2} \partial_\mu h \partial^\mu h - 2 \frac{\lambda}{v^2} h^2 - \sqrt{2} \frac{\lambda}{v^2} h^3 - \frac{\lambda}{4} h^4$$

$$- \frac{\beta g^2}{4 A^4} \left( \partial_\mu h \partial_\nu h \partial_\alpha h + \partial_\mu h \partial_\nu h \partial_\nu A^3_\mu \right) \left( 1 + \frac{\sqrt{2} h}{v} + \frac{h^2}{2 v^2} \right)$$

$$- \frac{\beta g^3}{4 \sqrt{2} A^4} \left( h^2 + 3 \frac{h^2}{2 \sqrt{2} v} + \frac{h^3}{2 v^2} + \frac{h^4}{8 \sqrt{2} v^3} \right) \times$$

$$\times \left[ \left( A_\mu^a A^a_\mu \right) \partial_\nu A^3_\nu + A_\mu^a A^a_\nu \partial_\mu A^3_\nu + A_\mu^a A^3_\mu \partial_\nu A^a_\nu \right.$$

$$- A_\mu^a A^a_\nu \partial_\mu A^3_\nu - 2 A_\mu^a A^a_\nu \partial_\nu A^a_\mu \right].$$ (2.52)

Hence when the $vev$, $v \neq 0$, this set-up spontaneously breaks the non-abelian gauge symmetry. It not only provides a mass to the three gauge bosons but also ghost-free, derivative, self-interactions among them that corresponds to a non-abelian generalization of the vector theories of chapter 1. Moreover, it introduces new higher-dimensional couplings (with or without derivatives) between the Higgs field and the vector, proportional to the coupling constant $\beta$. 49
The Lagrangian (equation (2.52)) is expressed in unitary gauge: if we were to re-introduce the would-be Goldstone bosons, we would find new interactions between them and the Higgs field, that in an appropriate decoupling limit leads to a theory of multi-Galileons, generalizing the findings of the previous section.

In this chapter we presented a Higgs mechanism for spontaneously breaking a gauge symmetry, to obtain the non-linear derivative vector self-interactions discussed in chapter 1 and extended the discussion to a case with non-abelian symmetry. After symmetry breaking, the resulting theory contains the desired vector self-interactions, and in addition new ghost-free derivative interactions between the Higgs and the vector bosons. We studied some of the features of the resulting set-up. We showed that the Lagrangian controlling the would-be Goldstone boson of this theory obtains a Galileon structure in an appropriate decoupling limit. Interestingly, in the same limit the would-be Goldstone boson also acquires derivative couplings with the physical Higgs, that combine in such a way as to form a bi-Galileon system with fixed coefficients, determined by gauge invariance. This suggests that, once we introduce an appropriate source, a Vainshtein mechanism should actively screen it from both the longitudinal mode of the vector and the Higgs field of the full theory.
Chapter 3

Covariant Vector Galileons

The content of this chapter is based on my paper [77], ‘Covariantized vector Galileons’, published in the journal, ‘Physical Review D’.

In this chapter we discuss the effect of gravitational interactions on the stability of vector Galileons. This is a non-trivial question as these theories are constructed with non-linear, derivative self-interactions which necessarily include kinetic mixing with gravity due to the use of a Levi-Civita connection.

It might be argued that the previous statement is simply motivated through formalism as there is no physical reason to require that our theory is generally covariant. However, what is necessary is that we include gravitational interactions in a consistent way. Therefore, given that general relativity is formulated as a covariant theory of space-time, it proves beneficial to keep our derivatives covariant by using the unique metric compatible connection available to us, i.e. the Levi-Civita connection.

In section 3.2 we discuss the covariantisation of vector Galileons based on the structure of the Horndeski class of theories, which is the most general, Lorentz invariant, scalar-tensor theory with equations of motion up to second order in derivatives [36]. This ‘non-minimal’ covariantisation was first suggested by [59, 60].

\footnote{This theory was re-discovered by cosmologists thirty years later when considering the covariantisation of generalised Galileons [43, 78].}
via the identification of the Horndeski action with the covariantised longitudinal mode. After establishing the motivation for the structure of the theory, we then investigate its stability by analysing how the additional non-minimal counter terms cure the theory through the cancellation of certain ‘dangerous’ terms from the Lagrangian.

Recently, scalar-tensor theories, generalised beyond the class of Horndeski, have been shown to be consistent [79, 80, 81]. These theories typically have equations of motion with derivatives higher than second order. Motivated by the existence of these theories, in section 3.2.1 we examine the ‘minimal covariantisation’ of vector Galileons and find, up to an ansatz for the vector field, the same cancellation of the ‘dangerous terms’ identified in our analysis of the non-minimally covariantised theory. However, unlike the non-minimally covariantised theory, we find that removing our ansatz leads to terms that obstruct the cancellation, signalling that we should consider our theory carefully. We continue our analysis in chapter 4, where we develop a more sophisticated method of analysis.

3.1 The ADM formalism

Given a four-dimensional space-time $\mathcal{V}$, we may introduce a scalar field $t(x^\alpha)$ such that $t = \text{const}$ defines a family of non-intersecting, spacelike, three surfaces $\Sigma_t$. This allows us to introduce a foliation of the four dimensional space-time such that the metric, $g_{\mu \nu}$, can be decomposed in terms of components normal and tangent to the three surfaces $\Sigma_t$.

$$ds^2 = g_{\mu \nu} dx^\mu dx^\nu = -(N^2 - N_i N^i) dt^2 + 2 N_i dt dx^i + \gamma_{ij} dx^i dx^j,$$

(3.1)

where we introduce the time vector flow $t^\mu = \partial / \partial t$ decomposed as

$$t^\mu = N n^\mu + N^\mu,$$

(3.2)

and $n^\mu$ is the unit normal vector to the $t = \text{const}$. hypersurface, $\Sigma_t$, $N$ the lapse function and $N^\mu$ the shift vector orthogonal to the normal vector.

We write the unit normal to the three surface as $n^\mu = ( - \frac{1}{N}, \frac{N^i}{N} )$ with corresponding one-form $n_\mu = ( - N, 0 )$. Furthermore, equivalently we may write: $g_{00} =$
\[-(N^2 - N_iN^i), \ g_{0i} = N_i, \text{ and } g_{ij} = \gamma_{ij}. \] The associated four dimensional inverse metric’s components may be written as: \[g^{00} = -\frac{1}{N^2}, \ g^{0i} = \frac{N^i}{N^2}, \text{ and } g^{ij} = \frac{\gamma^{ij} - \frac{N^iN^j}{N^2}}{N^2}. \]

The corresponding metric determinants are associated by: \(\sqrt{-g} = N\sqrt{\gamma}\).

The constant time hypersurface, \(\Sigma_t\), is then characterised by the following three quantities:

\[n^\mu, \ h^\mu_\nu, \ K^\mu_\nu, \quad (3.3)\]

where \(h^\mu_\nu = \delta^\mu_\nu + n^\mu n_\nu\) is the projection tensor related to \(\gamma_{ij}\) on the hypersurface\(^2\), \(\Sigma_t\), and \(K^\mu_\nu\) the associated extrinsic curvature

\[K_{\mu\nu} = \frac{1}{2N} (\dot{h}_{\mu\nu} - D_{(\mu}N_{\nu)}) \quad (3.4)\]

With “dot” we mean the Lie derivative with respect to \(t^\mu\), \(D_\mu\) is the 3D covariant derivative on the constant time hypersurface and the parenthesis \((\ldots)\) on the indices denote symmetrisation.

### 3.2 Covariantisation of vector Galileons

As explained in the introduction, the aim of this chapter is to use an ADM approach to reconsider more carefully the consistency of the covariant couplings of vector Galileons with gravity by making use of the analogy with the scalar Galileon counterparts. We start in this section by discussing non-minimal couplings of vector Galileons with gravity, studying the conditions to avoid the propagation of ghosts. Furthermore, the existence of various forms for the vector Galileon on Minkowski space raises questions about whether there is any freedom to choose the form of any additional non-minimal couplings.

#### 3.2.1 Covariantisation of vector Galileons via the use of non-minimal couplings

In this section we address the problem by identifying the potentially unstable terms that are generated when we naively covariantise the derivative interactions

\(^2\)We often relax our formalism and refer to this quantity as the three dimensional quantity.
in the vector Galileon system. Indeed, the fact that the system is able to avoid producing Ostrogradsky ghosts relies on the fact that partial derivatives commute together with the antisymmetric sum over the indices. Minimal covariantisation of the derivatives spoils their, seemingly essential, commutative property and generates extra interaction terms that, a subset of which, appear to be potentially unstable. It is exactly the need to eliminate these extra terms that fixes the form of the non-minimal coupling. In this sense we can view the non-minimal couplings as counter terms that cure the theory from unstable gravitational interaction terms. In the following we present the previously proposed ‘generalised Proca’ system of [59, 60] and then use the ADM formalism to investigate its consistency by studying the restrictions imposed on its non-minimal couplings of vectors with gravity.

The generalised Proca system

The non-minimally covariantised vector Galileons, otherwise known as the ‘generalised Proca system’, were first presented in [59],[60],[82] and can be written in the form resembling the Horndeski system for scalar-tensor theories,

\[ \mathcal{L}_F = -\frac{1}{4} \sqrt{-g} F^{\mu\nu} F_{\mu\nu}, \] (3.5)
\[ \mathcal{L}_{vH}^{(2)} = \sqrt{-g} G_2(X), \] (3.6)
\[ \mathcal{L}_{vH}^{(3)} = \sqrt{-g} G_3(X) A_\mu^\mu, \] (3.7)
\[ \mathcal{L}_{vH}^{(4)} = \sqrt{-g} G_4(X) R + \sqrt{-g} G_{4,X} e^{\mu\rho\lambda_1\lambda_2} e^{\nu\sigma\lambda_1\lambda_2} (A_{\mu\nu} A_{\rho\sigma} + c_2 F_{\mu\rho} F_{\nu\sigma}) , \] (3.8)
\[ \mathcal{L}_{vH}^{(5)} = \sqrt{-g} G_5(X) A_{\mu\nu} G^{\mu\nu} - \frac{1}{6} G_{5,X} e^{\mu\rho\gamma\lambda} e^{\nu\sigma\xi} (A_{\mu\nu} A_{\rho\sigma} A_{\gamma\xi} + d_3 A_{\mu\nu} F_{\rho\gamma} F_{\sigma\xi}) , \] (3.9)

where \( X \equiv -\frac{1}{2} A_\mu A^\mu, \) \( G_{N,X} = \frac{\partial G_N}{\partial X} \) and \( A_{\mu\nu} = \nabla_{(\mu} A_{\nu)} \).

Whereas the earlier mentioned work motivated this system via its similarity in

\(^3\)See also [63] for a discussion of an effective field theory for vectors.
\(^4\)We use the same notation as in section 1.80 to denote the same symmetric and antisymmetric combinations formed with covariant derivatives. The extra bi-parameter freedom also extends to the covariantisation of that system.
construction to Horndeski theory, in the following sections, we analyse the consistency of the covariantised model by focussing on the role of the non-minimal couplings.

**Non-minimal covariantisation of the quartic vector Galileon**

In this section we examine how the inclusion of a specific non-minimal coupling term is able to ‘cure’ a potential instability arising from the covariantisation of the derivatives in the quartic vector Galileon, given by equation (3.10) below. Although not necessary, we begin by choosing a certain special ansatz for the vector field and find that within the ADM formalism and at the level of the action, this possible instability is related to the existence of terms of the form, $A_0 \dot{N}$ and $\dot{A}_0 N$. Such terms in the action can produce dynamics for the lapse, $N$ or $A_0$, which would be an extra degree of freedom which does not exist in general relativity nor electromagnetism, where the equations of motion for $N$ and $A_0$ appear as a constraint. Typically, this extra degree of freedom is identified as a propagating ghost.

Note however that the existence or non-existence of these terms by themselves do not guarantee that this is or isn’t a true classical instability. It was recently shown in [83] that the existence of time derivatives of the lapse does not necessarily mean we have a pathology. Indeed, they showed that one could start with an Einstein-Hilbert action, perform a transformation that results in a theory with time derivatives of the lapse but as long as the transformation is regular and invertible, the number of degrees of freedom remain invariant.

Whilst the theories might be equivalent in the vacuum (around Minkowski), it is not true that they remain equivalent once metric is allowed to have dynamics. Therefore, in this case, a full Hamiltonian analysis needs to be performed to confirm the number of propagating degrees of freedom. Furthermore, it is true that other terms of the form $A_0 \dot{N}_i$ and $\dot{A}_0 N_i$, could also produce classically unstable dynamics, however, if they were to exist, such terms would represent a far more dramatic increase in the number of degrees of freedom of the theory and therefore as a first step we do not consider this possibility.
We choose an ansatz for the vectors by setting the spatial components of the vector to zero, \( A_\mu = (A_0, 0) \). This choice of ansatz not only ensures that we are able to recover a homogeneous and isotropic background cosmology but also conveniently selects a direction for which we can ignore any spatial mixing. We consider the covariantisation of a simplified form for the quartic vector Galileon \(^5\) which is given by,

\[
\mathcal{L}^{(4)}_{vH} = \sqrt{-g} A_\sigma A_\lambda g^{\lambda \mu} \left( \nabla^\mu A_\nu \nabla^\nu A_\sigma - \nabla^\mu A_\nu \nabla^\nu A_\mu - \frac{1}{4} g^{\mu \nu} g^{\nu \rho} R \right). \tag{3.10}
\]

Focussing on the derivative terms we find that after cancellations we are left with a term which contains,

\[
\mathcal{L}^{(4)}_{vH}|_A = 2 \sqrt{-g} A_0 A_0 g^{\mu \nu} \left( \nabla^\mu A_0 \nabla^\nu A_0 - \nabla^\mu A_0 \nabla^\nu A_0 \right), \tag{3.11}
\]

\[
\geq 2 \sqrt{\gamma} N A_0^2 \left( - \frac{1}{N^2} \right) \left( \dot{A}_0 - \frac{N}{A_0} \frac{KA_0}{N^3} \right),
\]

\[
\Rightarrow \mathcal{L}^{(4)}_{vH}|_A \geq - \frac{1}{2} \sqrt{\gamma} (A_0)^4 \left( \frac{\dot{K}}{N^4} \right). \tag{3.12}
\]

where we use the symbol ‘\( \geq \)’ to denote that \( \mathcal{L} \) contains this expression amongst other terms. With our ansatz this is the only term originating from the derivative structure to contain potential instabilities of the form \( \dot{A}_0 N \) and \( \dot{A}_0 N \). For the non-minimal term, we use the results from appendix B.2.3 to find that it contributes,

\[
\mathcal{L}^{(4)}_{vH}|_B = - \sqrt{-g} A_0 g^{\mu \nu} \left( \frac{1}{4} R - \frac{1}{4} N \sqrt{\gamma} (A_0)^4 \left( g^{\mu \nu} \right)^2 \right),
\]

\[
\geq \frac{1}{2} \sqrt{\gamma} (A_0)^4 \left( \frac{\dot{K}}{N^4} \right), \tag{3.13}
\]

which cancels the contribution from the previous derivative term. Interestingly, this contribution comes from what would have been a total divergence in the Einstein-Hilbert action (see equations (B.22) and (B.23) in appendix B.2.3). Notice indeed that in this case the Ricci scalar does not stand alone, but is weighted by the fourth power of the gauge field. The addition of a non-minimal coupling thus contributes a new derivative term that after integration by parts has the right

\(^5\)This form is found by performing an integration by parts and can be identified with \( \mathcal{L}^{(4)}_{vH} \) in equation (3.8) by setting \( c_2 = -\frac{1}{2} \) and \( G_4 = -X^2 \).
structure to cancel the time derivative of the lapse, ensuring that it remains an auxiliary variable.

At the level of operators, the relationship between covariant derivatives and curvature forms on a general space-time allows one to use the non-minimal coupling as a gravitational counter term\(^6\). From this point of view, the tuning for the functional form of the factors in the Horndeski theory ensures that these particular unstable operators do not appear in the action.

**Non-minimal covariantisation of the quintic vector Galileon**

In this section we apply the same analysis to the covariantised quintic vector Galileon given by,\(^7\)

\[
\mathcal{L}_{vH}^{(5)} = \mathcal{L}_{vH|A}^{(5)} + \mathcal{L}_{vH|B}^{(5)},
\]

\[
\mathcal{L}_{vH|A}^{(5)} = \sqrt{-g} A_\mu A_\nu g^{\mu \nu} g^{\rho \sigma} g^{\lambda \tau} g^{e \delta} \left( A_\rho A_\sigma A_\lambda A_\tau A_\epsilon A_\delta - 3A_\rho A_\sigma A_\lambda A_\tau A_\epsilon A_\delta + 2A_\rho A_\sigma A_\lambda A_\epsilon A_\tau A_\delta \right),
\]

\[
\mathcal{L}_{vH|B}^{(5)} = \frac{6}{4} \sqrt{-g} A_\mu A_\nu g^{\mu \nu} g^{\rho \sigma} g^{\lambda \tau} g^{e \delta} A_\rho A_\sigma A_\lambda A_\epsilon G_{\tau \delta}.
\]

We again use a special ansatz for which \(A_\mu = (A_0, \vec{0})\) and look for terms of the form \(\dot{A}_0 N\).

With this ansatz we have that \(A_00 = \dot{A}_0 - \Gamma_{00}^0 A_0 \geq \dot{A}_0 - \frac{N}{A_0} A_0\) and \(G_{ij} \geq \frac{2}{N} \mathcal{R}_{ij} K^{kl} K_{kl}\).

Since we are restricting our focus, we only need to consider the factors of these terms. \(\mathcal{L}_{vH|A}^{(5)}\) contributes,

\[
\mathcal{L}_{vH|A}^{(5)} \geq 6 \sqrt{N} A_0^2 g^{(0)} A_0 \left[g^{ij} g^{kl} - 2 g^{ij} g^{0k} g^{0l}\right] (A_{ij} A_{kl})
\]

but with our ansatz the last factor can be expressed as,

\[
A_{ij} A_{kl} = \frac{1}{2N^2} \left(K_{ij} K_{kl} - K_{ik} K_{jl}\right) A_0^2.
\]

Therefore we see that the contribution from \(\mathcal{L}_{vH|A}^{(5)}\) can be expressed as,

\[
\mathcal{L}_{vH|A}^{(5)} \geq \frac{6 \sqrt{N} A_0^4}{2N^3} \left(\dot{A}_0 - \frac{N}{A_0}\right) \left(K^2 - K_{ij} K^{ij}\right).
\]

\(^6\)This is called the ‘Ricci identity’ and is given by equation (1.3).

\(^7\)This can be identified with \(\mathcal{L}_{vH}^{(5)}\) in equation (3.9) by setting \(d_3 = 0\) and \(G_5 = 6X^2\).
We use the fact that in the ADM formalism we can write $G^{00} = \bar{R} + K^2 - K_{ij}K^{ij}$, to find that the contribution from $\mathcal{L}_{vH|B}^{(5)}$ is,

$$\mathcal{L}_{vH|B}^{(5)} \supseteq \frac{6\sqrt{7}A_0^4}{8N^5} (\dot{A}_0 - \frac{N}{N'}A_0) (K^2 - K_{ij}K^{ij}) + \frac{6\sqrt{7}}{4N^5}A_0^5 (KK - K_{ij}K^{ij}). \quad (3.20)$$

These three terms cancel up to a boundary term after performing an integration by parts on the last term. We again see that the Horndeski like tuning of the action depends on the fact that non-minimal couplings with the curvature tensors contain the same operator components as the non-linear combination of covariant derivatives. We will see in section 3.2.2 that this property motivates the utility of the epsilon tensor construction as such operators are generated with opposing signs thanks to the antisymmetry of the epsilon tensor and thus they are cancelled out.

### 3.2.2 Covariantisation via minimal substitution

In the previous section we saw that the non-minimal coupling term in the vector-Horndeski Lagrangian, once written in terms of ADM variables, contained terms that exactly cancelled the problematic terms containing the derivative of the lapse introduced by naive covariantisation. We can view this property as being inherited from using the Horndeski Lagrangian for the non-minimal covariantisation of the decoupled longitudinal mode.

In this section we take this analogy further where, motivated by the recent results concerning beyond Horndeski theories [79], [84], [80], we investigate the possibility that the vector-Galileons can be covariantised by minimal substitution. In order to do this we work with the fully expanded epsilon tensor construction given by equations (1.80) to (1.84), \footnote{In short, we construct $\mathcal{L}_N$ with 2N-2 space-time indices contracted over two epsilon tensors.} and minimally covariantise by substituting the derivatives for covariant derivatives.

We work with the same ansatz as before and again focus on searching for the existence of potentially unstable terms of the form $A_0\dot{N}$ and $\dot{A}_0N$. We find that in the covariantisation of this construction the terms involving the derivative of the
lapse are generated in an antisymmetric combination such that they automatically cancel without the use of an additional counter term.

**Minimally covariantised quartic vector Galileon**

The minimally covariantised quartic vector Galileon can be written as

\[
\mathcal{L}_{vG_{\text{min}}}^{(4)} = \sqrt{-g} A_\sigma A_\lambda g^{\sigma\lambda} \left( \nabla^\mu A_\mu \nabla^\nu A_\nu - \nabla^\mu A_\nu \nabla^\nu A_\mu \right) + 2 \sqrt{-g} g_{\mu \nu} g^{\mu \sigma} \left( \nabla^\rho A_\sigma \nabla^\nu A_\rho - \nabla^\nu A_\sigma \nabla^\rho A_\mu \right),
\]

which is the derivative term studied in section 3.2.1 combined with another derivative term stemming from the extra antisymmetric sum over the first two indices.

Working with the ADM formalism and with our ansatz we find that the second term contains,

\[
2 \sqrt{-g} A_\mu A_\nu g^{\mu \sigma} \left( \nabla^\rho A_\sigma \nabla^\nu A_\rho - \nabla^\nu A_\sigma \nabla^\rho A_\mu \right) \geq \frac{1}{2} \sqrt{\mathcal{T}(A_0)^4 \left( \frac{\mathcal{K}}{N^4} \right)},
\]

which cancels the contribution from the first term containing the derivative of the lapse given by equation (3.11) without the need of an additional non-minimal counter term.

An efficient way to realise the cancellation with this ansatz is found by utilising the antisymmetric structure of the Lagrangian. Indeed, the antisymmetric properties of the epsilon tensors make it straightforward to realise that these terms cancel without the use of non-minimal couplings such as those discussed in the previous

\[9\]Where for convenience we have chosen the extra free parameter to be \( c_2 = -1 \).
\[ \mathcal{L}^{(4)}_G = \sqrt{-g} \varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4} v^\lambda v_{\nu}^\gamma v_{\nu}^\delta A_{\mu_1} A_{\mu_2} (A_{\mu_3} v_4 A_{\mu_4} v_6), \]
\[ = N \sqrt{\gamma} \varepsilon^{0 \mu_3 \mu_5 \mu_7} v^0 v^6 \lambda (A_0)^2 A_{\mu_3} v_4 A_{\mu_5} v_6, \]
\[ = N \sqrt{\gamma} \varepsilon^{0 m_3 m_5 m_7} v^{0 n_4 n_6} (A_0)^2 A_{m_3} n_4 A_{m_5} n_6, \]
\[ = N \sqrt{\gamma} \varepsilon^{0 m_3 m_5 m_7} v^{0 n_4 n_6} (A_0)^4 \Gamma^0_{m_3 n_4} \Gamma^0_{m_5 n_6}, \]
\[ = \sqrt{\gamma} \varepsilon^{0 m_3 m_5 m_7} v^{0 n_4 n_6} (A_0)^4 K_{m_3 n_4} K_{m_5 n_6} K_{n_4 n_6}, \]

(3.23)

where we have used the results of appendix B.1 for the Christoffel symbols and latin indices denote spatial components.

With our ansatz we recover an antisymmetric combination of the extrinsic curvature, \( K_{ij} \). Inspecting its definition given by equation (3.4) reveals that there are no terms of the form \( \dot{A}_0 N \), or \( A_0 \dot{N} \) which provides evidence that the special antisymmetric structure of the vector Galileons allows them to be consistently covariantised by minimal substitution.

**Minimally covariantised quintic vector Galileon**

We find that the kind of cancellation of terms found above for the minimally covariantised quartic vector Galileon is possible for the quintic vector Galileon as well. As for the quartic, it is rather more efficient to make use of the antisymmetric properties of the epsilon tensors \(^{11}\),

\[ \mathcal{L}^{(5)}_G = \sqrt{-g} \varepsilon^{\mu_1 \mu_2 \mu_3 \mu_5 \mu_7} v^\lambda v_{\nu}^\gamma v_{\nu}^\delta v_{\nu}^\epsilon A_{\mu_1} A_{\mu_2} (A_{\mu_3} v_4 A_{\mu_5} v_6 A_{\mu_7} v_8), \]
\[ = N \sqrt{\gamma} \varepsilon^{0 \mu_3 \mu_5 \mu_7} v^0 v^6 v^8 \lambda (A_0)^2 A_{\mu_3} v_4 A_{\mu_5} v_6 A_{\mu_7} v_8, \]
\[ = N \sqrt{\gamma} \varepsilon^{0 m_3 m_5 m_7} v^{0 n_4 n_6 n_8} (A_0)^2 A_{m_3} n_4 A_{m_5} n_6 A_{m_7} n_8, \]
\[ = N \sqrt{\gamma} \varepsilon^{0 m_3 m_5 m_7} v^{0 n_4 n_6 n_8} (A_0)^4 \Gamma^0_{m_3 n_4} \Gamma^0_{m_5 n_6} \Gamma^0_{m_7 n_8}, \]
\[ = \sqrt{\gamma} \varepsilon^{0 m_3 m_5 m_7} v^{0 n_4 n_6 n_8} (A_0)^4 K_{m_3 n_4} K_{m_5 n_6} K_{m_7 n_8} K_{n_4 n_6 n_8}, \]

(3.24)

\(^{10}\)Here we chose the value of the free parameter to be \( c_2 = 0. \)

\(^{11}\)Here we chose the value for the free parameter to be \( d_3 = 0. \)
which shows that the quintic vector Galileon exhibits the same property as was found for the quartic. Specifically, we recover an antisymmetric combination of extrinsic curvature terms, $K_{ij}$, suggesting that it can also be minimally covariantised. In order to see the cancellation in detail we expand out the Lagrangian to find,

$$\mathcal{L}_{vG}^{(5)} := \mathcal{L}_{vG|A}^{(5)} + \mathcal{L}_{vG|B}^{(5)},$$

$$\mathcal{L}_{vG|A}^{(5)} = \sqrt{-g}A_\mu A_\nu g^{\mu \nu} g^{\rho \sigma} g^{\lambda \tau} g^{\epsilon \delta} \left( A_\rho A_\sigma A_\lambda A_\tau A_\epsilon A_\delta - 3A_\rho A_\sigma A_\lambda A_\tau A_\epsilon A_\delta + 2A_\rho A_\sigma A_\lambda A_\tau A_\epsilon A_\delta \right),$$

$$\mathcal{L}_{vG|B}^{(5)} = -6\sqrt{-g}A_\mu A_\nu g^{\mu \rho} g^{\nu \sigma} g^{\lambda \tau} g^{\epsilon \delta} \left( A_\rho A_\sigma A_\lambda [\tau A_\epsilon]_\delta + A_\rho [\tau A_\epsilon]_\lambda A_\sigma A_\delta \right).$$

(3.25)

First note that only $A_{00} \supseteq \dot{A}_0$ and $\dot{N}$. Therefore we only need to consider the factors of this term. We find that the first term contributes,

$$\mathcal{L}_{vG|A}^{(5)} = 6\sqrt{-g}A_0^2 g^{00} A_{00} \left[ g^{0i} A_i g^{kl} - 2g^{ij} g^{0k} g^{0l} \right] (A_i j A_{kl}),$$

(3.26)

which is exactly cancelled by the contribution from the second term,

$$\mathcal{L}_{vG|B}^{(5)} = -6\sqrt{-g}A_0^2 g^{00} A_{00} \left[ g^{0i} A_i g^{kl} - 2g^{ij} g^{0k} g^{0l} \right] (A_i j A_{kl}).$$

(3.27)

We have focussed our attention on the covariantisation of quartic and quintic vector Galileons as these are the only terms that come with counter terms in the vector Horndeski system. We have found evidence that, as with the ‘beyond Horndeski’ theories discussed in chapter 4, the vector Galileons can be consistently covariantised by minimal substitution.

**The effect of switching on the spatial components of $A_\mu$**

So far we have investigated the cancellation of dangerous terms for a special ansatz, albeit with a general metric. We have seen that the antisymmetric property of the epsilon tensor guarantees a cancellation. In this subsection we investigate what happens to this cancellation once we remove the restriction of our ansatz and switch on the spatial components of $A_\mu$. We start by examining the structure of
the quartic Lagrangian,

\[
\mathcal{L} = \sqrt{-g} e^{\mu_1 \mu_3 \mu_5 \lambda} e^{v_2 v_4 v_6} \mathcal{A}_{\mu_1 \mathcal{A}_{v_2}} \mathcal{N}_v \mathcal{A}_{v_6}, \\
= \sqrt{-g} e^{\mu_1 \mu_3 \mu_5 \lambda} e^{v_2 v_4 v_6} \mathcal{A}_{\mu_1 \mathcal{A}_{v_2}} \left[ \partial_{(\mu_3} A_{v_4)} - 2 \partial_{(\mu_3} A_{v_4)} \Gamma^\lambda_{\mu_5 v_6 A_\lambda} \right] + \Gamma_{\mu_5 v_4}^\lambda \mathcal{A}_{v_2} A_{v_1}, \\
\geq \sqrt{-g} e^{\mu_1 \mu_3 \mu_5 \lambda} e^{v_2 v_4 v_6} \mathcal{A}_{\mu_1 \mathcal{A}_{v_2}} \left[ \Gamma_{\mu_5 v_4}^\lambda A_{v_2} - 2 \partial_{(\mu_3} A_{v_4)} \right] \Gamma^\lambda_{\mu_5 v_6 A_\lambda}. \quad (3.28)
\]

First notice that the first term is just the ordinary quartic vector Galileon with commuting derivatives and thus we need only concentrate on the second and third terms. Furthermore, since only \( \Gamma^\mu_{00} \) contribute \( N \) terms, if we choose at least one of either \( \mu_1 \) or \( v_2 \) to be zero, or if the zero components are shared between the two Christoffel symbols, then we do not recover any of the ‘dangerous’ terms we are focussing on.

Therefore, we need only consider,

\[
\mathcal{L} = \sqrt{-g} e^{\mu_1 \mu_3 \mu_5 \lambda} e^{v_2 v_4 v_6} \mathcal{A}_{\mu_1 \mathcal{A}_{v_2}} \mathcal{N}_v \mathcal{A}_{v_6}, \\
\geq 2 \sqrt{-g} e^{m_1 m_3 0 \lambda} e^{0 v_4 v_6} \mathcal{A}_{m_1 A_{m_3 v_4}} A_{0 v_6}, \\
= 2 \sqrt{-g} g^{00} \left( g^{m_1 m_2} g^{m_3 v_4} - g^{m_1 m_2} g^{v_4 v_6} \right) A_{m_1 A_{m_2 v_4}} A_{0 v_6}, \\
\geq - 2 \sqrt{g} \left( \gamma^{m_1 m_2} \gamma^{m_3 v_4} - \gamma^{m_1 m_2} \gamma^{v_4 v_6} \right) A_{m_1 A_{m_2 v_4}} A_{0 v_6} \left( \frac{1}{N} A_0 + \partial_0 \left[ \frac{1}{N} A_0 \right] \right). \quad (3.29)
\]

Since, \( A_{m n} = \partial_{(m} A_{n)} - \Gamma^0_{m n} A_0 - \Gamma^\lambda_{m n} A_\lambda \geq \frac{1}{N} (A_0 - A_i N^i) K_{m n} \), we find an obstruction to the cancellation due to terms of the form (ignoring the contributions from the shift, \( N_t \)),

\[
\mathcal{L} \geq \frac{\sqrt{g}}{N^2} A_{i} A_{j} \left( \gamma^{i j} K - K^{i j} \right), \quad (3.30)
\]

which remain after integration by parts.

We find a cancellation for the form of the non-minimal coupling inspired by Horndeski\(^\text{12}\), where using the results of appendix B.2.3 gives us,

\[
\sqrt{-g} A_{i} A_{j} G^{i j} \geq N \sqrt{g} \left( \frac{1}{N^2} \right) A_{i} A_{j} A_{l} A_{l} \left( \gamma^{i j} K - K^{i j} \right), \\
= - \frac{\sqrt{g}}{N^2} A_{i} A_{j} A_{l} A_{l} \left( \gamma^{i j} K - K^{i j} \right). \quad (3.31)
\]

\(^{12}\text{See also [43] for the generalisation for scalar fields to D dimensions.}\)
Allowing for vector fields to have non-zero spatial components prevents us from relying on the epsilon tensor to provide a cancellation and this reduces the applicability of our analysis for the vector Galileons to the choice of a special ansatz.

Something similar happens for the beyond Horndeski theories where the cancellation of the terms involving the derivative of the lapse in the action can be seen when the scalar field is used to select a preferred frame in which it depends only on time [79, 84, 80]. This is the reason why we employ a more robust formalism in chapter 4 where we construct geometrical quantities based on the constant scalar hypersurface of the longitudinal mode, $\nabla_\mu \phi$. Furthermore, note that the special case of the minimally covariantised quartic Galileon has been shown to possess the correct number of degrees of freedom in all frames [85]. Therefore, it would appear that this failure of cancellation in the action for general frames is simply a complication rather than a true pathology.

For the vector theory the cancellation was for a chosen ansatz, rather than a preferred frame, however both rely on the absence of spatial components which spoil the cancellation by increasing the exponent of the lapse relative to the exponent for either $\dot{\phi}$ for the scalars or $A_0$ for the vectors. Given that the non-linear structure for the vectors seems to impart the same behaviour as that for the scalars, it seems likely that our choice of ansatz could be reinterpreted as a gauge choice for a preferred foliation. This could possibly be achieved by fixing $A_\mu$ to be parallel to the unit normal vector $v_\mu$ defined in section 3.1. In chapter 4 we employ our more advanced formalism and identify the longitudinal mode with the scalar field in the corresponding scalar-tensor theory. This allows us to examine the types of theory for which this ‘unitary gauge’ analysis is applicable.

Another way to see the apparent resolution of the pathology for the minimally covariantised scalar Galileons, is to focus on their field equations. For example, minimally substituting covariant derivatives into the quartic Galileon leads to third order derivatives of the metric and of the field appearing in the equations of motion [86]. However, the Bianchi identities can be used to find a second order constraint equation that allows one to replace the higher order derivatives to recover a second
order system [85]. That the vector Galileons share the same special cancellations as their scalar counterparts could suggest that a similar constraint exists for these theories as well. Note, however, that the process to find the second order system is not covariant and therefore it is unclear whether or not, the resultant theory is consistent.

In summary, in this chapter we have considered the effect of covariantising out vector Galileon theory. We choose to work with a special ansatz which is analogous to the ‘unitary’ gauge and found that there are multiple ways to covariantise our theory. Along these lines, we have provided some circumstantial evidence for the existence of a ‘beyond Horndeski’ version of the vector Galileon theory. If it exists, such a theory corresponds to a vector-tensor system that, in analogy with the scalar counterpart of [79], is free of ghosts thanks to second class constraints that avoid the propagation of additional dangerous degrees of freedom. Note that a sextic order operator has recently been shown to exist [87, 88]. This relies on mixing the transversal and longitudinal modes and so does not exist for the scalar Galileons. See [62, 89] for exploration of this new operator and the minimally covariantised, ‘beyond generalised Proca’.

The inconclusive results of this chapter lead us to conduct a more thorough analysis in chapter 4 where we will identify the existence of a primary constraint for the related scalar-tensor theory\textsuperscript{13}.

\textsuperscript{13}This system corresponds to a restricted sector of our theory were we consider only the pure longitudinal mode-gravity interactions.
Chapter 4

Constraint Analysis For Generalised Horndeski Theories

The content of this chapter is based on my paper [90], ‘Horndeski: beyond or not beyond?’ , published in the Journal of Cosmology and Astroparticle Physics.

4.1 The extended Horndeski class.

In this chapter we continue our analysis by examining the mathematical consistency of the related scalar-tensor theories popularly named ‘Horndeski’ and ‘beyond Horndeski’. These theories can be re-interpreted, within our context, as a restricted sector of our covariant vector Galileon theories, where we only consider the interactions of the longitudinal mode with gravity.

The Horndeski system and its extension to ‘beyond Horndeski’ were discovered in the attempt to find the most general scalar-tensor theory of gravity. This question of generality has featured significantly in the field of cosmology due to its importance when building physical models of inflation and dark energy [91]. Theories in the Horndeski class also feature heavily in the effective actions derived from string theory and brane-world models. The question of generality continues to be re-addressed and has recently led to the discovery of theories outside the general
In this chapter we restrict our discussion to the extended Horndeski system. As was mentioned in chapter 1, these theories are the covariant version of the most general, Lorentz invariant, scalar field system with equations of motion with derivatives of up to second order. This system is defined on Minkowski and is coined as the generalised Galileon. Due to its generality, this theory necessarily includes all canonical scalar field theories as well as those with non-linear, derivative, self-interactions. These latter theories remain consistent due to their anti-symmetric structure preventing any higher order partial derivatives from appearing in their equations of motion, thus avoiding the Ostrogradsky ghost. Covariantising the partial derivatives in theories with non-canonical kinetic terms typically leads to equations of motion with higher order derivatives in time. As was shown in section 3.2.1, for the Horndeski/Galileon class of theories these can be removed, covariantly, via counter terms to form a second order system [86]. Alternatively, it is possible that a certain subset of these theories remain consistent without these counter terms. Such issues were first discussed by [94, 79, 80, 81].

Through following their lead but working in more generality, we will find that, under certain conditions, we can have a consistent theory. Furthermore, we find that this is due to a non-trivial degeneracy for the kinetic matrix. This leads to a general result that applies to all multi-field theories: The Ostrogradsky theorem can be evaded through the careful construction of a degenerate kinetic matrix [95, 96].

4.1.1 The Horndeski class

In section 3.2.2 we conducted our analysis after choosing a special ansatz for our vector field. This analysis resembles what was first done in [79, 80], where the authors constructed theories in the so-called unitary gauge. In this gauge the scalar field depends only on time, which allows it to be used as a clock. In this section we reformulate the scalar theory using geometrical quantities defined with respect to a constant scalar field hypersurface.
The scalar tensor theory of Horndeski is given by the action

\[
H_S = \int d^4x \sqrt{-g} \left( H\mathcal{L}_2 + H\mathcal{L}_3 + H\mathcal{L}_4 + H\mathcal{L}_5 \right),
\]

(4.1)

where the Lagrangian densities are

\[
H\mathcal{L}_2 = K(\phi, X),
\]

(4.2)

\[
H\mathcal{L}_3 = G_3(\phi, X)\Box \phi,
\]

(4.3)

\[
H\mathcal{L}_4 = G_4(\phi, X)R - 2G_4X(\phi, X) \left[ (\Box \phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu} \right],
\]

(4.4)

\[
H\mathcal{L}_5 = G_5(\phi, X)G^{\mu\nu}\phi_{\mu\nu} + \frac{1}{3}G_{5X}(\phi, X) \left[ (\Box \phi)^3 - 3 \Box \phi \phi_{\mu\nu}\phi^{\mu\nu} + 2\phi_{\mu\nu}\phi^{\nu\rho}\phi_{\rho}^{\mu} \right].
\]

(4.5)

\[
\phi_{\mu\nu} \equiv \nabla_\mu \nabla_\nu \phi \quad \text{and} \quad K, G_{3,4,5} \text{ are arbitrary functions of } \phi \text{ and } X, \text{ defined as}
\]

\[
X \equiv \partial_\mu \phi \partial^\mu \phi.
\]

(4.7)

This is the most general, Lorentz invariant, scalar-tensor theory of gravity, involving a single scalar field, that leads to covariant second order equations of motion. In this subsection we focus on the quartic \(\mathcal{L}_4\), and the quintic \(\mathcal{L}_5\) Horndeski theories, where the tensor spin-2 degrees of freedom have their own kinetic terms. General Relativity is recovered by setting \(K = G_3 = G_5 = 0\), and \(G_4 = M_{Pl}^2/2\).

In order to study the dynamics of theories with second order derivatives in the action, it is convenient to adopt a field redefinition, originally introduced in Ref. [97], which identifies the scalar field with the longitudinal mode from section 3.2.1.

We therefore define a four vector

\[
A_\mu \equiv \nabla_\mu \phi.
\]

(4.8)

Notice that since we are only dealing with the longitudinal mode, the covariant derivative \(\nabla_\mu A_\nu \equiv \nabla_\mu \nabla_\nu \phi = A_{\mu\nu}\) is necessarily symmetric and therefore we will switch our notation without any ambiguity.

\[\text{This is the natural covariant extension of the one used by Ostrogradsky in his pioneering work (see [64]).}\]
Using $A_\mu$, the quartic and quintic Horndeski Lagrangians are rewritten as

$$H^{4L} = G_4(\phi, X) R - 2G_4X(\phi, X) \delta_{\mu\nu}^{\alpha\beta} \nabla_\alpha A^\mu \nabla_\beta A^\nu, \quad (4.9)$$

$$H^{5L} = G_5(\phi, X) G^\mu_\alpha \nabla_\alpha A^\mu + \frac{1}{3} G_5X(\phi, X) \delta_{\mu\nu\rho}^{\alpha\beta\gamma} \nabla_\alpha A^\mu \nabla_\beta A^\nu \nabla_\gamma A^\rho, \quad (4.10)$$

where

$$X = A_\mu A^\mu = A^2. \quad (4.11)$$

### 4.1.2 Beyond Horndeski

‘Beyond Horndeski’ is the first example of a general class of extended scalar-tensor theories with the basic characteristic of having equations of motions of higher order together with constraints that remove additional, undesired degrees of freedom. It is described by the following action

$$BH S = \int d^4x \sqrt{-g} \left( BH^{4L} + BH^{5L} \right), \quad (4.12)$$

where

$$BH^{4L} = F_4(\phi, X) \left[ X \left( (\Box)^2 - \phi_\mu \phi^\mu \right) - 2 \left( \Box \phi_\mu \phi_\nu \phi^\nu - \phi_\mu \phi_\nu \phi^\nu \phi^\rho \phi^\rho \right) \right], \quad (4.13)$$

$$BH^{5L} = F_5(\phi, X) \left[ X \left( (\Box)^3 - \phi_\mu \phi_\nu \phi^\mu \phi^\nu + 2 \phi_\mu \phi_\nu \phi_\rho \phi^\rho \phi^\rho \right) \right.
\quad \left. - 3 \left( (\Box)^2 \phi_\mu \phi_\nu \phi_\rho \phi^\rho \phi^\sigma \phi^\sigma + 2 \phi_\mu \phi_\nu \phi_\rho \phi^\rho \phi^\rho \phi^\sigma \phi^\sigma \right) \right], \quad (4.14)$$

$\phi_\mu = \delta_\mu \phi$ and $F_4, F_5$ are arbitrary functions of $\phi, X$. These actions reduce to the generalised Galileonic actions of chapter 1 in an appropriate decoupling limit, when gravity is turned off. That is they recover exactly the same system as Horndeski. However, away from this limit we find equations of motion with up to three time derivatives on the metric. Furthermore, using diffeomorphism invariance and selecting the unitary gauge, allows one to find hidden constraints that remove the unstable degrees of freedom. This suggests, therefore, that the dynamics of the theory is genuinely different from the Horndeski class and that two extra operators together with arbitrary coefficient functions may be included to create an
extended Horndeski class. However, these results were found to depend on the use of the unitary gauge [90]. In an alternative choice of gauge where we recover spatial derivatives of the scalar field, we find the re-appearance of the unstable dynamics, indicating that a more thorough analysis is necessary.

The key to our analysis is to introduce a suitable geometrical formulation for these Lagrangians that greatly simplifies the calculation. The basic point of this interpretation can be found in [80], but in this case the choice of the unitary gauge obscures the more useful construction.

The approach we take is to introduce quantities according to the constant scalar field hypersurface, \( \phi = \text{const.} \) This hypersurface, which we denote by \( \mathcal{H}_\phi \), is characterised by the following geometrical quantities

\[
\frac{A^\mu}{\sqrt{-A^2}}, \quad P^\mu, \quad \Phi^\mu, \quad F^\mu^\nu^\alpha
\]

where \(-A^\mu/\sqrt{-A^2}\) is the unitary normal vector,

\[
P^\alpha = \delta^\alpha - \frac{1}{A^2}A^\mu A^\alpha
\]

the projection tensor on \( \mathcal{H}_\phi \), and

\[
\Phi^\nu = -P^\mu P^\nu A^\beta
\]

is the extrinsic curvature of \( \mathcal{H}_\phi \) multiplied by \( \sqrt{-A^2} \). Notice that, in principle, these objects have nothing to do with the ones associated with the space-time foliation described in section 3.1.

The beyond Horndeski Lagrangians can be expressed in terms of the extrinsic curvature \( \Phi^\mu_\alpha \) in the following way:

\[
_{BH} \mathcal{L}_4 = X F_4(\phi, X) \delta^{\alpha\beta}_\mu \Phi^\mu_\alpha \Phi^\nu_eta, \quad \quad (4.18)
\]

\[
_{BH} \mathcal{L}_5 = -X F_5(\phi, X) \delta^{\alpha\beta\gamma}_\mu \Phi^\mu_\alpha \Phi^\nu_\beta \Phi^\rho_\gamma. \quad \quad (4.19)
\]

Using the expression for the extrinsic curvature (equation (4.17)), this can be written as

\[
_{BH} \mathcal{L}_4 = X F_4(\phi, X) \mathcal{H}^{\alpha\beta}_\mu A^\mu \nabla^\alpha A^\nu, \quad \quad (4.20)
\]

\[
_{BH} \mathcal{L}_5 = X F_5(\phi, X) \mathcal{H}^{\alpha\beta\gamma}_\mu A^\mu \nabla^\alpha A^\nu \nabla^\gamma A^\rho, \quad \quad (4.21)
\]
where
\[ \mathcal{M}^\alpha_\mu = p^\alpha_{\mu [\nu} p^\beta_{\nu]}, \quad \mathcal{M}^{\mu \nu}_\rho = p^\alpha_{[\mu} p^\beta_{\nu] \rho}. \] (4.22)

Notice that the matrix \( \mathcal{M} \) has a crucial property for the arguments that we are going to develop:
\[ \mathcal{M}^{\alpha - \gamma}_\mu A_\alpha A^\mu = \mathcal{M}^{\alpha - \gamma}_\mu A_\gamma A^\mu = 0. \] (4.23)

This property stems from the fact that \( \mathcal{M} \) is built in terms of the projection tensor (equation (4.16)). As we will show in section 4.3, this leads to the existence of a primary constraint necessary to avoid the propagation of an extra mode in beyond Horndeski, despite the fact that the equations of motion have higher derivatives.

We stress that the unitary gauge used in the literature [80, 84] is a very special choice of gauge where the constant time hypersurface of section 3.1, \( \Sigma_t \), and the constant scalar field hypersurface, \( \mathcal{H}_\phi \), coincide. The analysis in this special gauge could lead to misleading results, as was explicitly pointed out first in [85] and then in [97, 98]. Moreover, this also calls into question results for the vector Galileons as our special ansatz in section 3.2.2 mirrored the action of choosing a unitary gauge. For these reasons we will refrain from making this choice in what comes next.

4.2 Kinetic terms

In order to identify the kinetic terms and carry out the analysis of constraints, we go back to the ADM formalism in section 3.1 and perform a 3+1 decomposition. Following [97], we decompose \( A_\mu \) into the normal and transverse components with respect to the \( \Sigma_t \) hypersurface:
\[ A_\mu = -A_\ast n_\mu + \hat{A}_\nu h^\nu_\mu. \] (4.24)

The expression for the covariant derivative of \( A_\mu \) can be decomposed into various pieces depending on the derivatives of its components and of the metric:
\[ \nabla_\mu A_\nu = D_\mu \hat{A}_\nu - A_\ast K_\mu \nu + n_{(\mu K_\nu)\rho} \hat{A}^\rho - n_{(\mu D_\nu)A_\ast} + n_\mu n_\nu \left( V_\ast - \hat{A}_\rho d^\rho \right), \] (4.25)

The notation \([...]\] denotes an un-weighted anti-symmetrised product i.e. \( A_{[\mu_1...\mu_n]} = \delta_{[\mu_1...\mu_n]} A_{\nu_1...\nu_n} \).
where $a^\mu = n^\nu \nabla_\nu n^\mu$ is the acceleration vector. In equation (4.25), as well as for the whole (beyond) Horndeski Lagrangian, time derivatives appear only for the three dimensional metric $h_{\mu\nu}$ (inside the extrinsic curvature) and for the component $A_\nu$ (inside what we called $V_\nu$). $V_\nu$ plays for $A_\nu$ the same role that $K_{\mu\nu}$ plays for $h_{\mu\nu}$, i.e.

$$V_\nu \equiv n^\mu \nabla_\mu A_\nu = \frac{1}{N} \left( \dot{A}_\nu - N^\mu D_\mu A_\nu \right).$$  \hspace{1cm} (4.26)

With this decomposition we find that it is more convenient to work directly with the extrinsic curvature and $V_\nu$, instead of the real velocities $\dot{h}_{\mu\nu}$ and $\dot{A}_\nu$. We identify the former terms as the kinetic contributions to the action\(^3\). This allows us to treat the fields in a decomposed space-time while still remaining in a covariant form.

It is important to remember, however, that $V_\nu$ contains the second time derivative of the scalar field, hence it represents a potentially dangerous contribution that could lead to the propagation of the Ostrogradsky mode.

A further advantage of this procedure is that, unlike in our analysis in sections 3.2.1 and 3.2.2 the Lagrangian densities do not depend explicitly on the lapse and shift functions. This is because such quantities are implicitly included in $K^\mu_\nu$ and $V_\nu$. We could potentially reconcile these two descriptions by identifying the second order time derivatives of the scalar field in $V_\nu$ with $NA_0$. This is a huge simplification that considerably reduces the number of fields involved in the calculation.

Performing a standard ADM canonical analysis would be very complicated, as was already shown in [85] for the case of quartic beyond Horndeski only. On the other hand, (beyond) Horndeski Lagrangians are diffeomorphism invariant, so intuitively we do not expect any modification to the algebra of constraints associated with the lapse and shift, as they are the generators of such a symmetry. However, this is not straightforward to show and a general proof is still missing. Steps forward in this direction have been made in [98], where for degenerate quart-

\(^3\)To avoid confusion, with kinetic terms we indicate contributions to the Lagrangian that contain time derivatives; while for quartic (beyond) Horndeski the kinetic terms are at most bilinear in the time derivatives of the fields, for quintic Horndeski they are at most trilinear.
tic Lagrangians, it was shown that the dimension of the physical phase space is reduced by the primary and secondary constraints.

For our purposes, it is sufficient to retain in the Lagrangian only the highest order terms in the extrinsic curvature, so we obtain the following expressions (we adopt the notation used in [97]):

\[ \mathcal{L}^{\text{kin}}_4 = 2 B_{\mu} K_\mu K_\nu + \mathcal{X}^{\alpha\beta} K_\mu K_\nu, \]
\[ \mathcal{L}^{\text{kin}}_5 = 3 B_{\mu\nu} K_\mu K_\nu + \mathcal{X}^{\alpha\beta\gamma} K_\mu K_\nu K_\rho. \]

In the following, for simplicity, we also assume that the functions \( G_4, G_5, F_4 \) and \( F_5 \) depend only on \( X \), and not on \( \phi \).

For Horndeski, we obtain for the quantities \( B \) and \( K \) the following expressions

\[ H^{B_{\mu}} = 0, \quad H^{B_{\mu\nu}} = 0, \]
\[ H^{K_{\mu\nu}} = -G_4 h^\alpha_{[\mu} h^\beta_{\nu]} + 2G_4 X \left( A^2 h^\alpha_{[\mu} h^\beta_{\nu]} - \hat{A}^2 \hat{\rho}_{[\mu} \hat{\rho}_{\nu]} \right), \]
\[ H^{K_{\mu\nu}\rho} = \frac{1}{3} G_5 X A_{\mu} \left( A^2 h^\alpha_{[\mu} h^\beta_{\nu]} h^\gamma_{\rho]} - \hat{A}^2 \hat{\rho}_{[\mu} \hat{\rho}_{\nu]} \hat{\rho}_{\rho]} \right), \]

where

\[ \hat{\rho}_{\mu} = \frac{1}{A^2} \hat{A}_{\mu} \hat{A}^\alpha \]

is the three-dimensional projection tensor defined in terms of \( \hat{A}_\mu \).

Note that there are no terms containing \( V_{\nu} \), since the \( B \) quantities of equation (4.29) vanish. This ensures that the equations of motion are second order and that the Horndeski Lagrangian only propagates at most three degrees of freedom. This is achieved by the special Horndeski tuning between the (non-minimal) coupling of gravity to the derivatives of the scalar field.

The relevant kinetic terms for beyond Horndeski are

\[ BH^{B_{\mu}} = F_4 A_{\alpha} \hat{A}^{2} \hat{\rho}_{\mu}^{\alpha}, \quad BH^{B_{\mu\nu}} = -F_3 A_{\alpha} \hat{A}^{2} \hat{\rho}_{[\mu} \hat{\rho}_{\nu]}^{\alpha}, \]
\[ BH^{K_{\mu\nu}} = -F_4 \left[ A^4 h^\alpha_{[\mu} h^\beta_{\nu]} - A^2 \hat{A}^2 \hat{\rho}_{[\mu} \hat{\rho}_{\nu]} + 2 \hat{A}^4 \hat{\rho}_{[\mu} \hat{\rho}_{\nu]} \hat{\rho}_{\mu} \hat{\rho}_{\nu} \right] \]
\[ -\hat{A}^{4} \left\{ \hat{\rho}_{[\mu} \hat{\rho}_{\nu]} + \hat{\rho}_{[\mu} \hat{\rho}_{\nu]} \right\}, \]
\[ BH^{K_{\mu\nu}\rho} = F_5 A_{\alpha} \left[ A^4 h^\alpha_{[\mu} h^\beta_{\nu]} h^\gamma_{\rho]} - A^2 \hat{A}^2 \hat{\rho}_{[\mu} \hat{\rho}_{\nu]} \hat{\rho}_{\rho]} \right. \]
\[ + \hat{A}^{4} \left\{ \hat{\rho}_{[\mu} \hat{\rho}_{\nu]} \left( \hat{\rho}_{\rho} - h^\gamma_{\rho} \right) + \text{sym.} \right\}, \]
where “sym.” in equation (4.37) stands for the symmetric permutation of doublets of vertical indices (e.g. the last term in equation (4.35)). Since the mixing term only appears in beyond Horndeski theory, we will mostly omit the superscript “BH” from $\mathcal{P}_\mu^\alpha$ and $\mathcal{P}_{\mu\nu}^{\alpha\beta}$.

For beyond Horndeski the quantities in equation (4.33) do not vanish, therefore there are potentially dangerous mixings between $V_\nu$ (containing the second derivative of the scalar field) and $K_{\mu}^\nu$ (containing the first derivative of the spatial metric). Such contributions to the action lead to higher order equations of motion. This suggests, but not necessarily implies, the presence of additional propagating degrees of freedom. We now study the existence of a primary constraint that could prevent the propagation of an additional (ghost) mode.

### 4.3 Primary constraints

The natural tool for counting the number of degrees of freedom is the (Dirac) canonical analysis of constraints. Here we do not go as far as making a complete analysis but instead concentrate on studying the existence of primary constraints. For the covariant scalar-tensor theories that we are considering, we expect to find three dynamical degrees of freedom due to the number of helicity eigenvalues: two tensor modes and one scalar mode. This is due to the fact that, whilst in General Relativity we have a symmetric metric and therefore we find 20 phase space variables, 16 of these are removed by 8 first class constraints leaving us with four phase space variables and therefore 2 dynamical degrees of freedom. Trivially including an additional scalar field just adds one degree of freedom.

Since in our non-trivial scalar-tensor theories we find healthy dynamics despite having third order derivatives of both the metric and the scalar field, we also expect to find additional constraints removing an Ostrogradsky mode. Indeed, in the case for the quartic Lagrangian, a complete analysis has been recently performed in [98], confirming that a secondary constraint does indeed exist.

---

4See [99] and [100] for a review.
Primary constraints exist when, passing over to the Hamiltonian formalism, all the velocities cannot be expressed in terms of the fields and their conjugate momenta. This translates to relations (constraints) between the fields and momenta that need to be added to the canonical Hamiltonian through Lagrangian multipliers. This is in contrast to secondary constraints which are found from the time evolution of the primary constraints. Furthermore, it is possible to define the non-primary constraints into a sequence where, for example, the time evolution of the secondary constraints give us tertiary constraints and so on. Each constraint removes either one or two degrees of freedom, depending on whether they are of the first or second class respectively. To be first class, a constraint needs to commute with all the other constraints, otherwise it is second class. Furthermore, if a constraint is first class, then it’s commutation with the other constraints is strongly (as opposed to ‘weakly’\(^5\)) equal to a linear function of those constraints.

For example, in General Relativity the metric is a symmetric tensor which can be represented as a symmetric matrix or order four. Therefore we have 10 variables, which in turn, provides 20 phase space variables. We also find four primary constraints, represented as,

\[ \pi_\mu \approx 0 \quad (4.38) \]

The time evolution of these constraints provides us with four more constraints,

\[ \{H, \pi_\mu\} \approx 0 \quad (4.39) \]

were \(H\) is the Hamiltonian of the system and \(\{A, B\}\) denotes the ‘Poisson Bracket’ of \(A\) with \(B\). These constraints are all first class and thus remove 16 phase space variables. The four remaining phase space variables translate into two physical degrees of freedom.

We therefore see that the existence of a primary constraint is not enough to remove physical degrees of freedom, nevertheless we see that this would be the first necessary condition for it. Moreover, to our knowledge, there are no known Lorentz invariant theories that propagate half degrees of freedom: this would be the case

---

\(^5\)Weak equality, denoted by “\(\approx\)” means equality on the phase space determined by the constraints.
if there are an odd number of second class constraints.

As explained in section 4.2, instead of working with the true velocities, we work with closely related quantities and therefore define the conjugate momenta accordingly:

\[
\pi_s = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta V_s}, \quad \pi^\alpha_\mu = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta K^\mu_\alpha}.
\]  

(4.40)

Notice that this definition also differs from the usual one due to the presence of the factor \(1/\sqrt{-g}\); this helps to completely remove the lapse from the relations.

In Horndeski theory, the primary constraint needed to remove the Ostrogradsky ghost is automatically enforced since, by construction, there are no terms containing \(V^\mu\). The \(\mathcal{B}\) quantities of equation (4.29) vanish and we simply get \(\pi_s \approx 0\).

### 4.4 Beyond Horndeski

The case for beyond Horndeski is more involved. Using the expression for the theories given in equations (4.20) and (4.21), together with the expression of \(V^\alpha A^\mu\) given in equation (4.25), the conjugate momenta are obtained as

\[
\pi_s = 2XF_\alpha \mathcal{M}^{\alpha \beta}_{\mu \nu} n_\alpha n^\mu A^\nu_{\beta} + 3XF_5 \mathcal{M}^{\alpha \beta \gamma}_{\mu \nu \rho} n_\alpha n^\mu A^\nu_{\beta} A^\rho_{\gamma},
\]

(4.41)

\[
\hat{A}^\alpha \hat{A}^\beta \pi^\alpha_\mu = 2XF_\alpha \mathcal{M}^{\alpha \beta}_{\mu \nu} \left(-A_s \hat{A}^\alpha A^\beta + \hat{A}^2 n_\alpha \hat{A}^\beta\right) A^\nu_{\beta} + 3XF_5 \mathcal{M}^{\alpha \beta \gamma}_{\mu \nu \rho} \left(-A_s \hat{A}^\alpha A^\beta + \hat{A}^2 n_\alpha \hat{A}^\beta\right) A^\nu_{\beta} A^\rho_{\gamma}.
\]

(4.42)

Using the properties of the matrix \(\mathcal{M}\)

\[
\mathcal{M}^{\alpha \beta}_{\mu \nu} n_\alpha A^\mu = \mathcal{M}^{\alpha \beta}_{\mu \nu} \hat{A}^\alpha A^\mu = 0,
\]

(4.43)

which stem from the fact that \(\mathcal{M}\) is constructed from the projection tensor \(P^\alpha_{\mu}\) and \(A^\mu = -A_s n^\mu + \hat{A}^\mu\), we can derive the following identities

\[
\mathcal{M}^{\alpha \beta}_{\mu \nu} \hat{A}^\alpha A^\mu = A_s^2 \mathcal{M}^{\alpha \beta}_{\mu \nu} n_\alpha n^\mu, \quad \mathcal{M}^{\alpha \beta}_{\mu \nu} n_\alpha \hat{A}^\mu = A_s \mathcal{M}^{\alpha \beta}_{\mu \nu} n_\alpha n^\mu.
\]

(4.44)

Hence, we find a primary constraint by constructing a linear combination of the conjugate momenta \(\pi_s\) and \(\pi^\alpha_\mu\) and is of the form

\[
A_s \left(2\hat{A}^2 - A_s^2\right) \pi_s - \hat{A}^\mu \hat{A}_\alpha \pi^\alpha_\mu \approx 0.
\]

(4.45)
With the formulation of beyond Horndeski theories given in section 4.1.2, it is therefore very easy to show the existence of the primary constraint necessary to remove the Ostrogradsky ghost. It is important to notice that since the constraint (equation (4.45)) is a linear combination of the conjugate momenta $\pi_v$ and $\pi^\mu_{\alpha}$, it is possible to remove $V_v$ by a suitable field redefinition and the system can be recast as a second order one. However there is no guarantee that the new system will be Lorentz invariant.

In order to demonstrate how the primary constraint is constructed from a linear combination of the conjugate momenta (equations (4.41) and (4.42)) we focus only on the highest order terms in the extrinsic curvature as given in equations (4.27) and (4.28). The conjugate momenta simplify to

$$
\pi_v = 2 \mathcal{B}^\alpha_{\mu} K^\mu_{\alpha} + 3 \mathcal{B}^\alpha_{\mu} K^\mu_{\alpha} K^\nu_{\beta}, \quad (4.46)
$$

$$
\pi^\mu_{\alpha} = \left( 2 \mathcal{B}^\alpha_{\mu} + 6 \mathcal{B}^\alpha_{\mu} K^\nu_{\beta} \right) V_v + 2 \mathcal{K}^\alpha_{\mu v} K^\nu_{\beta} + 3 \mathcal{K}^\alpha_{\mu v} K^\nu_{\beta} K^\rho_{\gamma}, \quad (4.47)
$$

From equations (4.46) and (4.47) it becomes easy to see that the key property of these momenta is that $V_v$ appears only in $\pi^\alpha_{\mu}$. This implies that to build a constraint we need to eliminate $V_v$ by taking a suitable linear combination of components of $\pi^\alpha_{\mu}$. Using the properties of the three dimensional projection tensor, i.e.

$$
\rho^\mu_{\alpha} A^\alpha_{\mu} = 0, \quad h^\alpha_{\mu v} h^\beta_{\nu|\rho} A^\alpha_{\mu} = \hat{A}^2 \rho^\beta_{\nu}, \quad h^\alpha_{\mu v} h^\beta_{\nu|\rho} \hat{A}^\alpha_{\mu} = \hat{A}^2 \rho^\beta_{\nu |\rho}, \quad (4.48)
$$

we can show the following relations for $\mathcal{B}$ and $\mathcal{K}$ in beyond Horndeski theories

$$
\mathcal{B}^\mu_{\alpha} \hat{A}^\alpha_{\mu} = \mathcal{B}^\mu_{\alpha} \hat{A}^\alpha_{\mu} = 0, \quad (4.49)
$$

$$
BH \mathcal{K}^\alpha_{\mu v} \hat{A}^\alpha_{\mu} = A_\ast (2 \hat{A}^2 - A_\ast^2) \mathcal{B}^\beta_{\nu}, \quad (4.50)
$$

$$
BH \mathcal{K}^\alpha_{\mu v} \hat{A}^\alpha_{\mu} = A_\ast (2 \hat{A}^2 - A_\ast^2) \mathcal{B}^\beta_{\nu |\rho}, \quad (4.51)
$$

These results imply that we can eliminate $V_v$ by contracting $\pi^\mu_{\alpha}$ with $\hat{A}^\alpha_{\mu}$

$$
\hat{A}^\mu \hat{A}^\alpha \pi^\mu_{\alpha} = A_\ast (2 \hat{A}^2 - A_\ast^2) \left( 2 \mathcal{B}^\mu_{\alpha} K^\mu_{\alpha} + 3 \mathcal{B}^\mu_{\alpha} K^\mu_{\alpha} K^\nu_{\beta} \right). \quad (4.52)
$$

Then, it is straightforward to verify the primary constraint (see equation (4.46)). To conclude, let us give the redefinition of the extrinsic curvature that eliminates
the cross term between $V_\ast$ and the extrinsic curvature,

$$K^\mu_\alpha = \bar{K}^\mu_\alpha - \frac{V_\ast}{A_\ast(2\bar{A}^2 - A_\ast^2)} \dot{A}_\alpha \dot{A}^\mu. \quad (4.53)$$

In this chapter we have presented arguments that strongly suggest that the extended Horndeski system is free from the Ostrogradsky instability. We demonstrated that reformulating the theory in terms of certain geometrical objects defined on the constant scalar field hypersurface, $\mathcal{H}_\phi$, allows us to simply read off the primary constraint. Furthermore, this formalism proves to be highly effective even for the more complicated beyond Horndeski theory. We then showed how a primary constraint for this class of theory can be constructed from a linear combination of the conjugate momenta. Ultimately, understanding the structure of the constraint led us to the unique field re-definition of the extrinsic curvature that results in an second order theory.
Chapter 5

Covariantisation Of The Galileonic Higgs

5.1 Covariantisation of the Galileonic Higgs system

Vector Galileons are a self derivative extension of the Proca action and therefore explicitly break any gauge symmetry associated with the kinetic term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. Up until now we have focussed on massive vectors associated with an explicitly broken abelian gauge invariance. Here we will discuss an extension to the usual complex scalar Higgs mechanism for spontaneously generating these terms. Although interesting for its relevance to possible applications to particle physics and superconductivity, the existence of such a mechanism is necessary if we wish to treat theories with a non-abelian gauge symmetry as the Higgs mechanism is known to cure issues with unitarity and raise the strong coupling scale up to the Planck mass. Another motivation comes from the cosmological phenomenology of the non-minimally covariantised vector Galileons. It has been shown that although interesting cosmological applications can be found, there is the possibility that they might suffer from strong coupling issues around non-trivial backgrounds [59, 82]. It is possible that the additional Galileon scalar inherited from the Higgs dynamics might alleviate these strong coupling issues and enhance the cosmo-

\footnote{See [101] for an excellent discussion.}
logical applicability of these models. Moreover, possible connections with the general scenario of Higgs inflation (see e.g. [102] for a recent review) could also be developed.

In chapter 2 we showed how to extend the Higgs mechanism with a Galileonic symmetry to generate the vector-Galileons spontaneously. Interestingly, we found that this Galileonic Higgs theory recovers a bi-Galileon system in its decoupling limit which, given the existence of non-renormalisation theorems [39, 48], could further improve the phenomenological attractiveness of this set of theories.

5.1.1 The effect on gauge invariance.

In this section we consider the effects of covariantising the Galileonic Higgs system on general space-times. Since we have constructed the system using the same non-linear structure as that of the Galileons, it would appear that after minimal substitution with covariant derivatives the terms up to cubic order remain ghost free whereas the terms of quartic and higher order could introduce ghosts and therefore need careful consideration.

It has been proven in [85] that the pure scalar sector of $\mathcal{L}_{(12)}$, which corresponds to a quartic Galileon, can be consistently covariantised via minimal substitution. Furthermore, in section 3.2 and chapter 4 we found evidence that this is also a consistent way to write a covariant theory for the vector Galileons and hence the pure vector sector of $\mathcal{L}_{(12)}$. We are able to reproduce these sectors if we re-write the above expressions in terms of new covariantised operators constructed from replacing partial derivatives with covariant derivatives:

\[ \tilde{L}_{\mu} := L_{\mu}|_{\partial \rightarrow \nabla} \equiv L_{\mu}, \]
\[ \tilde{P}_{\mu} := P_{\mu}|_{\partial \rightarrow \nabla} \quad \text{and} \quad \tilde{Q}_{\mu} := Q_{\mu}|_{\partial \rightarrow \nabla}. \]

However, it is not immediately clear whether such a process interferes with the $U(1)$ gauge invariance of the operators. Indeed, for terms with only one partial derivative or an undifferentiated gauge vector, there will be no change and therefore as the notation suggests, there is no problem with the $L$ operator. However, the two remaining operators depend upon $\phi^* \mathcal{D}_\mu \mathcal{D}_\nu \phi$
and therefore we should check the effect of covariantising the partial derivatives,

\[ \phi^* \mathcal{D}_\mu \mathcal{D}_\nu \phi = \phi^* \nabla_\mu \partial_\nu \phi - i \phi^* \phi \nabla_\mu A_\nu - i \phi^* A_\nu \partial_\mu \phi - i \phi^* A_\mu \partial_\nu \phi - A_\mu A_\nu \phi^* \phi. \]  

(5.1)

Although we find two additional terms due to the covariantisation of the partial derivatives, this does not spoil the gauge invariance as for \( \phi \to \phi e^{i \xi} \) and \( A_\mu \to A_\mu + \partial_\mu \xi \) we find,

\[ -\Gamma^\lambda_{\mu \nu} \left\{ \partial_\lambda \phi - i A_\lambda \phi \right\} \to -e^{i \xi} \Gamma^\lambda_{\mu \nu} \left\{ \partial_\lambda \phi - i A_\lambda \phi \right\}, \]  

(5.2)

where the multiplicative factor of \( e^{i \xi} \) is cancelled by the contribution coming from \( \phi^* \to e^{-i \xi} \phi^* \).

We have established that generalising our operators for curved spacetimes does not interfere with their \( U(1) \) gauge invariance. However, it is not yet clear how we should approach the mixing terms that would also be generated as in this case we lose the utility of choosing a special ansatz. This is also true for the decoupling limit of the theory, which is a system of bi-Galileons, whose minimal covariantisation, if its consistency were to be established, would resemble a multi-field generalisation of the beyond Horndeski theory.

In the next section, we investigate the guaranteed way to remove the problematic terms involving the derivative of the lapse by introducing non-minimal couplings.

### 5.1.2 \( U(1) \) invariant non-minimal couplings

In this section we investigate the form of the possible non-minimal couplings we could add to the theory. These should be compatible with \( U(1) \) invariance and valid in all frames.

In section 3.2 we presented a consistent non-minimal covariantisation for vector Galileons. This result together with the Horndeski system [36] provides a consistent non-minimal covariantisation for both the pure scalar and vector sectors of the Galileonic Higgs. However, the consistency and effectiveness of the non-minimal counter terms for the mixed scalar vector sector still needs to be addressed. Subsequently, we comment on the correspondence of the decoupling limit of this theory.
to the multi-field generalisation of the Horndeski system proposed by [103].
For inspiration we start with analysing the scalar sector and focus on the non-minimal covariantisation of $\mathcal{L}_a^{(12)}$.

**Non-minimal coupling for $\mathcal{L}_{(12)}$**

First we set $g = 1$, define $\varphi \dot{A}_\mu := A_\mu$ and expand out the terms from equations (2.24), (2.25) and (2.26).

Notice that the antisymmetry of the epsilon tensors guarantees that we cannot have more than two vector fields. Therefore we can write $\mathcal{L}_{(12)}$ as,

$$\mathcal{L}_{(12)} \equiv \sqrt{-g} \frac{\alpha_{(12)}}{\Lambda^8} \epsilon^{\mu_1 \mu_2 \mu_3 \lambda} \epsilon_{v_2 v_4 v_6} \lambda \left\{ \left( \varphi_{\mu_1} \varphi_{v_2} + A_{\mu_1} A_{v_2} \right) \varphi^2 \varphi_{\mu_3} v_4 \varphi_{\mu_5} v_6 \right\} \left( A \right)$$

$$- \varphi \varphi_{\mu_1} \varphi_{v_2} \left( A_{\mu_3} A_{v_3} \varphi_{\mu_5} v_6 + \varphi_{\mu_5} v_4 A_{\mu_3} A_{v_6} \right) \right\} \left( B \right).$$

(5.3)

The contribution to the equation of motion for $\varphi$ from term (B) does not produce any higher order derivatives on the metric, however the contribution from term (A) does. In order to find a consistent non-minimal covariantisation we follow the method demonstrated by [43] and find a term that mixes the derivatives of the scalar with the curvature tensor. This type of non-minimal coupling to gravity, called *kinetic gravity braiding*, introduces interesting cosmological phenomenology to our model. However, we must also guarantee that our action remains U(1) invariant. We should therefore gauge covariantise the derivatives of the scalar and thus introduce a non-minimal coupling between the vectors and the curvature tensor.

By taking a variation with respect to the scalar field, a higher derivative term is derived from (A) and can be expressed as,

$$\sqrt{-g} \frac{\alpha_{(12)}}{2 \Lambda^8} \epsilon^{\mu_1 \mu_3 \mu_5 \lambda} \epsilon_{v_2 v_4 v_6} \lambda \delta \varphi \varphi^2 \left( \varphi_{\mu_1} \varphi_{v_2} + A_{\mu_1} A_{v_2} \right) \nabla^\lambda \varphi R_{v_4 v_6 \mu_3 \mu_5 \lambda}.$$

(5.4)
In order to remove this we are required to have the following additional term in the Lagrangian,
\[ \sqrt{-g} \frac{\alpha_{(12)}}{4\Lambda^8} \epsilon^{\mu_1 \mu_3 \nu_5 \lambda} \epsilon \varepsilon^v \varepsilon^w v_6 \varepsilon \lambda \phi^2 \nabla^2 \phi \nabla_{[\lambda} \phi (\phi_{\mu_1} \phi_{\nu_2} + A_{\mu_1} A_{\nu_2}) R_{v_4 v_6} \mu_3 \mu_5, \] (5.5)
which cures both the pure scalar sector and the scalar vector cross terms. However this is not U(1) invariant. We can make this U(1) invariant by ‘gauge covariantising’ the covariant derivatives: \( \nabla_{[\lambda} \phi \rightarrow D_{[\lambda} \phi = \nabla_{[\lambda} \phi - i A_{[\lambda} \phi \). We can then write this in terms of the operators in our theory as,
\[ \mathcal{L}^{\alpha NMC}_{(12)} = \sqrt{-g} \frac{\alpha_{(12)}}{4\Lambda^8} \epsilon^{\mu_1 \mu_3 \nu_5 \lambda} \epsilon \varepsilon^v \varepsilon^w v_6 \varepsilon \lambda \phi^* \phi LL_{\mu_1 v_2} R_{\mu_3 v_4 v_6} \phi. \] (5.6)

In order to get a better intuitive picture about this term we expand out the sum,
\[ \epsilon^{\mu_1 \mu_3 \nu_5 \lambda} \epsilon \varepsilon^v \varepsilon^w v_6 \varepsilon \lambda L_{\mu_1 v_2} R_{\mu_3 v_4 v_6} = LR - LR_{\rho \sigma}^{\rho \sigma} \]
\[ + L_{\mu \nu} \left(R_{\rho \mu}^{\rho \nu} - R_{\rho \nu}^{\rho \mu} + R_{\rho \mu}^{\sigma \nu} - R_{\rho \nu}^{\sigma \mu} \right), \]
\[ = 2LR - 4L_{\mu \nu} R_{\mu \nu}, \]
\[ = 4L_{\mu \nu} \left(\frac{1}{2} g_{\mu \nu} R - R_{\mu \nu} \right), \]
\[ = -4L_{\mu \nu} G_{\mu \nu}. \] (5.7)

We can now write our non-minimal coupling term as,
\[ \mathcal{L}^{\alpha NMC}_{(12)} = -\sqrt{-g} \frac{\alpha_{(12)}}{\Lambda^8} \phi^* \phi LL_{\mu \nu} G_{\mu \nu}. \] (5.8)

Notice that we now have extra cross terms coming from the need to make our counter term U(1) gauge invariant. Although this might seem at first to be pathological as it introduces new higher derivative terms, we will see contrary to this, that a solution can be found in the form of a unique theory.

We now go through the same process to find the counter term for the vector sector. Again we concentrate on the quartic and expand the term factored by \( \beta_{(12)} \) which contains the quartic vector Galileon plus mixed scalar and vector terms,
\[ \mathcal{L}_{(12)} \supset \sqrt{-g} \frac{\beta_{(12)}}{\Lambda^8} \epsilon^{\mu_1 \mu_3 \nu_5 \lambda} \epsilon \varepsilon^v \varepsilon^w v_6 \varepsilon \lambda (\phi_{\mu_1} \phi_{\nu_2} + A_{\mu_1} A_{\nu_2}) \phi^2 A_{\mu_3} A_{\nu_4} A_{\mu_5} A_{\mu_6}. \] (5.9)
We know from section 3.2.2 that for this particular form of the vector sector we require a counter term of the form,

\[ \sim -\sqrt{-g}A^2 (\partial_\mu \varphi \partial_\nu \varphi + A_\mu A_\nu) G^{\mu \nu}, \]  

(5.10)

which cures both the pure vector sector and the scalar vector cross terms. To ensure our counter term is \( U(1) \) invariant, we form the trace of the gauge invariant operator, \( \text{tr} L_{\mu \nu} = L \), out of the \( A^2 \) factor. Thus we end up with the same form of counter term as we found for the scalar sector,

\[ \mathcal{L}^{\text{B NMC}}_{(12)} = -\sqrt{-g} \frac{\beta_{(12)}}{\Lambda^8} \phi^* \phi L L_{\mu \nu} G^{\mu \nu}. \]  

(5.11)

Here we conclude that the combination of both covariantisation and \( U(1) \) gauge invariance has ensured that we recover the same form for the non-minimal coupling for both branches of the quartic Galileonic Higgs.

**Cross terms**

We have found that \( U(1) \) gauge invariance requires the counter terms constructed for both the scalar and vector sectors of the quartic to be identical. In order to be satisfied that this is the correct choice of counter term we must also check whether we recover the right form for the mixed vector scalar terms. We begin by expanding out the gauge invariant operators in \( \mathcal{L}^{\alpha}_{(12)} + \mathcal{L}^{\beta}_{(12)} + \mathcal{L}^{\text{B NMC}}_{(12)} \) and examining the cross terms,

\[ \mathcal{L}^{\alpha}_{(12)} + \mathcal{L}^{\beta}_{(12)} + \mathcal{L}^{\text{B NMC}}_{(12)} \]

\[ \approx \sqrt{-g} \frac{\alpha_{(12)}}{\Lambda^8} \epsilon^{\mu_1 \mu_2 \mu_3 \mu_5} \epsilon_{\nu_1 \nu_2 \nu_4 \nu_6} (\varphi_{\mu_1} \varphi_{\nu_2} + A_{\mu_1} A_{\nu_2}) \varphi^2 \varphi_{\mu_3} \varphi_{\nu_5} \]

\[ + \sqrt{-g} \frac{\beta_{(12)}}{\Lambda^8} \epsilon^{\mu_1 \mu_2 \mu_3 \mu_5} \epsilon_{\nu_1 \nu_2 \nu_4 \nu_6} (\varphi_{\mu_1} \varphi_{\nu_2} + A_{\mu_1} A_{\nu_2}) \varphi^2 A_{\mu_3} A_{\nu_5} \]

\[ - \sqrt{-g} \frac{\alpha_{(12)}}{\Lambda^8} \phi^* \phi (\varphi_{\lambda} \varphi^\lambda + A_{\lambda} A^\lambda) (\varphi_{\mu} \varphi_{\nu} + A_{\mu} A_{\nu}) G^{\mu \nu}. \]  

(5.12)

The non-minimal coupling factored by the term labelled \( I \) was necessary for the consistency of \( \mathcal{L}^{\alpha}_{(12)} \) but \( U(1) \) gauge invariance forced us to include the term...
factored by $II$. This extra term, however, turns out to be exactly the form of non-minimal coupling necessary for the consistency of $\mathcal{L}^\beta_{(12)}$. This suggests that in order to generate the correct combination of terms for the pure vector and scalar sectors as well as the mixing terms we simply need to factorise each term with the appropriate dimensionless parameter. On the other hand, these parameters cannot be independent as this would be inconsistent with $U(1)$ gauge invariance. Therefore a consistent non-minimal covariantisation of the quartic Galileonic Higgs is with $\alpha_{(12)} = \beta_{(12)} = \gamma_{(12)}$.

$$\mathcal{L}_{(12)} = \mathcal{L}^\alpha_{(12)}|_{\alpha=\gamma} + \mathcal{L}^\beta_{(12)}|_{\beta=\gamma} + \mathcal{L}^{\gamma_{NMC}}_{(12)}, \quad (5.13)$$

where $\mathcal{L}^{\gamma_{NMC}}_{(12)}$ is that of equation (5.8) with $\alpha_{(12)}$ substituted with a new dimensionless parameter, $\gamma_{(12)}$.

In the process of constructing a non-minimal covariantisation of the quartic Galileonic Higgs, we have found that, although around Minkowski we can have two separate sectors of the theory parametrised by $\alpha_{(12)}$ and $\beta_{(12)}$, on generally curved space-times consistency with $U(1)$ gauge invariance requires them to be equal. Furthermore, we find that the form of the unique counter term simultaneously compatible with both generally curved space-times and gauge invariance is closely related to that suggested for the generalised multi-field quartic by [103]. Indeed, in an appropriate decoupling limit, we find that this counter term would exactly resemble that for the covariantised quartic bi-Galileon. This is consistent with the covariantisation of the decoupling limit of our theory as around Minkowski space we find in such a limit that the Galileonic Higgs reduces to a bi-Galileon system.

In this chapter we have discussed how we can consistently introduce gravitational interactions to the Galileonic Higgs system we developed in chapter 2. We have found that we can construct consistent theories, via the use of non-minimal counter terms, up to the quartic level. Moreover, the requirement that any counter term be $U(1)$ gauge covariant, fixed the couplings of the separate purely scalar and vector sectors to be equal.

Finally, we mention that there appears to be an obstruction for the quintic case where the counter term necessary for one sector, when made $U(1)$ gauge covari-
ant, reintroduces pathologies in the other sector\textsuperscript{2}.

In chapter 6 we consider the cosmological applications of these theories and analyse the stability of their perturbations around viable backgrounds.

\textsuperscript{2}Appendix D contains a brief discussion on one of these terms.
Chapter 6

Cosmological Applications

In chapter 2 we reviewed the mechanism presented in [65] that generates vector Galileons via spontaneous symmetry breaking. Since we are interested in cosmological applications, it is important to ascertain whether our theory remains covariant after including gravitational interactions. We discussed these issues, first presented in [77], in chapter 5.1. In the case where, like above, we only consider operators that lead to at most the cubic vector system, we covariantise the system by simply applying minimal substitution. That is, we replace each partial derivative with a covariant one.

In this chapter we will discuss the cosmological applications of the covariant Galileonic Higgs. The results of related theories indicate that this system is likely to have a rich cosmological phenomenology. For example, in chapter 2 we saw that around Minkowski space, the Galileonic Higgs can be related to a non-linear system of two scalars characterised by bi-Galileon interactions. Bi-Galileon systems possess examples of ‘self accelerating models that are simultaneously free from ghosts, tachyons and tadpoles, able to pass solar system constraints through Vainshtein screening…’ [69].

This ‘decoupling limit’ example hints that our model has the potential to have interesting cosmological applications. Moreover, focussing separately, on the purely scalar and vector sectors of the Galileonic Higgs, we find further examples of models with ghost free self-acceleration and Vainshtein screening.
We start in section 6.1 by discussing the general cosmological perturbation theory necessary for investigating the phenomenology of our model. We then, in section 6.2, discuss the properties of models related to the decoupling limit of the Galileonic Higgs by first briefly introducing an interesting cosmological application for the cubic Galileon and then presenting in more depth the cosmological phenomenology of the cubic vector Galileon with a tachyonic mass. In section 6.3 we solve a specific example of the Galileonic Higgs system and discuss the stability of the system. Finally in section 6.4 we discuss possible extensions to our model that could lead to a more phenomenologically viable model.

### 6.1 Perturbations around FLRW

In order to study the phenomenology of the Galileonic Higgs and related models, we must expand the action up to at least second order in perturbations. For example, to solve for the background, we must vary the first order action with respect to the perturbations. Having solved for a background, we then can find the equations of motion for the dynamical fields by substituting the solutions into the second order action and varying with respect to the perturbations. However, before embarking on this exercise for our system, we first define the different kinds of fields acting on our space-time.

#### 6.1.1 Scalar and vector perturbations

In a general space-time we classify quantities by the way they transform under diffeomorphisms, (general coordinate transformations). Let $x^\mu \rightarrow \tilde{x}^\mu = x^\mu + \xi^\mu$ be an infinitesimal coordinate transformation, where $\xi^\mu = \xi^\mu (x)$ is an infinitesimal vector fluctuation. Then, we also have $\tilde{x}^\mu = \tilde{x}^\mu - \xi^\mu$. Since scalars quantities are defined to be those with a form which does not transform under a change of coordinates, to first order we have, (by Taylor expanding),

$$\tilde{\phi}(\tilde{x}) = \phi(x) \quad \Rightarrow \quad \tilde{\phi}(x) = \phi(x) - \xi^\mu \partial_\mu \phi(x).$$

\[ (6.1) \]
For a perturbation of the form, \( \tilde{\phi}(\tilde{x}) = (0)\phi(\tilde{x}) + \delta \tilde{\phi} \), we have,

\[
\delta \tilde{\phi} = \delta \phi - \xi^\sigma \partial_\sigma (0)\phi. \tag{6.2}
\]

Vectors transform with one derivative which gives us,

\[
\tilde{A}_\mu(\tilde{x}) = A_\alpha(x) \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} = A_\alpha(x) \left( \delta_\mu^\alpha - \partial_\mu \xi^\alpha \right) = A_\alpha(x) - A_\alpha(x) \frac{\partial \xi^\alpha}{\partial \tilde{x}^\mu}. \tag{6.3}
\]

Using, the first order Taylor expansion, \( \tilde{A}_\mu(\tilde{x}) = \tilde{A}_\mu(x) + \xi^\sigma \partial_\sigma \tilde{A}_\mu(x) \), we have that the infinitesimal transformation for the vector is given by,

\[
\delta \tilde{A}_\mu = \delta A_\mu - (0)A_\alpha \partial_\mu \xi^\alpha - \xi^\sigma \partial_\sigma (0)A_\mu \tag{6.4}
\]

where \((0)A_\mu\) is the background quantity.

We select a homogeneous and isotropic ansatz for our background,

\[
(0)A_\mu(t, \tilde{x}) = (\tilde{A}_0(t), 0) \tag{6.6}
\]

and write our perturbations as,

\[
\delta A_\mu = (\pi, \partial_\mu \chi_V). \tag{6.7}
\]

This reduces the number of non-zero components in equation (6.5) and simplifies our expression to,

\[
\delta \tilde{A}_0 = \pi - \partial_t [\tilde{A}_0(t) \xi^0] \tag{6.8}
\]

\[
\delta \tilde{A}_i = \partial_i \chi_V - \tilde{A}_0(t) \partial_i \xi^0 \tag{6.9}
\]

### 6.1.2 Metric perturbations

Under the same infinitesimal coordinate transformation, \( x^\mu = \tilde{x}^\mu - \xi^\mu \), rank two tensors transform with two derivatives giving us,

\[
\tilde{g}_{\alpha\beta}(\tilde{x}) = g_{\mu\nu}(x) \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial x^\nu}{\partial \tilde{x}^\beta} \approx g_{\mu\nu}(x) \left( \delta^\mu_\alpha - \partial_\alpha \xi^\mu \right) \left( \delta^\nu_\beta - \partial_\beta \xi^\nu \right) = g_{\alpha\beta}(x) - g_{\mu\beta}(x) \partial_\alpha \xi^\mu - g_{\alpha\nu}(x) \partial_\beta \xi^\nu \tag{6.10}
\]
Also, we have that \( \tilde{g}_{\alpha\beta}(x) = \tilde{g}_{\alpha\beta}(x) + \xi^\rho \partial_\rho \tilde{g}_{\alpha\beta}(x) \) which gives us,

\[
\tilde{g}_{\alpha\beta}(x) = g_{\alpha\beta}(x) - g_{\mu\beta}(x) \partial_\alpha \xi^\mu - g_{\alpha\nu}(x) \partial_\beta \xi^\nu - \xi^\lambda \partial_\lambda \tilde{g}_{\alpha\beta}(x)
\]  \hspace{1cm} (6.11)

For a perturbation of the form, \( \tilde{g}_{\alpha\beta}(x) = \,^{(0)}g_{\alpha\beta}(x) + \delta \tilde{g}_{\alpha\beta} \) we find,

\[
\delta \tilde{g}_{\alpha\beta} = \delta g_{\alpha\beta} - \xi^\lambda \partial_\lambda \,^{(0)}g_{\alpha\beta} - \,^{(0)}g_{\lambda\beta} \partial_\alpha \xi^\lambda - \,^{(0)}g_{\alpha\lambda} \partial_\beta \xi^\lambda
\]  \hspace{1cm} (6.12)

where \( \,^{(0)}g_{\alpha\beta}(x) \) denotes the background quantity.

We assume that the background is spatially homogeneous and isotropic so that at the linear level, we can classify the perturbations of the metric via their behaviour under spatial rotations.

\[
g_{\mu\nu} = \begin{bmatrix}
-1 & 0 \\
0 & a^2 \delta_{ij}
\end{bmatrix}
\]  \hspace{1cm} (6.13)

The perturbations are classified into tensors of rank 2, rank 1 (vectors) and rank 0 (scalars). We further (Helmholtz) decompose these objects into transversal and longitudinal modes.

We write the most general linear metric perturbation in terms of these modes as,

\[
ds^2 = \left\{ - (1 + \epsilon N)^2 dt^2 + 2\epsilon \left( \frac{1}{2} \partial_i \psi - \chi_i \right) dr dx^i \\
+ a^2 [(1 + 2\epsilon \zeta) \delta_{ij} + \epsilon \gamma_{ij} + 2\epsilon E_{ij} + 2\epsilon F_{i,j}] dx^i dx^j \right\}
\]  \hspace{1cm} (6.14)

where,

- **Rank 0:** \( N, \psi, \zeta \) and \( E \).

- **Rank 1:** \( \chi_i \) and \( F_i \), where \( \partial_i \chi_i = 0 \) and \( \partial_i F_i = 0 \).

- **Rank 2:** \( \gamma_{ij} \), where \( \gamma_{ii} = 0 \) and \( \partial_i \gamma_{ij} = 0 \).

### 6.1.3 Gauge choices and scalar perturbations

We have written expressions for the perturbations of each kind of field but since we are working on a dynamical space-time we can now simplify our expressions by choosing a gauge. This is analogous to choosing a special foliation of
the space-time in accordance with the metric decomposition detailed in equation (6.14). This special foliation effectively removes the chosen matter perturbation by encoding it in terms of a metric perturbation of the same type. Note that without any other fields, we should find only two dynamical degrees of freedom for the metric perturbations. These correspond to the helicity states of the graviton. Therefore when we have other non-gravitational degrees of freedom propagating in the space-time, the large degeneracy of non-physical degrees of freedom in this formalism allows us to re-express those degrees of freedom in terms of a metric perturbation and any remaining non-physical degrees of freedom are removed as constraints.

In the case when we have theories with their own gauge symmetries, we can also choose a gauge to reduce the complexity of the calculation. For example, the Galileonic Higgs has a $U(1)$ gauge symmetry and therefore we chose to remove some of the non-physical degrees of freedom by making a gauge choice. The Higgs field is complex scalar and we can write it in polar form as (see chapter 2),

$$\Phi(x) = \rho(x)e^{i\theta(x)}$$

(6.15)

In section 6.3 we chose to work in the unitary gauge which in our present notation is given by $\theta = 0$. This is useful because we wish to calculate the quadratic action for the scalar perturbations and this choice of gauge minimises the number of scalars involved in the calculation.

Given our choice of gauge for the Higgs, we decompose the perturbations of the gauge field and the scalar field as,

- Rank 0: $\rho$ and $\chi_V$.
- Rank 1: $V_i$ where $\partial_i V_i = 0$.

In sections 6.2 and 6.3, we calculate the quadratic action for the scalar perturbations of the vector Galileon and the Galileonic Higgs respectively. In order to do this, we find that it is convenient to work in a gauge where,

$$E = 0 \quad \text{and} \quad \chi_V = 0.$$
6.2 Decoupling limit cosmology

6.2.1 The KGB model

The first example we discuss features quite often in the cosmology literature and is sometimes referred to as the ‘KGB model’ for Kinetic-Gravity-Braiding [104, 105, 106]. The action is formed of the Einstein-Hilbert action plus a scalar field action related to the cubic Galileon. In particular, it is a subset of Horndeski formed out of a linear combination of operators of up to cubic order. Note that discussions of related set-ups can be found in [107, 108, 109, 110]. The KGB action is given by,

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{16\pi G} R + K(\phi, X) + G(\phi, X) \times B \right]$$

(6.17)

where $X = -\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi$ and $B = g^{\mu\nu} \nabla_\mu \nabla_\nu \phi \equiv \Box \phi$.

The expanding background solutions found for $KGB$ [104] and the original $G$-inflation [105] both rely on choosing the ‘wrong sign’ for the scalar field. That is, for example, where the scalar field is ghostly on the Minkowski background ($X = 0$)[104].

We follow the example of [104] and choose $K$ and $G$ for the action given in equation (6.17), to be,

$$K = -X$$

(6.18)

$$G = \alpha X$$

(6.19)

where $\alpha$ is a constant with inverse mass dimension to the third power.

Note that this action is symmetric under constant shifts of $\phi \rightarrow \phi + c$. Therefore there is an associated Noether current,

$$J_\mu = (B - 1) \nabla_\mu \phi - \nabla_\mu X$$

(6.20)

such that the scalar equation of motion equates with $\nabla_\mu J^\mu = 0$. In a homogeneous and isotropic background we have,

$$\dot{J} + 3HJ = 0$$

(6.21)
giving that the shift charge in a comoving volume is constant. This equation integrates to give us,

\[ J = \dot{\phi}(3H\dot{\phi} - 1) = \frac{\text{const}}{a(t)^3}. \]  

(6.22)

Therefore, an expanding universe, \( J \) tends to zero. The de Sitter solution is an attractor \([104]\) with a background profile of,

\[ \dot{\phi} = -\frac{1}{3H} \]  

(6.23)

Note that we will find a similar scenario for the cubic vector Galileon with a tachyonic mass.

The choice of \( K \) implies that the dynamics for the scalar could be unstable. However, for this background, the gravitational interactions contribute a counter term which ‘cures’ the overall sign and we recover a healthy theory.

### 6.2.2 The tachyonic vector Galileon

Our second example follows on from the first by introducing apparent tachyonic mass for the non-trivial vector interactions. Theories of this type were first considered in \([111]\). In the following, we restrict our attention to the cubic vector Galileon. The Lagrangian for the vector is given by,

\[
\mathcal{L}_A = \sqrt{-g} \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m_A^2 A^\mu A^\mu - \beta A^\mu A^\nu \nabla_\nu A^\mu \right)
\]  

(6.24)

where \( F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \), \( \beta \) is a dimensionless constant and \( m_A \) is the mass of the vector.

On Minkowski, the longitudinal mode in the decoupling limit of this theory recovers the previous example’s ghostly dynamics. However, away from the decoupling limit we see that there are subtle differences between the covariant vector Galileon and the previous scalar-tensor system.

#### Background solutions

We wish to understand the phenomenology of our theory in the context of cosmology. Therefore, we choose a metric which is consistent with a statistically
homogeneous and isotropic background. This is the FLRW metric from chapter 1 given by equation (eq:FLRWmetric). Furthermore, for the purpose of simplifying the calculation, we make the additional assumption that the Hubble parameter, $H$ is a positive constant such that we are on a de Sitter background where $a(t) = e^{Ht}$. In order to be consistent with an isotropic background we use the ansatz for the vector from section 6.1; $\mathbf{A}_\mu = (A_0(t), \vec{0})$. We work in the gauge from equation (6.16). Expanding the action to first order and varying with respect to the remaining scalar perturbations, gives us three background equations.

Solving for $A_0(t)$, we recover a non-trivial field profile that on de Sitter is constant and denoted by $\tilde{A}_0$. With this, the background equations of motion reduce to,

\begin{align}
0 &= -\tilde{A}_0(m_A^2 + 6\tilde{A}_0 H \dot{\beta}) \\
0 &= \tilde{A}_0^2 m_A^2 - 6H^2 M_{Pl}^2 \\
0 &= \tilde{A}_0^2 m_A^2 + 6H^2 M_{Pl}^2 + 12\tilde{A}_0^3 H \dot{\beta}
\end{align}

where $H$ is the Hubble parameter. Notice that the first equation shows that we have both a trivial $\tilde{A}_0 = 0$ and a non-trivial background solution for $\tilde{A}_0 = -m_A^2/6H\dot{\beta}$. Furthermore, notice that the tachyonic signature for the mass terms allows us to recover sensible solutions. Indeed, substituting the solution for the $\tilde{A}_0$ profile into the remaining two equations leaves us with one equation,

$$6H^2 M_{Pl}^2 - \frac{m_A^6}{36H^2 \dot{\beta}^2} = 0$$

We find two solutions: one contracting and one expanding. This confirms that our de Sitter background was a consistent ansatz. Choosing the expanding branch we have,

$$\tilde{A}_0 = -\frac{m_A^2}{6H\dot{\beta}}$$

$$H = \frac{m^{3/2}}{6^{3/4}\sqrt{M_{Pl}\dot{\beta}}}$$

Therefore we have found an accelerating solution for the cubic vector Galileon. Note that due to the tachyonic signature for the mass, this theory is unstable.
on Minkowski. However, the de Sitter solution is an attractor [111]. Furthermore, for the theory to have self-accelerating solutions whilst also being stable on Minkowski, we can include the higher order operators studied in chapter 3 [111].

**Scalar perturbations**

In order to study the character of the propagating scalar degrees of freedom, we expand the action to second order in $\varepsilon$ and again vary with respect to $\zeta$, $\pi$, $N$ and $\psi$. Three out of the four Euler equations are actually constraint equations for $N$, $\pi$ and $\psi$, which leaves us with one propagating degree of freedom: $\zeta(t,\vec{x})$. The equations of motion for $N$, $\pi$, $\psi$ and $\zeta$, respectively are given by,

\[
\begin{align*}
\epsilon H_t (3HM \pi N + 3M_\pi \pi) - 2M_\pi \partial^2 \zeta &= 0 \\
-\sqrt{6} \partial^2 \pi + 2M_\pi \partial^2 \psi + \epsilon H_t m_\pi (6HM \pi N + 3M_\pi \pi - M_\pi \dot{\zeta}) &= 0 \\
\sqrt{6} m_\pi \partial^2 \pi + 6M_\pi \partial^2 \zeta &= 0 \\
-2M_\pi \left( 2(\partial^2 N + \partial^2 \zeta + \partial^2 \psi) + H \partial^2 \psi \right) \right. \\
&\left. + \epsilon H_t \left( 3\sqrt{6}H m_\pi \pi + 2\sqrt{6}m_\pi \dot{\pi} + 9HM_\pi \dot{\zeta} + 6M_\pi \dot{\zeta} \right) \right) &= 0
\end{align*}
\]  

We plug in the background solutions and solve the constraints,

\[
\begin{align*}
\pi &= -\frac{\sqrt{6}M_\pi \dot{\zeta}}{m_\pi} \\
\psi &= -\frac{2\zeta}{H} \frac{3\dot{\zeta}}{m_\pi^2} \\
N &= \frac{2}{H} \dot{\zeta} + \frac{2e^{-Ht} \partial^2 \zeta}{3H^2}
\end{align*}
\]  

With these expressions we can now re-write the quadratic scalar action in terms of just one field, $\zeta(t,\vec{x})$. After a few integrations by parts we find that the quadratic action can be written,

\[
\mathcal{L}_{\zeta^2} = \frac{3\epsilon H t M_\pi^2}{m_\pi^2} k^2 \dot{\zeta}^2 - \frac{e^{-Ht} M_\pi^2}{6H^2} k^4 \zeta^2
\]  

where the dimensionless field, $\tilde{\zeta}(t,\vec{k})$, is the fourier transform of $\zeta(t,\vec{x})$. Although written in a non-canonical way due to the dependence on $k^2$, we find...
that we have a massless scalar which propagates with the correct sign such that
we have neither a ghost, nor a gradient instability. Moreover, note that whereas
the bare mass for the vector is tachyonic, its effective mass around de Sitter is in
fact positive. This is an important feature of the non-linear vector theory; for a
dynamical metric, the additional gravitational interactions drive up the effective
mass of the vector such that we recover healthy dynamics for the transversal and
longitudinal modes.

We have discussed the cosmological applications of, what can be considered as,
two special limiting cases of the Galileonic Higgs model. That we find, healthy,
self-accelerating solutions for both examples only increases the potential attract-
tiveness of our model. We therefore discuss in section 6.3 our attempt at replicat-
ing these results with the Galileonic Higgs.

### 6.3 Ghost Galileonic Higgs

We wish to find non-trivial cosmological solutions with positive expansion. For
the covariant cubic vector, we found that this requires inputting a bare tachyonic
mass for the vector (see equation (6.24)). This allows us to find, for example, a
de Sitter solution. Interestingly, however, when considering the perturbations of
the system around this background de Sitter solution, we recover a non-tachyonic
mass for the propagating mode. That is, what would be a tachyonic instability
around Minkowski without gravity, turns out to be perfectly stable on de Sitter.

In this section we aim to generate the interesting cosmological properties of this
theory spontaneously via the covariant Galileonic Higgs mechanism. In chapter
2 we discussed how the mass for the vector is generated through its interactions
with the Higgs. These interactions are contained in the gauge covariantised kinetic
term for the Higgs. Expanding the kinetic term gives,

\[
\pm (\mathcal{D}_\mu \Phi)(\mathcal{D}^\mu \Phi) = \pm (\partial_\mu \phi - iq\phi A_\mu)(\partial^\mu \phi^* + iq\phi^* A^\mu),
\]

(6.39)

\[
= \pm \partial_\mu \phi \partial^\mu \phi^* \pm q^2 \phi \phi^* A_\mu A^\mu,
\]

(6.40)

\[
\rightarrow \pm q^2 v^2 A_\mu A^\mu,
\]

(6.41)
where $v$ is the vacuum expectation value of the Higgs. Therefore we see that choosing a ghost signature for the kinetic term of the Higgs enables us to generate a tachyonic mass for the the gauge field via the Higgs acquiring a vacuum expectation value. The action for the model we work with is given by,

$$
\int d^4x \sqrt{-g} \left[ \frac{1}{2} (M^2_{PL} R - 2\Lambda) - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} + (\mathcal{D}_\mu \Phi)(\mathcal{D}^\mu \Phi)^\dagger + \mathcal{L}_{(8)} + V(\Phi) \right]
$$

where $\mathcal{L}_{(8)}$ is discussed in chapter 2 and is given by equation (2.20). Although we have introduced a ghost signature for the Higgs, the hope is that once gravitational interactions are taken into account, the effective signature will be non-ghostly. The following analysis makes use of the general perturbative analysis for modified gravity presented in [112].

### 6.3.1 Background solutions and constraint equations

#### Background solutions

Since we have chosen to perturb around an FLRW universe, we again choose a homogeneous and isotropic ansatz for the gauge field and put the Higgs, which is in the unitary gauge, in its vev:

$$
A_\mu(t, \vec{x}) = (\tilde{A}_0(t) + \varepsilon \pi(t, \vec{x}), \vec{0}) \text{ and }
$$

$$
\Phi(t, \vec{x}) = \langle \rho \rangle + \varepsilon \rho(t, \vec{x}),
$$

where $\langle \rho \rangle = \mu / \sqrt{\lambda} = v$ is the vev of the Higgs and quantities factored by $\varepsilon > 0$ are perturbations. Furthermore, to simplify the discussion, we again choose $H$ to be constant so that our background is de Sitter and work in the gauge given by equation (6.16). We expand the action to first order in $\varepsilon$ and find the background equations of motion by varying with respect to the remaining scalar degrees of freedom, $N, \zeta, \psi, \pi$ and $\rho$. The equation for $\psi$ does not contribute at this level.
The remaining equations are,

\[
-3H^2M_{\text{Pl}}^2 - q^2\langle \rho \rangle^2 \tilde{A}_0(t)^2 + 9Hq^3\beta \langle \rho \rangle^4 \tilde{A}_0(t)^3 = 0 \tag{6.45}
\]

\[
-3H^2M_{\text{Pl}}^2 + q^2\langle \rho \rangle^2 \tilde{A}_0(t)^2(1 + 3q\beta \langle \rho \rangle^2 \dot{\tilde{A}}_0(t)) = 0 \tag{6.46}
\]

\[
\tilde{A}_0(t)(-2 + 9Hq\beta \langle \rho \rangle^2 \dot{\tilde{A}}_0(t)) = 0 \tag{6.47}
\]

\[
\langle \rho \rangle \tilde{A}_0(t)^2 \left( 2 + 9H^2\alpha \langle \rho \rangle^2 - 12Hq\beta \langle \rho \rangle^2 \tilde{A}_0(t) + 6H\alpha \langle \rho \rangle^2 \dot{\tilde{A}}_0(t) \right) = 0 \tag{6.48}
\]

Solving equation (6.47) we find a non-trivial profile for \( \tilde{A}_0(t) \) that is dependent on \( H \),

\[
\tilde{A}_0 = \frac{2}{9Hq\beta \langle \rho \rangle^2} \tag{6.49}
\]

Therefore, for constant \( H \) we find that \( \tilde{A}_0 \) must also be a constant and our system reduces to,

\[
9Hq^3\beta \langle \rho \rangle^4 \tilde{A}_0^3 = 3H^2M_{\text{Pl}}^2 + q^2\langle \rho \rangle^2 \tilde{A}_0^2 \tag{6.50}
\]

\[
q^2\langle \rho \rangle^2 \tilde{A}_0^2 = 3H^2M_{\text{Pl}}^2 \tag{6.51}
\]

\[
2 + 9H^2\alpha \langle \rho \rangle^2 = 12Hq\beta \langle \rho \rangle^2 \tilde{A}_0 \tag{6.52}
\]

We solve for solutions consistent with the de Sitter background and find,

\[
\tilde{A}_0 = \frac{2}{9Hq\beta \langle \rho \rangle^2} \tag{6.53}
\]

\[
H = \frac{\sqrt{2/3}}{3\sqrt{\alpha \langle \rho \rangle}} \tag{6.54}
\]

\[
\beta = \frac{\sqrt{3\alpha \langle \rho \rangle}}{M_{\text{Pl}}} \tag{6.55}
\]

\[
\langle \rho \rangle = \frac{\mu}{\sqrt{\lambda}} \tag{6.56}
\]

where \( \beta = \beta_{(8)}/\Lambda^4 \) and \( \alpha = \alpha_{(8)}/\Lambda^4 \), with \( \alpha_{(8)} \) and \( \beta_{(8)} \) being dimensionless parameters. Note that when solving for \( \langle \rho \rangle \) we encounter a complicated algebraic expression with multiple solutions. Here we choose the solution with the simplest expression. Moreover, we should add that the choice of background expanded around can affect the overall stability of the perturbations.
Constraint equations

We set \( \alpha = 2\lambda/27H^2\mu^2 = (2/27) \cdot 1/(H\langle \rho \rangle)^2 \) and expand the action to second order in \( \epsilon \). We then vary with respect to the scalar fields to derive the equations of motion. We find three constraint equations leaving us with two propagating scalars. Substituting in the background solutions will allow us to solve the constraints in terms of these remaining variables, \( \zeta \) and \( \rho \).

\[
-54H^2\sqrt{\lambda}\mu M_P N + 81H^2\lambda M_P \rho + 18\sqrt{3}Hq\mu^2\pi \\
+ 9e^{-2Ht}M_P\sqrt{\lambda}\mu \partial^2z + e^{-2Ht}M_P\lambda \partial^2\rho - 9H\lambda M_P \dot{\rho} = 0 \quad (6.57)
\]

\[
-\frac{e^{-2Ht}}{6Hq} (27\lambda H \partial^2\pi + 2\sqrt{3}\lambda M_P q(9H \mu \partial^2 \psi + 2\sqrt{\lambda} \partial^2\rho)) \\
+ 9q\mu^2\pi + \sqrt{3}\lambda M_P (9\mu \dot{\zeta} - 18H\mu N + 2\sqrt{\lambda}(9H \rho + \dot{\rho})) = 0 \quad (6.58)
\]

\[
3\sqrt{3}q^2\mu^2 \partial^2\pi + 9HM_P\lambda \partial^2\rho - 9M_P\sqrt{\lambda}\mu \partial^2\dot{\zeta} + M_P\lambda \partial^2\dot{\rho} = 0 \quad (6.59)
\]

Solutions

We find the following solutions:

\[
\pi = -\frac{M_P}{3\sqrt{3}q\mu^2} \left( 9H\lambda \rho - 9\sqrt{\lambda}\mu \dot{\zeta} + \lambda \dot{\rho} \right) \quad (6.60)
\]

\[
\psi = \frac{3}{Hq^2\mu^3} \left[ \sqrt{\lambda}(9H^2\lambda - \frac{2}{3}q^2\mu^2)\rho - 6q^2\mu^3\zeta + H(\lambda^{3/2} \rho - 9\lambda \mu \dot{z}) \right] \quad (6.61)
\]

\[
N = \frac{1}{54H^2\mu e^{2Ht}} \left( 9\mu \partial^2\dot{\zeta} - \sqrt{\lambda} \partial^2 \rho + 3e^{2Ht}H[18\mu \dot{\zeta} + \sqrt{\lambda}(9H \rho + \dot{\rho})] \right) \quad (6.62)
\]

6.3.2 Quadratic action

Now that we have solved the constraints in terms of the remaining two propagating scalar fields, we substitute the solutions into the action at \( O(\epsilon^2) \) to recover the quadratic action for our perturbations. We then extract the time derivatives of the kinetic terms and diagonalise.
Substitute the solutions for $\pi$, $\psi$ and $N$.

After substituting the above solutions to the constraint equations into the action, we find after many integrations by parts that we have a Lagrangian with kinetic mixing. For the purpose of illustration, before diagonalising, we decompose the Lagrangian via the way each term depends on the fields and list them with increasing dependence on the squared momentum, $k^2$. Note that we have taken the Fourier transform but neglect complicating the notation further by choosing, for example, $\zeta(t)$ to represent the time component of $\tilde{\zeta}(t, \vec{k})$.

\[
\mathcal{L}_{e^2}[\zeta, \rho] = \mathcal{L}_{e^2}[\zeta] + \mathcal{L}_{e^2}[\rho] + \mathcal{L}_{e^2}[\zeta \rho],
\]

Where,

- $O(k^0)$:
  \[
  \mathcal{L}_{e^2}[\rho] = -e^{3Ht} \frac{(18\mu^2 + 7M_{Pl}^2\lambda)}{54\mu^2} \dot{\rho}^2 + e^{3Ht} \frac{(2\mu^4 - H^2M_{Pl}^2\lambda)}{\mu^2} \rho^2
  \]
  - Tachyonic ghost
  - Mass term

- $O(\geq k^2)$:
  \[
  \mathcal{L}_{e^2}[\zeta] = \frac{3e^{2Ht}k^2M_{Pl}^2\lambda}{2q^2\mu^2} \dot{\zeta}^2 - e^{-Ht}k^4M_{Pl}^2\lambda \zeta^2
  + e^{Ht}k^4(M_{Pl}^2(216\mu^2 + 59q^2\mu^2)) \rho^2
  \]
  - Kinetic mixing

We find that we have terms dependent strictly on $\rho(t)$ that are or $O(0)$ in $\vec{k}$ and with a tachyonic ghost signature. Furthermore, we see that $\rho(t)$ propagates with a mass whereas $\zeta(t)$ is massless.
Construction of matrices

We have a system of two fields that we aim to diagonalise. For this purpose, we first re-write our system in terms of matrices. For example, the time components of our original kinetic term can be written as, $\dot{\mathcal{Z}}^T K_t \dot{\mathcal{Z}}$ where our vector is given by, $\mathcal{Z} := \{ \zeta, \rho \}$ and the notation $T$ in $\mathcal{Z}^T$ is used to represent its transpose.

Formally, we write our action as,

$$L_{\phi^2} = a(t)^3 \left\{ \mathcal{Z}^T K_t \mathcal{Z} + \frac{k^2}{a(t)^2} \mathcal{Z}^T G \mathcal{Z} - \mathcal{Z}^T M \mathcal{Z} - \mathcal{Z}^T \mathcal{B} \mathcal{Z} \right\}$$  \hspace{1cm} (6.64)

We now use this formal expression as a framework to construct our matrices. The entries for the kinetic matrix $K_t$, are formed of the time-time components of the above kinetic terms. We ensure we construct a symmetric matrix such that we able to diagonalise our system,

$$K_t = \begin{bmatrix} \frac{3e^{-2Hl(k^2)M_{\phi^2}\lambda}}{2q^2\mu^2} & \frac{-e^{-2Hl(k^2)M_{\phi^2}\lambda^{3/2}}}{6q^2\mu^3} \\ \frac{-e^{-2Hl(k^2)M_{\phi^2}\lambda^{3/2}}}{6q^2\mu^3} & \frac{e^{-2Hl(k^2)M_{\phi^2}\lambda^2-q^2\mu^2(7M_{\phi^2}\lambda+18\mu^2)}}{54q^2\mu^3} \end{bmatrix}$$  \hspace{1cm} (6.65)

We also construct the other matrices in the same way. The mass matrix, $M$, is built out of terms that are not dependent on $k^2/a^2$:

$$M = \begin{bmatrix} 0 & 0 \\ 0 & \frac{(H^2M_{\phi^2}\lambda-2\mu^4)}{\mu^4} \end{bmatrix}$$  \hspace{1cm} (6.66)

The $k^2$ dependent factors of $\zeta^2$, $\rho^2$ and $\zeta \rho$ form the entries to the $(k^2/a^2)G$ matrix,

$$\mathcal{G} = \begin{bmatrix} -\frac{e^{-2Hl(k^2)M_{\phi^2}}}{(2H^2+e^{-2Hl(k^2)M_{\phi^2}/\lambda})} & \frac{(24H^2+e^{-2Hl(k^2)M_{\phi^2}/\lambda})M_{\phi^2}/\lambda}{54H^2\mu} \\ \frac{e^{-2Hl(k^2)M_{\phi^2}q^2\lambda\mu^2+3H^2(216H^2M_{\phi^2}\lambda^2+q^2\mu^2)(59M_{\phi^2}\lambda+90\mu^2)}}{486H^2q^2\mu^4} & 0 \end{bmatrix}$$  \hspace{1cm} (6.67)

and the $\mathcal{B}$ matrix is formed of remaining terms which are factors of $\dot{\zeta} \rho$:

$$\mathcal{B} = \begin{bmatrix} 0 \\ \frac{e^{-2Hl(k^2)M_{\phi^2}/\lambda(27H^2\lambda-q^2\mu^2)}}{9H^2q^2\mu^4} \end{bmatrix}$$  \hspace{1cm} (6.68)

We will see that after using a field redefinition to diagonalise the kinetic matrix, the $\mathcal{G}$ matrix will receive an extra contribution coming from the terms that originally formed this $\mathcal{B}$ matrix.
6.3.3 Diagonalisation

Since our system forms a symmetric order two matrix, it is particularly simple to diagonalise via a field redefinition found by completing the square.

First we define the factors \( \{c_1, c_2, c_3\} \) via \( c_1 \dot{\zeta}^2 + c_2 \dot{\zeta} \dot{\rho} + c_3 \rho^2 \) such that our kinetic matrix is given by,

\[
K_t = \begin{bmatrix} c_1 & \frac{1}{2} c_2 \\ \frac{1}{2} c_2 & c_3 \end{bmatrix}
\]  
(6.69)

By examining our Lagrangian in equation (6.63) we find

\[
c_1 = \frac{3e^{-2Ht}k^2M_{Pl}^2 \lambda}{2q^2 \mu^2}
\]  
(6.70)

\[
c_2 = -\frac{e^{-2Ht}k^2M_{Pl}^2 \lambda^{3/2}}{3q^2 \mu^3}
\]  
(6.71)

\[
c_3 = \frac{e^{-2Ht}k^2M_{Pl}^2 \lambda^2}{54q^2 \mu^4} - \frac{q^2 \mu^2(7M_{Pl}^2 \lambda + 18 \mu^2)}{54q^2 \mu^4}
\]  
(6.72)

We now complete the square by

\[
c_1(\dot{\zeta} + c_2 \dot{\rho} / 2c_1)^2 + (c_3 - c_2^2 / 4c_1) \rho^2
\]  
(6.73)

Therefore we define a new field,

\[
\sigma(t) := \zeta(t) + \left( \frac{c_2}{2c_1} \right) \rho(t) = \zeta(t) - \frac{\sqrt{\lambda}}{9\mu} \rho(t)
\]  
(6.74)

This field redefinition can be written as a translation in field space given by,

\[
P = \begin{bmatrix} 1 & c_2 / 2c_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{\sqrt{\lambda}}{9\mu} \\ 0 & 1 \end{bmatrix}
\]  
(6.75)

We define the new vector, \( \Sigma \equiv P^T \mathcal{P} = (\sigma(t), \rho(t))^T \) and find that the diagonalised kinetic action is given by,

\[
(P^T P)^T \tilde{K}_t (P^T P) = \frac{3e^{-2Ht}k^2M_{Pl}^2 \lambda}{2q^2 \mu^2} \sigma(t)^2 - \frac{(7M_{Pl}^2 \lambda + 18 \mu^2)}{54q^2 \mu^4} \dot{\rho}(t)^2
\]  
(6.76)

Therefore the diagonalised kinetic matrix is given by,

\[
\tilde{K}_t = \begin{bmatrix} \frac{3e^{-2Ht}k^2M_{Pl}^2 \lambda}{2q^2 \mu^2} & 0 \\ 0 & -\frac{(7M_{Pl}^2 \lambda + 18 \mu^2)}{54q^2 \mu^4} \end{bmatrix}
\]  
(6.77)
From these expressions, we see that the kinetic term for the $\rho(t)$ field has the wrong sign, indicating that we have a ghost. Therefore, we conclude that the ghost we put in by hand is, in this case, not cured by the gravitational interactions. Notice, however, that the longitudinal mode associated with the effective mass of the vector boson has acquired healthy dynamics from the gravitational interactions. Therefore, despite retaining unstable dynamics for the Higgs, we have successfully recovered the phenomenology of the tachyonic cubic vector Galileon discussed in section 6.2.2.

**Masses**

Since the $\zeta$ field is massless, the mass matrix, $\mathcal{M}$, is already diagonal. Furthermore, since the $\rho(t)$ field is not transformed via the field redefinition we find,

$$
\mathcal{M} = M = \begin{pmatrix} 0 & 0 \\
0 & \frac{(H^2M^2_P\lambda - 2\mu^4)}{\mu^2} \end{pmatrix}.
$$

(6.78)

In order to canonically normalise $\sigma(t)$ we factor this expression with the corresponding entry in the kinetic matrix. Therefore the squared mass of the $\sigma(t)$ field is given by,

$$
\frac{1}{2} m^2_\sigma = \frac{54(H^2M^2_P\lambda - 2\mu^4)}{7M^2_P\lambda + 18\mu^2}.
$$

(6.79)

**Gradient and cross terms**

Since we are performing a field redefinition, some of the factors for the time derivative cross terms represented by the entries of the $\mathcal{B}$ matrix will, after integrations by parts, contribute to the $\mathcal{G}$ matrix. Indeed, first after the field redefinition we find that the matrix is invariant $\mathcal{B} = P^T\mathcal{B}P = \mathcal{B}$. Therefore we have,

$$
\Sigma^T \mathcal{B} \Sigma = \frac{e^{-2Ht}k^2M^2_P\sqrt{\lambda}(27H^2\lambda - q^2\mu^2)}{81Hq^2\mu^4}\rho(t)\left[\sqrt{\lambda}\dot{\rho}(t) + 9\mu\dot{\sigma}(t)\right].
$$

(6.80)

We identify a term of the form: $F(t)\dot{\rho}(t)\rho(t)$. We integrate this term by parts to receive a contribution to the $\mathcal{G}$ matrix of the form: $-\frac{1}{2}F(t)\rho(t)^2$. Removing this
term leaves us with,

\[ \Sigma^T \tilde{B} \Sigma = \frac{e^{-2Ht}k^2M_{Pl}^2\sqrt{\lambda}(27H^2\lambda - q^2\mu^2)}{9Hq^2\mu^3} \rho(t) \sigma(t). \]  

(6.81)

Therefore the resultant cross term matrix, \( \tilde{B} \), has just one non-zero entry and is identical to the original matrix \( B \) in equation (6.68),

\[ \tilde{B} = B = \begin{bmatrix} 0 & 0 \\ \frac{e^{-2Ht}k^2M_{Pl}^2\sqrt{\lambda}(27H^2\lambda - q^2\mu^2)}{9Hq^2\mu^3} \\ 0 \end{bmatrix}. \]  

(6.82)

We now apply the field redefinition to the terms associated with the \( G \) matrix and then also include the term removed from \( \tilde{B} \). This leaves us with,

\[ \tilde{G} = \begin{bmatrix} \frac{e^{-2Ht}k^2M_{Pl}^2}{6H^2} \\ \frac{243H^2M_{Pl}^2\lambda^2 + 2q^2\mu^2(37M_{Pl}^2\lambda + 45\mu^2)}{162q^2\mu^3} \\ \frac{4M_{Pl}^2\sqrt{\lambda}}{9\mu} \end{bmatrix}. \]  

(6.83)

Using these matrices we can now write the diagonalised Lagrangian as,

\[ \mathcal{L}_{e^2} = a(t)^3 \left\{ \Sigma^T \tilde{G} \Sigma + \frac{k^2}{a(t)^2} \Sigma^T \tilde{B} \Sigma - \Sigma^T \tilde{M} \Sigma - \Sigma^T \tilde{B} \Sigma \right\}. \]  

(6.84)

We have derived the quadratic action for the scalar perturbations around de Sitter and found that, unlike what we saw for the cubic Galileon, the ghost sign for the Galileonic Higgs is not cured by the gravitational interactions on a de Sitter background. However, the system does successfully reproduce the dynamics of the tachyonic vector Galileon via the Higgs acquiring a vacuum expectation value.

### 6.3.4 Stability analysis

**Ghost instability**

In our derivation of the quadratic action for the scalar perturbations, we found that one of the dynamical fields is a tachyonic ghost. The existence of such ghost terms indicates that there is a severe instability in the system. In effective field theories it is common to ignore these issues if the mass of the ghost is above the cut-off of the theory.
The mass for the ghost is given by equation (6.79) and is therefore of the order the Hubble scale, \( m_\sigma \sim O(H) \). From equations (6.54) and (6.55), we can relate this to our naïve cut-off scale, \( \Lambda \),

\[
M_p H^2 \sim \Lambda^3
\]  

(6.85)

This suggests that the strong-coupling scale is far higher than the Hubble scale which is what we expect given that the aim was to calculate the cosmological applications of the theory. On the other hand, this also implies that the mass of the ghost is far below the cut-off, signalling that our theory is not healthy. Note that in order to ascertain the true value for \( \Lambda \), it would be necessary to expand our action to cubic order, solve the constraints for our background and examine when the remaining cubic operators are of order one.

Gradient instability

We have established that we have a ghost which indicates that our theory is kinetically unstable. However, it is also important to determine the existence of any gradient instability in the theory. This is an instability caused by the spatial gradient term acquiring the wrong sign. The presence of such an instability would signal that the theory has no regime of validity.

In our cosmological context, we compute the sound speed for the perturbations, \( c_s \), and determine the condition on which it is real valued. If we write, for example \( \xi(t) \sim \exp(i \omega t) \), the sound speed is given by the wave dispersion relation,

\[
\omega^2 = c_s^2 k^2 / a^2
\]

For a system of fields, there are multiple sound speeds. These are found by first writing the system in terms of matrices and then solving the following equation for \( \omega^2 \),

\[
\det[\omega^2 \mathbf{K} - i \omega (\mathbf{B} + \mathbf{\dot{B}}) + \left( \frac{k^2}{a^2} \right) \mathbf{G} + \mathbf{M} + \mathbf{\dot{B}}] = 0
\]  

(6.86)

The wave dispersion relation is then used to calculate the squared sound speed \( c_s^2 \). However, note that gradient instabilities are dominated by the highest order in \( \tilde{k} \). Therefore, since the kinetic and gradient terms are higher order in \( \tilde{k} \) than the mass and cross terms, we find that in the large \( |\tilde{k}| \) limit, the sound speed can be...
calculated by solving the following for \( \omega^2 \),

\[
\det[\omega^2 \tilde{K}_t + (\frac{k^2}{a^2}) \tilde{G}] = 0. \tag{6.87}
\]

By substituting the solutions for \( \omega^2 \) into the wave dispersion relation, we find the following two sound speeds,

\[
c^2_{s,\sigma} = \frac{q^2 \mu^2}{9H^2 \lambda} \tag{6.88}
\]

\[
c^2_{s,\rho} = \frac{243H^2 M^2_\text{Pl} \lambda^2 + 74M^2_\text{Pl} q^2 \lambda \mu^2 + 90q^2 \mu^4}{21M^2_\text{Pl} q^2 \lambda \mu^2 + 54q^2 \mu^4} \tag{6.89}
\]

Note that both solutions are positive which indicates that we avoid any gradient instability for our theory.

6.4 Discussion

In this chapter we first introduced the perturbation theory necessary to investigate the properties of the Galileonic Higgs and related theories in a cosmological context. We then discussed the cosmological applications of theories related to our model. In particular we highlighted how these theories can have healthy scalar dynamics on self-accelerating backgrounds. This provided motivation for our investigation into the cosmological applications of the Galileonic Higgs.

Unfortunately, we found that, although the simplest version of our model has a self-accelerating background, it does not admit healthy dynamics as we find that a ghost propagates around this background. Note that the purpose of introducing a ghost signature for the Higgs was to ensure that the vector acquired a bare tachyonic mass. Indeed, the phenomenology of the vector sector of our theory resembles that of the cubic vector Galileon. The scalar sector, however, remains unstable and therefore we do not have a consistent theory.

One possible remedy is to recognise that we chose to expand around a particularly simple background field value for the Higgs. That is, instead of simply expanding around \( \langle \rho \rangle = \mu / \sqrt{\lambda} \) we could choose to expand around other solutions which depend on, in a complicated way, the profile of \( A_0 \). It is possible that we might find
an energetically stable theory if we were to expand around one of these alternative background profiles.

Another possibility is to include additional non-minimal couplings that effect only the Higgs and not the vector. For example, we could include a simple conformal coupling between the Higgs and the Ricci scalar such that the gravitational action of our system takes the form,

\[
\int d^4x \sqrt{-\tilde{g}} (\xi \Phi^4 \Phi R - \Lambda)
\]  

(6.90)

where \(\xi\) is a dimensionless parameter. Note that this modification is typically encountered in models of Higgs inflation where it is argued that such a term is a natural extension to the Standard Model of particle physics [102].

For our purposes, including this term is beneficial as we find that the gravitational interactions from the non-minimal coupling are able to raise the sign of the Higgs’s kinetic term such that we recover healthy dynamics for both sets of scalar perturbations.

Alternatively, we could explore other couplings that could introduce a richer phenomenology. However, as we explained in chapter 5, in order to keep a consistent theory, any non-minimal coupling that we add should preserve the gauge invariance.

One example we could consider is \(L_{\mu \nu} G^{\mu \nu}\), where \(L_{\mu \nu}\) is gauge invariant operator defined in chapter 2. The gravitational effects from this kind of non-minimal coupling would also act on the vector, so it is unclear, without a thorough analysis, what benefits there would be from including such a term. Note that the scalar part of this coupling has been explored in the context of Higgs inflation [113].
Chapter 7

Conclusions and Discussion

There is overwhelming evidence that the Universe is undergoing a period of accelerated expansion. Most notably, observations of type Ia supernovae together with measurements of the CMB constrain the cosmological density parameter for dark energy to be roughly 70%. Relaxing the assumption that General Relativity remains an accurate description of gravity on cosmological scales opens up the opportunity to consider alternative mechanisms for cosmic expansion. Modifications to Einstein’s field equations are able to yield interesting results. For example, theories with non-linear, derivative self-interactions are able to reproduce the accelerating background without the use of a cosmological constant. Furthermore, these same interactions ensure that the theories remain consistent with observations and tests of the solar system via the Vainshtein screening mechanism.

Galileons and the associated vector Galileons possess these properties. Furthermore, vector Galileons have been show to possess interesting cosmological applications beyond those of the scalars. In particular, in [62] it was found that the existence of intrinsic vector modes allows the possibility for reducing $G_{\text{eff}}$ such that it is even smaller than the Newton gravitational constant $G$ in the late cosmological epoch. This is an important feature to explore in light of the recent tension between the data of redshift-space distortions and CMB.

Vector Galileons involve an explicit breaking of the gauge symmetry associated with Maxwell’s Lagrangian. However, in nature we see examples of symmetries
broken spontaneously. In chapter 2 we discussed an extension to the Higgs mechanism for the spontaneous generation of vector Galileons. This involved defining new operators from gauge covariant derivatives and utilising the special structure of Galileons to ensure that no ghost instabilities were introduced. We found that in a suitable decoupling limit, we recovered a bi-Galileon system of interactions between the Higgs and the longitudinal mode of the vector. This is a desirable property as such structures are associated with powerful non-renormalisation theorems that ensure the theories are protected from radiative corrections.

We also discussed how the Galileonic Higgs mechanism was extended to non-abelian symmetries. This is an important result which improves the scope of these theories.

In order to understand the cosmological applications of these theories, it is important to check whether they can be consistently covariantised. This is especially true in the case of theories with non-linear, derivative interactions. This is because, on a dynamical space-time, these higher order derivatives necessarily introduce new derivative interactions between the scalar/vector and the metric. In chapter 3 we examined the effect that the interactions with the metric had on the special cancellation of terms associated with instabilities. In 4 we continued our analysis by examining the constraint structure for the restricted case of the longitudinal mode. We found that the special structure of the Galileons ensured that the kinetic matrix of our theories was degenerate and therefore the ghost instability was unable to propagate.

We took a more conservative step in chapter 5 and used non-minimal counter terms for the covariantisation of the Galileonic Higgs. That is, we constructed our covariant theory such that it had equations of motion with derivatives of up to second order. However, in this case, removing higher order derivatives is not enough; it is also important for consistency to ensure that the theory remains gauge invariant. The quartic was shown to possess a consistent counter term, however there appears to be an obstruction to finding a suitable counter term for the quintic.

In chapter 6 we investigated the cosmological applications of the Galileonic Higgs. First we introduced the cosmological perturbation theory necessary for our calcu-
lations, then we reviewed the interesting applications of related models. These are examples of scalar and vector theories that despite being unstable with a ghost on Minkowski, are able to produce, consistent, self-accelerating backgrounds whilst possessing healthy dynamics.

Inspired by these examples we considered the cosmological applications of the Galileonic Higgs with a ghost instability on Minkowski space. This choice of model possesses a similar self-accelerating background solution. However, after expanding the action to quadratic order in linear perturbations we discovered that a ghost instability remains in the model.

One known solution to this problem is to conformally couple the Higgs with the Ricci curvature scalar. This is a well known coupling that is often invoked by models of Higgs inflation. Its use in alleviating the problem with our theory is that the coupling generates a dynamical term exclusively for the Higgs, which for an appropriate choice of coupling constant can remove the ghost instability in our theory. It is important that this kind of term does not contribute a term for the vector as our system depended on the time component of the vector having a non-trivial profile in the background. It would be interesting to explore what other non-minimal terms we could introduce without preventing the existence of interesting background solutions.

Another resolution to the ghost instability could come from expanding around a different solution for the background profile of the Higgs. We simply expanded around its vacuum expectation value but it would be interesting to see whether expanding around another solution has an effect.

Lastly, we could consider higher order operators. In our example, we only consider operators up to the cubic level. It would be interesting to see whether we could find a consistent cosmological solution for the more general action. In particular, we would have the option of including the non-minimal counter term we found for the quartic.

We have chosen to focus on the application to the accelerated expansion of the present (late) Universe. However, it is possible that these models could have applications to the accelerated expansion in the early Universe. For example, we
could consider our model as a generalisation of Higgs inflation. In this scenario, inflation is driven by the potential of the Higgs. To fit observational data, we wish to achieve a scenario resembling slow-roll inflation, so typically a conformal coupling similar to that given in equation (6.90) is added to the standard model of particle physics. This has the effect of flattening the potential in the early epoch. The higher order operators of our model have been shown to extend the slow-roll parameters [114] of Higgs inflation. This is because they act to slow the roll of the inflaton down its potential. Alternatively, we could examine the scenario where the non-trivial profile of the vector drives the accelerated expansion in the early universe. In this case we could explore the characteristic difference between accelerated expansion driven by the potential compared to those driven by the non-trivial profile of the vector.
Appendix A

Constraining Galileon Inflation

In this appendix I include my paper [115], ‘Constraining Galileon Inflation’, published in the Journal of Cosmology and Astroparticle Physics (JCAP).

A.1 Introduction

Recent microwave background data from the Planck satellite suggest that the pattern of density fluctuations in our universe is consistent with a canonical, single-field, slow-roll inflationary model [116]. To test for deviations from this paradigm we typically search for signatures in the the $n$-point functions of the microwave background anisotropies. At the time of writing, meaningful constraints have been obtained for the cases $n = 2$ and $n = 3$—respectively, the power spectrum and the bispectrum, corresponding to spectral decompositions of the variance and skewness.

In this paper we focus on searches using the bispectrum, usually conducted by comparing fixed ‘templates’ to the data. This is useful in a discovery phase, where the relevant question is only whether evidence exists for the amplitude of some template to be inconsistent with zero. However, because templates do not accurately explore the range of shapes produced in a specific model, it would be more satisfactory to search for evidence for the model as a whole, rather than focusing on separate templates.
How should this be done? A given experiment measures each angular com-ponent of the bispectrum with varying signal-to-noise, depending on its instrumental characteristics. Therefore different experiments are sensitive to differing contributions to the bispectrum. For any chosen experiment, a typical model will predict contributions to which it is highly sensitive and others to which it is comparatively blind. We can expect to constrain only those parameters of a model which contribute to regions of sufficient sensitivity. Fitting our models to these regions simultaneously gives a balanced picture of the goodness-of-fit associated with the experiment. Fitting separate templates may not produce such a balanced picture if it fails to take all experimentally-sensitive regions into account.

Byun & Bean have used this approach to develop forecasts for a Planck-like experiment [117]. More recently, some of us applied similar reasoning to the WMAP 9-year dataset and a very general model described by the effective theory of single-field inflation [118]. This theory describes the most general pattern of fluctuations which can be realized in a Lorentz-invariant field theory, assuming Lorentz invariance to be spontaneously broken by a nearly de Sitter background. Because it can describe any adiabatic fluctuation with sufficiently smooth statistical properties it can be regarded as a weak prior—and, as for any prior, more stringent constraints can be obtained by strengthening it. One reason for doing so is to explore how the interpretation of the data changes as we vary our assumptions. Another is to study how the constraints improve when we commit to a particular model, rather than allowing for the most general range of possibilities.

In this paper we focus on a particular prior for the nonlinear stochastic properties of the inflationary density perturbation—that it was generated during an era of ‘Galileon’ inflation [119]; see also Refs. [120, 105, 121, 78, 122]. Galileons are scalar fields with highly constrained self-interactions which contain higher-order time derivatives. These cancel in the equations of motion [35], yielding stable second-order field equations. When quantized this implies that the theory is ghost-free, and therefore maintains unitarity and stability.

These stability properties are preserved by quantum fluctuations around flat Minkowski space. At present it is apparently unclear whether the ghost-free the-
ory can arise as an effective description of a theory with an ultraviolet completion [123], which would require the special Galileon self-interactions to be unaccompanied by other higher-derivative operators which would generate a ghost. It is also unknown whether the ghost-free property survives on a cosmological background or in the field created by a heavy source. But if these possible complications can be evaded and a field with Galileon-like interactions were dynamically important during inflation, then it is possible that their special nature could leave interesting signatures in the stochastic properties of the density perturbation [124, 125, 126, 127].

Our principal result is a constraint on the importance of the Galileon self-interactions which would generate these signatures. For this purpose the Galileon model is particularly interesting because it allows just three 3-body interactions compared to the eleven allowed by the unconstrained effective field theory. In Ref. [118] we argued that the WMAP 9-year dataset is sensitive to three or (at most) four characteristic contributions to the bispectrum. Unless we are unlucky and the Galileon 3-body interactions contribute to these regions in a degenerate way, we can expect to obtain constraints on all three couplings.

More generally, the operators of the Galileon Lagrangian form a subset of the class of single-field effective theories of inflation. The symmetries of the Galileon model impose a relationship between the coefficients of the possible Lagrangian terms. In this paper we exploit this relationship to present the strongest possible constraints on the Galileon paradigm.

**Notation.**—In section A.2 we briefly describe the action for the Galileon inflationary model up to third order, and in section A.3 we use it to derive the bispectrum. In section A.4 we describe the procedure used to obtain our constraints. This is a summary of the approach developed in Ref. [118]. We obtain constraints on the couplings in the Lagrangian for the cases of one, two or three (the most general possibility) independent third-order couplings. Finally, in section A.5 we carry out a Bayesian model comparison for the various incarnations of the Galileon model. We give our conclusions in section A.6.
A.2 Overview of Galileon inflation

The original Galileon model constructed by Nicolis et al. was based on a field-space generalization of the Galilean shift-symmetry of classical mechanics, $\phi \rightarrow \delta_g \phi = \phi + b_\mu x^\mu + c$ where $x^\mu$ is a spacetime coordinate and $b_\mu$ and $c$ are constants [48]. Nicolis et al. worked in flat spacetime and constructed four operators, labelled $\mathcal{L}_i$ for $2 \leq i \leq 5$, which yielded an action satisfying this symmetry and produced second-order equations of motion. On a curved background it is not possible to retain both properties [86, 43, 128]. Insisting on second-order equations of motion and accepting a break of the shift symmetry proportional to the background curvature yields the ‘covariantized’ formulation, with action

$$S \supset \int d^4x \sqrt{-g} [c_2 \mathcal{L}_2 + c_3 \mathcal{L}_3 + c_4 \mathcal{L}_4 + c_5 \mathcal{L}_5],$$

and $\mathcal{L}_i$ defined by

$$\mathcal{L}_2 = \frac{1}{2} (\nabla \phi)^2$$

(A.2a)

$$\mathcal{L}_3 = \frac{1}{\Lambda^3} \Box \phi (\nabla \phi)^2$$

(A.2b)

$$\mathcal{L}_4 = \frac{(\nabla \phi)^2}{\Lambda^6} \left[ (\Box \phi)^2 - \nabla_\mu \nabla_\nu \phi \nabla^{\mu} \nabla^{\nu} \phi - \frac{R}{4} (\nabla \phi)^2 \right]$$

(A.2c)

$$\mathcal{L}_5 = \frac{(\nabla \phi)^2}{\Lambda^9} \left[ (\Box \phi)^3 - 3 \Box \phi \nabla_\mu \nabla_\nu \phi \nabla^{\mu} \nabla^{\nu} \phi + 2 \nabla_\mu \nabla_\nu \phi \nabla^{\nu} \phi \nabla^\rho \phi \nabla^{\mu} \phi 
- 6 G_{\mu \nu} \nabla^{\rho} \phi \nabla^{\nu} \phi \nabla_\rho \phi \right].$$

(A.2d)

Here, $G_{\mu \nu}$ is the Einstein tensor, $R$ denotes the scalar curvature of the background and $\Lambda$ is a mass scale at which the higher-order operators $\mathcal{L}_3$, $\mathcal{L}_4$ and $\mathcal{L}_5$ become comparable to the Gaussian term $\mathcal{L}_2$.

These are not the only nonlinear operators which yield an action invariant under the shift symmetry in flat space—for example, any function of $\Box \phi$ is automatically invariant—but they are the only combinations which produce second-order equations of motion. Consistency of the model requires that unwanted combinations such as $(\Box \phi)^2 / M^2$ (which would generally indicate the presence of a ghost

---

1The $\mathcal{L}_2$ themselves need not be invariant, provided that the transformation shifts them only by a total derivative which vanishes on integration.
at the scale $M$) are not generated by renormalization-group running for any $M$ at which we wish to trust the predictions of the effective theory.\footnote{It is inconsequential if a ghost is generated at scales which are not intended to be described by the effective Lagrangian: this happens generically in any effective field theory. To understand whether the putative ghost really exists in the spectrum we would need details of the ultraviolet completion.} This does not happen in the vacuum, where the $\phi$ fluctuations are massless and do not generate renormalization-group evolution. It is not yet known whether a ghost will appear for non-vacuum field configurations [123].

In this paper we assume that the action (A.1) can be used to describe fluctuations on a quasi-de Sitter background representing an inflationary phase, and that there is a regime for which quantum effects do not cause ghost modes to become excited. Working in the ‘decoupling’ limit where gravitational degrees of freedom can be ignored, it was shown in Ref. [119] that the action for small fluctuations can be written

$$S \supset \int d^4x a^3 \left[ \alpha \left( \pi^2 - \frac{c_s^2}{a^2} (\partial \pi)^2 \right) + g_1 \pi^3 + \frac{g_3}{a^2} \pi (\partial \pi)^2 + \frac{g_4}{a^4} (\partial \pi)^2 \partial^2 \pi \right]. \quad (A.3)$$

The decoupling limit applies when the higher-order terms $L_3$, $L_4$ and $L_5$ are relevant, making Galileon self-interactions stronger than gravitational interactions. But if pushed too far these self-interactions require a rapidly evolving field configuration which risks spoiling the de Sitter background. A complete understanding of what happens in this regime would require an analysis of the associated ‘cosmological’ Vainshtein effect, which has not yet been carried out. We assume that these complications can be evaded by having the higher-order terms sufficiently relevant that they dominate gravitational corrections, but not so relevant that they destabilize the inflationary era. The coefficients $\alpha$, $c_s$, $g_2$, $g_3$ and $g_4$ are given by various combinations of $c_i$, $H$, $\phi$ and the nonlinear scale $\Lambda$. For precise formulae, or further discussion of the role of the nonlinear terms, we refer to Burrage et al. [119]. On superhorizon scales the curvature perturbation is given by $\zeta = H \pi$, and is conserved.

Constraints on this model take the form of limits on the parameters $c_i$. Some limits exist based on short-distance gravitational effects in the late universe; see,
for example, Ref. [129]. There is no particular requirement for the Galileon-like fields relevant during the early and late universe to have the same identity (although this may be the case in certain models), so these limits need not apply during inflation. In this paper we obtain limits on the $c_i$ from the bispectrum of the inflationary density perturbation without making use of any late-universe data.

Note that the fluctuations generated in certain $k$-inflation and Horndeski models may be controlled by the same action [130]. Therefore our results can equally be interpreted as constraints on these models, although we do not identify them explicitly. Within this large class of theories, Galileons are algebraically special in that they require only three independent cubic operators, as in Eq. (A.3). A generic $k$-inflation or Horndeski model may require up to four independent operators [130, 125].

### A.3 Bispectrum Shapes

The bispectrum, $B_\zeta$, is defined by the three-point correlation function of the curvature perturbation,

$$\langle \zeta(k_1)\zeta(k_2)\zeta(k_3) \rangle = (2\pi)^3 \delta(k_1 + k_2 + k_3) B_\zeta(k_1, k_2, k_3).$$

**Wavefunctions.**—A complication of the most general models studied in Ref. [118] is that fourth-derivative operators can appear in the quadratic term, leading to a very complex form for the elementary functions. While a solution can be found in closed form, the subsequent vertex integrations appearing in the Feynmann rules cannot be performed analytically. The Galileon fluctuation Lagrangian (A.1) belongs to the class of models considered Ref. [118] but does not possess the problematic fourth-derivative operators. Neglecting slow-roll corrections the elementary wavefunction is

$$u(k, \tau) = \frac{iH}{2\sqrt{\alpha}} \frac{1}{(kc_s)^{3/2}} (1 - ikc_s \tau) e^{ikc_s \tau}.$$  

On superhorizon scales where $|kc_s \tau| \ll 1$ the power spectrum can be written

$$P_\zeta(k) = \frac{H^4}{4\alpha c_s^3} \frac{1}{k^3}.$$  

116
Momentum dependence.—The necessary calculations were described by Mizuno & Koyama [122], and given to next-order in slow-roll parameters in Refs. [119, 125]. Similar calculations were performed by Kobayashi, Yamaguchi and Yokoyama [121]. There is one contribution to the bispectrum from each cubic operator in (A.3), which we write in the form

$$B_z(k_1, k_2, k_3) = \frac{3}{5} \sum_{\alpha=1}^{3} \lambda_\alpha B_\alpha(k_1, k_2, k_3).$$  \hspace{1cm} (A.7)$$

The $B_\alpha$ are normalized so that $B_\alpha(k, k, k)/6P_z(k)^2 = 1$ at the equilateral point. In terms of the couplings $g_\alpha$ in the fluctuation Lagrangian these means that that $\lambda_\alpha$ correspond to

$$\lambda_1 = \frac{5}{81} \frac{g_1}{c_\alpha}, \quad \lambda_2 = -\frac{85}{324} \frac{g_3}{c_\alpha^2}, \quad \lambda_3 = -\frac{65}{162} \frac{g_4 H}{c_\alpha^4}. \hspace{1cm} (A.8)$$

Focusing only on the momentum dependence (prefactors can be inferred from the normalization convention if required), the bispectra can be written

$$B_1(k_1, k_2, k_3) \sim \frac{1}{\prod_i k_i} \frac{1}{k_i^3}, \hspace{1cm} (A.9a)$$

$$B_2(k_1, k_2, k_3) \sim \frac{1}{\prod_i k_i} k_i^2 (k_2 \cdot k_3) \left( \frac{1}{k_t} + \frac{k_2 + k_3}{k_t^2} + \frac{2k_2 k_3}{k_t^3} \right) + 1 \to 2 + 1 \to 3, \quad (A.9b)$$

$$B_3(k_1, k_2, k_3) \sim \frac{1}{\prod_i k_i} k_i^2 (k_2 \cdot k_3) \left( \frac{1}{k_t} + \frac{K^2}{k_t^3} + \frac{3k_1 k_2 k_3}{k_t^4} \right) + 1 \to 2 + 1 \to 3, \quad (A.9c)$$

where $k_t = k_1 + k_2 + k_3$, and $K^2 = k_1 k_2 + k_1 k_3 + k_2 k_3$.

A.4 Estimating Galileon Parameters

We now aim to estimate the $\lambda_\alpha$ using CMB data. Given a model, and therefore knowledge of the parameters $\alpha$ and $c_s$, this enables the $g_\alpha$ to be determined. Given knowledge of the background field configuration this enables constraints to be placed on the $c_i$. 

117
Estimation methodology.—Our methodology for estimating the $\lambda_\alpha$ was described in Ref. [118]. Following Fergusson, Liguori & Shellard we write each of (A.9a)–(A.9c) as a sum over some basis $B_n$, giving $B_\alpha = \sum_n \alpha_n B_n$ [131]. We extract multipole coefficients for the temperature anisotropy $\Delta T$ using,

$$\Delta T(\hat{n})/T = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\hat{n}),$$

(A.10)

and define the angular bispectrum $b_{\ell_1 \ell_2 \ell_3}$ to satisfy,

$$\langle a_{\ell_1 m_1} a_{\ell_2 m_2} a_{\ell_3 m_3} \rangle = b_{\ell_1 \ell_2 \ell_3} C_{m_1 m_2 m_3},$$

(A.11)

where $C_{m_1 m_2 m_3}$ is the Gaunt integral. After using the primordial perturbation $\zeta$ to seed fluctuations in the radiation era, and accounting for radiative transfer and projection onto the sky, each $B_n$ will yield some angular bispectrum $b_n$. We introduce a further basis $b_n^{A \ell_1 \ell_2 \ell_3}$ and write $b_n^{A \ell_1 \ell_2 \ell_3} = \Gamma_A^n b_n^{\ell_1 \ell_2 \ell_3}$. Then, given a choice $\lambda_\alpha$, it follows that the observable angular bispectrum can be written

$$b_{\ell_1 \ell_2 \ell_3} = \sum_A \beta_A b_n^{A \ell_1 \ell_2 \ell_3},$$

(A.12)

where the coefficients $\beta_A$ are defined by

$$\beta_A \equiv \lambda_\alpha \alpha_n \Gamma_A^n.$$  

(A.13)

The advantage of this basis decomposition is that the transfer matrix $\Gamma_A^n$ can be computed relatively easily [132]. It encodes details of the cosmology, together with the processes of radiative transfer which connect primordial times to observation.

A given microwave background experiment makes measurements of the $\beta_A$ associated with our last-scattering surface. We write these estimates $\hat{\beta}_A$. Assuming Gaussian experimental errors, and given our prior, the likelihood of an experiment returning some particular set of values can be written

$$\mathcal{L}(\hat{\beta}_A | \lambda_\alpha) = \frac{1}{\sqrt{2\pi \det \hat{\mathcal{C}}}} \exp \left( -\frac{1}{2} \sum_{A, B} (\hat{\mathcal{C}}^{-1})^{AB} \Delta \hat{\beta}_A \Delta \hat{\beta}_B \right),$$

(A.14)

where $\Delta \hat{\beta}_A \equiv \hat{\beta}_A - \beta_A$ and the covariance matrix is defined by $C_{AB} = \langle \Delta \hat{\beta}_A \Delta \hat{\beta}_B \rangle$. We estimate it using Gaussian simulations, accounting for realistic WMAP beam
and noise properties and the effects of masking. For all quantitative details we refer to Ref. [118].

Eq. (A.14) yields a maximum likelihood estimator for each parameter $\lambda_{\alpha}$ corresponding to

$$\hat{\lambda}_{\alpha} = \sum_{\beta} \hat{b}^\beta (\hat{\mathcal{F}}^{-1})_{\beta\alpha}, \quad (A.15)$$

where $\hat{b}^\alpha \equiv \sum_{A,B,n} \hat{b}_A (\hat{\mathcal{F}}^{-1})^{AB} \alpha_n \Gamma_B^n$ and the Fisher matrix $\hat{\mathcal{F}}$ satisfies

$$\hat{\mathcal{F}}^{\alpha\beta} = \sum_{A,B,m,n} \Gamma_A^m \alpha_m (\hat{\mathcal{F}}^{-1})^{AB} \alpha_n \Gamma_B^n \quad (A.16)$$

Application to 9-year WMAP data.—We use the WMAP 9-year dataset to estimate the amplitudes $\hat{b}_A$ [133, 134], and use these to constrain subcases of the fluctuation Lagrangian (A.3). The most general case includes all three operators $g_1$, $g_3$ and $g_4$ and yields the weakest constraints. This would be expected where correlations among the operators exist in the regions to which WMAP is most sensitive. In this case rather more is true: Renaux-Petel pointed out [130, 135] that there is an approximate degeneracy (spoilt by boundary terms which become irrelevant at late times) which allows the $g_4$ contribution to be absorbed into renormalizations of the other couplings,

$$g_4 \rightarrow g'_4 = g_4 + g_4 H / c_s^4, \quad g_3 \rightarrow g'_3 = g_3 + 2 g_4 H / c_s^2. \quad (A.17)$$

We can leave $g_4$ in the analysis, accounting for the correlation in shape, or eliminate $g_4$ using (A.17) at the outset. In what follows we will give constraints for both choices. Finally, we consider the most restrictive subcase in which only one parameter is allowed to be nonzero. This corresponds most directly with the standard approach of fitting individual templates to the data. It gives optimistic constraints unless we are prepared to commit to a scenario in which two operators are subdominant compared to the third.

Case 1: General scenario (three free parameters).—Using the relationship between the parameters $\lambda_{\alpha}$ and the coefficients of the Lagrangian given in Eq. (A.8), we find
The quoted uncertainties represent 1σ errors bars, with marginalization over the other two parameters. Each constraint is consistent with zero to within 1σ. Note that the uncertainties are rather large, due to exploration of the entire parameter space.

**Case 2: Two free parameters.**—It was explained above that the three-parameter case is perhaps too pessimistic, because of correlations between the bispectra produced by the three cubic operators. In this section we obtain constraints on the subcase where two couplings are allowed to vary with the third fixed at zero. In the case where \( g_4 \) is held fixed Eq. (A.17) can be used to map these constraints to the Lagrangian obtained by elimination of the \( g_4 \) term. The results are

<table>
<thead>
<tr>
<th>Fixed parameter</th>
<th>Variable</th>
<th>Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g_1/\alpha )</td>
<td>( \hat{g}_3/c_s^2 \alpha )</td>
<td>(-11,000 \pm 6,770)</td>
</tr>
<tr>
<td></td>
<td>( \hat{g}_4H/c_s^4 \alpha )</td>
<td>( 7,760 \pm 4,870)</td>
</tr>
<tr>
<td>( g_3/c_s^2 \alpha )</td>
<td>( \hat{g}_1/\alpha )</td>
<td>( 11,000 \pm 6,660)</td>
</tr>
<tr>
<td></td>
<td>( \hat{g}_4H/c_s^4 \alpha )</td>
<td>( 3,380 \pm 2,240)</td>
</tr>
<tr>
<td>( g_4H/c_s^4 \alpha )</td>
<td>( \hat{g}_1/\alpha )</td>
<td>( 15,300 \pm 9,470)</td>
</tr>
<tr>
<td></td>
<td>( \hat{g}_3/c_s^2 \alpha )</td>
<td>( 2,870 \pm 1,900)</td>
</tr>
</tbody>
</table>

Each constraint is consistent with zero to within ~ 1.5σ, matching our expectations from constraints on individual templates using the 9-year WMAP dataset [134]. Similar results have been reported from the Planck [116] 2013 data release. The formalism used here ensures that the entire available parameter space is explored, rather than inferring these constraints from the overlap with a selection of templates.
Case 3: One free parameter.—Finally, we consider the constraints where only one parameter is allowed to vary. We obtain

\[ \hat{g}_1 / \alpha = 1120 \pm 1280, \quad \hat{g}_3 / c_s^2 \alpha = -260 \pm 390, \quad \hat{g}_4 H / c_s^4 \alpha = -160 \pm 280. \]  

(A.18)

We may also obtain a constraint from the amplitude of the power spectrum, which gives \( H^4 / (\alpha c_s^2) = (190 \pm 8) \times 10^{-9} \) at the pivot scale \( k = 0.002 \text{Mpc}^{-1} \). Together this gives 4 constraints for 6 parameters, \( \{H, \alpha, c_s, g_3, g_4\} \). Breaking the degeneracy would require constraints on the scalar tilt, \( n_s \) or the tensor to scalar ratio, \( r \).

A.5 Bayesian Model Comparison

While our results in the previous section are useful in determining best-fit values for the parameters, we wish also to perform a model comparison. One method with which to quantify the evidence for or against a model is through calculation of the ‘Bayes’ factor. In Ref. [118] this was first applied to the comparison of non-Gaussian models. We briefly recapitulate the description here.

Given a data set \( D \) and a pair of models \( M_1 \) and \( M_2 \), with respective parameter sets \( \{\lambda_1\} \) and \( \{\lambda_2\} \), the Bayes factor is defined as the ratio of the likelihoods of the respective models,

\[ K_{12} = \frac{P(D|M_1)}{P(D|M_2)} = \frac{\int P(D|\{\lambda_1\}, M_1)P(\{\lambda_1\}|M_1)\,d\lambda_1}{\int P(D|\{\lambda_2\}, M_2)P(\{\lambda_2\}|M_2)\,d\lambda_2}. \]  

(A.19)

It should be noted that the integrals here are over the entire parameter space of each model. These may be of different dimensionalities. The prior probabilities \( P(\{\lambda_i\}|M_i) \) represent the probability that a particular parameter choice occurs. To determine our priors we use the requirement that the bispectrum generated by each operator must not dominate the power spectrum, and therefore that each parameter is constrained by \( |\lambda_i| \lesssim 10^4 \). However, given that we have no reason\(^3\) to prefer any

\(^3\)We cannot use constraints from CMB experiments to choose our prior, because we are using the CMB as our dataset.
scale we choose a Jeffries prior with \( P(\lambda_i) \propto 1/|\lambda_i| \), with \( |\lambda_i| \in [1, 10^4] \). The cutoff is chosen to avoid a divergence at \( \lambda_i = 0 \), and our results show little dependence on its precise value. The Bayes factor does not become independent of the prior, so to study its dependence for different choices we also compute values for a flat prior in the range \([-1, 1]\). The probability \( P(D|\{\lambda_i\}, M_1) \) represents the likelihood for a particular choice of parameters \( \{\lambda_i\} \) and may be computed using Eq. (A.14).

We interpret our results using the Kass & Raftery scale [136]. In this scheme, \(| \ln K | \) in the range \((0, 1)\) is ‘indecisive’, in the range \((1, 3)\) represents ‘evidence in favour of \( M_1 \)’, in the range \((3, 5)\) represents ‘strong evidence in favour of \( M_1 \)’, and larger values are ‘decisive’. A similar scale applies to \( K^{-1} \) with \( M_2 \) substituted for \( M_1 \).

**Results**

- Comparing the Gaussian model (i.e. with all \( \lambda_i = 0 \)) with the case of just one non-zero parameter, we find the Bayes factor is given by \( \ln K \approx 0.7 \), such that the data is indecisive in distinguishing these scenarios.

- Next we compare the Gaussian model and the case of two free parameters, with the result that \( \ln K \approx 1.3 \). This indicates (weak) evidence against the Galileon model with two free parameters, and is mainly due to an Ockham penalty which disfavors addition of extra parameters without sufficient support from the data. Adding a further parameter and comparing the Gaussian model to the most general Galileon model with three free parameters gives a Bayes factor \( \ln K \approx 2 \). In this case there is an even stronger preference for the simpler description.

- Comparing the single free parameter case with the two parameter and three parameter cases giving \( \ln K \approx 0.6 \) and \( \ln K \approx 1.3 \), respectively. The data is indecisive in the former case, but shows preference for the single parameter case in the latter.

In summary, the data shows little power to discriminate between the Gaussian model and a Galileon model with one extra free parameter. However, for mod-
els with two or more extra parameters the WMAP 9-year data exhibits a weak preference for the simpler description.

A.6 Conclusions

In this paper we have utilised the formalism developed in Ref. [118] to constrain the Galileon inflationary model using the bispectrum. Our constraints show that the couplings of the cubic terms in the fluctuation Lagrangian are consistent with zero to within $1.5\sigma$. We have separately considered the cases of one, two and three free parameters in the fluctuation Lagrangian.

The formalism can be used to carry out a Bayesian model comparison. This establishes that the data weakly disfavours models requiring two extra free parameters, but is inconclusive between a Gaussian model and the case of a Galileon model with a single extra coupling. It is possible that carrying out the analysis using Planck data [116] may lead to a stronger result.
Appendix B

Formulae in the ADM formalism

B.1 Christoffel symbols

The four dimensional connection is given by, \( \Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} \{ g_{\nu\sigma,\rho} + g_{\rho\sigma,\nu} - g_{\nu\rho,\sigma} \} \).

We will find it useful to first collect the results for all the connection components in one place,

\[
\begin{align*}
\Gamma_{ij0} &= \Gamma_{i0j} = -NK_{ij} + D_jN_i, \quad \text{(B.1)} \\
\Gamma_{ijk} &= \tilde{\Gamma}_{ijk}, \quad \text{(B.2)} \\
\Gamma^0_{00} &= \frac{1}{N} (\dot{N} + N^i \partial_i N - N^i N^j K_{ij}), \quad \text{(B.3)} \\
\Gamma^0_{0i} &= \Gamma^0_{i0} = \frac{1}{N} (\partial_i N - N^j K_{ij}), \quad \text{(B.4)} \\
\Gamma^0_{0j} &= \Gamma^0_{j0} = -\frac{1}{N} (\dot{N} + N^i \partial_i N - N^i N^k K_{ik} + D_j N^i), \quad \text{(B.5)} \\
\Gamma^i_{ij} &= \tilde{\Gamma}^i_{ij} + \frac{N^i}{N} K_{ij}, \quad \text{(B.6)} \\
\Gamma^i_{jk} &= \tilde{\Gamma}^i_{jk} + \frac{N^i}{N} K_{jk}, \quad \text{(B.7)} \\
\Gamma^i_{00} &= \frac{\dot{N} N^i}{N} + \gamma^j N_j + \frac{1}{2N^2} N^i N_k N_l \gamma^{kl} \\
&\quad + \frac{1}{2} \left( \gamma^j - \frac{N^i N_j}{N^2} \right) \partial_j (N^2 - N_k N^k). \quad \text{(B.8)}
\end{align*}
\]
B.2  Formula for the Riemann tensor

The curvature components are given by:

\[ R_{\nu \alpha \beta} = \partial_\alpha \Gamma^\mu_{\nu \beta} - \partial_\beta \Gamma^\mu_{\alpha \nu} + \Gamma^\mu_{\alpha \rho} \Gamma^\rho_{\beta \nu} - \Gamma^\mu_{\beta \rho} \Gamma^\rho_{\alpha \nu} \]  

(B.9)

This has the following symmetry properties:

\[ R_{\alpha \beta \mu \nu} = -R_{\beta \alpha \mu \nu} = -R_{\alpha \nu \beta \mu} = R_{\mu \nu \alpha \beta} \]  

(B.10)

Which means we only need to compute: \( R_{ijkl}, R_{0ijk} \) and \( R_{00ij} \).

We use the above formulae to first find the components of \( R_{ijkl} \):

\[
R_{ijkl} = g_{i\mu} \partial_k \Gamma^\mu_{lj} - g_{j\mu} \partial_l \Gamma^\mu_{ik} + \Gamma^\mu_{ik \rho} \Gamma^\rho_{lj} - \Gamma^\mu_{il \rho} \Gamma^\rho_{kj}
\]

\[
= -N_i \partial_k \left( \frac{1}{N} K_{lj} \right) + \gamma_{km} \partial_k \left( \Gamma^m_{lj} + \frac{N^m}{N} K_{lj} \right) - \frac{1}{N} K_{lj} \left( -NK_{ik} + D_k N_l \right)
\]

\[
+ \Gamma_{ikm} \left( \Gamma^m_{lj} + \frac{N^m}{N} K_{lj} \right) - (k \leftrightarrow l)
\]

\[
= \tilde{R}_{ijkl} + K_{ik} K_{jl} - K_{il} K_{jk}
\]

(B.11)

where \( \tilde{R}_{ijkl} \) is the three dimensional Riemann curvature tensor formed from the
three metric $\gamma_{ij}$ and its compatible derivative $D_i$. The components of $R_{0i0j}$ are:

$$R_{0i0j} = g_{\mu0} \left( \partial_0 \Gamma^\mu_{ij} - \partial_j \Gamma^\mu_{0i} + \Gamma^\mu_{0p} \Gamma^p_{ij} - \Gamma^\mu_{jp} \Gamma^p_{0i} \right)$$

$$= - \left( N^2 - N_k N^k \right) \left\{ \partial_0 \left( - \frac{K_{ij}}{N} \right) - \partial_j \left( \frac{\partial_i N}{N} - \frac{N^l K_{il}}{N} \right) \right\}$$

$$- \frac{1}{N^2} \left( \dot{N} + N^l \partial_i N - N^l N^m K_{lm} \right) K_{ij}$$

$$+ \frac{1}{N} \left( \partial_i N - N^m K_{im} \right) \left( \dot{\Gamma}^l_{ij} + \frac{N^l}{N} K_{ij} \right)$$

$$- \frac{1}{N^2} \left( \partial_j N - N^m K_{jm} \right) \left( \partial_i N - N^m K_{in} \right)$$

$$+ K_{ji} \left( - \frac{N^l \partial_i N}{N^2} - \left( N^k \frac{N^l}{N^2} \right) K_{ik} + D^l_i N \right)$$

$$+ N_k \left\{ \partial_0 \left( \dot{\Gamma}^k_{ij} + \frac{N^k}{N} K_{ij} \right) \right\}$$

$$- \partial_j \left( - \frac{N^k \partial_i N}{N} - N \left( \gamma^{jk} - \frac{N^k N^l}{N^2} \right) K_{li} + D_l i N k \right)$$

$$- \frac{K_{ij}}{N} \left( - \frac{N^k N^l}{N^2} + \dot{\gamma}^{kl} N_l + \frac{N^i N^m N^l}{2N^2} \gamma^{ml} \right)$$

$$+ \frac{1}{2} \left( \gamma^{kl} - \frac{N^k N^l}{N^2} \right) \partial_i \left( N^2 - N_m N^m \right)$$

$$+ \left\{ - \frac{N^k \partial_i N}{N} - N \left( \gamma^{km} - \frac{N^k N^m}{N^2} \right) K_{ml} + D_l i N k \right\} \left[ \dot{\Gamma}^l_{ij} + \frac{N^l}{N} K_{ij} \right]$$

$$- \left\{ - \frac{N^k \partial_i N}{N^2} - \left( \gamma^{kj} - \frac{N^k N^l}{N^2} \right) K_{jl} + D_j i N k \right\} \left( \partial_i N - N^m K_{im} \right)$$

$$- \left[ \dot{\Gamma}^k_{ji} + \frac{N^k}{N} K_{ji} \right] \left( - \frac{N^l \partial_i N}{N} - N \left( \gamma^{ml} - \frac{N^m N^l}{N^2} \right) K_{mi} + D_l i N \right) \right\} \quad (B.12)
Next we compute the components for $R_{kli}$:

\[
R_{kli} = g_{\mu k} \left( \partial_0 \Gamma^\mu_{ij} - \partial_j \Gamma^\mu_{0i} + \Gamma^\mu_{0p} \partial^p_{ij} - \Gamma^\mu_{jp} \Gamma^p_{0i} \right) \\
= N_k \left\{ \partial_0 \left( - \frac{K_{ij}}{N} \right) - \partial_j \left( \frac{1}{N} (\partial_i N - N^k K_{ik}) \right) \right. \\
- \frac{1}{N^2} \left( \hat{N} + N^l \partial_l N - N^l N^m K_{lm} \right) K_{ij} \\
+ \frac{1}{N} (\partial_k N - N^l K_{kl}) \left( \Gamma^l_{ij} + \frac{N^k}{N} K_{ij} \right) \\
\left. - \frac{1}{N^2} (\partial_j N - N^l K_{jl}) (\partial_i N - N^k K_{ik}) \right\} \\
- K_{jk} \left( - \frac{N^k \partial_i N}{N^2} + \frac{D_k N^i}{N} \right) \\
+ \left( \gamma_{kl} - \frac{N^k N^l}{N^2} \right) K_{jk} K_{li} \right\} \\
+ \gamma_{mk} \left\{ \partial_0 \left( \Gamma^m_{ij} + \frac{N^m}{N} K_{ij} \right) + \partial_j \left( \frac{N^m \partial_i N}{N} \right) \right. \\
- \partial_j \left( N \left( \gamma^m_{nk} - \frac{N^m N^k}{N^2} \right) K_{ki} + D_i N^m \right) \\
- \left[ - \frac{N^m \partial_j N}{N^2} + \frac{D_j N^m}{N} \right] (\partial_i N - N^l K_{il}) \\
- K_{ij} \left( - \frac{N^m N^i}{N^2} + \gamma^{mn} N_n + N^i N_i N_j \gamma^{jl} \right) \\
\left. - K_{ij} \left( \gamma^{mk} - \frac{N^m N^k}{N^2} \right) \partial_k (N^2 - N_i N^i) \right\} \\
+ \left( - \frac{N^m \partial_j N}{N} + D_j N^m \right) \left[ \Gamma^l_{ij} + \frac{N^l}{N} K_{ij} \right] \\
- N \left( \gamma^{mk} - \frac{N^m N^k}{N^2} \right) K_{kl} \left[ \Gamma^l_{ij} + \frac{N^l}{N} K_{ij} \right] \\
+ \left( \gamma^{mk} - \frac{N^m N^k}{N^2} \right) K_{ij} (\partial_i N - N^l K_{il}) \\
- \left( \Gamma^m_{jl} + \frac{N^m}{N} K_{jl} \right) \left( - \frac{N^l \partial_i N}{N} + D_i N^l \right) \\
+ \left( \Gamma^m_{jl} + \frac{N^m}{N} K_{jl} \right) N \left( \gamma^{jk} - \frac{N^l N^k}{N^2} \right) K_{ki} \right\} \quad (B.13)
\]
B.2.1 Time derivatives in the Riemann tensor.

From above we find that $R_{0i0j}$ contains the following time derivatives:

\[
R_{0i0j} \equiv (N^2 - N_k N^k) \left\{ \partial_0 \left( -\frac{K_{ij}}{N} \right) - \frac{NK_{ij}}{N^2} \right\} + N_k \left\{ \partial_0 \tilde{\Gamma}_{ij}^k + \partial_0 \left( \frac{N^k}{N} K_{ij} \right) + \tilde{N} N^k K_{ij} - \frac{\gamma^j_\ell N_i \tilde{N}_{ij}}{N} \right\} = NK_{ij} + N_k \left[ \partial_0 \tilde{\Gamma}_{ij}^k + \frac{N^k K_{ij}}{N} - \frac{\gamma^j_\ell N_i \tilde{N}_{ij}}{N} \right] \tag{B.14}
\]

Similar cancelations show that $R_{kiloj}$ contains:

\[
R_{kiloj} \equiv \gamma_{mk} \left[ \partial_0 \tilde{\Gamma}_{ij}^m + \frac{K_{ij}}{N} (N^m - \delta^m_k \tilde{N}_j) \right] \tag{B.15}
\]

B.2.2 Components of the Einstein tensor

We compute the expression for $G^{00} = R^{00} - \frac{1}{2}g^{00}R = R^{00} + \frac{R}{2N^2}$ which is used in the text. First notice that,

\[
R^{00} = g^{00} \left( g^{ij} g^{0j} - g^{i0} g^{0j} \right) R_{0i0j} + 2g^{00} g^{i0} g^{k0} \tilde{R}_{ijk0} + g^{ij} g^{k0} g^{l0} R_{likj}
\]

\[
= \frac{\gamma^j_\ell}{N^2} R_{0i0j} + \left( \gamma^j_\ell - \frac{N^i N^{\ell} N^j}{N^2} \right) \frac{N^k N^l}{N^2 N^2} R_{likj} \tag{B.16}
\]

which gives us,

\[
\Rightarrow R^{00} = \frac{\gamma^j_\ell}{N^2} (R_{0i0j} + N^j N^k R_{ikjl}) \tag{B.17}
\]

Also we have for the second term,

\[
\frac{R}{2N^2} = \frac{1}{N^2} \left( g^{ij} g^{0j} - g^{i0} g^{0j} \right) R_{0i0j} + \frac{1}{2N^2} (\gamma^j_\ell \gamma^k_\ell - 2\gamma^j_\ell \frac{N^k N^l}{N^2}) R_{ikjl}
\]

\[
= \frac{1}{N^2} \left( -\frac{1}{N^2} \right) (\gamma^j_\ell - \frac{N^i N^{\ell} N^j}{N^2}) R_{0i0j} + \frac{1}{N^2} \left( \frac{1}{2} \gamma^j_\ell \gamma^k_\ell - \frac{\gamma^j_\ell N^k N^l}{N^2} \right) R_{ikjl} \tag{B.18}
\]

which gives us,

\[
\Rightarrow G^{00} = \frac{\gamma^j_\ell \gamma^k_\ell}{2N^2} R_{ikjl} \tag{B.19}
\]
Substituting the above expression for $R_{ikjl}$ gives us,

$$G^{00} = \frac{\gamma^i g^{kl}}{2N^2} \left( \tilde{R}_{ikjl} + K_{ij} K_{kl} - K_{il} K_{jk} \right)$$  \hspace{1cm} (B.20)$$

and finally we have,

$$\Rightarrow \quad G^{00} = (\tilde{R} + K^2 - K_{ij} K^{ij})$$  \hspace{1cm} (B.21)$$

### B.2.3 Unstable terms from the curvature tensor components

In the ADM formalism the Ricci scalar, $R$ is given by,

$$R = g^{\mu \nu} g^{\alpha \beta} R_{\mu \alpha \nu \beta} = \gamma^k \gamma^{il} R_{ijkl} - 2 n^\mu n^\nu \gamma^j R_{\mu l \nu j},$$

$$= \tilde{R} + K^{ij} K_{ij} + K_{i}^{i} K_{j}^{j} - \frac{2}{N} \dot{K}_{i}^{i} + \frac{2}{N} N_{j} D_{j} K_{i}^{i} - \frac{2}{N} D^{2} N.$$  \hspace{1cm} (B.22)$$

We see that it contains,

$$R \geq - \frac{2}{N} \dot{K}_{i}^{i},$$  \hspace{1cm} (B.23)$$

which from the definition of $K_{ij}$ from equation (3.4) must contain $N$.

The spatial components of the Ricci curvature tensor contains,

$$R^{ij} \geq \frac{N_{i} N_{j} \dot{K}}{N^{3}} - \frac{\gamma^{jk} \gamma^{il}}{N} \dot{K}_{kl}.$$  \hspace{1cm} (B.24)$$

With these we find that the spatial components of the Einstein tensor, $G^{ij}$ therefore contains,

$$G^{ij} \geq \frac{1}{N} \left( \gamma^{ij} \dot{K} - \gamma^{jk} \gamma^{il} \dot{K}_{kl} \right).$$  \hspace{1cm} (B.25)$$

Which contains time derivatives of the lapse, $N$. 

129
Appendix C

Expansion Of Higher Order Lagrangians

The content of this chapter is based on my paper [65], ‘A Higgs Mechanism for

C.1 Ghost free scalar-vector interactions

C.1.1 bi-Galileons

We wish to find ghost free derivative couplings between a scalar $\pi$ and a vec-
tor field $A_\mu$. In order to achieve this, we will find it useful to first consider ‘bi-
Galileon’ interactions. Bi-Galileons are an extension to two scalar fields of the
original Galileon theory. They were first introduced in a general setting in [72]
and were treated in depth in [73]. The action for the two scalar fields $\pi$ and $h$,
is invariant under separate Galilean transformations: $\pi \rightarrow \pi + b^{(\pi)}_\mu x^\mu + c^{(\pi)}$ and
$h \rightarrow h + b^{(h)}_\mu x^\mu + c^{(h)}$. Furthermore, the equations of motion for both fields are
exactly second order in their derivatives. We use the notation introduced above
and follow the methods outlined in [72].

First, we enforce a symmetry relation. That is, $\mathcal{L}_h = \mathcal{L}_\pi$ with $h \leftrightarrow \pi$. I.e.

$$\epsilon^{\mu\nu\rho\lambda} \epsilon_{\alpha\beta\gamma\lambda} \pi_{\mu} h^{\alpha} (h_{\nu} h_{\rho}) \rightarrow \epsilon^{\mu\nu\rho\lambda} \epsilon_{\alpha\beta\gamma\lambda} h_{\mu} \pi^{\alpha} (\pi_{\nu} \pi_{\rho})$$

(C.1)
It will be important to remember this choice when we substitute the vector for one of the Galileons.

The general Lagrangian can be written as the sum of the following sub-Lagrangians:

\( \mathcal{E}_{(8)} : \)
\[
\alpha_{(5,0)} \mathcal{L}_{(5,0)} = \alpha_{(5,0)} \mathcal{E}_{(8)} h_1 h_2 (h_{34} h_{56} h_{78}) \quad \text{(C.2)}
\]
\[
\alpha_{(4,1)} \mathcal{L}_{(4,1)} = \alpha_{(4,1)} \mathcal{E}_{(8)} h_1 \pi_2 (h_{34} h_{56} h_{78}) \quad \text{(C.3)}
\]
\[
\alpha_{(3,2)} \mathcal{L}_{(3,2)} = \alpha_{(3,2)} \mathcal{E}_{(8)} h_1 \pi_2 (\pi_{34} h_{56} h_{78}) \quad \text{(C.4)}
\]

\( \mathcal{E}_{(6)} : \)
\[
\alpha_{(4,0)} \mathcal{L}_{(4,0)} = \alpha_{(4,0)} \mathcal{E}_{(6)} h_1 h_2 (h_{34} h_{56}) \quad \text{(C.5)}
\]
\[
\alpha_{(3,1)} \mathcal{L}_{(3,1)} = \alpha_{(3,1)} \mathcal{E}_{(6)} h_1 \pi_2 (h_{34} h_{56}) \quad \text{(C.6)}
\]
\[
\alpha_{(2,2)} \mathcal{L}_{(2,2)} = \alpha_{(2,2)} \mathcal{E}_{(6)} h_1 \pi_2 (\pi_{34} h_{56}) \quad \text{(C.7)}
\]

\( \mathcal{E}_{(4)} : \)
\[
\alpha_{(3,0)} \mathcal{L}_{(3,0)} = \alpha_{(3,0)} \mathcal{E}_{(4)} h_1 h_2 (h_{34}) \quad \text{(C.8)}
\]
\[
\alpha_{(2,1)} \mathcal{L}_{(2,1)} = \alpha_{(2,1)} \mathcal{E}_{(4)} h_1 \pi_2 (h_{34}) \quad \text{(C.9)}
\]

\( \mathcal{E}_{(2)} : \)
\[
\alpha_{(2,0)} \mathcal{L}_{(2,0)} = \alpha_{(2,0)} \mathcal{E}_{(2)} h_1 h_2 \quad \text{(C.10)}
\]
\[
\alpha_{(1,1)} \mathcal{L}_{(1,1)} = \alpha_{(1,1)} \mathcal{E}_{(2)} h_1 \pi_2 \quad \text{(C.11)}
\]

\( \mathcal{E}_{(0)} : \)
\[
\alpha_{(1,0)} \mathcal{L}_{(1,0)} = \alpha_{(1,0)} \mathcal{E}_{(0)} h \quad \text{(C.12)}
\]

Where for each sub-Lagrangian we have the corresponding symmetrical exchange of the two fields: \( \beta_{(m,n)} \mathcal{L}_{(m,n)} = \beta_{(m,n)} \mathcal{E}_{(2(m+n-1))} \pi_1 h_2 (\pi_{34} \ldots) \).
C.1.2 Bi-vectors and the scalar-vector Lagrangian

The above bi-Galileon terms can be identified as the decoupling limit of an interaction between a scalar and a vector. Due to their special properties, these interactions cannot induce a ghostly fourth mode. We construct these interaction terms by first considering the products of two vectors, \( X_\mu = \{ A_\mu, B_\mu \} \) with their derivatives, \( X_\mu \nabla \equiv \partial_\mu X_\nu = \{ \partial_\mu A_\nu, \partial_\mu B_\nu \} \) and then substituting \( B_\mu \equiv \partial_\mu h \):

\[
\mathcal{L}_{\text{bi-vector}} = \varepsilon(2n)X_1 X_2 \cdots X_{[2n-1, 2n]} \tag{C.13}
\]

Where we use \{ \} := ( ) or [ ] to indicate symmetric and anti-symmetric combinations respectfully.

When we constructed the Galileons above (see section 1.3.1), we relied on the fact that the indices associated with the partial derivatives acting on the scalar field commute (i.e. \( \pi_{\mu \nu} = \pi_{\nu \mu} \)). As a consequence of this, we had that any two indices falling on one scalar field would necessarily have had to be summed with a different epsilon tensor as otherwise we would just recover a combination that sums to zero. For vectors, however, this is not exactly true as the indices associated with the vector cannot be commuted (anti-commuted) with the indices associated with the partial derivative (i.e. \( \partial_\mu A_\nu \neq \partial_\nu A_\mu \)) and thus we need to take into account the new combinations that are possible. This subtlety was discussed for a single gauge field in [60] where they find that one extra parameter is needed for both the quartic and quintic vector Galileons. For example, in the quartic we can have a term like \( A^2 \partial_\mu A_\nu \partial_\mu A_\nu \) and another like \( A^2 \partial_\mu A_\nu \partial_\nu A_\mu \) whereas if we go to the decoupling limit we find that they are related to the same expression for the longitudinal mode \( \pi \), i.e. \( (\partial \pi)^2 (\partial_\mu \partial_\nu \pi)^2 \). Furthermore, it is easy to show that the difference between these two expressions is proportional to \( A^2 F_{\mu \nu} F^{\mu \nu} \). This means we can always re-express the sum of these two terms as one of the terms plus some factor of the previous term. For example,

\[
A^2 \partial_\mu A_\nu \partial_\mu A_\nu = \frac{1}{2} A^2 (F_{\mu \nu} F^{\mu \nu} + \partial_\mu A_\nu \partial_\nu A_\mu) \tag{C.14}
\]

For the reasons outlined in the above discussion, although we add some redundancy due to some terms differing only by a total derivative, it is convenient to
construct our Lagrangian by choosing \( X_n := aA_n + bB_n \) and \( X_{nm} := aA_{nm} + bB_{nm} \).

In order to make an effective demonstration, in the following we consider only terms up to cubic order in the fields. In such a case, we find that the terms with \( X_{nm} \) cancel and we have:

\[
\mathcal{L}^{(3)}_{\text{bi-vector}} = \mathcal{E}(4)X_1X_2(X_{34}) = \mathcal{E}(4)(aA_1 + bB_1)(aA_2 + bB_2)(aA_{34} + bB_{34}) = \mathcal{E}(4)\left\{ a^3A_1A_2(A_{34}) + a^2b[A_1A_2(B_{34}) + 2A_1B_2(A_{34})] + \text{exchange } \{aA_n, aA_{nm}\} \leftrightarrow \{bB_n, bB_{nm}\} \right\} \tag{C.15}
\]

Substituting \( \partial_\mu h \) for \( B_\mu \) gives us the cubic scalar-vector interactions:

\[
\alpha_{(3,0)} \mathcal{L}^{sv}_{(3,0)} = \alpha_{(3,0)} \mathcal{E}(4)A_1A_2(A_{34}) \tag{C.16}
\]
\[
\alpha_{(2,1)} \mathcal{L}^{sv}_{(2,1)} = \alpha_{(2,1)} \mathcal{E}(4)A_1A_2(h_{34}) \tag{C.17}
\]
\[
\alpha_{(2,1)}' \mathcal{L}^{sv}_{(2,1)}' = \alpha_{(2,1)}' \mathcal{E}(4)A_1h_2(A_{34}) \tag{C.18}
\]
\[
\beta_{(0,3)} \mathcal{L}^{sv}_{(0,3)} = \beta_{(0,3)} \mathcal{E}(4)h_1h_2(h_{34}) \tag{C.19}
\]
\[
\beta_{(1,2)} \mathcal{L}^{sv}_{(1,2)} = \beta_{(1,2)} \mathcal{E}(4)h_1h_2(A_{34}) \tag{C.20}
\]
\[
\beta_{(1,2)}' \mathcal{L}^{sv}_{(1,2)}' = \beta_{(1,2)}' \mathcal{E}(4)h_1A_2(h_{34}) \tag{C.21}
\]

Where,

\[
\begin{align*}
\alpha_{(n,m)} &= a^n b^m \text{ and } \alpha_{(n,m)'} = 2a^n b^m \quad \text{if } n > m, \\
\beta_{(n,m)} &= a^n b^m \text{ and } \beta_{(n,m)'} = 2a^n b^m \quad \text{if } n < m.
\end{align*}
\tag{C.22}
\]

These interactions return to the above cubic bi-Galileon terms in the appropriate decoupling limit, or rather, under substituting \( A_\mu \) with \( \partial_\mu \pi \).

## C.2 Proof that the Higgs self-interactions are ghost-free

Using the Higgs field decomposition in norm and phase it is straightforward to show that the higher order operators defined in equations (2.20)-(2.22), collect-
tively called $\mathcal{L}_i$, are ghost-free, since the equations of motion for the fields involved contain at most two time derivatives. For example, making the redefinition: $\phi \tilde{A}_\mu := \tilde{A}_\mu$ with $g = 1$, we find after some integrations by part that we always generate combinations of the ghost free terms found from equation (C.15) and its extension. For example, for the dimension eight operator we find:

$$\mathcal{L}(8) = \frac{1}{2! \Lambda^4} \epsilon_{\mu_1 \mu_2 \gamma \nu} e^{v_1 v_2 \gamma \nu} (\partial^{\mu_1} \phi \partial_{v_1} \phi + \tilde{A}^{\mu_1} \tilde{A}_{v_1}) \left\{ \alpha_{(8)}(\phi \partial^{\mu_2} \partial_{v_2} \phi - \tilde{A}^{\mu_2} \tilde{A}_{v_2}) - 2\beta_{(8)}(\partial^{\mu_2} \phi \tilde{A}_{v_2} + \partial_{v_2} \phi \tilde{A}^{\mu_2} + \phi (\partial^{\mu_2} \tilde{A}_{v_2} + \partial_{v_2} \tilde{A}^{\mu_2})) \right\} \quad (C.23)$$

We can integrate by parts the terms factored by $\alpha_{(8)}$ and then notice that for $n \geq 3$, terms schematically like $\sim e e \tilde{A}^n$ and $\sim e e (\partial \phi)^n$, which also appear in the first two terms factored by $\beta_{(8)}$, do not contribute. Subsequently, we find that,

$$\mathcal{L}(8) = \frac{1}{2! \Lambda^4} \epsilon_{\mu_1 \mu_2 \gamma \nu} e^{v_1 v_2 \gamma \nu} \phi \left\{ \alpha_{(8)} [\partial^{\mu_1} \phi \partial_{v_1} \phi (\partial^{\mu_2} \partial_{v_2} \phi) + 2 \tilde{A}^{\mu_1} \tilde{A}_{v_1} (\partial^{\mu_2} \partial_{v_2} \phi)] + \partial_{v_1} \phi \tilde{A}_{v_2} \partial^{\mu_1} \tilde{A}^{\mu_2} + \partial_{v_1} \phi \tilde{A}^{\mu_1} \partial^{\mu_2} \tilde{A}_{v_2} \right\} - 2\beta_{(8)}(\partial^{\mu_1} \phi \partial_{v_1} \phi + \tilde{A}^{\mu_1} \tilde{A}_{v_1}) [\partial^{\mu_2} \tilde{A}_{v_2} + \partial_{v_2} \tilde{A}^{\mu_2}] \quad (C.24)$$

The term inside the curly braces in equation (C.24) corresponds to a combination of terms found in equation (C.15) which are ghost free by construction. Note that the presence of an additional function of $\phi$ multiplying these terms does not prevent them from being ghost free.

The quartic and quintic Lagrangians can also be found in a similar fashion and are

---

$^1$This rescaling operation does not change the number of derivatives in the equations of motion, hence does not change the usual arguments about the presence, or absence, of ghosts.
given by,

\[
\mathcal{L}_{(12)} = \frac{1}{\Lambda^8} \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4} e^{v_1 v_2 v_3 v_4} \varphi^2 \left\{ \alpha_{(12)} \left[ \phi^{\mu_1} \phi_{\nu_1} \left( \phi^{\mu_2} \phi^{\mu_3} \phi^{\mu_4} \right) + 2 \bar{A}^{\mu_1} A^{\nu_1} \left( \phi^{\mu_2} \phi^{\mu_3} \phi^{\mu_4} \right) \\
+ \phi^{\mu_1} \bar{A}^{\nu_1} \left( \phi^{\mu_2} \partial_{\nu_3} A^{\mu_3} \right) + \phi^{\mu_1} A^{\nu_1} \left( \phi^{\mu_2} \partial_{\nu_3} \bar{A}^{\mu_3} \right) \right] \\
+ \beta_{(12)} \left[ \phi^{\mu_1} \phi_{\nu_1} \left( \bar{A}^{\mu_2} A^{\nu_2} \right) + \bar{A}^{\mu_1} A^{\nu_1} \left( \bar{A}^{\mu_2} A^{\nu_2} \right) \right] \right\} \tag{C.25}
\]

and

\[
\mathcal{L}_{(16)} = \frac{1}{\Lambda^8} \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4} e^{v_1 v_2 v_3 v_4} \varphi^3 \left\{ \alpha_{(16)} \left[ \phi^{\mu_1} \phi_{\nu_1} \left( \phi^{\mu_2} \phi^{\mu_3} \phi^{\mu_4} \right) \\
- \frac{2}{3} \bar{A}^{\mu_1} A^{\nu_1} \phi^{\mu_2} \phi^{\mu_3} \phi^{\mu_4} \right] - \frac{1}{3} \phi^{\mu_1} \bar{A}^{\nu_1} \phi^{\mu_2} \phi^{\nu_3} \phi^{\mu_4} \\
+ \frac{1}{3} \phi^{\mu_1} \bar{A}^{\nu_1} \phi^{\mu_2} \phi^{\nu_3} \phi^{\mu_4} \partial_{\nu_3} \bar{A}^{\mu_3} \right] \\
+ \beta_{(16)} \left[ \phi^{\mu_1} \phi_{\nu_1} \left( \bar{A}^{\mu_2} A^{\nu_2} \bar{A}^{\mu_3} \right) + \bar{A}^{\mu_1} A^{\nu_1} \left( \bar{A}^{\mu_2} A^{\nu_2} \bar{A}^{\mu_3} \right) \right] \right\} \tag{C.26}
\]

where \( \phi_\mu \equiv \partial_\mu \varphi \) and \( \bar{A}_{\mu \nu} \equiv \partial_{(\mu} A_{\nu)} \).

Again we see that they are indeed free of ghosts as the terms inside the curly braces correspond to combinations of the ghost free terms found from substituting \( \partial_\mu \varphi \) for \( B_\mu \) in equation \( (C.13) \) with \( \{ \} := (\) and \( n = 4 \& 5 \) respectively.

**Expansion about the vacuum** As we discussed in chapter 2 the phenomenon of spontaneous symmetry breaking relies on the Higgs achieving a non-zero vacuum expectation value, \( v \). In addition to this, the Higgs field develops non-trivial dynamics via fluctuations about the vacuum. In order to understand this we expand the field about the vacuum, \( v \) with a small perturbation \( h \):

\[
\varphi := \left( v + \frac{h}{\sqrt{2}} \right) \tag{C.27}
\]

135
With this definition of $\phi$ the expressions for our operators $L_{\mu\nu}, P_{\mu\nu} & Q_{\mu\nu}$ and $V(\phi)$, given in equations 2.3, 2.24, 2.25 & 2.26 become,

$$L_{\mu\nu} \rightarrow \frac{1}{2} \partial_{\mu} h \partial_{\nu} h + g^2 (v + h/\sqrt{2})^2 \hat{A}_{\mu} \hat{A}_{\nu}$$  \hspace{1cm} (C.28)
$$P_{\mu\nu} \rightarrow \frac{\sqrt{2}}{2} (v + h/\sqrt{2}) \partial_{\mu} h \partial_{\nu} h - g^2 (v + h/\sqrt{2})^2 \hat{A}_{\mu} \hat{A}_{\nu}$$  \hspace{1cm} (C.29)
$$Q_{\mu\nu} \rightarrow g \sqrt{2} (v + h/\sqrt{2}) \{ \partial_{\mu} h \hat{A}_{\nu} + \frac{\sqrt{2}}{2} (v + h/\sqrt{2}) \hat{A}_{\mu\nu} \}$$  \hspace{1cm} (C.30)
$$V(\phi) \rightarrow -\frac{\lambda}{2} v^4 + \frac{\lambda}{v} h^2 - \frac{\lambda}{8} h^4$$  \hspace{1cm} (C.31)

We now expand out the Lagrangian $\mathcal{L}_{\text{tot}}$ around the background of the Higgs’s vev, we use the $E_{(2n)}$ notation in order to keep the terms from the higher order operators, $\mathcal{L}_{(8),(12),(16)}$ as simple as possible,

$$\mathcal{L}_{\hat{A},h} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - (m_A^2 - \sqrt{2} g m_A h - \frac{1}{2} g^2 h^2) \hat{A}^2$$
$$- \frac{1}{2} (\partial h)^2 + \frac{\lambda}{2} v^4 - \frac{1}{2} m_A^2 h^2 - \frac{\sqrt{2}}{2} m_A h^3 - \frac{\lambda}{8} h^4$$
$$+ \sum_{n=2}^{n=4} \frac{1}{(2n) \delta_{(2n)}} (v + \frac{\sqrt{2}}{2} h)^{n-1} \left( \frac{1}{2} h_1 h_2 + g^2 (v + \frac{\sqrt{2}}{2} h)^{\hat{A}_1 \hat{A}_2} \right)$$
$$\cdot \left\{ \alpha_{(4n)} (h_{34} - g^2 (v + \frac{\sqrt{2}}{2} h) \hat{A}_{3} \hat{A}_{4}) \ldots \right\}$$
$$+ \beta_{(4n)} (\sqrt{2} g)^{n-1} (h_{(3} \hat{A}_{4)} + \frac{2}{\sqrt{2}} (v + \frac{\sqrt{2}}{2} h) \hat{A}_{3} \hat{A}_{4}) \ldots$$
$$\ldots (h_{(2n-1} \hat{A}_{2n)} + \frac{2}{\sqrt{2}} (v + \frac{\sqrt{2}}{2} h) \hat{A}_{2n-1} \hat{A}_{2n}) \right\}$$  \hspace{1cm} (C.32)

with,

$$m_A \equiv g v,$$  \hspace{1cm} (C.33)
$$m_h \equiv \sqrt{2} \lambda v,$$  \hspace{1cm} (C.34)

The expansion about the Higgs’s vev and the resulting Lagrangian, $\mathcal{L}_{\hat{A},h}$ in equation (C.32), describes a fully $U(1)$ invariant theory of four degrees of freedom:
two scalars $\pi, h$ and a massless vector $A_\mu$. However, in general, we must go to the
unitary gauge ($\pi = 0$) to reveal the true physical degrees of freedom. Using this
gauge we find that our theory describes a physical system with two interacting
fields: a scalar field $h$, representing the Higgs, interacting with a massive vector
field $\hat{A}_\mu$. Furthermore, we see that indeed not only does the vev of the Higgs
produce a mass for the vector boson but also new higher dimensional operators
appear with the structure of those studied in [59, 60] (see section 1.3.4). In order
to connect with those results, we expand out the terms in equation (C.32) which
are both $O(h^0)$ and $O(A)$ and above. Notice that in this case, the terms factored
by $\alpha_{(4n)}$ do not contribute and we have,

$$L_{\hat{A}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - m_A^2 \hat{A}^2 + \sum_{n=2}^{n=4} \hat{\beta}_{(4n)} \left( e_{(2n)} \hat{A}_1 \hat{A}_2 \hat{A}_3 \ldots \hat{A}_{2n-1} \right)$$

(C.35)

where,

$$\hat{\beta}_{(4n)} = \frac{\beta_{(4n)} g_{n+1} v_{2n}}{\Lambda^{2n}}$$

(C.36)

Which shows we have indeed recovered the extended Proca theory.

**The decoupling limit** As was discussed in section 1.3.4, there exists a frame of
reference (the infinite momentum gauge) in which the Stückelberg field becomes
equivalent to the (decoupled) longitudinal mode of the massive vector. Following
this example, we choose the parameters in our theory to have the following limits
[65]:

$$g \to 0 \quad , \quad \lambda \to 0 \quad , \quad \alpha_{(4n)} \to 0 \quad , \quad \beta_{(4n)} \to 0 \quad , \quad v \to \infty ,$$

(C.37)

such that

$$m_A \to 0 \quad , \quad m_h \to 0 \quad , \quad \tilde{\alpha}_{(4n)} \to 0 \quad , \quad \tilde{\beta}_{(4n)} \to 0 ,$$

$$\frac{\tilde{\alpha}_{(4n)}}{m_A^{n+1}} = \text{fixed} \equiv \frac{1}{\Lambda^{n+1}_{\alpha_{(4n)}}} , \quad \frac{\tilde{\beta}_{(4n)}}{m_A^{n+1}} = \text{fixed} \equiv \frac{1}{\Lambda^{n+1}_{\beta_{(4n)}}} ,$$

(C.38)

where we have defined,

$$\tilde{\alpha}_{(4n)} \equiv \frac{\alpha_{(4n)} g_{n+1} v_{2n}}{\Lambda^{2n}}$$

(C.39)
and where $\Lambda_{\alpha(4n)}$ and $\Lambda_{\beta(4n)}$ are mass scales that, as we will see in a moment, are associated with the strength of the Galileon interactions. Notice, however that,

$$\frac{\Lambda_{\alpha(4n)}}{\Lambda_{\beta(4n)}} = \left( \frac{\beta(4n)}{\alpha(4n)} \right)^{\frac{1}{p+1}} \tag{C.40}$$

furthermore, the previous limits also imply that $g/m_A = 1/\nu \to 0$. We find that in order for the kinetic term of the longitudinal mode to have the correct normalisation we must rescale the field and define $\pi = \hat{\pi}/(\sqrt{2}m_A)$ [65]. Thus in the limits (equations (C.37 and (C.38)), the total Lagrangian in equation (C.32), $\mathcal{L}_{A,h} \to \mathcal{L}_{A,\pi,h}$ results:

$$\mathcal{L}_{A,\pi,h} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_{n=2}^{n=4} \delta(2n) \left( h_1 h_2 + \hat{\pi}_1 \hat{\pi}_2 \right) \left\{ \frac{1}{\Lambda_{\alpha(4n)}} h_3 h_4 \ldots h_{2n-12n} \right\}$$

\[+ \left( -\sqrt{2} \right)^{n-1} \frac{\beta(4n)}{m_A^{n+1}} \hat{\pi}_3 \ldots \hat{\pi}_{2n-12n} \right\} \tag{C.41}\]

\[= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_{n=2}^{n=4} \delta(2n) \left( h_1 h_2 + \hat{\pi}_1 \hat{\pi}_2 \right) \left\{ \frac{1}{\Lambda_{\alpha(4n)}} h_3 h_4 \ldots h_{2n-12n} \right\} \]

\[+ \left( -\sqrt{2} \right)^{n-1} \frac{\beta(4n)}{\Lambda_{\beta(4n)}} \hat{\pi}_3 \ldots \hat{\pi}_{2n-12n} \right\} \tag{C.42}\]

Thus we can clearly see that in the decoupling limit our theory recovers the bi-Galileon structure discussed in section C.1.1. This is a theory where not only the Goldstone boson describing the longitudinal mode of the vector polarisation but the Higgs field is Galileonic. In addition, they are coupled together in such a way that their interactions form a bi-Galileon theory. The recovery of this special structure could bode well for the quantum radiative stability of our theory as powerful non-renormalisation theorems exist for Galileonic theories (which is ultimately related to the fact that they are Wess-Zumino terms\(^2\)).

\(^2\)This is because the quantum effective action must be built out of operators that are strictly invariant under the relevant symmetries whereas Wess-Zumino terms only shift by a total derivative.[34]
Finally, note that in this section we have discussed how a Galileonic-Higgs mechanism works for a theory with an abelian gauge symmetry, however with a slight modification the discussion can easily be generalised to theories with non-abelian symmetries, see [65] for details.

C.3 Consistency of our Higgs higher-dimensional interactions

In this section we would like to develop some arguments aimed to show that the Higgs interactions contained in Lagrangians equations (2.20-2.22) are consistent, in the sense that they are free of ghost degrees of freedom. We specialise to the case of abelian symmetry breaking, but the same arguments can be straightforwardly extended to the non-abelian case. The interactions in equations (2.20-2.22) are built in terms of totally antisymmetric $\varepsilon$-tensors. Once expanding the covariant derivatives acting on the Higgs field, and decomposing the Higgs in norm and phase as in the main text, we find that there can arise three kinds of possibly dangerous combinations:

$$\varepsilon^{a_1 a_2 \ldots} \varepsilon_{\beta_1 \beta_2 \ldots} \partial_{a_1} \partial_{\beta_1} \phi \partial_{a_2} \phi \partial_{\beta_2} \phi \ldots \quad (C.43)$$

$$\varepsilon^{a_1 a_2 \ldots} \varepsilon_{\beta_1 \beta_2 \ldots} \partial_{a_1} \phi \partial_{\beta_1} \phi \partial_{a_2} \phi \partial_{\beta_2} \phi \ldots \quad (C.44)$$

$$\varepsilon^{a_1 a_2 \ldots} \varepsilon_{\beta_1 \beta_2 \ldots} A_{a_1} A_{\beta_1} \partial_{a_2} \phi \partial_{\beta_2} \phi \ldots \quad (C.45)$$

where the dots contain additional pieces, of the same type as the above, or other contributions that contain single or no derivatives of $\phi$ – always contracted with the $\varepsilon$-tensor. Interactions as the ones listed in equations (C.43-C.45), when appearing in the Lagrangian, are a priori dangerous because they contain second derivatives acting on the scalar $\phi$, and/or the gauge potential $A_\mu$. We have to ensure that the corresponding equations of motion do not contain more than two space-time derivatives of the fields involved. Moreover, the equation of motion for $A_0$ should not contain time derivatives acting on $A_0$ itself, so to ensure that $A_0$ is a constraint. These requirements, together with the positivity of the kinetic terms, are sufficient to ensure the absence of ghosts.
Interactions as equation (C.43) are the familiar scalar Galileon interactions [35]: the structure of the $\varepsilon$-tensors does not allow them to generate higher space-time derivatives in their equations of motion. Indeed, the equations of motion for a scalar field $\phi$ can certainly lead to derivatives acting on the first part, $\partial_{\alpha_1} \partial^{\beta_1} \phi$, of (C.43) – as for example contributions like

$$
\varepsilon^{\alpha_1 \alpha_2 \ldots} \varepsilon_{\beta_1 \beta_2 \ldots} \partial_{\alpha_3} \partial^{\beta_3} \partial_{\alpha_4} \partial^{\beta_4} \phi \ldots \quad \text{or} \quad \varepsilon^{\alpha_1 \alpha_2 \ldots} \varepsilon_{\beta_1 \beta_2 \ldots} \partial_{\alpha_3} \partial_{\alpha_4} \partial^{\beta_3} \phi \ldots \quad (C.46)
$$

But the $\varepsilon$-tensor makes them vanishing: the operator $\partial_{\alpha_1} \partial_{\alpha_2}$ is symmetric on its indexes, and gives zero when contracted with the $\varepsilon^{\alpha_1 \alpha_2 \ldots}$. This fact is familiar and was developed in [43]. Similar arguments can be made to show that equation (C.44) and equation (C.45) cannot contribute to the equation of motion for $A_0$ with terms containing the time derivative of $A_0$ itself (see also [59, 60]). Since $A_\mu$ is always contracted with the $\varepsilon$-tensor, it is simple to convince oneself that the only possibly dangerous contributions from the equation of motion of $A_0$ – that is the ones that might have time derivatives acting on $A_0$ – are pieces that contain first or second derivatives acting on the gauge potential, as

$$
\varepsilon^{0 \alpha_2 \ldots} \varepsilon_{\beta_1 \beta_2 \ldots} \partial^{\beta_1} A^{\beta_2}, \quad \text{or} \quad \varepsilon^{0 \alpha_2 \ldots} \varepsilon_{\beta_1 \ldots} \partial^{\beta_1} A_{\alpha_2}, \quad \text{or} \quad \varepsilon^{0 \alpha_2 \ldots} \varepsilon_{\beta_1 \beta_2 \ldots} \partial_{\alpha_2} \partial^{\beta_1} A^{\beta_2}.
$$

(C.47)

In the first option, the index $\beta_1$ and $\beta_2$ can not simultaneously take the value zero, due to the antisymmetric property of the $\varepsilon$-tensor, hence this contribution vanishes for the possibly dangerous case. A similar argument exists for the second and third option. The crucial fact is that one of the indexes of the $\varepsilon$-tensor is already fixed to be zero since we are evaluating the equation of motion for $A_0$; hence, $\alpha_2 \neq 0$ and we cannot have time derivatives acting on $A_0$. 

140
Appendix D

Non-minimal couplings

D.1 The necessity of non-minimal couplings

In section 3.2 we saw that the need for non-minimal couplings (NMC) for the covarianisation of the vector Galileons depended on whether we had made any integrations by parts before covariantisation. Indeed, for the previously proposed ‘vector-Horndeski’ class, we found that the form of the non-minimal coupling was greatly restricted by the need for a cancellation whereas for the general vector Galileons no non-minimal term was needed as the cancellation was provided by the antisymmetry of the epsilon tensors. It is instructive to look at what results if we reverse the process and covariantise before integrating by parts.

Focussing on the generally contracted quartic vector Galileon with minimal substitution, we find that due to the antisymmetric properties of the epsilon tensors, repositioning the derivatives by integration by parts cannot lead to terms with more than one derivative.

\[
\mathcal{L}_{vG|ms}^{(4)} = \frac{1}{2} \sqrt{-g} \epsilon_{\mu_1 \mu_3 \mu_5 \lambda} \epsilon_{\nu_2 \nu_4 \nu_6} \nabla_{\nu_4} (A_{\mu_1} A_{\nu_2} A_{\mu_3} A_{\nu_6}) A_{\mu_5} \\
\rightarrow - \frac{1}{2} \sqrt{-g} \epsilon_{\mu_1 \mu_3 \mu_5 \lambda} \epsilon_{\nu_2 \nu_4 \nu_6} \nabla_{\nu_4} A_{\mu_1} A_{\nu_2} A_{\mu_3} A_{\mu_5} \\
= - \frac{1}{2} \sqrt{-g} \epsilon_{\mu_1 \mu_3 \mu_5 \lambda} \epsilon_{\nu_2 \nu_4 \nu_6} \nabla_{\nu_4} (A_{\mu_1} A_{\nu_2} A_{\mu_3} A_{\mu_5}) \\
= \frac{1}{2} \sqrt{-g} \epsilon_{\mu_1 \mu_3 \mu_5 \lambda} \epsilon_{\nu_2 \nu_4 \nu_6} \nabla_{\nu_4} A_{\mu_1} A_{\nu_2} A_{\mu_3} A_{\mu_5} \\
\]

(D.1) (D.2) (D.3) (D.4)
where the term with second order derivatives cancelled and the last term resulted from the antisymmetry acting on $\mu_1$ and $\mu_3$. We can re-write the Lagrangian as,

$$L^{(4)}_{vG|ms} = \frac{1}{2} \sqrt{-g} e^{\mu_1 \mu_3 \lambda \kappa} e^{\nu_2 \nu_4} \lambda \kappa \{ A_\rho A^\rho A_{\mu_1 v_2 A_{\mu_3 v_4}} - 2 A_\rho A_{\mu_1} A_{v_2 A_{\mu_3 v_4}} \}$$  \hspace{1cm} (D.5)

Each term transforms via integrations by parts as:

$$(A) \rightarrow -2 A_\rho A_{\mu_1} A_{v_2 A_{\mu_3 v_4}} - A_\rho A^\rho A_{\mu_1 A_{\mu_3 v_4} v_2}$$  \hspace{1cm} (D.6)

$$(B) \rightarrow A_\rho A^\rho [ A_{\mu_1 v_2 A_{\mu_3 v_4} + A_{\mu_1} A_{\mu_3 v_4} v_2 ]$$  \hspace{1cm} (D.7)

As expected from above, the form of the generally contracted quartic vector Galileon is invariant under simultaneous integrations by parts. Therefore if we only consider term (A) it can be rewritten as,

$$\frac{1}{2} \sqrt{-g} e^{\mu_1 \mu_3 \lambda \kappa} e^{\nu_2 \nu_4} \lambda \kappa A_\rho A^\rho A_{\mu_1 v_2 A_{\mu_3 v_4}}$$  \hspace{1cm} (D.8)

$$- 2 A_\rho A_{\mu_1 v_2 A_{\mu_3 v_4}}$$  \hspace{1cm} (D.9)

The second term can has two covariant derivatives acting on a vector and so can be written in terms of the Riemann tensor:

$$A_2 \rightarrow - \frac{1}{4} \sqrt{-g} e^{\mu_1 \mu_3 \lambda \kappa} e^{\nu_2 \nu_4} \lambda \kappa A_\rho A^\rho A_{\mu_1} A_{v R^{\sigma}_{\mu_3 v_4} v_2} = - \frac{1}{2} \sqrt{-g} A_\rho A^\rho A_{\mu_1} A_{v R^{\mu v}_{\mu_3 v_4} v_2}$$  \hspace{1cm} (D.10)

Which reveals a ghost degree of freedom due to the presence of a coupling of the vectors with a total divergence. We see that there are two ways to cure this:

1. Remove this term by reconstructing the cancellation found in the generally contracted epsilon tensor form of the quartic vector Galileon,

$$L^{(4)}_{vG|ms} = \frac{1}{2} \sqrt{-g} e^{\mu_1 \mu_3 \lambda \kappa} e^{\nu_2 \nu_4} \lambda \kappa A_\rho A^\rho A_{\mu_1 v_2 A_{\mu_3 v_4}} + \frac{1}{2} A_{\mu_1} A_{\mu_3 v_4 v_2}$$  \hspace{1cm} (D.11)

$$- \frac{1}{4} \sqrt{-g} e^{\mu_1 \mu_3 \lambda \kappa} e^{\nu_2 \nu_4} \lambda \kappa A_\rho A^\rho A_{\mu_1 v_2 A_{\mu_3 v_4}} - 2 A_\rho A_{\mu_1} A_{v_2 A_{\mu_3 v_4}}$$  \hspace{1cm} (D.12)
2. Or include the non-minimal coupling \( \frac{1}{3} \sqrt{-g} A^2 A_\mu A_\nu g^{\mu \nu} R \) to construct the ghost free combination: \( -\frac{1}{2} \sqrt{-g} A^2 A_\mu A_\nu G^{\mu \nu} \).

To see that this combination is ghost free we can re-express it in terms of geometric quantities: \( A^2 A_\mu A_\nu G^{\mu \nu} \sim -X^2 n_\mu n_\nu G^{\mu \nu} \) but \( 2 n_\mu n_\nu G^{\mu \nu} = \bar{R} + K^2 - K_\mu \nu K^{\mu \nu} \) which does not contain any time derivatives of the lapse. Thus we see that path 1) gives us the minimally covariantised vector Galileon class, whereas path 2) gives us the vector-Horndeski class. Furthermore, we see that adding the ghost free non-minimal coupling, \( \sqrt{-g} A^2 A_\mu A_\nu G^{\mu \nu} \) to the minimally covariantised class takes us to the vector-Horndeski class and vice versa.

Focussing on \( \sqrt{-g} A^2 A_\mu A_\nu G^{\mu \nu} \) we find after integration by parts,

\[
\sqrt{-g} A^2 A_\mu A_\nu G^{\mu \nu} = \sqrt{-g} A^2 A_\mu A_\nu \left( R^{\mu \nu} - \frac{1}{2} g^{\mu \nu} R \right) \tag{D.13}
\]

\[
= \sqrt{-g} A^2 \epsilon^{\mu_1 \mu_2 \lambda \kappa} \epsilon_{\nu_2 \nu_3 \nu_4} \lambda \kappa A_{\mu_1 A_{\mu_3 \nu_4 v_2} - \sqrt{-g} \frac{1}{2} A^4 R} \tag{D.14}
\]

I.P \Rightarrow \begin{align*}
= \sqrt{-g} \epsilon^{\mu_1 \mu_2 \lambda \kappa} \epsilon_{\nu_2 \nu_3 \nu_4} \lambda \kappa \left( A^2 A_{\mu_1 v_2 A_{\mu_3 v_4} - 2 A_\rho A_{\mu_1 A_{\mu_3 v_2} A_{\mu_3 v_4}} \right) & - \sqrt{-g} \frac{1}{2} A^4 R \tag{D.15} \\
& - 2 \sqrt{-g} A^2 \epsilon^{\mu_1 \mu_2 \lambda \kappa} \epsilon_{\nu_2 \nu_3 \nu_4} \lambda \kappa A_{\mu_1 A_{\mu_3 v_4 v_2} - \sqrt{-g} \frac{1}{2} A^4 R} \tag{D.16} \\
& + \sqrt{-g} \epsilon^{\mu_1 \mu_2 \lambda \kappa} \epsilon_{\nu_2 \nu_3 \nu_4} \lambda \kappa \left( A^2 A_{\mu_1 A_{\mu_3 v_2 v_4} - 2 A_\rho A_{\mu_1 A_{\mu_3 v_2} A_{\mu_3 v_4}} \right)
\end{align*}

Therefore, we see that we have confirmed,

\[
\frac{1}{2} \sqrt{-g} A^2 A_\mu A_\nu G^{\mu \nu} = (\text{vector-Horndeski}) + (\text{vector Galileon}) \tag{D.17}
\]

Furthermore, we see that this also provides a root to proving the general ghost free property of the vector Galileon. That is, since we ‘know’ that vector-Horndeski is ghost free, then we can show that the ‘vector Galileon’ term is ghost free by examining the properties of the non-minimal coupling.
D.1.1 Non-minimal couplings for a single vector field

Since $L_{acb} = G_{ab}$, all the ghost free NMC terms that we have investigated have been of the form: $f(A^2)A_a A_b (B_{cd} + g_{cd}) L^{acbd}$. Given that there are no more terms for the Galileons in four dimensions, we could ask whether this expression exhausts the possibilities for the NMC’s? Indeed, since the number of indices on the epsilon tensor is limited by the number of dimensions, it would appear so. I.e. for three dimensions we would expect to have $f(A^2) \epsilon^{(6)}(A_1 A_2 + g_{12}) R_{3546}$ and for two dimensions $L_{acb} = G_{ab} = 0$ so we just have the Ricci scalar.

Furthermore, we already know that the Einstein-Hilbert action is formed from the dimensional continuation of the Euler-form, $S_{EH} = \int \theta^* \Omega$ and that we can reproduce the covariant Galileons by dimensionally reducing these forms via Kaluza-Klein compactification or via brane embeddings. Could we use this knowledge to suggest a systematic construction of ghost free non-minimal couplings for the vectors? We could envisage something of the form,

\begin{equation}
S_2 \sim \int \Omega_{ab} \tag{D.18}
\end{equation}

\begin{equation}
S_3 \sim \int \theta^*_a \Omega_{bc} \rightarrow \int f(A^2) \epsilon^{(6)}(A_1 A_2) R_{3546} \tag{D.19}
\end{equation}

\begin{equation}
S^E_{EH} \sim \int \theta^*_a \Omega_{cd} \rightarrow \int f(A^2) \epsilon^{(8)}(A_1 A_2 (g_{34} + B_{56}) R_{5768} \text{or,} \tag{D.20}
\end{equation}

\begin{equation}
S_4^G \sim \int \Omega_{ab} \Omega_{bc} \tag{D.21}
\end{equation}

\begin{equation}
S_5 \sim \int \theta^*_a \Omega_{bc} \Omega_{de} \rightarrow \int f(A^2) \epsilon^{(10)}(A_1 A_2 R_{3546} R_{798} \text{etc.} \tag{D.22}
\end{equation}

D.1.2 Non-minimal couplings for multiple vector fields

Non-minimal vector couplings motivated from covariant Galileons. Given a vector, $X_\mu$ non-minimally coupled to the double dual Riemann tensor $\mathcal{L}^{abcd}$ via,

\begin{equation}
f(X^2)X_a X_b X_{cd} \mathcal{L}^{acbd} \tag{D.23}
\end{equation}

we immediately find that its longitudinal mode $\partial_\mu \pi$, satisfies the NMC of the covariant quintic Galileon:

\begin{equation}
f(\Pi) \nabla_a \pi \nabla_b \pi \Pi_{cd} \mathcal{L}^{acbd} \tag{D.24}
\end{equation}

144
where $\Pi \equiv \nabla_\mu \pi \nabla^\mu \pi$ and $\Pi_{\mu \nu} \equiv \nabla(\mu \nabla_{\nu})\pi$. Now, we can set $X_\mu = A_\mu + B_\mu$ to give us the NMC for the bi-vector-Galileonic system:

$$f(X^2)X_aX_bX_{cd}\mathcal{L}^{abcd} \rightarrow f(X^2)(A + B)_a(A + B)_b(A + B)_{cd}\mathcal{L}^{abcd} \quad (D.25)$$

If, in addition, we denote the longitudinal modes of $A_\mu$ and $B_\nu$ by $\phi$ and $\chi$ respectively, then we recover multiple non-minimal coupling terms for the covariant bi-Galileon system by simply making the substitution $\pi = \phi + \chi$:

$$f(\Pi)\nabla_a\pi\nabla_b\pi\Pi_{cd}\mathcal{L}^{abcd} \rightarrow f(\Pi)\nabla_a(\phi + \chi)\nabla_b(\phi + \chi)\nabla_c(\phi + \chi)\mathcal{L}^{abcd}. \quad (D.26)$$

This gives us NMCs for two different covariant Galileons, $\phi$ and $\chi$ and 16 mixed NMCs. If we set $\partial_\mu \pi = \{\dot{\pi}, \ddot{\pi}\}$, then all 18 equations are of the form:

$$\mathcal{L}_{NMC} \sim \frac{1}{N^2} \phi^m \chi^n \Gamma_{ij}^{0(0)} \mathcal{L}^{ij} \quad \text{with} \quad (n + m) = 5 \quad \text{and} \quad 3 \geq n, m \geq 0. \quad (D.27)$$

Since all other terms are zero, we find that we can construct ghost free non-minimal coupling terms for bi-Galileons via using the double Riemann tensor, $\mathcal{L}^{abcd}$. Moreover, this construction can be straightforwardly generalised for multi-Galileons if we extend the field definitions to include more fields: $\pi \rightarrow \sum_i \pi_i$.

**Non-minimal couplings from the G-Higgs model.** The naïve covariantisation of the $\mathcal{L}^\alpha_{(16)}$ leads to a non-minimal coupling of the form,

$$\mathcal{L}^\alpha_{(16)} = \frac{3}{4} \phi^* \phi L_{E}(8) L_{12} (P_{34} - L_{34}) R_{5768}, \quad (D.28)$$

where $L_{nm} \equiv \Re[\mathcal{D}_m \phi \mathcal{D}_n \phi^*]$ and $P_{mn} = \Re[\phi^* \mathcal{D}_m \mathcal{D}_n \phi]$. We see that, as well as mixed scalar-vector terms, we also recover the pure scalar term of the previously discussed Lagrangian whereas the pure vector component cancels out due to the anti-symmetric properties of the Riemann tensor.

It is interesting to note that the naïve covariantisation of our Higgs construction naturally constructs a combination of both NMC terms discussed in [137] (where instead of $f$ being a constant, we have $f^2 \sim L\phi^2$). Indeed, since $\mathcal{L} \not\sim \mathcal{N}$, we find
that we can extend the types of non-minimal couplings beyond what was discussed for the pure vector case. For example we can include terms of the form,

\begin{align}
    f_6 \mathcal{E}(g_{12} + L_{12}) R_{3546} & \sim f_6 (g_{\mu\nu} + L_{\mu\nu}) G^{\mu\nu} \\
    f_8 \mathcal{E}(g_{34} + L_{34} + P_{34} + Q_{34}) R_{5768} & \sim f_8 L_{\alpha\beta} (g_{\mu\nu} + L_{\mu\nu} + P_{\mu\nu} + Q_{\mu\nu}) \Sigma^{\alpha\mu\beta\nu}
\end{align}

where \( f_i \) can be functions of \( f_i(\phi, \phi^*, L) \) etc. Thus we see that the longitudinal mode of our covariantised-Galileonic-Higgs theory can be embedded in a larger theory than the Generalised-multi-Galileons of [137]. Moreover, note that the term \( \mathcal{L}_* \) discussed in [137] originated from multifield-DBI-Galileons and corresponds to the decoupling limit, or rather, longitudinal mode of,

\begin{align}
    f_8 \mathcal{E}(g_{12} + L_{34}) R_{5768} & \sim f_8 \partial_\mu \varphi \partial_\nu \varphi A_\alpha A_\sigma \Sigma^{\mu\nu\rho\sigma}.
\end{align}

146
Bibliography


[28] Planck, “http://sci.esa.int/planck/,”.


154


161
Certificate of Ethics Review

<table>
<thead>
<tr>
<th>Project Title:</th>
<th>The role of vector fields in infrared modifications of gravity</th>
</tr>
</thead>
<tbody>
<tr>
<td>User ID:</td>
<td>712562</td>
</tr>
<tr>
<td>Name:</td>
<td>Matthew Dean Hull</td>
</tr>
<tr>
<td>Application Date:</td>
<td>27/05/2015 23:39:45</td>
</tr>
</tbody>
</table>

You must download your referral certificate, print a copy and keep it as a record of this review.

The FEC representative for the School of Engineering is Marco Bruni

It is your responsibility to follow the University Code of Practice on Ethical Standards and any Department/School or professional guidelines in the conduct of your study including relevant guidelines regarding health and safety of researchers including the following:

- University Policy
- Safety on Geological Fieldwork

It is also your responsibility to follow University guidance on Data Protection Policy:

- General guidance for all data protection issues
- University Data Protection Policy

SchoolOrDepartment: ICG
PrimaryRole: PostgraduateStudent
SupervisorName: Kazuya Koyama
HumanParticipants: No
PhysicalEcologicalDamage: No
HistoricalOrCulturalDamage: No
HarmToAnimal: No
HarmfulToThirdParties: No
OutputsPotentiallyAdaptedAndMisused: No
Confirmation-ConsideredDataUse: Confirmed
Confirmation-ConsideredImpactAndMitigationOfPotentialMisuse: Confirmed
Confirmation-ActingEthicallyAndHonestly: Confirmed
Supervisor Review

As supervisor, I will ensure that this work will be conducted in an ethical manner in line with the University Ethics Policy.

Supervisor signature:
Date: 28/5/2015
**Research Ethics Review Checklist**

Please include this completed form as an appendix to your thesis (see the Postgraduate Research Student Handbook for more information).

<table>
<thead>
<tr>
<th>Postgraduate Research Student (PGRS) Information</th>
<th>Student ID: UP712562</th>
</tr>
</thead>
<tbody>
<tr>
<td>PGRS Name: Matthew Dean Hull</td>
<td></td>
</tr>
<tr>
<td>Department: ICG</td>
<td>First Supervisor: Kazuya Koyama</td>
</tr>
<tr>
<td>Study Mode and Route: Part-time</td>
<td>MPhil</td>
</tr>
<tr>
<td></td>
<td>Full-time</td>
</tr>
<tr>
<td>Start Date: 01/10/2013 (or progression date for Prof Doc students)</td>
<td></td>
</tr>
<tr>
<td>Thesis Word Count: 30,000 (excluding ancillary data)</td>
<td></td>
</tr>
</tbody>
</table>

If you are unsure about any of the following, please contact the local representative on your Faculty Ethics Committee for advice. Please note that it is your responsibility to follow the University’s Ethics Policy and any relevant University, academic or professional guidelines in the conduct of your study.

Although the Ethics Committee may have given your study a favourable opinion, the final responsibility for the ethical conduct of this work lies with the researcher(s).

**UKRIO Finished Research Checklist:**

(If you would like to know more about the checklist, please see your Faculty or Departmental Ethics Committee rep or see the online version of the full checklist at: http://www.ukrio.org/what-we-do/code-of-practice-for-research/)

<table>
<thead>
<tr>
<th>Question</th>
<th>YES</th>
<th>NO</th>
</tr>
</thead>
<tbody>
<tr>
<td>a) Have all of your research and findings been reported accurately, honestly and within a reasonable time frame?</td>
<td>YES</td>
<td>NO</td>
</tr>
<tr>
<td>b) Have all contributions to knowledge been acknowledged?</td>
<td>YES</td>
<td>NO</td>
</tr>
<tr>
<td>c) Have you complied with all agreements relating to intellectual property, publication and authorship?</td>
<td>YES</td>
<td>NO</td>
</tr>
<tr>
<td>d) Has your research data been retained in a secure and accessible form and will it remain so for the required duration?</td>
<td>YES</td>
<td>NO</td>
</tr>
<tr>
<td>e) Does your research comply with all legal, ethical, and contractual requirements?</td>
<td>YES</td>
<td>NO</td>
</tr>
</tbody>
</table>

**Candidate Statement:**

I have considered the ethical dimensions of the above named research project, and have successfully obtained the necessary ethical approval(s).

**Ethical review number(s) from Faculty Ethics Committee (or from NRES/SCREC):** 687A-CFD1-12F5-36CF-605C-9F4D-05A9-8C3D

If you have not submitted your work for ethical review, and/or you have answered ‘No’ to one or more of questions a) to e), please explain below why this is so:

Signed (PGRS): [Signature]  
Date: 01/02/2017