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A NIEMYTZKY-TYCHONOFF THEOREM FOR ALL TOPOLOGICAL SPACES

ITTAY WEISS

Abstract. The classical Niemytzky-Tychonoff theorem characterises compactness of a metrisable topological space by means of the completeness of all of the metrics inducing the topology. Motivated by results of Kopperman and Flagg to the effect that every topological space is metrisable, as long as metrisability is suitably modified to allow the metric to take values more general than real numbers, we show that the Niemytzky-Tychonoff theorem remains true under this broader notion of metrisability, thus obtaining a metric characterisation of compactness valid for all topological spaces.

1. Introduction

For a topological space (X, τ), we say that a metric $d\colon X \times X \to [0, \infty]$ is *compatible* if its induced open ball topology is τ.

Theorem 1.1 (Niemytzki-Tychonoff, 1928, [9]). *A metrisable topological space is compact if, and only if, it is complete in every compatible metric.*

It was popularised in [6] that if the metric function is allowed to take values in structures more general than the non-negative reals, and if the metric axioms are slightly relaxed, then every topological space is metrisable. In the literature, this result can be obtained in (at least) two ways, depending on the axioms defining the codomain of the metric function. Consequently, it is natural to contemplate the validity of the classical

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Niemytzki-Tychonoff theorem under this broader notion of metrisability. The aim of this work is to prove that it holds verbatim, namely, since the adjective 'metrisable' becomes redundant, we present a suitable notion of the meaning of completeness and of compatibility so that the following holds.

Theorem 1.2. *A topological space is compact if, and only if, it is complete in every compatible metric.*

Versions of the Niemytzky-Tychonoff theorem, given in chronological order of appearance, can be found in [1] for uniform spaces, in [12] for quasi-uniform spaces, in [10] for quasi-guage spaces, and in [3, 7, 11] for pseudo-quasi-metric spaces, i.e., for $[0, \infty]$-valued metric spaces.

2. Value quantale preliminaries

The study of metric spaces as categories enriched in the non-negative extended real numbers $[0, \infty]$ was initiated by Lawvere in [8] and influenced signfinicant further research of a quantalic enrichment treatment of topology, a project involving numerous authors which is, in some sense, summarised by [4].

Coming from somewhat different perspectives, Kopperman in [5] introduced value semigroups (as part of a study of the first order properties of topological spaces), and Flagg in [2] introduced value quantales (as part of studies in domain theory), each leading to a definition of continuity space, namely a metric space valued in, respectively, a value semigroup or a value quantale. We briefly recount the details, following Flagg.

A *value quantale* $(L, +)$ is a complete lattice L, with $0 \neq \infty$, together with an associative and commutative binary operation $+$ on L, such that the conditions

- $a + 0 = a$, for all $a \in L$
- $a = \bigwedge\{b \in L \mid b \succ a\}$, for all $a \in L$
- $a + \bigwedge S = \bigwedge a + S$, for all $a \in L$ and $S \subseteq L$
- $a \wedge b \succ 0$ for all $a, b \in L$ with $a, b \succ 0$

hold. Here 0 is the least element in L (and ∞ is the largest element), $a + S = \{a + s \mid s \in S\}$, and the meaning of $b \succ a$ is that whenever $a \geq \bigwedge T$, for a subset $T \subseteq L$, there exists $t \in T$ with $b \geq t$.

Of the elementary properties regarding value quantales, to be found in [2], we shall make use of:

- If $\varepsilon \succ 0$ and $\bigwedge S = 0$, then there exists $s \in S$ with $\varepsilon \succ s$.
- For all $\varepsilon \succ 0$ there exists $\delta \succ 0$ with $\delta + \delta \prec \varepsilon$.
- If $b \succ a$, then there exists $\delta \succ 0$ with $b \succ a + \delta$.
- Either one of $a \leq b \prec c$ or $a \prec b \leq c$ implies $a \prec c$.

Flagg then defines a *continuity space* (X, L, d) to be a set X, a value quantale L, and a function $d\colon X \times X \to L$ satisfying $d(x,x) = 0$ and $d(x,z) \leq d(x,y) + d(y,z)$, for all $x, y, z \in X$. We shall call (X, L, d) a *metric space valued in* L. For such an L-valued metric space (X, L, d), an $\varepsilon \succ 0$ in L, and $x \in X$, the *open ball* of radius ε and centre x is the set $\mathrm{B}_\varepsilon(x) = \{y \in X \mid d(x,y) \prec \varepsilon\}$. Declaring a set $S \subseteq X$ to be *open* if for all $x \in S$ there exists $\varepsilon \succ 0$ in L with $\mathrm{B}_\varepsilon(x) \subseteq S$, it is easily seen that every open ball is an open set, and that the open sets constitute a topology, called the *open ball topology*, denoted by $\mathcal{O}(d)$.

For the metrisation result it is convenient to introduce, for every set S, the value quantale $\Omega(S)$, described in [2, Example 1.1, Example 2.8], as follows. $\Omega(S) = \{a \subseteq \Downarrow S \mid A \in a \implies \Downarrow A \subseteq a\}$, where, for all sets A, $\Downarrow A$ denotes the set of all finite subsets of A. The ordering on $\Omega(S)$ is by reverse set inclusion and $+$ is given by intersection. Clearly then, $0 = \Downarrow S$, $\infty = \emptyset$, and $\bigwedge = \bigcup$. It is straightforward to show that $\Omega(S)$ is a value quantale. A particular property of it which we require is the following.

Lemma 2.1. *For all sets S, an element $a \in \Omega(S)$ satisfies $a \succ 0$ if, and only if, there exists a finite set $F \subseteq S$ with $a \geq \Downarrow F$.*

Proof. Noticing that $F \in \Downarrow F$ for all $F \in \Downarrow S$, it follows that $\bigwedge_{F \in \Downarrow S} \Downarrow F = \Downarrow S = 0$, and thus if $a \succ 0$, then $a \geq \Downarrow F$ for some $F \in \Downarrow S$. Conversely, if $a \geq \Downarrow F$ for a given $F \in \Downarrow S$, and $\bigwedge T = 0$ for some $T \subseteq \Omega(S)$, then $F \in \Downarrow S = \bigcup T$, so that $F \in t$ for some $t \in T$. But then $\Downarrow F \subseteq t$, namely $t \leq \Downarrow F \leq a$, as is required to show that $a \succ 0$. □

If (X, τ) is a topological space, then taking $L = \Omega(\tau)$ and setting $d(x,y) = \Downarrow\{U \in \tau \mid x \in U \implies y \in U\}$ yields an L-valued metric space with $\mathcal{O}(d) = \tau$, which is Flagg's metrisability result [2, Theorem 4.15]. For our purposes we require a slight generalisation. For a basis $\mathcal{B}$ for a topological space (X, τ), and points $x, y \in X$, let $\mathcal{B}_{x \to y} = \{B \in \mathcal{B} \mid x \in B \implies y \in B\}$.

Theorem 2.2. *Let (X, τ) be a topological space and $\mathcal{B}$ a basis for it. Then the function $d\colon X \times X \to \Omega(\mathcal{B})$ given by $d(x,y) = \Downarrow(\mathcal{B}_{x \to y})$ is a metric function and $\mathcal{O}(d) = \tau$.*

Proof. Clearly $\mathcal{B}_{x \to x} = \mathcal{B}$, and so $d(x,x) = \Downarrow \mathcal{B} = 0$. For the triangle inequality, given $x, y, z \in X$, it is clear that $\mathcal{B}_{x \to y} \cap \mathcal{B}_{y \to z} \subseteq \mathcal{B}_{x \to z}$, and thus $\Downarrow \mathcal{B}_{x \to z} \supseteq \Downarrow(\mathcal{B}_{x \to y} \cap \mathcal{B}_{y \to z}) = \Downarrow \mathcal{B}_{x \to y} \cap \Downarrow \mathcal{B}_{y \to z}$, as required.

It is easily seen that generally in $\Omega(\mathcal{B})$, if F is a finite subset of $\mathcal{B}$, then $\Downarrow F \succ \Downarrow F$ (rendering computations easier, since showing $a \leq \Downarrow F$ immediately implies the stronger claim $a \prec \Downarrow F$). Note that if $F = \{B_1, \ldots, B_n\} \subseteq \mathcal{B}$, and $x \in B_1 \cap \cdots \cap B_k$ while $x \notin B_{k+1} \cup \cdots \cup B_n$, then, for all $y \in X$, $F \subseteq \mathcal{B}_{x \to y}$ precisely when $y \in B_1 \cap \cdots \cap B_k$. In other words, $\mathrm{B}_{\Downarrow F}(x) = B_1 \cap \cdots \cap B_k$. We now deduce that $\mathcal{O}(d) \subseteq \tau$, since if $x \in U$ and U is open in the open ball topology, then $B_\varepsilon(x) \subseteq U$, for some $\varepsilon \succ 0$. Then $\varepsilon \succ \Downarrow F$, for some $F = \{B_1, \dots, B_n\} \subseteq \mathcal{B}$ as above, and then $B_1 \cap \cdots \cap B_k = \mathrm{B}_{\Downarrow F}(x) \subseteq \mathrm{B}_\varepsilon(x) \subseteq U$, so $U \in \tau$. For the reverse inclusion, given an open set $U \in \tau$ and an arbitrary $x \in U$, let $B \in \mathcal{B}$ satisfy $x \in B \subseteq U$. Then $\Downarrow\{B\} \succ 0$ and $\mathrm{B}_{\Downarrow\{B\}}(x) = B \subseteq U$, so $U \in \mathcal{O}(d)$. $\square$

A convenient result is [2, Theorem 4.6]: a set $C \subseteq X$ in an L-valued metric space is closed in the induced open ball topology if, and only if, $d(x, C) = 0$, where $d(x, C) = \bigwedge_{y \in C} d(x, y)$. In particular, the closure of a set S is $\overline{S} = \{x \in X \mid d(x, S) = 0\}$.

Remark 2.3. We note that this last result does not hold within Kopperman's notion of continuity spaces valued in value semigroups, and that more generally this result, which is crucial to the arguments below, is not automatically true in the quantalic approach to topology.

In [13] the above metrisation result is shown to extend functorially to yield an equivalence of categories between the category **Top** of topological spaces and the category $\mathbf{Met_c}$ of all L-valued metric spaces, where L varies over all value quantales, with morphisms $f\colon (X, L, d) \to (X', L', d)$ the *continuous functions* in the usual sense: $f\colon X \to Y$ is continuous if for all $x \in X$ and $\varepsilon \succ 0$ in L' there exists $\delta \succ 0$ in L such that $d(fx, fy) \prec \varepsilon$ for all $y \in X$ with $d(x, y) \prec \delta$. The equivalency is given by the induced open ball topology functor $\mathcal{O}\colon \mathbf{Met_c} \to \mathbf{Top}$, which is surjective on objects. In particular, L-valued metric spaces are models for topology of theoretical equal strength as the usual models in terms of abstract open sets. The practical utility of the metric formalism is seen in [14, 15] where, respectively, connectedness is treated in a unified fashion and various topological invariants are constructed through a metric mechanism. This work is another result toward establishing the metric approach for topology as a convenient general formalism.

Definition 2.4. Let (X, τ) be a topological space. The class $\mathcal{O}^{-1}(X, \tau) = \{(X, L, d) \in \mathbf{Met_c} \mid \mathcal{O}(d) = \tau\}$ is the class of metrics *compatible* with τ.

Note that the value quantale L is not fixed! Typically, every topological space admits many pairs of isomorphic copies of compatible metrics.

Finally, to prepare for the proof of the Niemytzky-Tychonoff theorem, we require the following definition.

Definition 2.5. Let (X, L, d) be an L-valued metric space. For a subset $S \subseteq X$, its *diameter* is defined to be $\mathrm{diam}(S) = \bigvee_{x,y \in S} d(x, y)$. A filter $\mathcal{F}$ on X is *Cauchy* if for all $\varepsilon \succ 0$ in L there exists $F \in \mathcal{F}$ with $\mathrm{diam}(F) \leq \varepsilon$. The filter $\mathcal{F}$ *converges* if there exists $x \in X$ such that $\mathrm{B}_\varepsilon(x) \in \mathcal{F}$, for all $\varepsilon \succ 0$. (X, L, d) is *complete* if every Cauchy filter in X converges.

3. The proof

We first show that if (X, τ) is compact and (X, L, d) is a compatible L-valued metric on X, then (X, L, d) is complete, for which we assume a Cauchy filter $\mathcal{F}$ is given. Since $\mathcal{F}$, being a filter, satisfies the finite intersection property, compactness implies the existence of a point $x \in X$ common to all $\overline{F}$, $F \in \mathcal{F}$, and we claim that $\mathcal{F}$ converges to x. Indeed, given $\varepsilon \succ 0$ in L let $\delta \succ 0$ with $\delta + \delta \prec \varepsilon$, and let $F \in \mathcal{F}$ satisfy $\operatorname{diam}(F) \leq \delta$. Since $x \in \overline{F}$ we have that $d(x, F) = 0 \prec \delta$ and thus $d(x, y) \prec \delta$, for some $y \in F$. Then, for all $z \in F$, we have $d(x, z) \leq d(x, y) + d(y, z) \leq \delta + \delta \prec \varepsilon$. It follows that $F \subseteq \mathrm{B}_\varepsilon(x)$, as required.

For the converse implication, assume that (X, τ) is a topological space, and that $\mathcal{U}$ is a cover of X by open sets admitting no finite sub-covering. We may assume that for all $V \in \tau$, if $V \subseteq U$ for some $U \in \mathcal{U}$, then $V \in \mathcal{U}$, so in particular $\mathcal{U}$ is a basis for τ. We may further assume that $\mathcal{U}$ is closed under finite unions. Then $\hat{\mathcal{U}} = \{\hat{U} \mid U \in \mathcal{U}\}$, with $\hat{U} = X \setminus U$, is a filter base, and let $\mathcal{F}$ be the generated filter. Let $L = \Omega(\mathcal{U})$, and define $d(x, y) = \Downarrow\mathcal{U}_{x \to y}$, where $\mathcal{U}_{x \to y} = \{U \in \mathcal{U} \mid x \in U \implies y \in U\}$. By Theorem 2.2 (X, L, d) is a metric space compatible with (X, τ). Further, we show that $\mathcal{F}$ is Cauchy, for which we fix $\varepsilon \succ 0$. By Lemma 2.1 $\varepsilon \geq \Downarrow\{U_1, \ldots, U_n\}$ for some finitely many $U_1, \ldots, U_n \in \mathcal{U}$ and let $U = U_1 \cup \cdots \cup U_n$. Then $\hat{U} \in \mathcal{F}$, and we claim that $\operatorname{diam}(\hat{U}) \leq \varepsilon$. Given $x, y \in \hat{U}$, it suffices to show that $d(x, y) \leq \Downarrow\{U_1, \ldots, U_n\}$, namely that $\{U_1, \ldots, U_n\} \in d(x, y)$. But for each $1 \leq k \leq n$, clearly $x \notin U_k$, and thus $U_k \in \mathcal{U}_{x \to y}$, so that $\{U_1, \ldots, U_n\} \subseteq \mathcal{U}_{x \to y}$ and the claim follows.

It now follows by assumption that $\mathcal{F}$ converges, say to x. Fix $F \in \mathcal{F}$. Then, for all $\varepsilon \succ 0$, $F \cap \mathrm{B}_\varepsilon(x) \neq \emptyset$, so that $d(x, F) \leq \varepsilon$, and as $\varepsilon \succ 0$ is arbitrary it follows that $d(x, F) = 0$. In particular, for all $U \in \mathcal{U}$, $d(x, \hat{U}) = 0$, and since $\hat{U}$ is closed, we conclude that $x \in \hat{U}$. But then $x \notin \bigcup \mathcal{U}$, a contradiction.

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Department of Mathematics; University of Portsmouth, PO1 3HF, UK
E-mail address: ittay.weiss@port.ac.uk